Quantum HadroDynamics and applications to nuclear matter.

Dimitris Stefanopoulos

Aristotle University of Thessaloniki

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Overview

1. Nuclear Physics
   - History and Motivation
   - Models and Observables
2. Classical fields
   - Noether Theorem
   - Energy-Momentum Tensor
3. Quantum fields
   - Relativistic Quantum Mechanics
     - Klein-Gordon
     - Dirac
   - Canonical Quantization
     - Dirac,Klein-Gordon,Maxwell fields
4. Quantum Hadrodynamics
   - The Lagrangian
   - Relativistic Mean Field Approximation
   - Expectation values and observables

Nuclear Physics

History:

- Discovery of the nucleus(Rutherford 1911 Gold foil experiment.)
- Applications of Quantum Mechanics →Atomic and Nuclear Physics
- Different models describe different properties(Shell Model,Liquid Drop Model)
- Interaction via Meson(Yukawa) → Quantum field description for interactions.
- Chromodynamics(Strong Nuclear Force) is too complicated at hadron scales → Quantum Hadrodynamics Model(Walecka 1974) is an Effective Field Theory
- We can't use the same approximation techniques as in other quantized theories(perturbative approximation of the coupling strenghts will diverge)→ Relativistic Mean Field Approximation

Nuclear Physics

From the Liquid Drop Model to phenomenology:

- Nuclear matter as an idealized system stems from the liquid drop model. These are the properties that we will consider as observables.
- Saturation density($\sim 0.15 fm^{-3}$): It follows from the fact that the strong force is attractive and short ranged in general, but it becomes repulsive at $< 0.4 fm$.(At this density the system is stable with $P = 0$)
- Binding energy per nucleon($\sim 15 MeV$): the energy expended or required to form a system. At saturation density the binding energy is in its most stable state.
- Symmetry energy($\sim 30 MeV$): The symmetry energy is defined by a Taylor series expansion of the energy density in terms of the asymmetry $(N - Z)/A$. By expanding around saturation density ρ_0, the symmetry energy can be expressed as:

$$a_4 = \frac{1}{2}\left(\frac{\partial^2}{\partial t^2}\frac{\epsilon}{\rho}\right)_{t=0} \quad \left(t = \frac{\rho_n - \rho_p}{\rho}\right) \tag{1}$$

Classical Fields

- Usually the Lagrangian density of the field is of the form:

$$\mathcal{L}(\phi_a(x^\mu), \partial_\nu \phi_a(x^\mu)) \tag{2}$$

while the equations of motion can be calculated by the Euler-Lagrange equations(least action principal):

$$\partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi_a)} \right) - \frac{\partial \mathcal{L}}{\partial \phi_a} = 0 \tag{3}$$

- As in particle mechanics we find the conjugate momentum:$\pi^a = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_a}$ and the Hamiltonian density: $\mathcal{H} = \pi^a \dot{\phi}_a - \mathcal{L}$
- Noether's theorem: Every continuous symmetry of the Lagrangian gives rise to a conserved current j^μ such that: $\partial_\mu j^\mu = 0$
- Translational symmetry $x^\nu \to x'^\nu = x^\nu + \epsilon^\nu$ gives rise to the energy momentum tensor:

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial^\nu \phi_a - \eta^{\mu\nu} \mathcal{L}. \tag{4}$$

Relativistic quantum mechanics

We are trying to create a relativistic Lorentz invariant version of the Schrodinger equation with the use of the Einstein energy formula:

$$\hat{E} = \pm\sqrt{-\hbar^2 c^2 \nabla^2 + m^2 c^2} \tag{5}$$

- If we square both sides of the equation we find:

  $$(\partial_\mu \partial^\mu + m^2)\phi(x^\nu) = 0 \tag{6}$$

  which is the Klein-Gordon equation that describes spinless bosons. It accepts plane-wave solutions: $\phi(x^\mu) = Ne^{i(\vec{k}\cdot\vec{r}-\omega t)}$
- If we don't, we demand that:

  $$\sqrt{p^2 + m^2} \rightarrow \mathbf{a} \cdot \mathbf{p} + \beta m \tag{7}$$

  and we find that the solution is an N-component spin wave function. The simplest combination of a and b that satisfies the demand is:

  $$a_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \ \beta = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \tag{8}$$

  meaning N=4. So the Dirac spinor has 4 components while one possible solution is:

  $$\psi(x) = \omega e^{-ip^\mu x_\mu} \tag{9}$$

Relativistic quantum mechanics

By defining the γ-matrices:

$$\begin{aligned} \gamma^0 &= \beta \\ \gamma^i &= \beta a^i = \gamma^0 a^i \end{aligned} \tag{10}$$

we can write in a covariant form the DIrac equation:

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0 \tag{11}$$

- The important thing is that for both equations and plane wave solutions we have negative energy solutions.
- Dirac Interpretation: To prevent positive energy particles from spontaneously decaying to negative energy states he postulated that in the vacuum state all the negative energy states are filled (Dirac sea).
- Later Feynman interpreted the negative energy as positive energy particle propagating backwards in time or as an anti-particle propagating forward in time.
- Note that the phase symmetry $\psi \to e^{-ia}\psi$ gives rise to the current $j^\mu = \bar{\psi}\gamma^\mu\psi$ with the time like component:$\rho = \bar{\psi}\gamma^0\psi = \psi^\dagger\psi$

Canonical quantization

- We write down a classical Lagrangian density in terms of fields. This the creative part.
- We calculate the momentum density and work out the Hamiltonian Density in terms of field.
- Now treat the fields and the momentum density as operators. Impose commutation relations on them to make them quantum mechanical.
- Expand the fields in terms of creation/annihilation operators in order to use occupation numbers.
- The theory is ready(normal ordering)

The Dirac field

- The Dirac Lagrangian is:

$$\mathcal{L} = \bar{\psi}(x)[i\gamma^{\mu}\partial_{\mu}\psi(x) - m]\psi(x) \tag{12}$$

- The conjugate field will be:

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i\psi^{\dagger} \tag{13}$$

while the momentum density:

$$\mathcal{H} = \psi^{\dagger}(-i\boldsymbol{a} \cdot \boldsymbol{\nabla}) + \beta m)\psi \tag{14}$$

- We promote spinors into field operators and we impose the equal-time anticommutation relations to ensure the correct spin-statistics are satisfied:

$$\begin{array}{ccc} \psi(x) & \longrightarrow & \hat{\psi}(x) \\ \psi^{\dagger}(x) & \longrightarrow & \hat{\psi}^{\dagger}(x), \end{array} \tag{15}$$

$$\begin{array}{ccc} \left\{\hat{\psi}_a(t,\boldsymbol{x}), \hat{\psi}_b^{\dagger}(t,\boldsymbol{x'})\right\} & = & \delta_{ab}\delta(\boldsymbol{x} - \boldsymbol{x'}) \\ \left\{\hat{\psi}_a(t,\boldsymbol{x}), \hat{\psi}_b(t,\boldsymbol{x'})\right\} & = & \left\{\hat{\psi}_a^{\dagger}(t,\boldsymbol{x}), \hat{\psi}_b^{\dagger}(t,\boldsymbol{x'})\right\} = 0 \end{array} \tag{16}$$

- We expand the fields in terms of creation/annihilation operators:

$$\hat{\psi}(t,\boldsymbol{x}) = \sum_s \int \frac{d\boldsymbol{p}}{(2\pi)^{3/2}} \sqrt{\frac{m}{\omega_p}} \left(\hat{b}(p,s)\ u(p,s)\ e^{-i\boldsymbol{p}\cdot\boldsymbol{x}} + \hat{d}^{\dagger}(p,s)\ v(p,s)\ e^{+i\boldsymbol{p}\cdot\boldsymbol{x}}\right) \tag{17}$$

Our fields

The reason we mentioned all of these is to work with Lagrangian Densities as ansatz.

- The relativistic real scalar mass fields are described by:

$$\mathcal{L} = \frac{1}{2}\partial^{\mu}\phi(x)\partial_{\mu}\phi(x) - \frac{1}{2}m^2\phi(x)^2 \tag{18}$$

- The Dirac (spin 1/2) particles are governed by

$$\mathcal{L} = i\psi^{\dagger}(x)\frac{\partial}{\partial t}\psi(x) + i\psi^{\dagger}(x)a \cdot \nabla\psi(x) - m\psi^{\dagger}(x)\beta\psi(x) \tag{19}$$

where $\psi^{\dagger}$ is the conjugate field while with the γ-matrices can be written as:

$$\mathcal{L} = \bar{\psi}(x)[i\gamma^{\mu}\partial_{\mu}\psi(x) - m]\psi(x) \tag{20}$$

- Finally we will use the massive vector boson field Langrangian:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_{\mu}A^{\mu} - j_{\mu}A^{\mu} \tag{21}$$

which describes spin one bosons.

QHD-I or $\sigma - \omega$ model

Assumptions:

- Three fields: Baryon, Scalar Meson, Vector Meson.
- No charged mesons are included(i.e. electric properties of the baryons are not considered)
- The masses of the proton and the nutron are taken to be equal.
- The scalar mesons couple to the scalar density of the baryon field and the vector mesons couple to the conserved baryon current.

$$\begin{aligned}\mathcal{L} =& \bar{\psi}(x)\Big[\gamma_\mu\Big(i\partial^\mu - g_v V^\mu(x)\Big) - \Big(M - g_s\phi(x)\Big)\Big]\psi(x) \\ &+ \frac{1}{2}\Big(\partial_\mu\phi(x)\partial^\mu\phi(x) - m_s^2\phi^2(x)\Big) - \frac{1}{4}V_{\mu\nu}V^{\mu\nu} + \frac{1}{2}m_w^2 V_\mu(x)V^\mu(x)\end{aligned} \tag{22}$$

- V^μ denotes the vector meson field.
- ϕ denotes the scalar field.
- m_w, m_s are the different meson masses and M is the nucleon mass
- g_s,g_v are the vector and the scalar coupling constants
- $V_{\mu\nu} = \partial_\mu V_\nu(x) - \partial_\nu V_\mu(x)$

QHD-I or $\sigma - \omega$ model

From the Euler-Lagrange equations follows:

$$
\begin{aligned}
\partial_\mu \partial^\mu \phi(x) + m_s^2 \phi(x) &= g_s \bar{\psi}(x)\psi(x) && (23)\\
\partial_\mu V^{\mu\nu} + m_w^2 V^\nu(x) &= g_v \bar{\psi}(x)\gamma^\nu \psi(x) && (24)\\
\Big[\gamma_\mu\big(i\partial^\mu - g_v V^\mu(x)\big) - \big(M - g_s\phi(x)\big)\Big]\psi(x) &= 0 && (25)
\end{aligned}
$$

It's obvious that these equations can not be solve analytically. However we can motivate the relativistic mean-field theory approximation.

Relativistic Mean field

Assumptions:

- We have B baryons occupying a box of volume V
- $T = 0$ and the system is considered static.

Implications:

- Since the system is static, the baryon flux given by $\bar{\psi}(x)\gamma^i\psi(x)$ will be zero.
- If the baryon density B/V is increased the source terms will become large. If the source terms are large we can approximate the meson field operators by their ground state expectation values.
- In order to do that we have to replace the meson field operators with their ground state expectation values.

$$\phi \mapsto \langle\Phi|\phi|\Phi\rangle = \langle\phi\rangle = \phi_0, V_\mu \mapsto \langle\Phi|V_\mu|\Phi\rangle = \langle V_\mu\rangle = \delta_{\mu 0}V_0 \quad (26)$$

- Thus the fields are now constants independent of space and time.

Relativistic Mean Field

Finally we have:

$$m_s^2 \phi_0 = g_s \langle \bar{\psi} \psi \rangle \tag{27}$$

$$m_w^2 V_0 = g_v \langle \bar{\psi} \gamma_0 \psi \rangle \tag{28}$$

$$\Big[i\gamma_\mu \partial^\mu - g_v \gamma_0 V_0 - \big(M - g_s \phi_0 \big) \Big] \psi = 0 \tag{29}$$

$$\mathcal{L}_{RMF} = \bar{\psi} \big[i\gamma_\mu \partial^\mu - g_v \gamma^0 V_0 - (M - g_s \phi_0) \big] \psi = \frac{1}{2} m_s^2 \phi_0^2 + \frac{1}{2} m_w^2 V_0^2 \tag{30}$$

So now we can easily calculate:

$$\epsilon = \langle T^{00} \rangle = \langle i \bar{\psi} \gamma_0 \partial_0 \psi \rangle + \frac{1}{2} m_s^2 \phi_0^2 - \frac{1}{2} m_w^2 V_0^2 \tag{31}$$

$$P = \frac{1}{3} \langle T^{ii} \rangle = \frac{1}{3} \langle i \bar{\psi} \gamma^i \partial_i \psi \rangle - \frac{1}{6} m_s^2 \phi_0^2 + \frac{1}{6} m_w^2 V_0^2 \tag{32}$$

Expectation values

Since $\bar{\psi}$ and ψ are still operatores their expectation values have to be calculated to obtain sensible information.
We have a static system of fermions thus we can expand ψ as we did at the free single-particle solution for a Dirac particle.
From simple algebra we get:

$$\left\langle \psi^\dagger \psi \right\rangle = \rho = \frac{\lambda}{6\pi^2} k_F^3, \tag{33}$$

$$\langle \bar{\psi}\psi \rangle = \frac{\lambda}{2\pi^2} \int_0^{k_F} dk \; \frac{k^2 m^*}{\sqrt{k^2 + m^{*2}}} \tag{34}$$

$$\left\langle \psi^\dagger(-i\boldsymbol{a} \cdot \boldsymbol{\nabla} + \beta m^* + g_v V_0)\psi \right\rangle = \frac{\lambda}{(2\pi)^3} \int_0^{k_F} d\boldsymbol{k} \; \sqrt{\boldsymbol{k}^2 + m^{*2}} + g_v V_0 \rho, \tag{35}$$

$$\left\langle \psi^\dagger(-i\boldsymbol{a} \cdot \boldsymbol{\nabla})\psi \right\rangle = \frac{\lambda}{(2\pi)^3} \int_0^{k_F} d\boldsymbol{k} \; \frac{\boldsymbol{k}^2}{\sqrt{\boldsymbol{k}^2 + m^{*2}}} \tag{36}$$

in which we have calculated the integral to the Fermi momentum upper bound while($m^* = M - g_s\phi_0$).

State equation and observables

We easily substitute:

$$\epsilon = \frac{1}{2} m_s^2 \phi_0^2 + \frac{1}{2} m_w^2 V_0^2 + \frac{\lambda}{2\pi^2} \int_0^{k_F} dk \; k^2 \; \sqrt{k^2 + m^{*2}} \tag{37}$$

$$P = -\frac{1}{2} m_s^2 \phi_0^2 + \frac{1}{2} m_w^2 V_0^2 + \frac{1}{3} \left(\frac{\lambda}{2\pi^2} \int_0^{k_F} dk \; \frac{k^4}{\sqrt{k^2 + m^{*2}}} \right) \tag{38}$$

which are the energy density and the numerical density of the system.

- Having these, we can easily again calculate: the symmetry energy:

$$a_4 = \frac{k_F^2}{6\sqrt{k_F^2 + m^{*2}}} \tag{39}$$

  at saturation density: $\rho_0 = \frac{2}{\pi^2} \int_0^{k_F} dk \; k^2 = \frac{2k_F^3}{3\pi^2}$ and of course the binding energy per nucleon.

Thanks!