Value At Risk Using Historical and Monte Carlo Simulation

1st Shivam Agarwal
201601062 DA-IICT
Gandhinagar, Gujarat
201601062@daiict.ac.in

2nd Shikhar Dubey
201601144 DA-IICT
Gandhinagar, Gujarat
201601144@daiict.ac.in

***Abstract*—In our B.Tech Project, we have used two different approaches to find Value At Risk (VaR) of a portfolio. The portfolio consists of four different stocks. The two approaches used are namely Historical simulation and Monte Carlo simulation using Geometric Brownian motion.**

***Index Terms*—stock, portfolio, value at risk (VaR)**

I. Introduction

A financial institution usually calculates delta, gamma, and vega for describing different aspects of the risk in a portfolio of derivatives.They calculate each of these measures each day for every market variable to which it is exposed. A very large number of different risk measures are produced each day by this delta gamma vega analysis. These risk analysis might provide valuable information for the financial traders but they do no provide them with total risk of the portfolio to which the institution is exposed.

Value at Risk (VaR) provides a single number summarizing the total risk in a portfolio of financial assets. It is widely used by financial institutions, corporate treasurers, and fund managers. There are two main approaches for calculating VaR. These are known as the historical simulation approach and the model-building approach. The model-building approach can be implemented using Monte Carlo simulation to generate the probability distribution.

In our B.Tech project we first explored the the approaches to calculate Value at risk and then implemented them associated with the portfolio of 4 different stocks namely Apple, Facebook, Citigroup and Walt Disney by using both approaches: Historical Simulation and Monte Carlo simulation. In the Monte Carlo simulation, we have assumed the stock prices follow a Geometric Brownian motion.

II. Value At Risk

When an analyst uses Value At Risk (VaR), he uses the statement of the type:

"I am X percent certain there will not be a loss of more than V dollars in the next N days."

The variable V is the VaR of the portfolio. It is a function of two parameters: the time horizon (N days) and the confidence level (X percent). It is the loss level over N days that has a probability of only (100 - X)% of being exceeded.

Over the next N days, where N days is the time horizon and X% is the confidence level, VaR is the loss corresponding to the (100 - X)th percentile of the distribution of the gain in the value of the portfolio .

III. Measures to calculate VaR

A. Historical Simulation

In Historical Simulation way of calculating VaR, the data from the past is used to find the future of a stock. Past data is used to create different scenarios of what can happen on the coming day. Then the VaR from these scenarios are calculated and ranked in descending order of losses. The Value at risk of the portfolio is calculated by using the confidence level.

Suppose that we want to calculate VaR for a portfolio using a one-day time horizon, a 99% confidence level, and 101 days of data. Data are collected over the most recent 101 days in the market. This will provide 100 alternative scenarios for what can happen between today and tomorrow. Denote the first day for which we have data as Day 0, the second day as Day 1, and so on. Scenario 1 is where the percentage changes in the values of all variables are the same as they were between Day 0 and Day 1, Scenario 2 is where they are the same as between Day 1 and Day 2, and so on. For each scenario, the dollar change in the value of the portfolio between today and tomorrow is calculated. This defines a probability distribution for daily loss (gains are negative losses) in the value of our portfolio. The 99th percentile of the distribution can be estimated as the second highest loss.

Define v_{i} as the value of a market variable on Day i and then suppose that today is Day n. The value in i^{th} scenario of the historical simulation approach assumes that the value of the market variable tomorrow will be

$$Value\ under\ i_{\mathrm{th}}\ scenario\ =\ v_{\mathrm{n}} \times \frac{v_{\mathrm{i}}}{v_{\mathrm{i}} - 1} \tag{1}$$

This is algebraic approach to calculate the value in different scenarios and then to finally find the Value at Risk.

B. Model Building Approach

Model Building approach is the the other method to calculate Value at risk. In this method we will use Daily volatilities. So before discussing this approach I will discuss Daily Volatilities.

1) Daily Volatilities: The volatility of an asset are usually quoted as a "volatility per year" but in the model-building approach to calculate VaR for market risk, we usually use time measured in days and then the volatility of an asset is usually quoted as a "volatility per day."

σ_{year} is defined as the volatility of year and σ_{day} is defined as the volatility of a day. Then supposing 252 working days in a year, daily volatility can be calculated as

$$\sigma_{\text{day}} \;=\; \frac{\sigma_{\text{year}}}{\sqrt{252}} \tag{2}$$

2) Single-Asset Case: We will discuss how to calculate VaR of a single stock using the model-building approach where the the position in a single stock: S$. We suppose that N is time horizon and X is confidence percent, so that we are interested in the loss level over N days that we are X% confident will not be exceeded. Initially, we consider a 1-day time horizon.

Assume volatility is σ_{day} per day. The standard deviation of daily changes in the value of the position is $\sigma_{\text{day}}\%$ of S.

We assume that the change is normally distributed. From the Excel NORMSINV function, we calculate,

$$k \;=\; N^{\text{-}1}(0.01 \times (100 - X)) \tag{3}$$

This means that there is a (100-X)% probability that a normally distributed variable will decrease in value by more than k standard deviations. The 1-day X% VaR for our portfolio consisting of a $S position is therefore

$$1 - day\ VaR \;=\; k \times standarddeviation \tag{4}$$

The N-day VaR is calculated as $\sqrt{N}$ times 1-day VaR.

3) Two-Asset Case: Now we will discuss how to calculate VaR of a single stock using the model-building approach. Let the two stock be X and Y. We will calculate the standard deviation σ_{X} and σ_{Y} respectively as discussed in single-asset case. Let the coefficient of correlation between them equal to ρ. Then the standard deviation of X + Y is given by

$$\sigma_{\text{X+Y}} \;=\; \sqrt{\sigma_{\text{X}}{}^2 \;+\; \sigma_{\text{Y}}{}^2 \;+ 2\rho\sigma_{\text{X}}\sigma_{\text{Y}}} \tag{5}$$

Now we can calculate 1-day VaR and N-day VaR of two assset case as was calculated in single asset case using $\sigma_{\text{X+Y}}$.

4) Linear Model: Now we will discuss about portfolio worth P consisting of n assets. The portfolio will have amount α_{i} being invested in asset i (1≤i≤n). Let $\Delta\text{x}_{\text{i}}$ be defined as the return on asset i in one day. Then α_{i} $\Delta\text{x}_{\text{i}}$ will be change in the value of our investment in asset i in one day.

Let change in the value of the whole portfolio in one day be ΔP. It can be written as:

$$\Delta\text{P} \;=\; \sum_{i=1}^{n} \alpha_{\text{i}}\Delta\text{x}_{\text{i}} \tag{6}$$

In this equation we have assumed $\Delta\text{x}_{\text{i}}$ as multivariate normal and hence ΔP is normally distributed. To calculate VaR it is to be assumed that expected value (mean) of each $\Delta\text{x}_{\text{i}}$ is zero which also means that mean of ΔP is also zero.

i_{th} asset have daily volatility as σ_{i} and assets i and j have between them coefficient of correlation as ρ_{ij}. And so standard deviation of $\Delta\text{x}_{\text{i}}$ is σ_{i} and coefficient of correlation between $\Delta\text{x}_{\text{i}}$and $\Delta\text{x}_{\text{j}}$ is ρ_{ij}.

Then we can write variance of ΔP which is σ_P^2 as

$$\sigma_P^2 \;=\; \sum_{i=1}^{n}\sum_{j=1}^{n} \rho_{\text{ij}}\alpha_{\text{i}}\alpha_{\text{j}}\sigma_{\text{i}}\sigma_{\text{j}} \tag{7}$$

This equation can also be written as

$$\sigma_P^2 \;=\; \sum_{i=1}^{n} \alpha_i^2\sigma_i^2 \;+\; \sum_{i=1}^{n}\sum_{j<i} \rho_{\text{ij}}\alpha_{\text{i}}\alpha_{\text{j}}\sigma_{\text{i}}\sigma_{\text{j}} \tag{8}$$

For change over N days, $\sigma_{\text{P}}\sqrt{N}$ is the standard deviation. For X percent confidence, VaR over N day horizon is given by

$$Var \;=\; k \times \sigma_{\text{P}}\sqrt{N} \tag{9}$$

where k is as described in equation (3).

Let $\omega_{\text{i}} = \alpha_{\text{i}}/\text{P}$ be portfolio's i^{th} investment weight. Then ΔP/P will be portfolio return in one day. Variance of this is given by:

$$variance \;=\; \sum_{i=1}^{n}\sum_{j=1}^{n} \rho_{\text{ij}}\omega_{\text{i}}\omega_{\text{j}}\sigma_{\text{i}}\sigma_{\text{j}} \tag{10}$$

C. Monte Carlo Simulation

To generate probability distribution for ΔP, Monte Carlo Simulation is an alternate to model building approach. For those processes in which probability of different outcomes cannot be predicted due to intervention of different variables, this model is used. Monte Carlo simulation lets you see all the possible outcomes of your decisions and assess the impact of risk, allowing for better decision making under uncertainty [3].

Monte Carlo simulation model is also referred to as multiple probability simulation. It can be used in almost every field such as science, finance, engineering, supply chain to tackle a range of problems. This model is used to estimate the probability of cost overruns and likelihood that an asset price will move in a certain way in large projects [2].

For any factor that has uncertainty, this simulation performs risk analysis by using the building models of possible results by substituting a range of values. Then this model calculates results over and over and each time using a different set of random values from the probability functions. Monte Carlo simulation could involve thousands or tens of thousands of recalculations before it is complete, depending upon the number of uncertainties and the ranges specified for them . Monte

Carlo simulation produces distributions of possible outcome values [3].

The procedure to calculate 1-day VaR for a portfolio using Monte Carlo Simulation is as follows:

- Value the portfolio today in the usual way using the current values of market variables.
- Sample once from the multivariate normal probability distribution of the Δx_i.
- Use the values of the Δx_i that are sampled to determine the value of each market variable at the end of one day.
- Revalue the portfolio at the end of the day in the usual way.
- Subtract the value calculated in Step 1 from the value in Step 4 to determine a sample ΔP.
- Repeat Steps 2 to 5 many times to build up a probability distribution for ΔP [1].

Δx is the percentage change in the stock price in 1 day.

The VaR is calculated as the appropriate percentile of the probability distribution of ΔP. For example, for 500 different samples of ΔP, 1-day 99% VaR is 5_{th} worst outcome of the value of ΔP.

D. Geometric Brownian Motion

Brownian motion is often used in corporate finance to explain the movement of asset prices. A common assumption for stock markets is that they follow Brownian motion, where asset prices are constantly changing often by random amounts.

Many people believe that stock market prices exhibit the theory of random walk. The idea of random walk theory is that stocks take a unpredictable and random path, and this make it nearly impossible to outperform the market without assuming additional risk. The Geometric Brownian Motion (GBM) model incorporates this idea of random walks in stock prices through its uncertain component, along with the idea that stocks maintain price trends over time as the certain component.

There are certain assumption in Geometric Brownian Motion model. These are:

- Returns over a short period are normally distributed.

$$\frac{\Delta \mathrm{S}}{\mathrm{S}} \sim \mathrm{N}(\mu \Delta \mathrm{t} \ \ \sigma^2 \Delta \mathrm{t}) \tag{11}$$

- Expected returns is independent of the stock price.

$$E(\frac{\Delta \mathrm{S}}{\mathrm{S}}) = \mu \Delta \mathrm{t} \tag{12}$$

- The volatility of the stock ($sigma$) is independent of the stock price.

The discrete time version of model is:

$$\frac{\Delta \mathrm{S}}{\mathrm{S}} = \mu \Delta \mathrm{t} + \sigma \mathrm{z} \sqrt{\Delta \mathrm{t}} \tag{13}$$

Here μ is mean, σ is volatility of stock and z is a randomly drawn number from a normal distribution with a mean of zero and a standard deviation of one, representing random volatility.

We can also calculate μ and σ by using historical stock data.

1) Estimating μ: We can use the fact that $E(\frac{\Delta S}{S} \backsimeq \mu)$ for estimating μ. The average of all $\Delta S/S$ values will give an estimate of μ.

2) Estimating σ: To estimate σ, define:

- n+1: number of observation
- S_i: Stock price at the end of interval i. i=0,1,2 and so on.
- τ: length of time interval in years

Then let,

$$\mu_i = \ln \frac{S_i}{S_{i-1}} \quad i = 1, 2, ... \tag{14}$$

$$s = \sqrt{\frac{1}{N-1} \sum_{i=1}^{n} (\mu_i - \bar{\mu})^2} \tag{15}$$

Then the estimate $\hat{\sigma}$ of σ is given by

$$\hat{\sigma} = \frac{s}{\sqrt{\tau}} \tag{16}$$

IV. COMPARING THE TWO APPROACHES[1]

To get to know the difference between Historical simulation and Monte Carlo Simulation using Geometric Brownian Motion, we did fin out the Var risk of a portfolio using both the approaches.

For calculating Value at Risk using Historical simulation:

- We imported the data of the portfolio.
- Then we calculated the daily returns.
- Then we sorted those returns in ascending order.
- Calculate the VaR for 90%, 95%, and 99% confidence levels using quantile function.

For calculating Value at Risk using Monte Carlo Simulation, we used the model of Geometric Brownian Motion, which is described in above section. Then after generating random trials, we did process the output. To use GBM model, we do need other parameters and steps to find VaR are as follows:

- We imported the data of the portfolio.
- Then we calculate the mean using the data.
- Then we calculate sigma for the portfolio.
- Now we simulate the stock prices using GBM formula as described in above section.
- Now we calculate and plot Value at Risk.

For our project, we did calculate the Value at Risk of a portfolio containing 4 companies. The companies which we used in portfolio are APPL (Apple Inc.), FB(Facebook Inc.), C (Citigroup), DIS (Walt Disney).

After comparing the plots we obtained by using both the approaches we observe the following:

- As we increase the Confidence Level Value, the difference between Value at Risk value we got for the portfolio using historical simulation and Monte Carlo Simulation

[1]Note: The code for the plots are uploaded and are available at https://github.com/jmulherkar/BTP2020/tree/master/ShikharDubey_ShivamAgarwal

using Geometric Brownian Motion model increases significantly.
- In historical simulation, the range of returns we got is very wide, while in Monte Carlo Simulation using Geometric Brownian Motion model the range of returns is very less as compared to historical simulation.
- At the same confidence level, the difference between VaR value we got for the portfolio using historical simulation and Monte Carlo Simulation using Geometric Brownian Motion model is significant and the gap only widens for value at risk for the following days.

As the difference in Value at risk using both approaches has big difference, it can be because return distribution is not normal. In the Geometric Brownian Motion model, we assumes that the market variables have a multivariate normal distribution but in real daily changes in market variables often have distributions with tails that are quite different from the normal distribution. While in historical simulation, historical data determine the joint probability distribution of the market variables.

The data for the portfolio is taken from Yahoo Finance.

V. Plots and Analysis

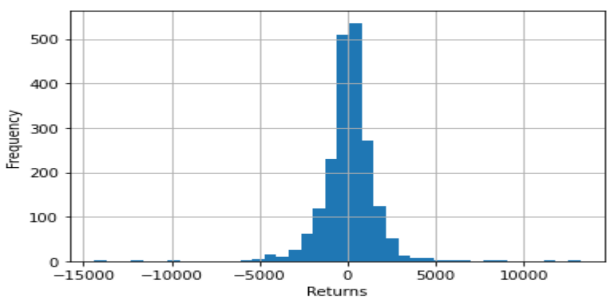

Fig. 1. Frequency vs Return graph obtained from Historical Simulation Method using the data of the portfolio from Yahoo finance starting from 1 Jan 2012 till present.

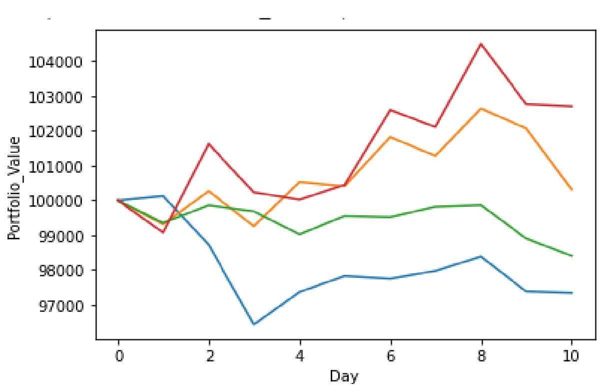

Fig. 2. Portfolio Value vs day graph obtained from Monte Carlo Simulation using Geometric Brownian Motion model using the data of the portfolio from Yahoo finance starting from 1 Jan 2012 till present.

The portfolio value in Fig. 2. has been obtained for 4 different simulations using random variables.

Confidence Level value	Hist_VaR	MC_VaR
90%	1484.68	1270.09
95%	2112.57	1669.88
99%	4173.52	2336.42

Fig. 3. For different confidence level, Value at Risk comparison for the portfolio using historical simulation and Monte Carlo Simulation using Geometric Brownian Motion model.

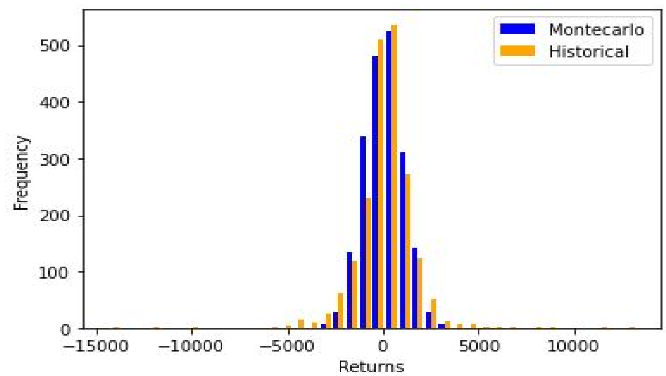

Fig. 4. Frequency vs Returns comparison of the portfolio using historical simulation and Monte Carlo Simulation using Geometric Brownian Motion model.

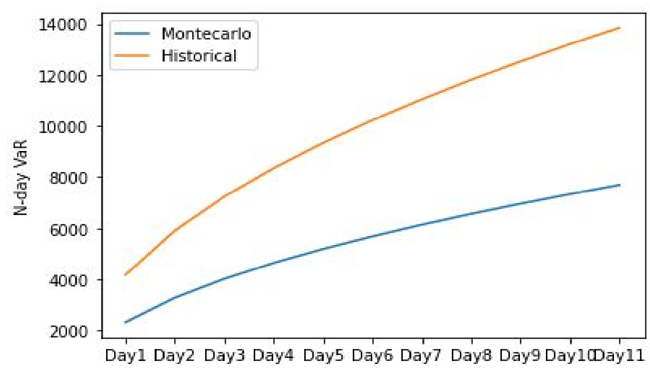

Fig. 5. N-day Value at risk comparison of the portfolio using historical simulation and Monte Carlo Simulation using Geometric Brownian Motion model.

Acknowledgement

We would like to thank Prof. Jaideep Mulherkar for his support and guidance throughout this project.

References

[1] J.C. Hull, Options, Futures, and other Derivatives, Ninth Edition

[2] W. Kenton, Monte Carlo Simulation Definition https://www.investopedia.com/terms/m/montecarlosimulation.asp

[3] https://www.palisade.com/risk/montecarlosimulation.asp

[4] K. Reddy, V. Clinton, Simulating Stock Prices Using Geometric Brownian Motion: Evidence from Australian Companies

[5] V. Singh, Calculating Value At Risk In Excel & Python https://blog.quantinsti.com/calculating-value-at-risk-in-excel-python/

[6] V. Divakar and R. Pachanekar, Random Walk: Introduction, GBM, Simulation https://blog.quantinsti.com/random-walk/

[7] D.R. Harper, How to use Monte Carlo simulation with GBM https://www.investopedia.com/articles/07/montecarlo.asp

[8] https://finance.yahoo.com/