Models for assymetric conflict with arbitrary beliefs: individual, group and sequential games

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Abstract

I model behaviour of players in an asymmetric conflict, where an aggressor attempts to acquire resources from the defender. I formulate a model for an inter-individual and an inter-group version of the conflict, as well as its sequential version. My model formulates the utility function of each individual not only as the function of his behaviour, but also of his beliefs about the other players. I use bayesian updating of the individual beliefs to model a sequential version of the game. As such, the models can be used to make predictions about behaviour and conflict dynamics for various types of first order beliefs.

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Game structure

At the beginning of each round of the game, the participant receives an endowment of En. The participant is then informed, that the opponent has been endowed with the same total sum. Both the player and his opponent(s) can contribute $0 < C_i < En$ of the endowment. The contributions of the two sides are then discarded. Function (1) represents the aggressor's final reward. If his contribution is higher than the defender's, he retains what remained of his endowment and takes what remained of the defender's. Otherwise, he only retains what remained of his endowment:

$$f(C_a, C_d) = \begin{cases} En - C_a + En - C_d, & \text{if } C_d < C_a \\ En - C_a, & \text{otherwise} \end{cases} \tag{1}$$

The defender payoff can be seen in (2). If the defender wins, he retains what remained of his endowment. If he loses, the attacker takes what remained of his endowment and the defender retains nothing:

$$f(C_d, C_a) = \begin{cases} En - C_d, & \text{if } C_d \geq C_a \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

A classical formal solution to this game would require computing the utility maximizing function of other players and use it in the utility maximizing function of the actor. I propose, that people instead formulate an arbitrary first order belief about the others (that is, a belief that can be partly or fully independent of the game structure) and then act on this belief. As this belief can involve varying levels of uncertainty, players face a game where they have to operate not only with values of the others behaviour, but also their respective probabilities.

Individual model

First model represents an individual playing against one or more opponents. The individual has a belief about possible contributions of the opponent(s). This belief is represented by a beta distribution with parameters α and β. Beta distribution is supported over values of $0 < \theta < 1$, where θ represents the proportion of En the opponent can contribute. Each θ is assigned a likelihood of occurring. See Figure 1.

The expected probability of winning against an opponent is equal to the expected probability of the opponent scoring bellow an individual's contribution. This can be computed by integrating the beta distribution over values of θ ranging from 0 to the proportion of individual's contribution $\frac{C_i}{En}$ (3). This indicates the expected probability that the opponent contributes bellow this contribution and therefore loses. See Figure 2.

$$P(Win_i|C_i) = \int_0^{\frac{C_i}{En}} Beta(x, \alpha_i, \beta_i)dx \tag{3}$$

The expected utility of any player can be expressed as the sum of products of his probability of winning or losing and their respective outcomes. If the defender loses, his outcome is 0 (2), so his utility function can be expressed the product of his probability of winning times what remains of his endowment (4).

$$U_d(C_d) = P(Win_d|C_d)(En - C_d) \tag{4}$$

The utility of aggressor (5) consists of the probability of aggressor winning, multiplied by his gains when winning (1) plus the probability of his loss, multiplied by the gain in case of losing.

$$U_a(C_a) = P(Win_a)(2En - C_a - EnE(C_d|Win_a)) + (1 - P(Win_a))(En - C_a) \tag{5}$$

If the aggressor wins, he receives the remaining endowment of the defender. The expected contribution of the opponent conditional on the individual winning is expressed as the mathematical expectation of the belief beta distribution over values ranging from 0 up to the point of $\frac{C_a}{En}$(6), that is, over all possible values that can result in the defender's defeat. See Figure 3.

$$E(C_d|Win_a) = \frac{\int_0^{\frac{C_a}{En}} xBeta(x, \alpha_a, \beta_a)dx}{\int_0^{\frac{C_a}{En}} Beta(x, \alpha_a, \beta_a)dx} \tag{6}$$

When the defender increases his contribution, he increases the probability of keeping the rest of his endowment. However, with increasing contributions, the remaining portion of the endowment decreases, so the player needs to find an optimal balance between maximal probability of winning and maximal retained value.

When the aggressor increases his contribution, he increases his probability of winning. But he faces the cost of his contribution and the increasing expected contribution of the opponent conditional on his victory.

For the aggressor and defender utility functions at various types of belief, see Figure 4, Figure 5 and Figure 6. While the defender always maximizes his utility with rather high contributions, the aggressor faces a dilemma between two local maxima. One is at 0, where he simply keeps his endowment, the other is at about one half of his endowment. For some types of belief, the second maximum becomes higher than the first for certain types of belief, indicating that belief might have a non-linear effect on aggressor contributions. In certain area, a small change in belief can result in a sudden switch from no contribution, to almost half of the endowment.

Inter-group model

In an inter-group version of the game, a player is facing a group of n opponents, himself playing with $n-1$ ingroup members (so both sides consist of n players). When the aggressors win, they equally divide the *plunder*, irrespective of individual contributions.

In the inter-group context, belief is not one-dimensional as in Figure 1, but consists of a joint distribution of beliefs about the ingroup and about the outgroup(7). An example of such joint distribution can be seen in Figure 7.

$$L(C_{in}, C_{out}) = Beta(C_{in}, \alpha_{in}, \beta_{in})Beta(C_{out}, \alpha_{out}, \beta_{out}) \tag{7}$$

A group wins, when its total contribution is higher than that of the opponent. The expected total in-group contribution can be expressed as the proportion of En contributed by ingroup members plus the proportion contributed by an individual, both weighted by the group size. The value of contribution can be can be computed by multiplying the weighted proportions by the endowment size (8).

$$C_{in-total} = En\left(\frac{n-1}{n}Beta(C_{in}, \alpha_{in}, \beta_{in}) + \frac{C_i}{nEn}\right) \tag{8}$$

Analogously to (3), the probability of ingroup winning is equal to the area of the joint distribution, where $C_{in} > C_{out}$. This can be computed by integrating the distribution accordingly (9):

$$P(Win_{in}|C_i) = \int_0^{\frac{(n-1)C_{in}}{n}+\frac{C_i}{nEn}} \int_0^1 L(C_{in}, C_{out})dC_{in}dC_{out} \tag{9}$$

The expected proportion of contribution of the (outgroup) defenders given the aggressors win is computed analogously to (6).

$$E(C_{out}|Win_a) = \frac{\int_0^{\frac{(n-1)C_{in}}{n}+\frac{C_i}{nEn}} \int_0^1 C_{out}L(C_{in}, C_{out})dC_{in}dC_{out}}{\int_0^{\frac{(n-1)C_{in}}{n}+\frac{C_i}{nEn}} \int_0^1 L(C_{in}, C_{out})dC_{in}dC_{out}} \tag{10}$$

The utility of defenders and aggressors then follows the same formulae as (4) and (5), respectively. Both types of player now face an incentive to free-ride on the expected contributions of their in-group, as well as a decreasing impact of their actions as the group size increases. While the players might have a motivation to free-ride, in a proper constellation of beliefs, they could have an incentive to contribute more, in order to tip the balance between two groups in their favour. Utility functions for two particular types of beliefs can be seen in Figure 8 and Figure 9.

Sequential inter-group model

The models discussed above are for one-shot games. In a sequential version of the game, players might update their beliefs based on the observed contributions of the ingroup and outgroup. The updating is represented by the posterior distribution of beliefs, given the observed contributions (11).

$$P\left(\theta_{group}|\frac{C_{group}}{En}\right) = \frac{P(\theta_{group})P\left(\frac{C_{group}}{En}|\theta_{group}\right)}{P\left(\frac{C_{group}}{En}\right)} \tag{11}$$

An easy way of representing the likelihood of representing the likelihood of a contribution $P\left(\frac{C_{group}}{En}|\theta_{group}\right)$ is treating the contribution as a function of binomial distribution with the parameter θ as the probability of contributing used for every single unit of En. As the binomial distribution is conjugate to beta distribution, the posterior for beta distribution with parameters α and β would equal (12).

$$P\left(\theta_{group}|\frac{C_{group}}{En}\right) = Beta(\theta, \alpha + C_{group}, \beta + (En - C_{group})) \tag{12}$$

As such updating would be highly sensitive to the endowment size, the updating parameters can be scaled down by En into (13). This way, the updating is still proporitonal to the observation, but the same for any value of En.[1] An example of such updating in three rounds of the game from the perspective of an aggressor can be seen in Figure 10.

$$P\left(\theta_{group}|\frac{C_{group}}{En}\right) = Beta(\theta, \alpha + \frac{C_{group}}{En}, \beta + \frac{En - C_{group}}{En}) \tag{13}$$

When all players update their beliefs, their utility functions change accordingly. In a sequential model of the game, each round, each player selects his utility maximizing option given his belief. This mechanism is used to generate contributions for that round. All players then use these contributions of others to update their beliefs, adjust utility functions and adapt behaviour in the subsequent round. Figure 11, Figure 12, Figure 13 and Figure 14 show conflict dynamics over time, where all players start with the same belief about both ingroup and outgroup. The figures show these dynamics for various group sizes $n_{in} = n_{out} = 2$ and $n_{in} = n_{out} = 4$. In all cases, over time, the players seem to arrive at a dynamic equilibrium, matching each other's contributions.

[1]Individuals can assigned varying evidential values to observation. This could be expressed by multiplying the updating values by a constant k: $P\left(\theta_{group}|\frac{C_{group}}{En}\right) = Beta(\theta, \alpha + \frac{kC_{group}}{En}, \beta + \frac{kEn - C_{group}}{En})$

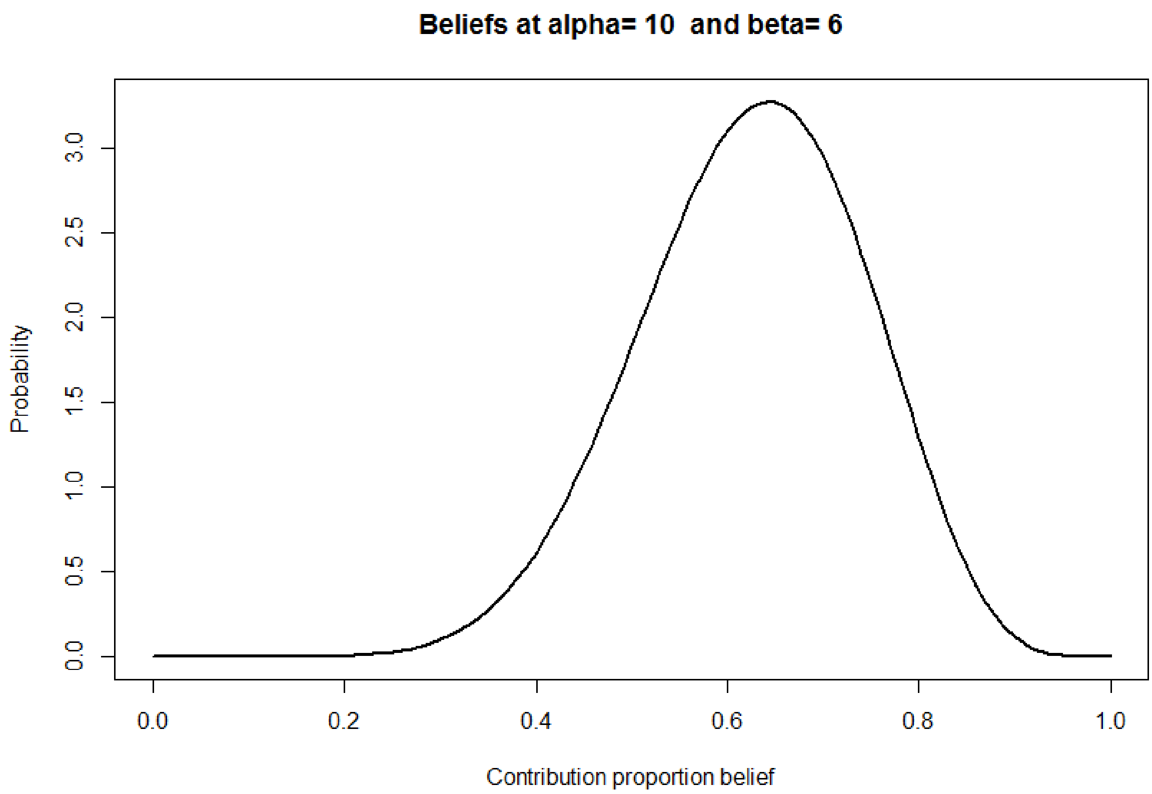

Figure 1. A beta distribution represents what contribution (as a proportion of En) the player expects from his opponent(s), with each proportion assigned a likelihood. The distribution integrates to 1.

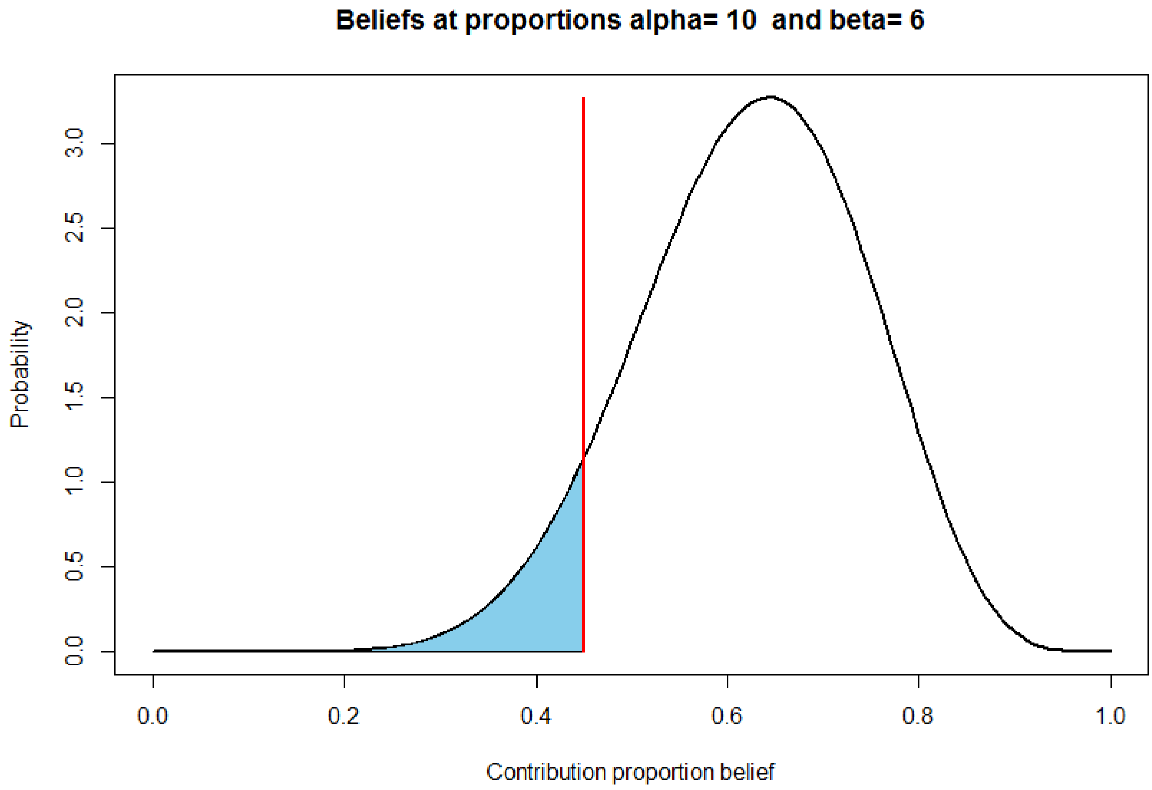

Figure 2. The red line represents the individual's contribution. The blue area represents the expected probability that the opponent's contribution is lower than the individual's.

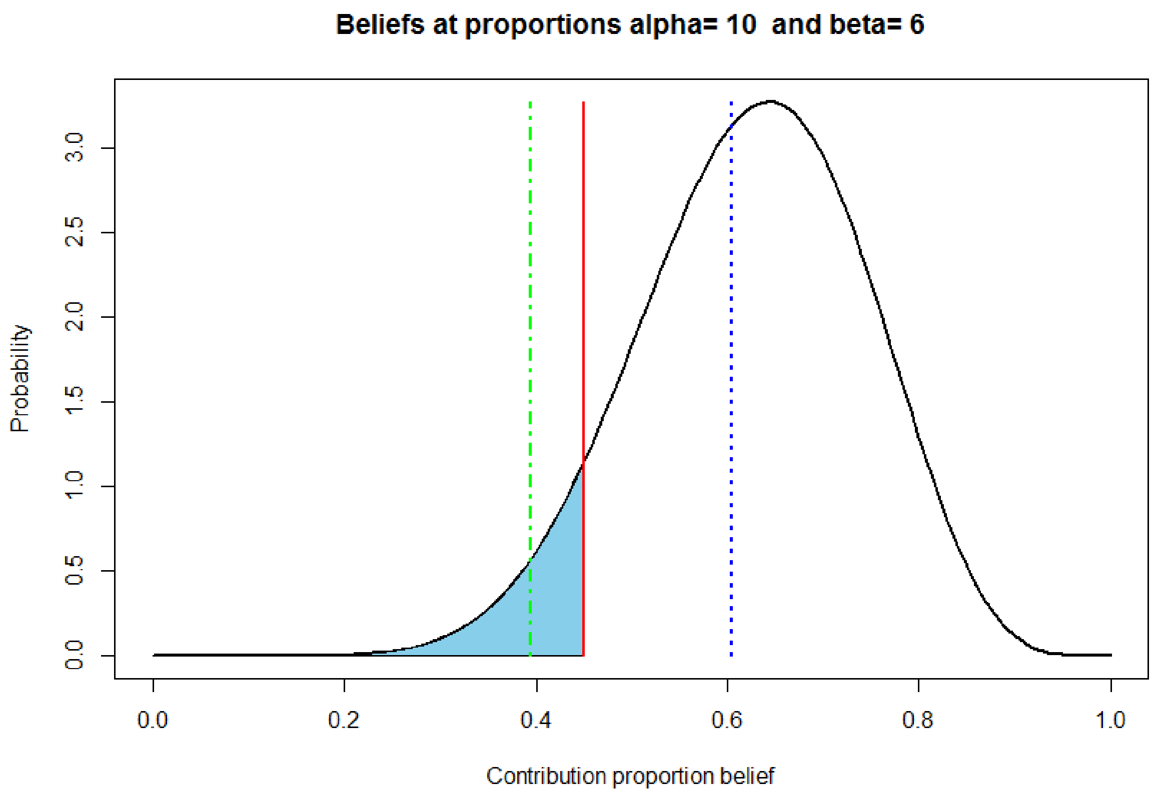

Figure 3. The green line represents the expected contribution of the opponent, given that the individual wins. The blue line represents what is expected to remain of his endowment in that case.

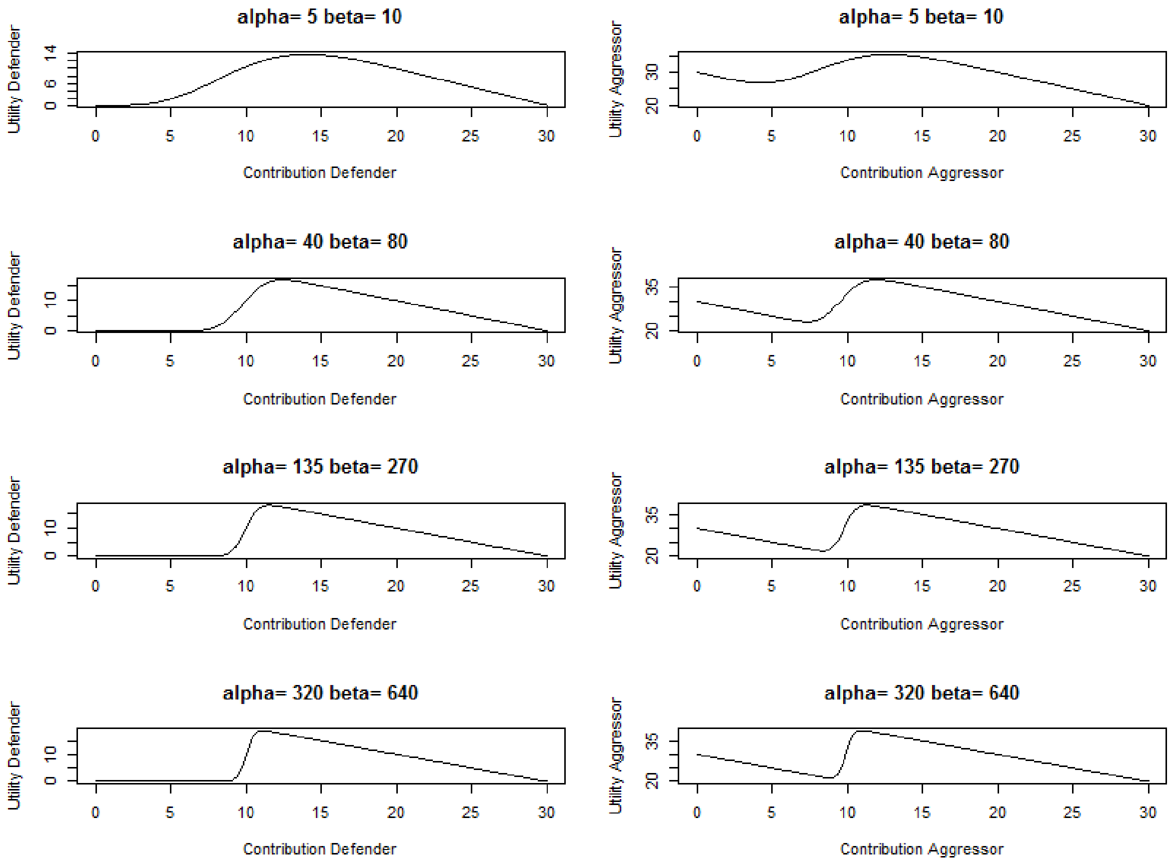

Figure 4. Defender (left) and aggressor (right) utility functions at $\frac{\alpha}{\beta} = \frac{5}{10}$.

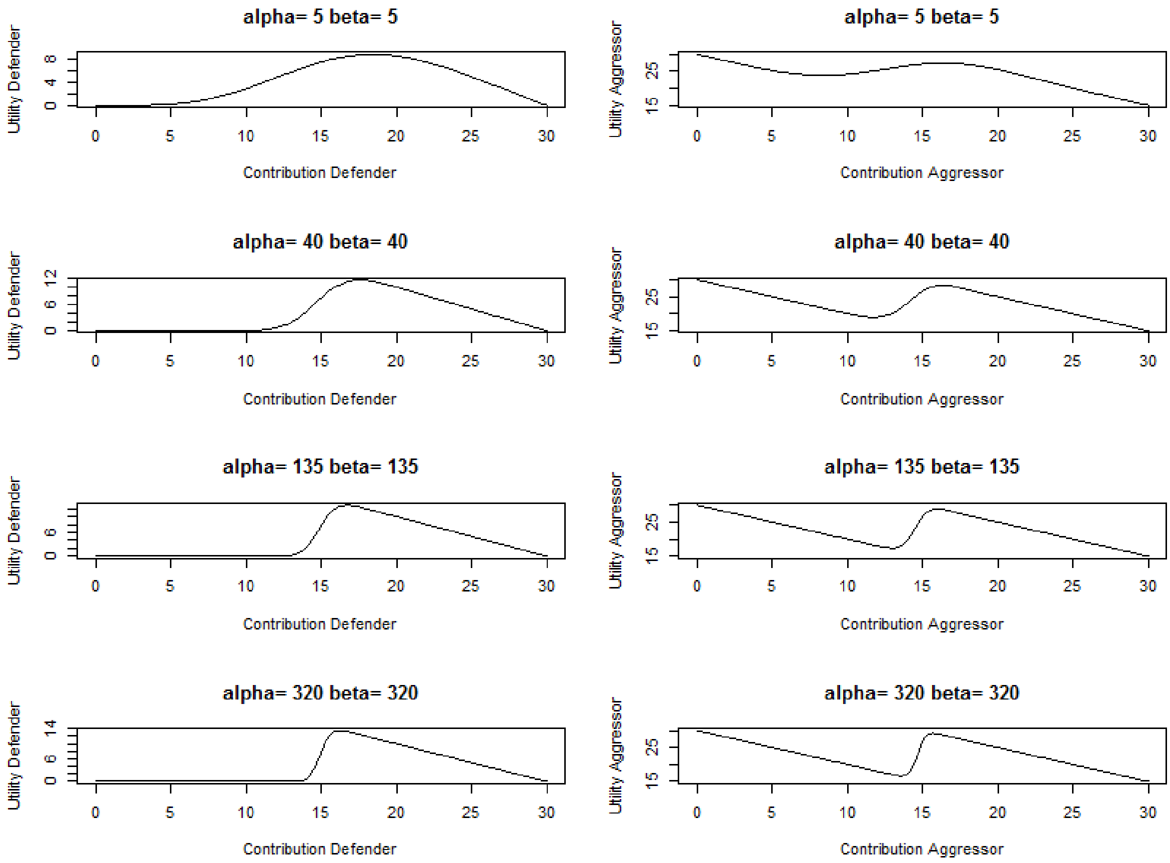

Figure 5. Defender (left) and aggressor (right) utility functions at $\frac{\alpha}{\beta} = \frac{5}{5}$.

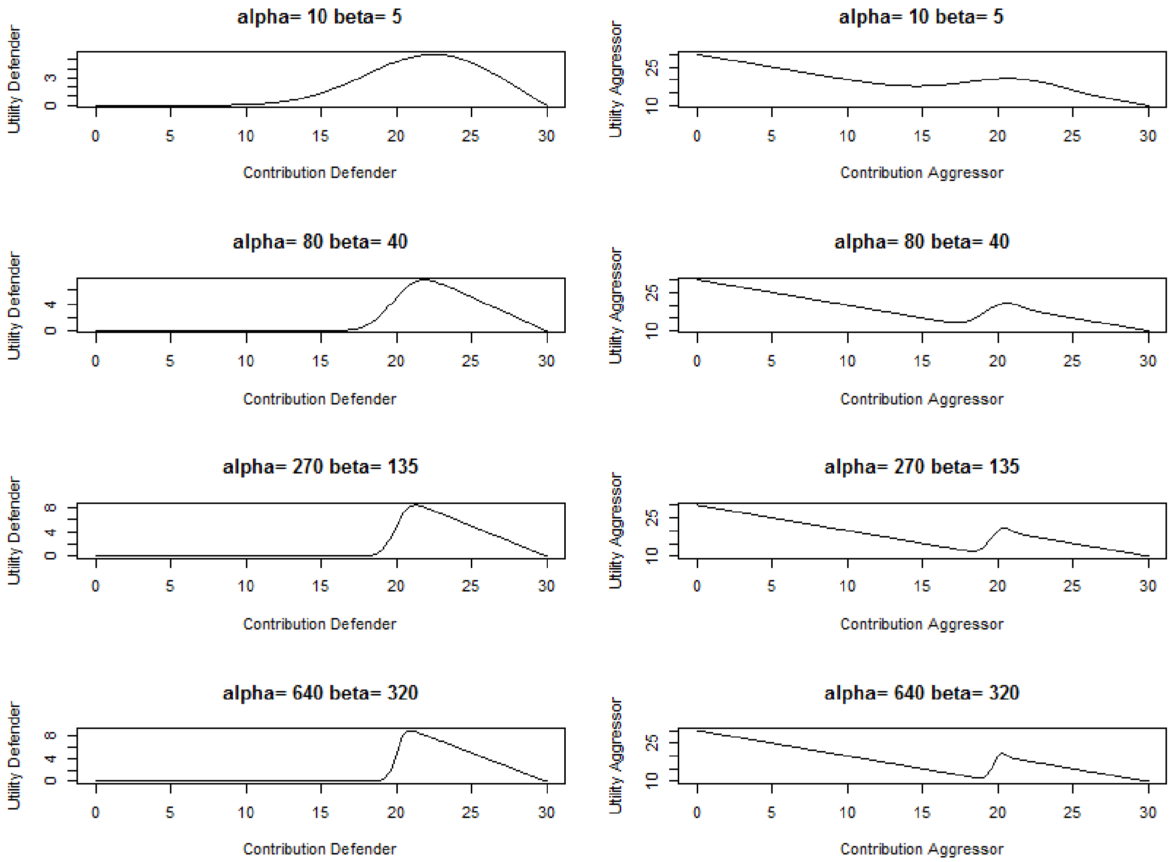

Figure 6. Defender (left) and aggressor (right) utility functions at $\frac{\alpha}{\beta} = \frac{10}{5}$.

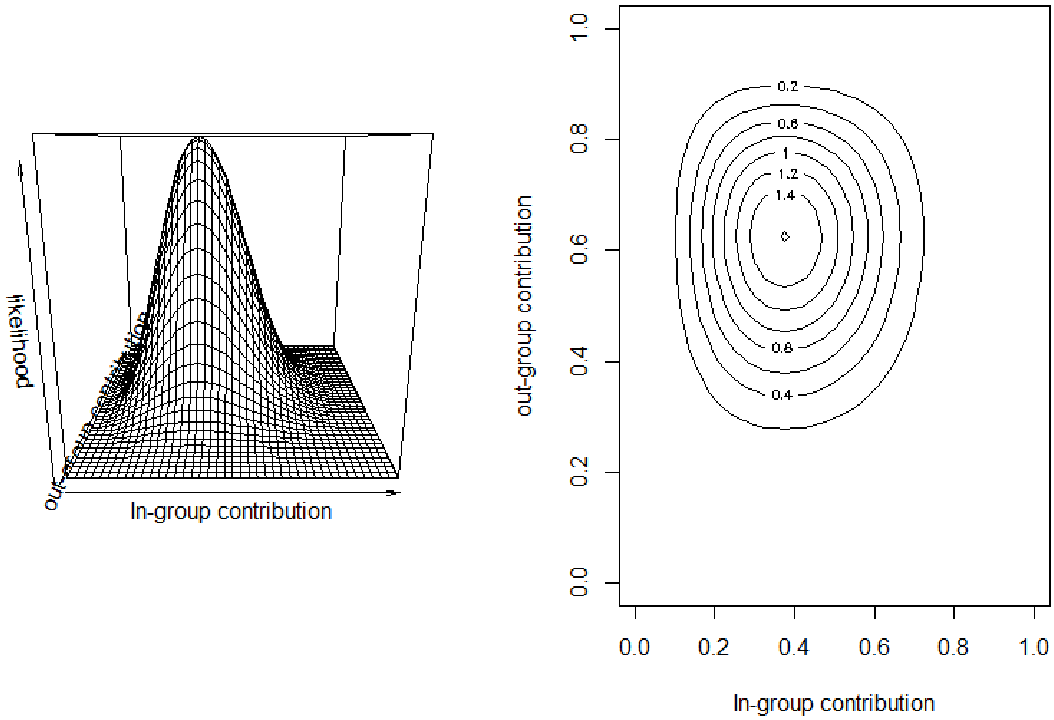

Figure 7. Two visualization of the same joint distribution. The x-axis shows contribution proportions by the in-group, y-axis contribution proportions of the out-group. The z-axis shows the likelihood of any of their combinations.

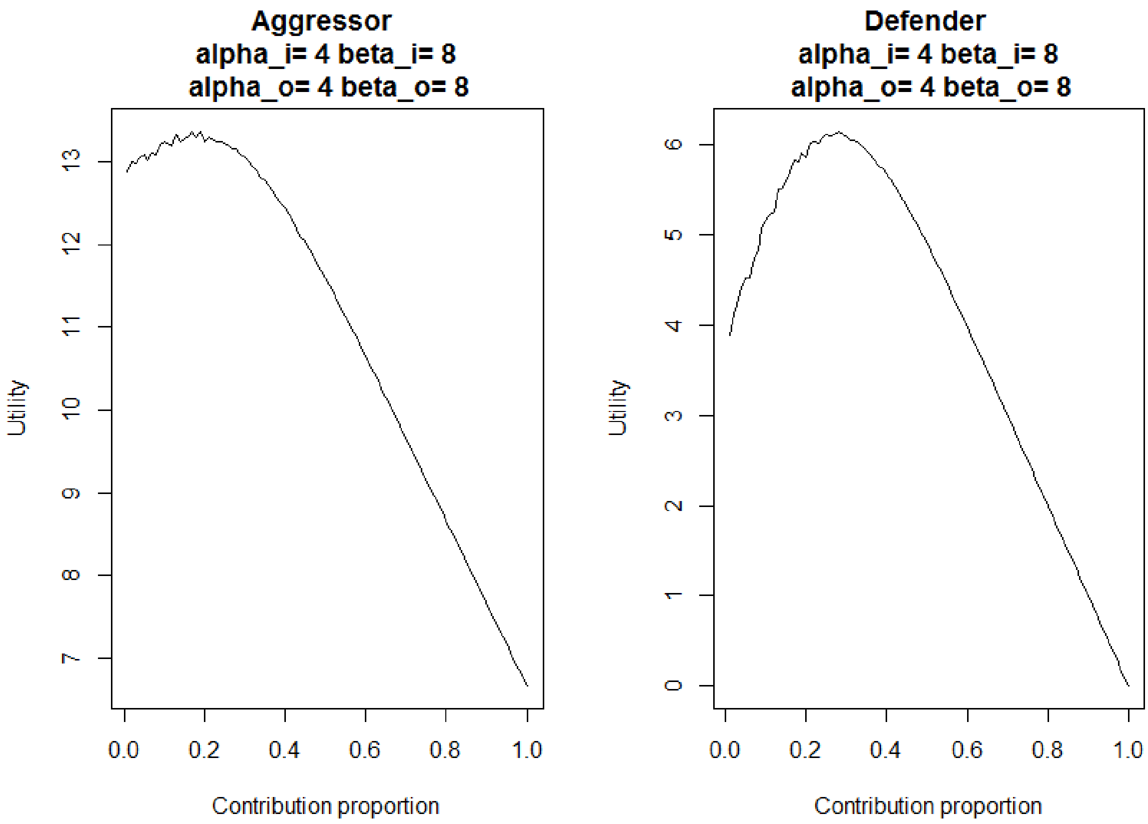

Figure 8. Utility function for an aggressor and defender with group size 6. Subscripts i and o stand for *ingroup* and *outgroup*, respectively.

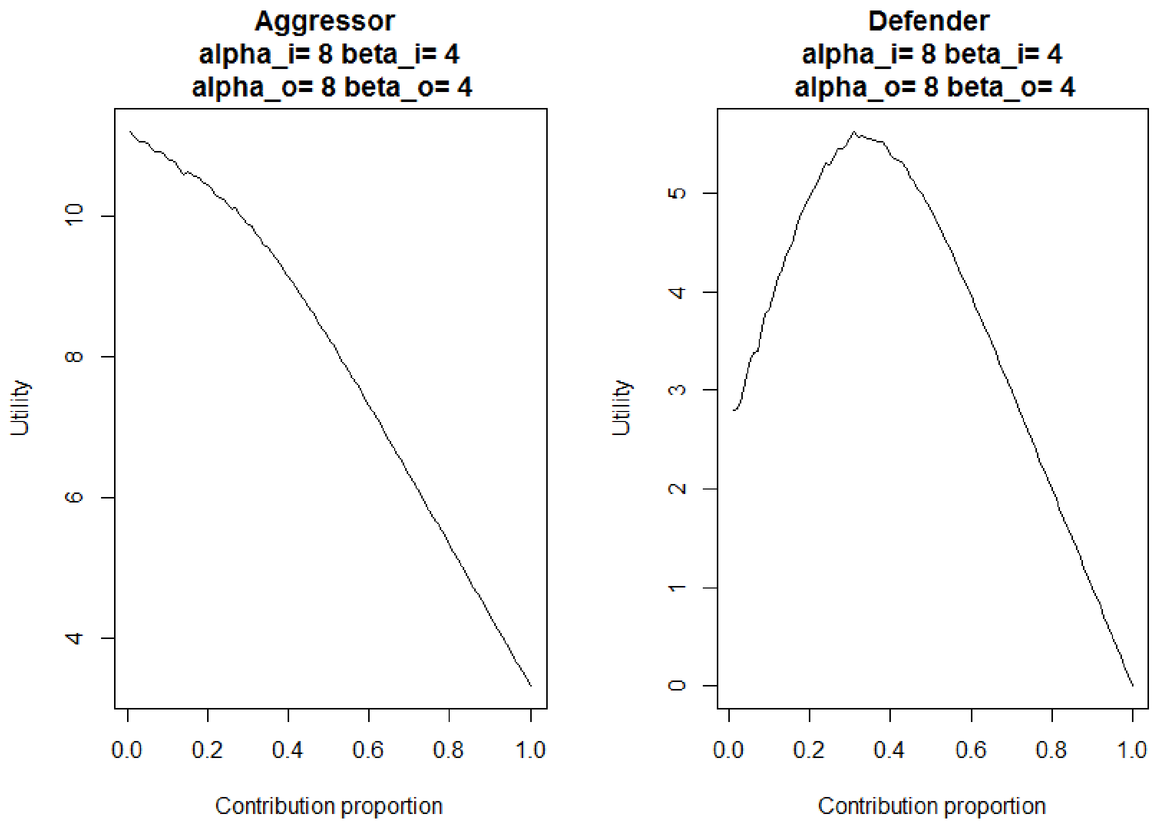

Figure 9. Utility function for an aggressor and defender with group size 6. Subscripts i and o stand for *ingroup* and *outgroup*, respectively.

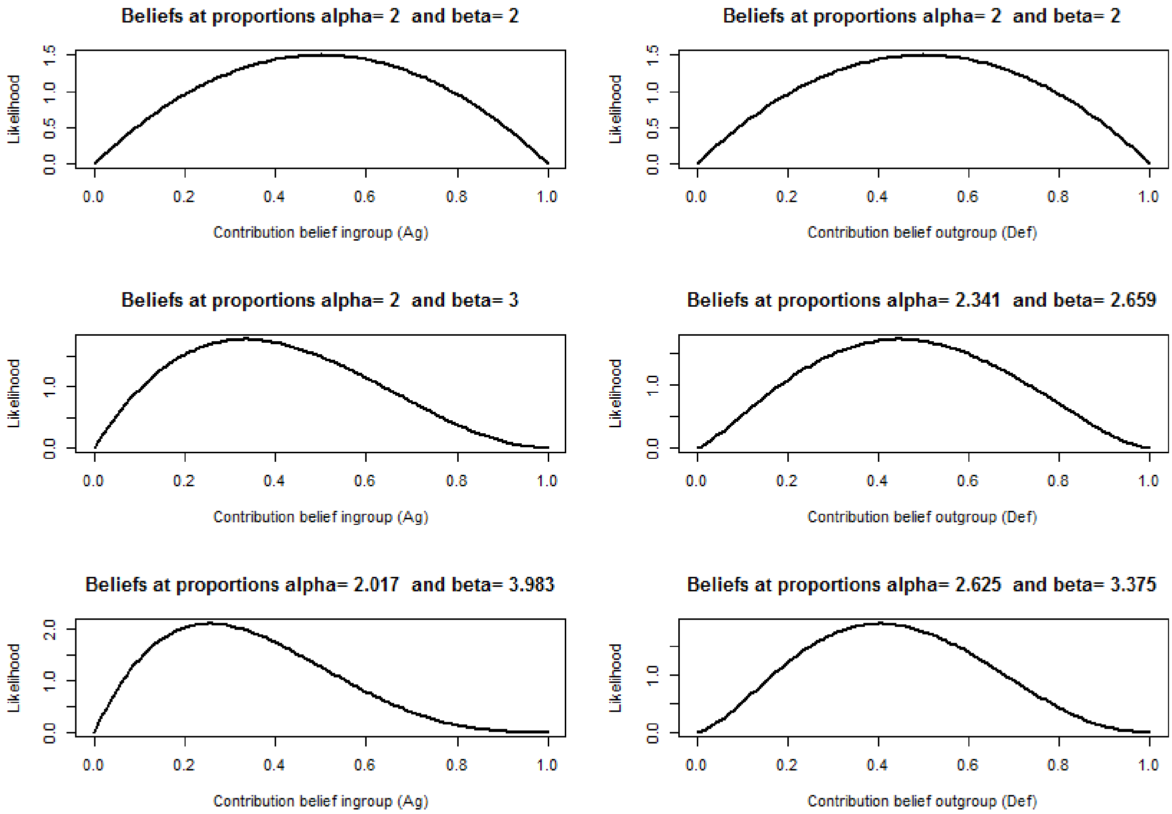

Figure 10. The aggressor starts with equal belief distribution for his *ingroup* and *outgroup*. He then updates these beliefs on the observed behaviour in three rounds of the game.

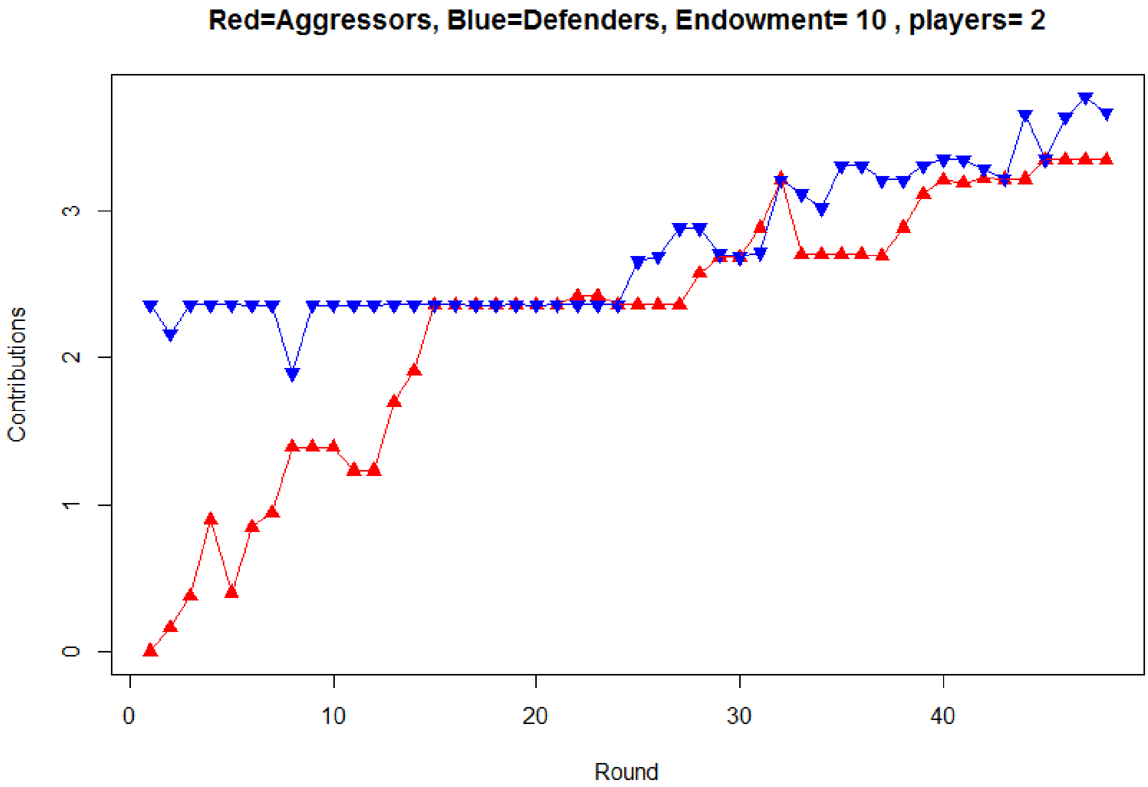

Figure 11. The figure shows contributions of an individual aggressor and defender both starting with the same belief about both ingroup and outgroup with parameters $\alpha = 1$ and $\beta = 1$. Both groups have the size $n = 2$

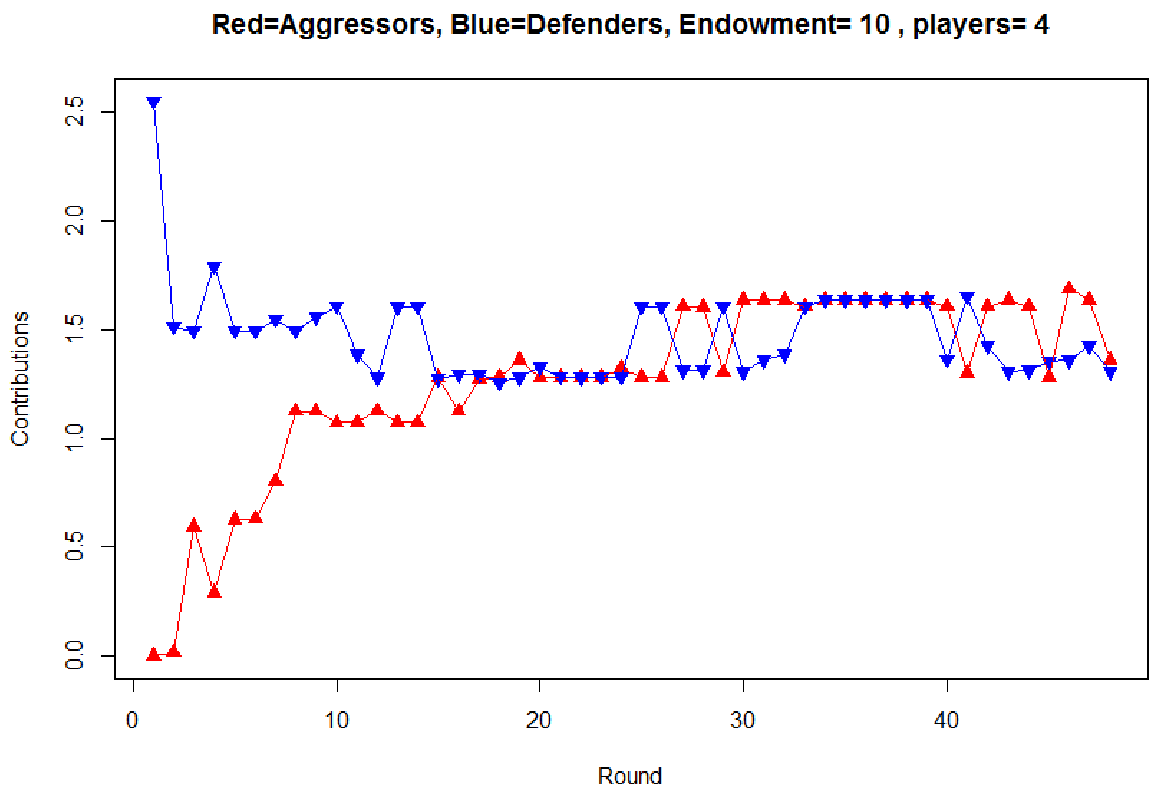

Figure 12. The figure shows contributions of an individual aggressor and defender both starting with the same belief about both ingroup and outgroup with parameters $\alpha = 1$ and $\beta = 1$. Both groups have the size $n = 4$

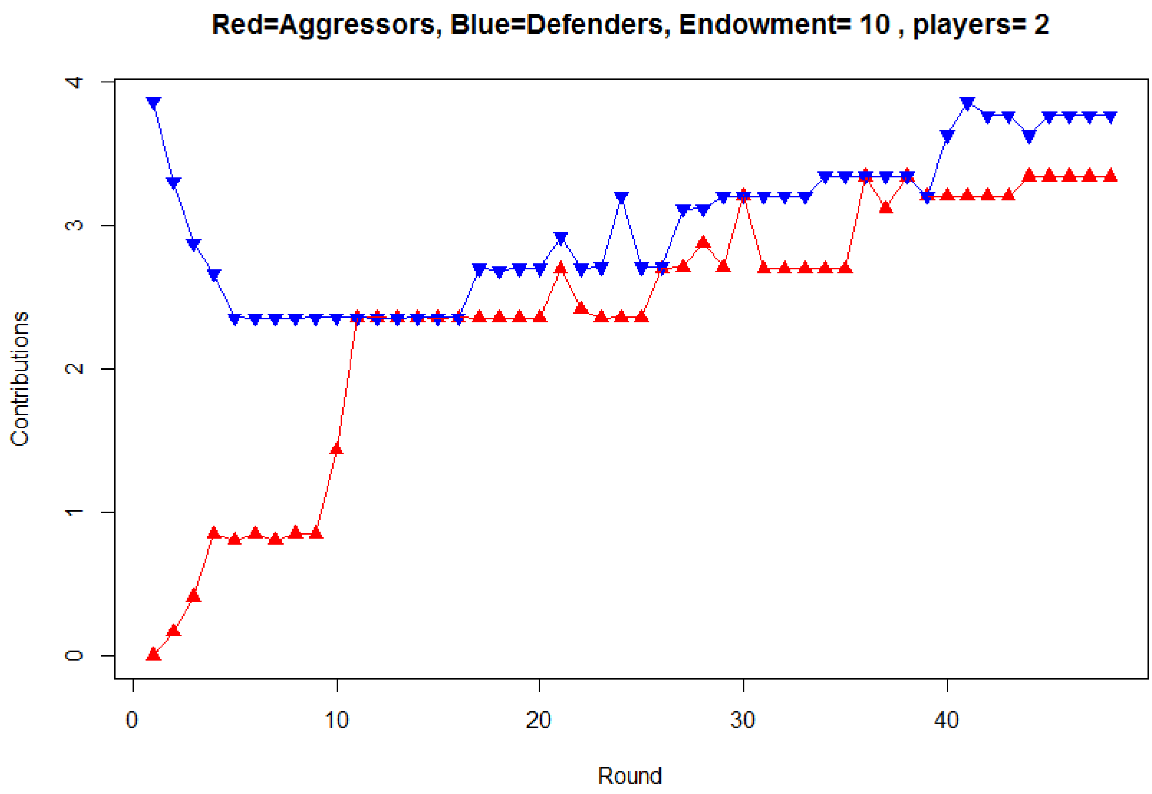

Figure 13. The figure shows contributions of an individual aggressor and defender both starting with the same belief about both ingroup and outgroup with parameters $\alpha = 2$ and $\beta = 2$. Both groups have the size $n = 2$

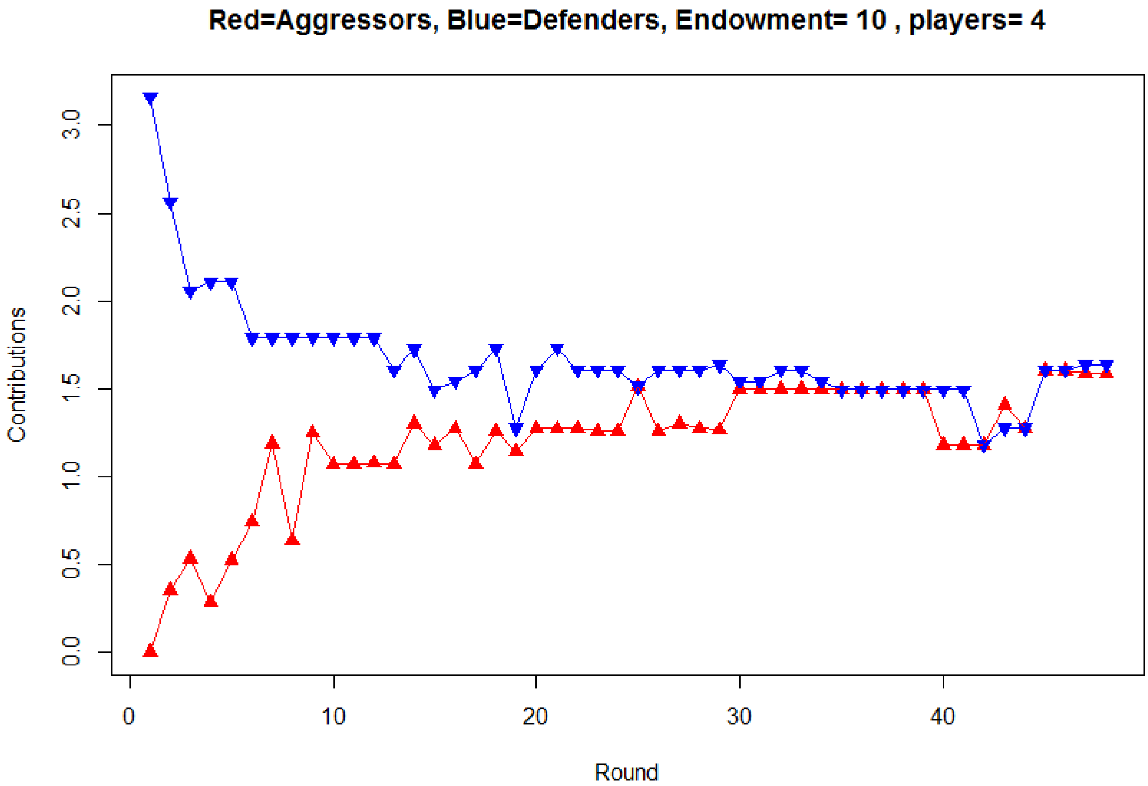

Figure 14. The figure shows contributions of an individual aggressor and defender both starting with the same belief about both ingroup and outgroup with parameters $\alpha = 2$ and $\beta = 2$. Both groups have the size $n = 4$