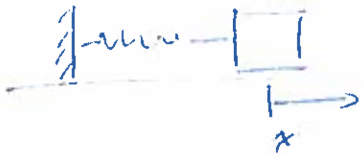
Harmonic Oscillator



Hooke's Law spring $F_s = -kx$

$F = ma$

$$-kx = m\frac{d^2x}{dt^2} = m\ddot{x}$$

$$\ddot{x} + \frac{k}{m}x = 0$$

① $\ddot{x} + \omega_0^2 x = 0$, $\omega_0 = \sqrt{\frac{k}{m}}$ → natural angular frequency

sol^{n} ↳ $x = A\cos(\omega_0 t + \phi)$

↑ amplitude ↑ phase constant

$f = \frac{\omega}{2\pi}$ frequency

$T = \frac{1}{f}$ period $= \frac{2\pi}{\omega}$

equivalent sol^{n} is

$x = B\sin\omega_0 t + C\cos\omega_0 t$ ← write here

In either case, two constants – A and ϕ or B and C – are determined by initial conditions. (can show equivalence using $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$)

Energy potential energy $U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega_0 t + \phi)$

kinetic energy $K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega_0^2 A^2\sin^2(\omega_0 t + \phi)$

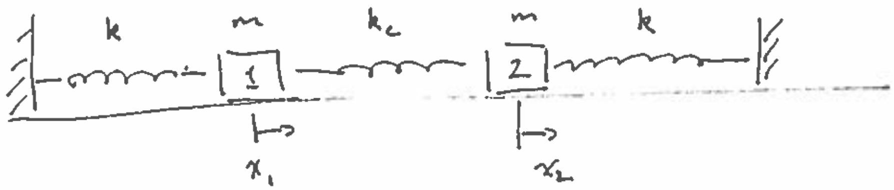
$= \frac{1}{2}kA^2\sin^2(\omega_0 t + \phi)$

$v = \dot{x} = -\omega_0 A\sin(\omega_0 t + \phi)$

total energy $E = U + K = \frac{1}{2}kA^2\left(\cos^2(\omega_0 t + \phi) + \sin^2(\omega_0 t + \phi)\right)$

$= \frac{1}{2}kA^2$ (constant)

Coupled Oscillators



x_i - displacement from equilibrium of block i

If the oscillators were not coupled through the middle spring, each would vibrate with ang. freq.

$\omega_0 = \sqrt{\frac{k}{m}}$

Consider forces on block 1

(1) $$m x_1'' = -k x_1 + k_c (x_2 - x_1)$$ ← stretch of middle spring

similarly for 2

(2) $$m x_2'' = -k x_2 - k_c (x_2 - x_1)$$

Eq^{ns} (1) and (2) are coupled, and in principle need to be solved simultaneously.

Notice, however

(1) + (2) → $$m (x_1'' + x_2'') = -k (x_1 + x_2)$$

call $q_1 = x_1 + x_2$

$$m q_1'' + k q_1 = 0$$

(Q1) $$q_1'' + \omega_1^2 q_1 = 0 \quad , \quad \omega_1 = \sqrt{\frac{k}{m}} = \omega_0$$

this is a single eq^n and can be solved independently

→ no mention of k_c

Also

① - ②

$$m(x_1'' - x_2'') = -k(x_1 - x_2) - 2k_c(x_1 - x_2)$$

call $q_2 = x_1 - x_2$

$$m q_2'' + k + 2k_c \; q_2 = 0$$

$$\text{(Q2)} \quad q_2'' + \omega_2^2 q_2 = 0 \quad , \quad \omega_2 = \sqrt{\frac{k + 2k_c}{m}}$$

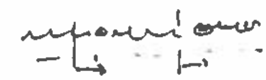
another eqⁿ that can be solved on its own.

$(= \sqrt{3}\, \omega_0 \text{ if } k_c = k)$

By finding q_1 and q_2, i.e. appropriate linear combinations of the original coordinates, we can solve the original coupled set of equations as a set of simpler, uncoupled equations.

Physical meaning:

consider initial conditions $x_1(0) = 1$, $x_1'(0) = 0$, $x_2(0) = 1$, $x_2'(0) = 0$



(both masses displaced the same amount, at rest)

with these initial conditions $q_1(0) = 2$, $q_1'(0) = 0$, $q_2(0) = 0$, $q_2'(0) = 0$

$$\left(\begin{array}{l} q_1 = x_1 + x_2 \\ q_2 = x_1 - x_2 \end{array} \right.$$

According to (Q2) $q_2(t) = 0$ for all time and therefore

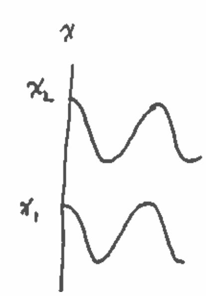
$$0 = x_1(t) - x_2(t) \rightarrow x_2(t) = x_1(t)$$

The displacements of each block will always be the same

$$\text{(Q1)} \longrightarrow \quad q_1(t) = 2\cos(\omega_1 t)$$

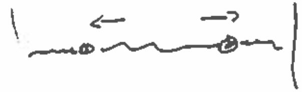
$$x_1(t) + x_2(t) = 2x_1(t) = 2\cos(\omega_1 t)$$

$$x_1(t) = \cos(\omega_1 t) = x_2(t)$$



For $x_1(0) = -1 \quad x_2(0) = 1$

$x_1'(0) = 0 \quad x_2'(0) = 0$



$q_1(0) = x_1(0) + x_2(0) = -1 + 1 = 0$

$q_1'(0) = 0$

$q_2(0) = -1 - 1 = -2$

$q_2'(0) = 0$

$Q_1 \rightarrow q_1(t) = 0$ for all t

$\rightarrow x_1(t) + x_2(t) = 0$ ← displacements are always opposite

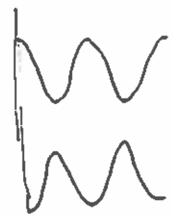
$x_1(t) = -x_2(t)$

$Q_2 \rightarrow q_2(t) = -2\cos(\omega_2 t)$

$x_1(t) - x_2(t) = -2x_2(t) = -2\cos(\omega_2 t)$

$x_2(t) = \cos(\omega_2 t)$

$x_1(t) = -\cos(\omega_2 t)$



These sets of initial conditions each pick out a particular independent (normal) mode of vibration. General initial conditions will give rise to a linear combination or superposition of the normal modes.

For any system of coupled harmonic oscillators, the motion can be understood as a superposition of independent normal modes.

The system will have a resonant response at each of the normal mode frequencies.

The trick is finding the normal coordinates $q_1, q_2 \ldots$ that decouple the equations

Matrix approach

for simplicity, consider $k_c = k$

(1) $\quad x_1'' = -2\frac{k}{m} x_1 + \frac{k}{m} x_2$

(2) $\quad x_2'' = \frac{k}{m} x_1 - 2\frac{k}{m} x_2$

calling $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $A = \frac{k}{m}\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$

(1),(2) become

$$x'' = A x$$

If A were diagonal, this would just be two independent H.O. equations

consider $q_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $q_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$q_1^T x = x_1 + x_2 \qquad q_2^T x = x_1 - x_2$

so q_1 and q_2 represent our normal modes

e.g. $q_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ ← if block 1 has displacement $x_1 = 1$ then block 2 has displacement $x_2 = -1$

q_1 and q_2 are <u>eigenvectors</u> of A, i.e. they satisfy

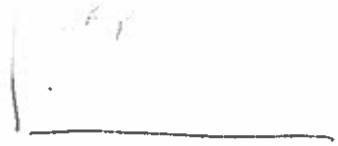
$A q_i = \lambda_i q_i$ where λ is ~~the~~ an <u>eigen value</u>

$$A q_2 = \frac{k}{m}\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{k}{m}\begin{pmatrix} -2-1 \\ 1+2 \end{pmatrix} = \frac{k}{m}\begin{pmatrix} -3 \\ 3 \end{pmatrix} = -\frac{3k}{m}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

so $\lambda_2 = -3\frac{k}{m}$ and $\omega_2 = \sqrt{-\lambda_2} = \sqrt{\frac{3k}{m}}$ ← ~~eigenmo~~ normal mode (eigenmode) ang. Frequency.

The problem of finding the normal modes and frequencies reduces to find the eigenvalues and eigenvectors of A.

Construct P from the normalized eigenvectors of A



$$P = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$\uparrow q_1 \quad \uparrow q_2$

$\leftarrow \tilde{q}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \tilde{q}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$P = (\tilde{q}_1, \tilde{q}_2)$

P is orthogonal (since eigenvectors are orthogonal i.e. $q_i \cdot q_j = 0$)

i.e. $P^T P = \cancel{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}} \; I \; \left(= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)$

The important thing is

$$AP = A(\tilde{q}_1 \;\; \tilde{q}_2) = (\lambda_1 \tilde{q}_1 , \lambda_2 \tilde{q}_2)$$

eigenvalues ~~eigenvector~~

$$= (\tilde{q}_1 \;\; \tilde{q}_2) D \qquad D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$= PD$$

or $P^{-1} A P = P^{-1} P D$

$P^T A P = D$

The matrix P transforms A into a diagonal matrix

also note $P^T x = \begin{pmatrix} \tilde{q}_1^T \\ \tilde{q}_2^T \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \end{pmatrix} \equiv q$

So we can transform our coupled equation

$x'' = A x$

$P^T x'' = P^T A \, P P^{-1} x \qquad (P P^{-1} = P P^T)$

$q'' = D q$

to an uncoupled one if we know the eigenvectors of A