Midpoint Scheme or Second order Runge Kutta

for solving $\frac{dy}{dt} = f(y,t)$, $y(0) = y_0$

Recall Euler: start with forward difference, solve for $y(t+\Delta t)$

$$\frac{dy}{dt} = \frac{y(t+\Delta t) - y(t)}{\Delta t} + O(\Delta t)$$

$$y(t+\Delta t) = y(t) + \frac{dy}{dt}\Delta t + O(\Delta t^2)$$

but $\frac{dy}{dt} = f(y,t)$

$$y(t+\Delta t) = y(t) + f(y,t)\,\Delta t + O(\Delta t^2)$$

↑ Recovers Taylor series to first order

Midpoint Scheme: start with centred difference

$$\frac{dy(t)}{dt} = \frac{y(t+\Delta t) - y(t-\Delta t)}{2\Delta t} + O(\Delta t^2)$$

$$y(t+\Delta t) = \cancel{\frac{dy(t)}{dt}}\; y(t-\Delta t) + 2\Delta t\,\frac{dy(t)}{dt} + O(\Delta t^3)$$

$$= y(t-\Delta t) + 2\Delta t\, f(y(t), t) + O(\Delta t^3)$$

↑ f is evaluated at the midpoint in time between $t-\Delta t$ and $t+\Delta t$

We can ~~wear~~ rewrite this ~~as~~

(1) $$y(t+\Delta t) = y(t) + \Delta t\, f\left(y\left(t+\tfrac{\Delta t}{2}\right), t+\tfrac{\Delta t}{2}\right) + O(\Delta t^3)$$

i.e. $\Delta t \to \frac{\Delta t}{2}$, then

shift $t \to t + \frac{\Delta t}{2}$

So, if we can estimate $f(y(t+\frac{\Delta t}{2}), t+\frac{\Delta t}{2})$, we can gain an order in our truncation error. To do this, we need to estimate $y(t+\frac{\Delta t}{2})$

trick: use an Euler half-step

$$y(t+\Delta t/2) = y(t) + \frac{\Delta t}{2} y'(t) + \mathcal{O}(\Delta t^2)$$

$$\{ y'(t) = f(y(t), t)$$

Is this good enough?

Consider, for simplicity $\frac{dy}{dt} = f(y)$

$$f(y(t+\tfrac{\Delta t}{2})) = f(\overbrace{y(t) + \tfrac{\Delta t}{2} y'(t) + \mathcal{O}(\Delta t^2)}^{\Delta y})$$

Taylor expansion

$$= f(y(t)) + \frac{df(y)}{dy}\underbrace{\left(\frac{\Delta t}{2} y'(t) + \mathcal{O}(\Delta t^2)\right)}_{\Delta y \sim \Delta t} + \mathcal{O}(\Delta y^2)$$

so $\Delta y^2 \sim \Delta t^2$

$$= f(y) + \frac{\Delta t}{2} y'(t) \frac{df}{dy} + \mathcal{O}(\Delta t^2)$$

chain rule

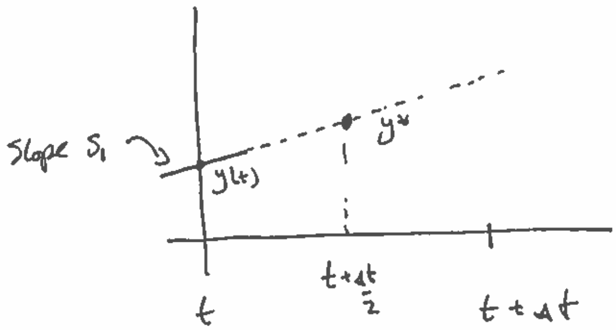
$$= f(y) + \frac{\Delta t}{2} \frac{dy}{dt} \frac{df}{dt} \frac{dt}{dy} + \mathcal{O}(\Delta t^2)$$

$$= f(y) + \frac{\Delta t}{2}\frac{df}{dt} + \mathcal{O}(\Delta t^2)$$

So ① becomes
$$y(t+\Delta t) = y(t) + \Delta t\, f(y(t+\tfrac{\Delta t}{2})) + \mathcal{O}(\Delta t^3)$$
$$= y(t) + \Delta t\, f(y) + \frac{\Delta t^2}{2}\frac{df}{dt} + \mathcal{O}(\Delta t^3)$$
$$= y(t) + \Delta t\, y'(t) + \frac{\Delta t^2}{2} y''(t) + \mathcal{O}(\Delta t^3)$$

with this procedure, we indeed recover the Taylor series up to second order in Δt

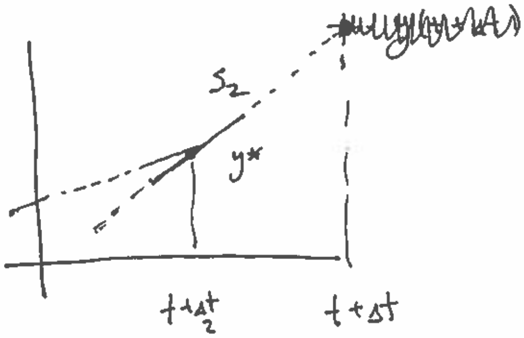
Therefore, our algorithm is $\frac{dy}{dt} = f(y,t)$, $y(0) = y_0$



$$S_1 = f(\,y(t),\,t\,)$$

use S_1 to estimate

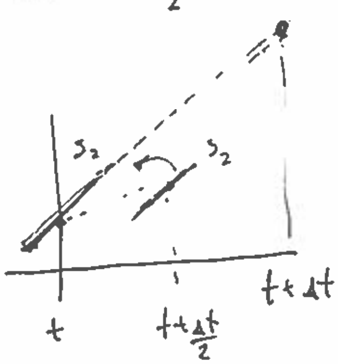
$$y^* \equiv y(t+\Delta t/2)$$
$$= y(t) + \frac{\Delta t}{2} S_1$$



use y^* to ~~estim~~ get

$$S_2 = f(\,y^*,\,t+\frac{\Delta t}{2}\,)$$

use S_2 to estimate $y(t+\Delta t) = y(t) + \Delta t\ S_2$



algorithm:

$$S_1 = f(y_n,\,t_n)$$
$$y^* = y_n + \frac{\Delta t}{2} S_1$$
$$S_2 = f(\,y^*,\,t_n+\frac{\Delta t}{2}\,)$$

$$y_{n+1} = y_n + \Delta t\ S_2$$

2 steps, but an order gain in accuracy

Example we are given the problem

$$m\frac{d^2y}{dt^2} = F_{net} \quad , \quad y(0)=y_0, \quad y'(0)=v_0$$

and we want $y(t)$ for $t\in[0,5]$

rewrite as

$$\frac{dv}{dt} = \frac{F_{net}}{m}$$

$$\frac{dy}{dt} = v$$

Let's say $F_{net} = -ky - bv + A\sin(\omega t)$ with A, k, b and ω given

Then in Mathematica:

(Initialize)

```
Δt = 0.1;
T = 5;   yinit = ___;   vinit = ___;
k = ___;   m = ___;   b = ___;   ω = ___;   A = ___;
nmax = Round[T/Δt];
fv[y_, v_, t_] := (-k y - b v + A Sin[ω t])/m
fy[y_, v_, t_] := v
y[0] = yinit;   v[0] = vinit;   t[0] = 0;
```

(Iterate)

```
Do[ t[n+1] = t[n] + Δt;
    s1v = fv[ y[n], v[n], t[n]];
    s1y = fy[ y[n], v[n], t[n]];
    vtmp = v[n] + Δt/2 s1v;
    ytmp = y[n] + Δt/2 s1y;
    s2v = fv[ ytmp, vtmp, t[n] + Δt/2 ];
    s2y = fy[ ytmp, vtmp, t[n] + Δt/2 ];
    v[n+1] = v[n] + Δt s2v;
    y[n+1] = y[n] + Δt s2y,   {n, 1, nmax} ]
```

(store in list)

```
ytab = Table[ {t[n], y[n]}, {n, 1, nmax+1} ];
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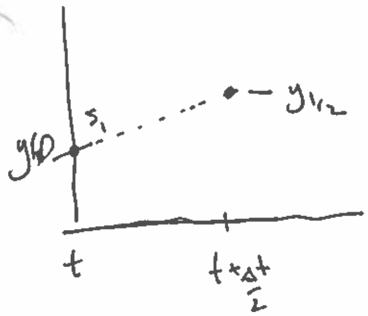
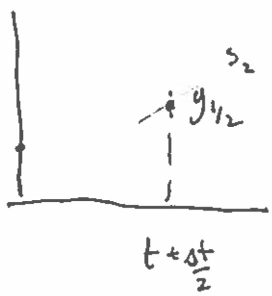
4th order Runge-Kutta

$$y'(t) = f(y, t) \quad , \quad y(0) = y_0$$



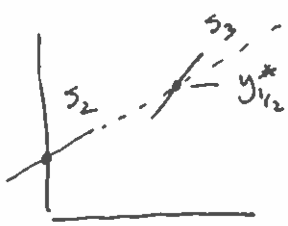
$$s_1 = f(y(t), t)$$

$$y_{1/2} = y(t) + \frac{\Delta t}{2} s_1 \qquad \text{(Euler half-step)}$$



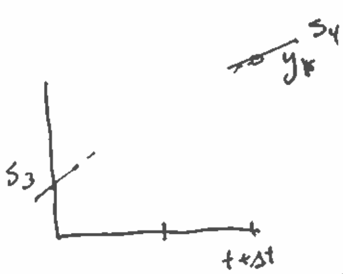
$$s_2 = f\left(y_{1/2}, t + \frac{\Delta t}{2}\right) \qquad \text{(Midpoint half step)}$$

$$y^*_{1/2} = y(t) + \frac{\Delta t}{2} s_2$$



$$s_3 = f\left(y^*_{1/2}, t + \frac{\Delta t}{2}\right)$$

$$y^* = y(t) + \Delta t \, s_3 \qquad \text{(estimate of } y(t+\Delta t)\text{)}$$



$$s_4 = f(y^*, t + \Delta t) \qquad \text{(estimate of slope at right end point)}$$

$$y(t+\Delta t) = y(t) + \frac{\Delta t}{6}\left(s_1 + 2 s_2 + 2 s_3 + s_4\right) + O(\Delta t^5)$$