Euler's Method for solving ODE's

consider a 1^{st} order ODE

① $$\frac{dy}{dt} = f(y,t)$$, ② $$y(t=0) = y_{init}$$

$f(y,t)$ gives the slope for any value of y and t

Eq. ① defines a family of functions and initial condition ② picks out one in particular.

E.g.

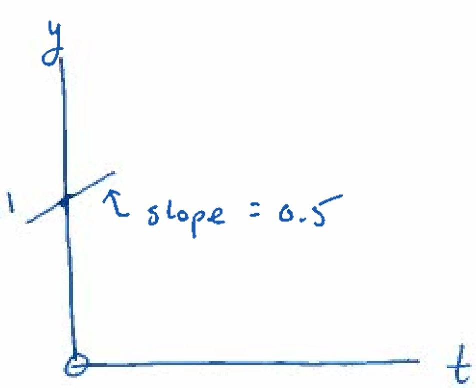
$$\frac{dy}{dt} = \frac{t+1}{y+1} \quad , \quad y(0) = 1$$

$$f(y,t) = \frac{t+1}{y+1}$$

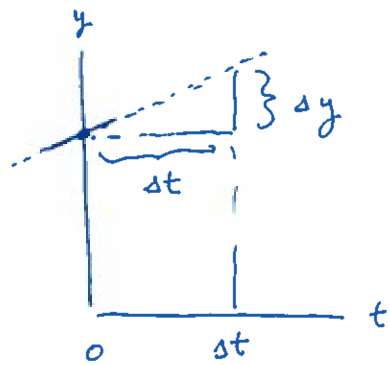
given our initial condition $y(0) = 1$

$f(y,t)$ tells us the slop $\frac{dy}{dt}$ initially as well

$$f(y=1, t=0) = \frac{0+1}{1+1} = \frac{1}{2}$$



We know from calculus that smooth functions behave locally as straight lines



we use this idea to extrapolate our function (y) a small step Δt away from the present t

$$\text{slope} = \frac{\Delta y}{\Delta t}$$

$$y(0+\Delta t) \approx y(0) + \Delta y$$

$$y(\Delta t) \approx y(0) + \text{slope} \cdot \Delta t$$

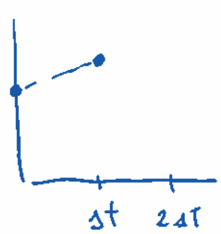
$$= y(0) + f(y(0), t=0)\, \Delta t$$

say $\Delta t = 0.1$

$$= y(0) + 0.5 \cdot 0.1$$

$$= 1 + 0.05$$

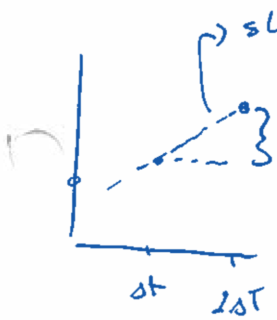
$$= 1.05$$



To get $y(2\Delta t)$, we repeat the procedure, using $y(\Delta t)$ as our starting point

slope at $t = \Delta t$ is $f(y(\Delta t), \Delta t) = \dfrac{\Delta t + 1}{y(\Delta t)+1}$

$$= \frac{1.1}{2.05} = \cancel{0.5365} \; 0.536585$$



$\Delta y = \text{slope} \cdot \Delta t$

$$y(\Delta t + \Delta t) = y(\Delta t) + \Delta y$$

$$y(0.2) = y(0.1) + (0.536585)\cdot(0.1)$$
$$= 1.05 + 0.0536585$$
$$= 1.10366$$

we keep repeating until we reach our desired final time.

In the end, we have a set of points

y	1	1.05	1.10366	...	
t	0	0.1	0.2	0.3	...

that approximate the function $y(t)$ at a discrete set of times

Notice that what we have done is equivalent to using the forward difference scheme for the derivative of a function

$$y'(t) \doteq \frac{y(t+\Delta t) - y(t)}{\Delta t}$$

solving for $y(t+\Delta t)$

$$y(t+\Delta t) = y'(t)\,\Delta t + y(t)$$

fn at current time

time step

fn value at next time step away

this is $f(y(t), t)$

our algorithm for stepping through time is

Euler $\quad y(t+\Delta t) = y(t) + f(y(t), t)\,\Delta t$

what kind of error are we making at each step?

The actual Taylor expansion is

Taylor $\quad y(t+\Delta t) = y(t) + y'(t)\,\Delta t + \dfrac{y''(t)}{2}\Delta t^2 + \ldots$

lead error term

since $y'(t)$ is $f(y(t), t)$

we see that Euler is the same as Taylor only up to the Δt term, and leaves off the Δt^2 and higher order terms.

So the error we make at each step is $\mathcal{O}(\Delta t^2)$

But how many steps do we take?

$$n_{steps} = \frac{t_{final} - t_{initial}}{\Delta t}$$

So after n_{steps} steps, our error estimate is

$$n_{steps} \cdot \mathcal{O}(\Delta t^2) \sim \frac{\mathcal{O}(\Delta t^2)}{\Delta t} = \mathcal{O}(\Delta t)$$

Therefore, we expect our overall error to scale as Δt.

Euler   Advantages: easy to implement

Disadvantages: need to make Δt small to get accurate results (slow)

- often unstable (depending on the problem, roundoff errors add up very quickly and cause the solution to "blow up")