1 Introduction

2 Hellinger-Reissner weak form for elastic thin plate problems

2.1 Kinematics

Consider a thin plate with the thickness h, as shown in figure ??, Accordance with Kirchhoff's hypothesis, the displacement for a thin plate denoted by $\hat{\boldsymbol{u}}$ is a linear function in deflection direction:

$$\begin{cases} \hat{u}_\alpha(\boldsymbol{x}) = u_\alpha(x_1, x_2) - x_3 w_{,\alpha} & \alpha = 1, 2 \\ \hat{u}_3(\boldsymbol{x}) = w(x_1, x_2) \end{cases} \tag{1}$$

with the xiaobianxing assumption, the components of strain tensor is given by:

$$\begin{cases} \hat{\varepsilon}_{\alpha\beta} = \frac{1}{2}(\hat{u}_{\alpha,\beta} + \hat{u}_{\beta,\alpha}) = \varepsilon_{\alpha\beta} + x_3\kappa_{\alpha\beta}, & \alpha, \beta = 1, 2 \\ \hat{\varepsilon}_{3i} = \hat{\varepsilon}_{i3} = 0, & i = 1, 2, 3 \end{cases} \tag{2}$$

with

$$\varepsilon_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}), \quad \kappa_{\alpha\beta} = -w_{,\alpha\beta} \tag{3}$$

under this circumstance, the potential prob the problem can be split into two independent problems, the traditional plane stress problem with variable u_α, and the thin plate problem with variable w.

$$\hat{\sigma}_{\alpha\beta} = \mathbb{C}_{\alpha\beta\gamma\eta}\hat{\varepsilon}_{\gamma\eta} = \mathbb{C}_{\alpha\beta\gamma\eta}(\varepsilon_{\gamma\eta} + x_3\kappa_{\gamma\eta}) \tag{4}$$

$$\begin{aligned} \int_{\hat{\Omega}} \frac{1}{2}\hat{\varepsilon}_{ij}\hat{\sigma}_{ij} d\Omega &= \int_\Omega \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{1}{2}(\varepsilon_{\alpha\beta} + x_3\kappa_{\alpha\beta})\mathbb{C}_{\alpha\beta\gamma\eta}(\varepsilon_{\gamma\eta} + x_3\kappa_{\gamma\eta}) dx_3 d\Omega \\ &= \int_\Omega \frac{1}{2}\varepsilon_{\alpha\beta}\sigma_{\alpha\beta} d\Omega + \int_\Omega \frac{1}{2}\kappa_{\alpha\beta}M_{\alpha\beta} d\Omega \end{aligned} \tag{5}$$

where

$$\sigma_{\alpha\beta} = \mathbb{C}_{\alpha\beta\gamma\eta}\varepsilon_{\gamma\eta}, \quad M_{\alpha\beta} = \frac{h^3}{12}\mathbb{C}_{\alpha\beta\gamma\eta}\kappa_{\gamma\eta} \tag{6}$$

2.2 Elasticity problems

For elasticity problems, the Hellinger-Reissner energy functional is given by:

$$\begin{aligned} \mathcal{L}(\boldsymbol{\sigma}, \boldsymbol{u}) = \int_\Omega \frac{1}{2}\boldsymbol{\varepsilon}(\boldsymbol{\sigma}) : \boldsymbol{\sigma} d\Omega - \int_{\Gamma_u} \boldsymbol{n} \cdot \boldsymbol{\sigma} \cdot \bar{\boldsymbol{u}} d\Gamma \\ - \int_{\Gamma_t} \boldsymbol{u} \cdot (\boldsymbol{\sigma} \cdot \boldsymbol{n} - \bar{\boldsymbol{t}}) d\Gamma + \int_\Omega \boldsymbol{u} \cdot (\boldsymbol{\sigma} \cdot \nabla + \bar{\boldsymbol{b}}) d\Omega \end{aligned} \tag{7}$$

where $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ stands for the strain and stress tensor respectively. Γ_u and Γ_t are the essential boundary and natural boundary satisfying that $\Gamma_u \cup \Gamma_t = \partial\Omega$, $\Gamma_u \cap \Gamma_t = \varnothing$. $\bar{\boldsymbol{u}}$ and $\bar{\boldsymbol{t}}$ are the prescribed displacement and traction on Γ_u and Γ_t. $\bar{\boldsymbol{b}}$ denotes to the prescribed body force in Ω. Introducing the standard variation argument to Eq.7 leads to the following HR weak form:

$$\text{find } \boldsymbol{\sigma}, \boldsymbol{u} \in H_1 \quad \begin{cases} a(\delta\boldsymbol{\sigma}, \boldsymbol{\sigma}) + b(\delta\boldsymbol{\sigma}, \boldsymbol{u}) = g(\delta\boldsymbol{\sigma}) & \forall \delta\boldsymbol{\sigma} \in H_1 \\ b(\boldsymbol{\sigma}, \delta\boldsymbol{u}) = f(\delta\boldsymbol{u}) & \forall \delta\boldsymbol{u} \in H_1 \end{cases} \tag{8}$$

where $a : L_2 \times L_2 \to \mathbb{R}$, $b : L_2 \times L_2 \to \mathbb{R}$ are bilinear forms:

$$a(\delta\boldsymbol{\sigma}, \boldsymbol{\sigma}) = \int_\Omega \boldsymbol{\varepsilon}(\delta\boldsymbol{\sigma}) : \boldsymbol{\sigma} d\Omega \tag{9}$$

$$b(\delta\boldsymbol{\sigma}, \boldsymbol{u}) = -\int_\Gamma \boldsymbol{u} \cdot \delta\boldsymbol{\sigma} \cdot \boldsymbol{n} d\Gamma + \int_\Omega \boldsymbol{u} \cdot \delta\boldsymbol{\sigma} \cdot \nabla d\Omega + \int_{\Gamma_u} \boldsymbol{u} \cdot \delta\boldsymbol{\sigma} \cdot \boldsymbol{n} d\Gamma \tag{10}$$

and g, f are the linear operators evaluated by

$$g(\delta\boldsymbol{\sigma}) = \int_{\Gamma_u} \boldsymbol{n} \cdot \delta\boldsymbol{\sigma} \cdot \bar{\boldsymbol{u}} d\Gamma \tag{11}$$

$$f(\delta\boldsymbol{u}) = -\int_{\Gamma_t} \delta\boldsymbol{u} \cdot \bar{\boldsymbol{t}} d\Gamma - \int_\Omega \delta\boldsymbol{u} \cdot \bar{\boldsymbol{b}} d\Omega \tag{12}$$

2.3 Thin plate problems

$$\begin{aligned} \mathcal{L}(\boldsymbol{M}, w) = &\int_\Omega \frac{1}{2} \boldsymbol{\kappa}(\boldsymbol{M}) : \boldsymbol{M} d\Omega \\ &- \int_{\Gamma_w} V_{\boldsymbol{n}}(\boldsymbol{M}) \bar{w} d\Gamma + \int_{\Gamma_\theta} M_{\boldsymbol{nn}}(\boldsymbol{M}) \bar{\theta}_{\boldsymbol{n}} d\Gamma - P(\boldsymbol{M}) \bar{w}|_{\Gamma_c} \\ &+ \int_{\Gamma_M} \theta_{\boldsymbol{n}}(w)(M_{\boldsymbol{nn}} - \bar{M}_{\boldsymbol{nn}}) d\Gamma - \int_{\Gamma_V} w(V_{\boldsymbol{n}} - \bar{V}_{\boldsymbol{n}}) d\Gamma \\ &- w(P - \bar{P})|_{\Gamma_p} + \int_\Omega w(\nabla \cdot \boldsymbol{M} \cdot \nabla + \bar{q}) d\Omega \end{aligned} \tag{13}$$

$$\text{find } \boldsymbol{M}, w \in H_2 \quad \begin{cases} a(\delta\boldsymbol{M}, \boldsymbol{M}) + b(\delta\boldsymbol{M}, w) = g(\delta\boldsymbol{M}) & \forall \delta\boldsymbol{M} \in L_2 \\ b(\boldsymbol{M}, \delta w) = f(\delta w) & \forall \delta w \in H_2 \end{cases} \tag{14}$$

$$a(\delta\boldsymbol{M}, \boldsymbol{M}) = \int_\Omega \boldsymbol{\kappa}(\delta\boldsymbol{M}) : \boldsymbol{M} d\Omega \tag{15}$$

3 MLS/RK meshfree approximation

In accordance with Moving Least Square approximation (MLS) [1] or Reproducing Kernel approximation (RK) [2], the domain Ω is discrete by a set of meshfree nodes $\{\boldsymbol{x}_I\}_{I=1}^{n_p}$, n_p is the total number of meshfree nodes. And then, a variable u in Ω can be approximated as follow:

$$u^h(\boldsymbol{x}) = \sum_{I=1}^{n_p} \Psi_I(\boldsymbol{x}) d_I \tag{16}$$

where Ψ_I and d_I are the meshfree shape function and the nodal coefficient associated with meshfree node $\boldsymbol{x}_I$. This RK shape function Ψ_I is constructed by the undetermined coefficient vector $\boldsymbol{c}$, the basis function vector $\boldsymbol{p}$ and the kernel function ϕ:

$$\Psi_I(\boldsymbol{x}) = \boldsymbol{c}^T(\boldsymbol{x})\boldsymbol{p}(\boldsymbol{x}_I - \boldsymbol{x})\phi(\boldsymbol{x}_I - \boldsymbol{x}) \tag{17}$$

For instance in 2D case, the basis function vector $\boldsymbol{p}$ contains the pth order complete monomials:

$$\boldsymbol{p}(\boldsymbol{x}) = \{1,\ x,\ y,\ x^2,\ \ldots,\ y^p\}^T \tag{18}$$

The kernel function ϕ determines the support size and continuity of shape function, and the cubic and quintic spline functions is chosen herein for elasticity problems and thin plate problems respectively.

$$\phi(\boldsymbol{x}_I - \boldsymbol{x}) = \varphi(\frac{x_I - x}{h_x})\varphi(\frac{y_I - y}{h_y}) \tag{19}$$

where h_i is the

- Cubic spline function:

$$\varphi(r) = \frac{1}{3!}\begin{cases} (2-2r)^3 - 4(1-2r)^3, & r \le \frac{1}{2} \\ (2-2r)^3, & \frac{1}{2} < r \le 1 \\ 0, & r > 1 \end{cases} \tag{20}$$

- Quintic spline function:

$$\varphi(r) = \frac{1}{5!}\begin{cases} (3-3r)^5 - 6(2-3r)^5 + 15(1-3r)^5, & r \le \frac{1}{3} \\ (3-3r)^5 - 6(2-3r)^5, & \frac{1}{3} < r \le \frac{2}{3} \\ (3-3r)^5, & \frac{2}{3} < r \le 1 \\ 0, & r > 1 \end{cases} \tag{21}$$

in which h_i is the support size in x_i axis direction.

The undetermined vector $\boldsymbol{c}$ can be attain by enforcing the following consistency condition:

$$\sum_{I=1}^{n_p} \Psi_I(\boldsymbol{x})\boldsymbol{p}(\boldsymbol{x}_I) = \boldsymbol{p}(\boldsymbol{x}) \tag{22}$$

or with a shift-form

$$\sum_{I=1}^{n_p} \Psi_I(\boldsymbol{x})\boldsymbol{p}(\boldsymbol{x}_I - \boldsymbol{x}) = \boldsymbol{p}(\boldsymbol{0}) \tag{23}$$

substituting Eq. 17 into the consistency condition leads to:

$$\boldsymbol{c}(\boldsymbol{x}) = \boldsymbol{A}^{-1}(\boldsymbol{x})\boldsymbol{p}(\boldsymbol{0}) \tag{24}$$

with moment matrix $\boldsymbol{A}$

$$\boldsymbol{A}(\boldsymbol{x}) = \sum_{I=1}^{n_p} \boldsymbol{p}^T(\boldsymbol{x}_I - \boldsymbol{x})\boldsymbol{p}(\boldsymbol{x}_I - \boldsymbol{x})\phi(\boldsymbol{x}_I - \boldsymbol{x}) \tag{25}$$

It is noted that, to ensure the invertibility of moment matrix, a well-posed meshfree nodal distribution should be required, i.e. there should be sufficient meshfree nodes be covered by their support, as shown in Fig. **??**. Under this circumstance,

$$\|u - u^i\|_{H_k} \leq Ch^{p-k+1}|u|_{H_{p+1}}, \quad \forall k \leq p+1 \tag{26}$$

In Hellinger-Reissner reproducing kernel gradient smoothing framework, the displacement $\boldsymbol{u}$ is approximated by traditional meshfree shape functions, namely $\boldsymbol{u}^h$:

$$\boldsymbol{u}^h(\boldsymbol{x}) = \sum_{I=1}^{n_p} \Psi_I(\boldsymbol{x})\boldsymbol{d}_I \tag{27}$$

On the other hand, the components of stress tensor is assumed as a polynomial in each background cells:

$$\sigma_{ij}^h(\boldsymbol{x}) = \boldsymbol{q}^T(\boldsymbol{x})\boldsymbol{c}_{ij}, \quad \text{in } \Omega_C \tag{28}$$

with

$$\boldsymbol{q}(\boldsymbol{x}) = \{1,\ x,\ y,\ \ldots,\ y^{p-1}\}^T \tag{29}$$

Theorem 1.

$$\sup_{\boldsymbol{\sigma}^h \in Q} \|\boldsymbol{\sigma} - \boldsymbol{\sigma}^h\|_e \leq Ch^p|\boldsymbol{\sigma}| \tag{30}$$

Proof.

$$\sigma_{ij}(\boldsymbol{x}) = \sigma_{ij}(\boldsymbol{0}) + x\sigma_{ij} \tag{31}$$

□

4 Error analysis for HR gradient smoothing meshfree formulation

Appendix A

In this appendix, we define some funcitional operator used in the main sections. Firstly, the energy norms for elasticity problems and thin plate problems are defined by:

$$\|\boldsymbol{\sigma}\|_e = \frac{1}{2}a_2(\boldsymbol{\sigma}, \boldsymbol{\sigma}) = \int_\Omega \frac{1}{2}\boldsymbol{\varepsilon}(\boldsymbol{\sigma}) : \boldsymbol{\sigma} d\Omega \tag{32}$$

$$\|\boldsymbol{M}\|_e = \frac{1}{2}a_4(\boldsymbol{M}, \boldsymbol{M}) = \int_\Omega \frac{1}{2}\boldsymbol{\kappa}(\boldsymbol{M}) : \boldsymbol{M} d\Omega \tag{33}$$

References

[1] T. Belytschko, Y. Y. Lu, L. Gu, Element-free Galerkin methods 37 (2) 229–256.

[2] W. K. Liu, S. Jun, Y. F. Zhang, Reproducing kernel particle methods 20 (8-9) 1081–1106.