# 1 Introduction

# 2 MLS/RK meshfree approximation

In accordance with Moving Least Square approximation (MLS) [1] or Reproducing Kernel approximation (RK) [2], the domain $\Omega$ is discrete by a set of meshfree nodes $\{\boldsymbol{x}_I\}_{I=1}^{n_p}$, $n_p$ is the total number of meshfree nodes. And then, a variable $u$ in $\Omega$ can be approximated as follow:

$$u^h(\boldsymbol{x}) = \sum_{I=1}^{n_p} \Psi_I(\boldsymbol{x}) d_I \tag{1}$$

where $\Psi_I$ and $d_I$ are the meshfree shape function and the nodal coefficient associated with meshfree node $\boldsymbol{x}_I$. This RK shape function $\Psi_I$ is constructed by the undetermined coefficient vector $\boldsymbol{c}$, the basis function vector $\boldsymbol{p}$ and the kernel function $\phi$:

$$\Psi_I(\boldsymbol{x}) = \boldsymbol{c}^T(\boldsymbol{x})\boldsymbol{p}(\boldsymbol{x}_I - \boldsymbol{x})\phi(\boldsymbol{x}_I - \boldsymbol{x}) \tag{2}$$

For instance in 2D case, the basis function vector $\boldsymbol{p}$ contains the $p$th order complete monomials:

$$\boldsymbol{p}(\boldsymbol{x}) = \{1,\ x,\ y,\ x^2,\ \ldots,\ y^p\}^T \tag{3}$$

The kernel function $\phi$ determines the support size and continuity of shape function, and the cubic and quintic spline functions is chosen herein for elasticity problems and thin plate problems respectively.

$$\phi(\boldsymbol{x}_I - \boldsymbol{x}) = \varphi(\frac{x_I - x}{h_x})\varphi(\frac{y_I - y}{h_y}) \tag{4}$$

where $h_i$ is the

- Cubic spline function:

$$\varphi(r) = \frac{1}{3!}\begin{cases} (2-2r)^3 - 4(1-2r)^3, & r \leq \frac{1}{2} \\ (2-2r)^3, & \frac{1}{2} < r \leq 1 \\ 0, & r > 1 \end{cases} \tag{5}$$

- Quintic spline function:

$$\varphi(r) = \frac{1}{5!}\begin{cases} (3-3r)^5 - 6(2-3r)^5 + 15(1-3r)^5, & r \leq \frac{1}{3} \\ (3-3r)^5 - 6(2-3r)^5, & \frac{1}{3} < r \leq \frac{2}{3} \\ (3-3r)^5, & \frac{2}{3} < r \leq 1 \\ 0, & r > 1 \end{cases} \tag{6}$$

in which $h_i$ is the support size in $x_i$ axis direction.

The undetermined vector $\boldsymbol{c}$ can be attain by enforcing the following consistency condition:

$$\sum_{I=1}^{n_p} \Psi_I(\boldsymbol{x})\boldsymbol{p}(\boldsymbol{x}_I) = \boldsymbol{p}(\boldsymbol{x}) \tag{7}$$

or with a shift-form

$$\sum_{I=1}^{n_p} \Psi_I(\boldsymbol{x})\boldsymbol{p}(\boldsymbol{x}_I - \boldsymbol{x}) = \boldsymbol{p}(\boldsymbol{0}) \tag{8}$$

substituting Eq. 2 into the consistency condition leads to:

$$\boldsymbol{c}(\boldsymbol{x}) = \boldsymbol{A}^{-1}(\boldsymbol{x})\boldsymbol{p}(\boldsymbol{0}) \tag{9}$$

with moment matrix $\boldsymbol{A}$

$$\boldsymbol{A}(\boldsymbol{x}) = \sum_{I=1}^{n_p} \boldsymbol{p}^T(\boldsymbol{x}_I - \boldsymbol{x})\boldsymbol{p}(\boldsymbol{x}_I - \boldsymbol{x})\phi(\boldsymbol{x}_I - \boldsymbol{x}) \tag{10}$$

It is noted that, to ensure the invertibility of moment matrix, a well-posed meshfree nodal distribution should be required, i.e. there should be sufficient meshfree nodes be covered by their support, as shown in Fig. **??**. Under this circumstance,

$$\|u - u^i\|_{H_k} \leq Ch^{p-k+1}|u|_{H_{p+1}}, \quad \forall k \leq p+1 \tag{11}$$

# 3 Hellinger-Reissner based RK gradient smoothing meshfree formulation

In this section, the Hellinger-Reissner reproducing kernel gradient smoothing meshfree formulations for second order elasticity problems and forth order thin plate problems are briefly introduced here.

## 3.1 Elasticity problems

For elasticity problems, the Hellinger-Reissner energy functional is given by:

$$\begin{aligned} \mathcal{L}(\boldsymbol{\sigma}, \boldsymbol{u}) = \int_\Omega \frac{1}{2}\boldsymbol{\varepsilon}(\boldsymbol{\sigma}) : \boldsymbol{\sigma} d\Omega - \int_{\Gamma_u} \boldsymbol{n} \cdot \boldsymbol{\sigma} \cdot \bar{\boldsymbol{u}} d\Gamma \\ - \int_{\Gamma_t} \boldsymbol{u} \cdot (\boldsymbol{\sigma} \cdot \boldsymbol{n} - \bar{\boldsymbol{t}}) d\Gamma + \int_\Omega \boldsymbol{u} \cdot (\boldsymbol{\sigma} \cdot \nabla + \bar{\boldsymbol{b}}) d\Omega \end{aligned} \tag{12}$$

where $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ stands for the strain and stress tensor respectively. $\Gamma_u$ and $\Gamma_t$ are the essential boundary and natural boundary satisfying that $\Gamma_u \cup \Gamma_t = \partial\Omega$, $\Gamma_u \cap \Gamma_t = \varnothing$. $\bar{\boldsymbol{u}}$ and $\bar{\boldsymbol{t}}$ are the prescribed displacement and traction on $\Gamma_u$

and $\Gamma_t$. $\bar{\boldsymbol{b}}$ denotes to the prescribed body force in $\Omega$. Introducing the standard variation argument to Eq.12 leads to the following HR weak form:

$$\text{find } \boldsymbol{\sigma}, \boldsymbol{u} \in H_1 \quad \begin{aligned} a(\delta\boldsymbol{\sigma}, \boldsymbol{\sigma}) + b(\delta\boldsymbol{\sigma}, \boldsymbol{u}) &= g(\delta\boldsymbol{\sigma}) \quad \forall \delta\boldsymbol{\sigma} \in H_1 \\ b(\boldsymbol{\sigma}, \delta\boldsymbol{u}) &= f(\delta\boldsymbol{u}) \quad \forall \delta\boldsymbol{u} \in H_1 \end{aligned} \tag{13}$$

where $a : L_2 \times L_2 \to \mathbb{R}$, $b : L_2 \times L_2 \to \mathbb{R}$ are bilinear forms:

$$a(\delta\boldsymbol{\sigma}, \boldsymbol{\sigma}) = \int_\Omega \boldsymbol{\varepsilon}(\delta\boldsymbol{\sigma}) : \boldsymbol{\sigma} d\Omega \tag{14}$$

$$b(\boldsymbol{\sigma}, \boldsymbol{u}) = -\int_\Gamma \boldsymbol{u} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{n} d\Gamma + \int_\Omega \boldsymbol{u} \cdot \boldsymbol{\sigma} \cdot \nabla d\Omega + \int_{\Gamma_u} \boldsymbol{u} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{n} d\Gamma \tag{15}$$

and $g, f$ are the linear operators evaluated by

$$g(\delta\boldsymbol{\sigma}) = \int_{\Gamma_u} \boldsymbol{n} \cdot \delta\boldsymbol{\sigma} \cdot \bar{\boldsymbol{u}} d\Gamma \tag{16}$$

$$f(\delta\boldsymbol{u}) = -\int_{\Gamma_t} \delta\boldsymbol{u} \cdot \bar{\boldsymbol{t}} d\Gamma - \int_\Omega \delta\boldsymbol{u} \cdot \bar{\boldsymbol{b}} d\Omega \tag{17}$$

In Hellinger-Reissner reproducing kernel gradient smoothing framework, the displacement $\boldsymbol{u}$ is approximated by traditional meshfree shape functions, namely $\boldsymbol{u}^h$:

$$\boldsymbol{u}^h(\boldsymbol{x}) = \sum_{I=1}^{n_p} \Psi_I(\boldsymbol{x}) \boldsymbol{d}_I \tag{18}$$

On the other hand, the components of stress tensor is assumed as a polynomial in each background cells:

$$\sigma_{ij}^h(\boldsymbol{x}) = \boldsymbol{q}^T(\boldsymbol{x}) \boldsymbol{c}_{ij}, \quad \text{in } \Omega_C \tag{19}$$

with

$$\boldsymbol{q}(\boldsymbol{x}) = \{1,\ x,\ y,\ \ldots,\ y^{p-1}\}^T \tag{20}$$

**Theorem 1.**

$$\sup_{\boldsymbol{\sigma}^h \in Q} \|\boldsymbol{\sigma} - \boldsymbol{\sigma}^h\|_e \leq Ch^p |\boldsymbol{\sigma}| \tag{21}$$

*Proof.*

$$\sigma_{ij}(\boldsymbol{x}) = \sigma_{ij}(\boldsymbol{0}) + x\sigma_{ij} \tag{22}$$

□

### 3.2 Thin plate problems

$$\begin{aligned}\mathcal{L}(\boldsymbol{M}, w) = &\int_\Omega \frac{1}{2}\boldsymbol{\kappa}(\boldsymbol{M}) : \boldsymbol{M} d\Omega \\ &- \int_{\Gamma_w} V_{\boldsymbol{n}}(\boldsymbol{M})\bar{w} d\Gamma + \int_{\Gamma_\theta} M_{\boldsymbol{nn}}(\boldsymbol{M})\bar{\theta}_{\boldsymbol{n}} d\Gamma - P(\boldsymbol{M})\bar{w}|_{\Gamma_c} \\ &+ \int_{\Gamma_M} \theta_{\boldsymbol{n}}(w)(M_{\boldsymbol{nn}} - \bar{M}_{\boldsymbol{nn}}) d\Gamma - \int_{\Gamma_V} w(V_{\boldsymbol{n}} - \bar{V}_{\boldsymbol{n}}) d\Gamma \\ &- w(P - \bar{P})|_{\Gamma_p} + \int_\Omega w(\nabla \cdot \boldsymbol{M} \cdot \nabla + \bar{q}) d\Omega\end{aligned} \tag{23}$$

$$\text{find } \boldsymbol{M}, w \in H_2 \qquad \begin{aligned} a(\delta\boldsymbol{M}, \boldsymbol{M}) + b(\delta\boldsymbol{M}, w) = g(\delta\boldsymbol{M}) \quad & \forall \delta\boldsymbol{M} \in L_2 \\ b(\boldsymbol{M}, \delta w) = f(\delta w) \quad & \forall \delta w \in H_2 \end{aligned} \tag{24}$$

$$a(\delta\boldsymbol{M}, \boldsymbol{M}) = \int_\Omega \boldsymbol{\kappa}(\delta\boldsymbol{M}) : \boldsymbol{M} d\Omega \tag{25}$$

# 4 Error analysis for HR gradient smoothing mesh-free formulation

# Appendix A

In this appendix, we define some funcitional operator used in the main sections. Firstly, the energy norms for elasticity problems and thin plate problems are defined by:

$$\|\boldsymbol{\sigma}\|_e = \frac{1}{2}a_2(\boldsymbol{\sigma}, \boldsymbol{\sigma}) = \int_\Omega \frac{1}{2}\boldsymbol{\varepsilon}(\boldsymbol{\sigma}) : \boldsymbol{\sigma} d\Omega \tag{26}$$

$$\|\boldsymbol{M}\|_e = \frac{1}{2}a_4(\boldsymbol{M}, \boldsymbol{M}) = \int_\Omega \frac{1}{2}\boldsymbol{\kappa}(\boldsymbol{M}) : \boldsymbol{M} d\Omega \tag{27}$$

# References


[1] T. Belytschko, Y. Y. Lu, L. Gu, Element-free Galerkin methods 37 (2) 229–256.

[2] W. K. Liu, S. Jun, Y. F. Zhang, Reproducing kernel particle methods 20 (8-9) 1081–1106.