1. Introduction

2. Hu-Washizu's formulation of complementary energy for thin shell

2.1. Kinematics for thin shell

$$\varepsilon_{\alpha\beta} = \frac{1}{2}(\boldsymbol{a}_\alpha \cdot \boldsymbol{v}_{,\beta} + \boldsymbol{a}_\beta \cdot \boldsymbol{v}_{,\alpha}) \tag{1}$$

$$\theta_{\boldsymbol{n}} = \boldsymbol{a}_3 \cdot \boldsymbol{v}_{,\alpha} n^\alpha \tag{2}$$

$$\kappa_{\alpha\beta} = (\Gamma^\gamma_{\alpha\beta}\boldsymbol{v}_{,\gamma} - \boldsymbol{v}_{,\alpha\beta}) \cdot \boldsymbol{a}_3 = -\boldsymbol{v}_{,\alpha}|_\beta \cdot \boldsymbol{a}_3 \tag{3}$$

$$\boldsymbol{t} = \boldsymbol{t}_N + \boldsymbol{t}_M \tag{4}$$

$$\boldsymbol{t}_N = \boldsymbol{a}_\alpha N^{\alpha\beta} n_\beta \tag{5}$$

$$\boldsymbol{t}_M = (\boldsymbol{a}_3 M^{\alpha\beta})|_\beta n_\alpha + (\boldsymbol{a}_3 M^{\alpha\beta} s_\alpha n_\beta)_{,\gamma} s^\gamma \tag{6}$$

$$M_{\boldsymbol{nn}} = M^{\alpha\beta} n_\alpha n_\beta \tag{7}$$

$$\boldsymbol{b} = \boldsymbol{b}_N + \boldsymbol{b}_M \tag{8}$$

$$\boldsymbol{b}_N = (\boldsymbol{a}_\alpha N^{\alpha\beta})|_\beta \tag{9}$$

$$\boldsymbol{b}_M = (\boldsymbol{a}_3 M^{\alpha\beta})_{,\alpha}|_\beta \tag{10}$$

$$P = -[[M^{\alpha\beta} s_\alpha n_\beta]] \tag{11}$$

2.2. Galerkin weak form for Hu-Washizu principle of complementary energy

In accordance with the Hu-Washizu variational principle of complementary energy [1], the corresponding complementary functional, denoted by Π, is listed as follow:

$$\begin{aligned}
&\Pi(\boldsymbol{v}, \varepsilon_{\alpha\beta}, \kappa_{\alpha\beta}, N^{\alpha\beta}, M^{\alpha\beta}) \\
&= \int_\Omega \frac{h}{2}\varepsilon_{\alpha\beta} C^{\alpha\beta\gamma\eta} \varepsilon_{\gamma\eta} d\Omega + \int_\Omega \frac{h^3}{24}\kappa_{\alpha\beta} C^{\alpha\beta\gamma\eta}\kappa_{\gamma\eta} d\Omega \\
&+ \int_\Omega \varepsilon_{\alpha\beta}(N^{\alpha\beta} - hC^{\alpha\beta\gamma\eta}\varepsilon_{\gamma\eta}) d\Omega + \int_\Omega \kappa_{\alpha\beta}(M^{\alpha\beta} - \frac{h^3}{12}C^{\alpha\beta\gamma\eta}\kappa_{\gamma\eta}) d\Omega \\
&- \int_{\Gamma_v} \boldsymbol{t} \cdot \bar{\boldsymbol{v}} d\Gamma + \int_{\Gamma_\theta} M_{\boldsymbol{nn}} \bar{\theta}_{\boldsymbol{n}} d\Gamma - (P\boldsymbol{a}_3 \cdot \bar{\boldsymbol{v}})_{\boldsymbol{x}\in C_w} \\
&+ \int_{\Gamma_M} \theta_{\boldsymbol{n}}(M_{\boldsymbol{nn}} - \bar{M}_{\boldsymbol{nn}}) d\Gamma - \int_{\Gamma_t} \boldsymbol{v} \cdot (\boldsymbol{t} - \bar{\boldsymbol{t}}) d\Gamma - \boldsymbol{v} \cdot \boldsymbol{a}_3 (P - \bar{P})_{\boldsymbol{x}\in C_P} \\
&- \int_\Omega \boldsymbol{v} \cdot (\boldsymbol{b} - \bar{\boldsymbol{b}}) d\Omega
\end{aligned} \tag{12}$$

Introducing a standard variational argument to Eq. (12), $\delta\Pi = 0$, and considering the arbitrariness of virtual variables, $\delta\boldsymbol{v}$, $\delta\varepsilon_{\alpha\beta}$, $\delta\kappa_{\alpha\beta}$, $N^{\alpha\beta}$, $M^{\alpha\beta}$ lead to the following weak form:

$$-\int_\Omega h\delta\varepsilon_{\alpha\beta}C^{\alpha\beta\gamma\eta}\varepsilon_{\gamma\eta}d\Omega + \int_\Omega \delta\varepsilon_{\alpha\beta}N^{\alpha\beta}d\Omega = 0 \tag{13a}$$

$$-\int_\Omega \frac{h^3}{12}\delta\kappa_{\alpha\beta}C^{\alpha\beta\gamma\eta}\kappa_{\gamma\eta}d\Omega + \int_\Omega \delta\kappa_{\alpha\beta}M^{\alpha\beta}d\Omega = 0 \tag{13b}$$

$$\begin{aligned}\int_\Omega \delta N^{\alpha\beta}\varepsilon_{\alpha\beta}d\Omega - \int_\Gamma \delta\boldsymbol{t}_N\cdot\boldsymbol{v}d\Gamma + \int_\Omega \delta\boldsymbol{b}_N\cdot\boldsymbol{v}d\Omega& \\ + \int_{\Gamma_v}\delta\boldsymbol{t}_N\cdot\boldsymbol{v}d\Gamma = \int_{\Gamma_v}\delta\boldsymbol{t}_N\cdot\bar{\boldsymbol{v}}d\Gamma&\end{aligned} \tag{13c}$$

$$\begin{aligned}\int_\Omega \delta M^{\alpha\beta}\kappa_{\alpha\beta}d\Omega - \int_\Gamma \delta M_{\boldsymbol{nn}}\theta_{\boldsymbol{n}}d\Gamma + \int_\Gamma \delta\boldsymbol{t}_M\cdot\boldsymbol{v}d\Gamma + (\delta P\boldsymbol{a}_3\cdot\boldsymbol{v})_{\boldsymbol{x}\in C} + \int_\Omega \delta\boldsymbol{b}_M\cdot\boldsymbol{v}d\Omega \\ + \int_{\Gamma_\theta}\delta M_{\boldsymbol{nn}}\theta_{\boldsymbol{n}}d\Gamma - \int_{\Gamma_v}\delta\boldsymbol{t}_M\cdot\boldsymbol{v}d\Gamma - (\delta P\boldsymbol{a}_3\cdot\boldsymbol{v})_{\boldsymbol{x}\in C_v} \\ = \int_{\Gamma_\theta}\delta M_{\boldsymbol{nn}}\bar{\theta}_{\boldsymbol{n}}d\Gamma - \int_{\Gamma_v}\delta\boldsymbol{t}_M\cdot\bar{\boldsymbol{v}}d\Gamma - (\delta P\boldsymbol{a}_3\cdot\bar{\boldsymbol{v}})_{\boldsymbol{x}\in C_v}\end{aligned} \tag{13d}$$

$$\begin{aligned}\int_\Gamma \delta\theta_{\boldsymbol{n}}M_{\boldsymbol{nn}}d\Gamma - \int_\Gamma \delta\boldsymbol{v}\cdot\boldsymbol{t}d\Gamma - (\delta\boldsymbol{v}\cdot\boldsymbol{a}_3P)_{\boldsymbol{x}\in C} + \int_\Omega \delta\boldsymbol{v}\cdot\boldsymbol{b}d\Omega \\ -\int_{\Gamma_\theta}\delta\theta_{\boldsymbol{n}}M_{\boldsymbol{nn}}d\Gamma + \int_{\Gamma_v}\delta\boldsymbol{v}\cdot\boldsymbol{t}d\Gamma + (\delta\boldsymbol{v}\cdot\boldsymbol{a}_3P)_{\boldsymbol{x}\in C_v} = -\int_{\Gamma_t}\delta\boldsymbol{v}\cdot\bar{\boldsymbol{t}}d\Gamma - \int_\Omega \delta\boldsymbol{v}\cdot\bar{\boldsymbol{b}}d\Omega\end{aligned} \tag{13e}$$

where the geometric relationships of Eq. () is used herein.

3. Mixed meshfree formulation for modified Hellinger-Reissner weak form

3.1. Reproducing kernel approximation for displacement

In this study, the displacement is approximated by traditional reproducing kernel approximation. As shown in Fig,

$$\boldsymbol{v}(\boldsymbol{\xi}) = \sum_{I=1}^{n_p} \Psi_I(\boldsymbol{\xi}) \boldsymbol{d}_I \tag{14}$$

3.2. Reproducing kernel gradient smoothing approximation for effective stress and strain

In Galerkin meshfree formulation, the mid-plane of thin shell Ω is split by a set of integration cells Ω_C's, $\cup_{C=1}^{n_e} \Omega_C \approx \Omega$. With the inspiration of reproducing kernel smoothing framework, the Cartesian and covariant derivatives of displacement, $\boldsymbol{v}_{,\alpha}$ and $-\boldsymbol{v}_{,\alpha}|_\beta$, in strains $\varepsilon_{\alpha\beta}$, $\kappa_{\alpha\beta}$ are approximated by $(p-1)$-th order polynomials in each integration cells. In integration cell Ω_C, the approximated derivatives and strains denoted by $\boldsymbol{v}^h_{,\alpha}$, $\varepsilon^h_{\alpha\beta}$ and $-\boldsymbol{v}^h_{,\alpha}|_\beta$, $\kappa^h_{\alpha\beta}$ can be expressed by:

$$\boldsymbol{v}^h_{,\alpha}(\boldsymbol{\xi}) = \boldsymbol{q}^T(\boldsymbol{\xi}) \boldsymbol{d}^\varepsilon_\alpha, \quad \varepsilon^h_{\alpha\beta}(\boldsymbol{\xi}) = \boldsymbol{q}^T(\boldsymbol{\xi}) \frac{1}{2} (\boldsymbol{a}_\alpha \cdot \boldsymbol{d}^\varepsilon_\beta + \boldsymbol{a}_\beta \cdot \boldsymbol{d}^\varepsilon_\alpha) \tag{15}$$

$$-\boldsymbol{v}^h_{,\alpha}|_\beta(\boldsymbol{\xi}) = \boldsymbol{q}^T(\boldsymbol{\xi}) \boldsymbol{d}^\kappa_{\alpha\beta}, \quad \kappa^h_{\alpha\beta}(\boldsymbol{\xi}) = -\boldsymbol{q}^T(\boldsymbol{\xi}) \boldsymbol{a}_3 \cdot \boldsymbol{d}^\kappa_{\alpha\beta} \tag{16}$$

where $\boldsymbol{q}$ is the $(p-1)$th order polynomial vector and has the following form:

$$\boldsymbol{q} = \{1,\ \xi^1,\ \xi^2,\ \dots, (\xi^2)^{p-1}\}^T \tag{17}$$

and the $\boldsymbol{d}^\varepsilon_\alpha$, $\boldsymbol{d}^\kappa_{\alpha\beta}$ are the corresponding coefficient vector tensors. For the conciseness, the mixed usage of tensor and vector is introduced in this study, for example, the component of coefficient tensor vector $\boldsymbol{d}^\varepsilon_{\alpha I}$, $\boldsymbol{d}^\varepsilon_\alpha = \{\boldsymbol{d}^\varepsilon_{\alpha I}\}$, is a three dimensional tensor, $\dim \boldsymbol{d}^\varepsilon_{\alpha I} = \dim \boldsymbol{v}$.

In order to meet the integration constraint of thin shell problem, the approximated stresses $N^{\alpha\beta h}$, $M^{\alpha\beta h}$ are assumed to be a similar form with strains, yields:

$$N^{\alpha\beta h}(\boldsymbol{\xi}) = \boldsymbol{q}^T(\boldsymbol{\xi}) \boldsymbol{a}^\alpha \cdot \boldsymbol{d}^N_\beta, \quad \boldsymbol{a}_\alpha N^{\alpha\beta h}(\boldsymbol{\xi}) = \boldsymbol{q}^T(\boldsymbol{\xi}) \boldsymbol{d}^N_\beta \tag{18}$$

$$M^{\alpha\beta h}(\boldsymbol{\xi}) = \boldsymbol{q}^T(\boldsymbol{\xi}) \boldsymbol{a}_3 \cdot \boldsymbol{d}^M_{\alpha\beta}, \quad \boldsymbol{a}_3 M^{\alpha\beta h}(\boldsymbol{\xi}) = \boldsymbol{q}^T(\boldsymbol{\xi}) \boldsymbol{d}^M_{\alpha\beta} \tag{19}$$

substituting the approximations of Eqs. (14), (15), (16), (18), (19) into Eqs. (13c), (13d) can express $\boldsymbol{d}^\varepsilon_\beta$ and $\boldsymbol{d}^\kappa_{\alpha\beta}$ by $\boldsymbol{d}$ as:

$$\boldsymbol{d}^\varepsilon_\beta = \boldsymbol{G}^{-1} \left(\sum_{I=1}^{n_p} (\tilde{\boldsymbol{g}}^\varepsilon_{\beta I} - \bar{\boldsymbol{g}}^\varepsilon_{\beta I}) \boldsymbol{d}_I + \hat{\boldsymbol{g}}^\varepsilon_\beta \right) \tag{20}$$

$$\boldsymbol{d}^\kappa_{\alpha\beta} = \boldsymbol{G}^{-1} \left(\sum_{I=1}^{n_p} (\tilde{\boldsymbol{g}}^\kappa_{\alpha\beta I} - \bar{\boldsymbol{g}}^\kappa_{\alpha\beta I}) \boldsymbol{d}_I + \hat{\boldsymbol{g}}^\kappa_{\alpha\beta} \right) \tag{21}$$

with

$$\boldsymbol{G} = \int_{\Omega_C} \boldsymbol{q}^T \boldsymbol{q} d\Omega \tag{22}$$

$$\tilde{\boldsymbol{g}}^{\varepsilon}_{\beta I} = \int_{\Gamma_C} \Psi_I \boldsymbol{q} n_\beta d\Gamma - \int_{\Omega_C} \Psi_I \boldsymbol{q}^* |_\beta d\Omega \tag{23a}$$

$$\bar{\boldsymbol{g}}^{\varepsilon}_{\beta I} = \int_{\Gamma_C \cap \Gamma_v} \Psi_I \boldsymbol{q} n_\beta d\Gamma \tag{23b}$$

$$\bar{\boldsymbol{g}}^{\varepsilon}_{\beta I} = \int_{\Gamma_C \cap \Gamma_v} \boldsymbol{q} n_\beta \bar{\boldsymbol{v}} d\Gamma \tag{23c}$$

$$\begin{aligned} \tilde{\boldsymbol{g}}^{\kappa}_{\alpha\beta I} = & \int_{\Gamma_C} \Psi_{I,\gamma} n^\gamma \boldsymbol{q} n_\alpha n_\beta d\Gamma - \int_{\Gamma_C} \Psi_I (\boldsymbol{q}^{**}|_\beta n_\alpha + (\boldsymbol{q} s_\alpha n_\beta)_{,\gamma} s^\gamma) d\Gamma \\ & + [[\Psi_I \boldsymbol{q} s_\alpha n_\beta]]_{\boldsymbol{x} \in C_C} - \int_{\Omega_C} \Psi \boldsymbol{q}^{**}_{,\alpha} |_\beta d\Omega \end{aligned} \tag{24a}$$

$$\begin{aligned} \bar{\boldsymbol{g}}^{\kappa}_{\alpha\beta I} = & \int_{\Gamma_C \cap \Gamma_\theta} \Psi_{I,\gamma} n^\gamma \boldsymbol{q} n_\alpha n_\beta d\Gamma - \int_{\Gamma_C \cap \Gamma_v} \Psi_I (\boldsymbol{q}^{**}|_\beta n_\alpha + (\boldsymbol{q} s_\alpha n_\beta)_{,\gamma} s^\gamma) d\Gamma \\ & + [[\Psi_I \boldsymbol{q} s_\alpha n_\beta]]_{\boldsymbol{x} \in C_C \cap C_v} \end{aligned} \tag{24b}$$

$$\begin{aligned} \hat{\boldsymbol{g}}^{\kappa}_{\alpha\beta I} = & \int_{\Gamma_C \cap \Gamma_\theta} \boldsymbol{q} n_\alpha n_\beta \boldsymbol{a}_3 \bar{\theta}_{\boldsymbol{n}} d\Gamma - \int_{\Gamma_C \cap \Gamma_v} (\boldsymbol{q}^{**}|_\beta n_\alpha + (\boldsymbol{q} s_\alpha n_\beta)_{,\gamma} s^\gamma) \bar{\boldsymbol{v}} d\Gamma \\ & + [[\boldsymbol{q} s_\alpha n_\beta \bar{\boldsymbol{v}}]]_{\boldsymbol{x} \in C_C \cap C_v} \end{aligned} \tag{24c}$$

4. Naturally variational enforcement for essential boundary conditions

5. Numerical examples

	Linear patch test		Quadratic patch test	
	L_2-Error	H_e-Error	L_2-Error	H_e-Error
GI-Penalty				
GI-Nitsche				
RKGSI-Penalty				
RKGSI-Nitsche				
RKGSI-HR				

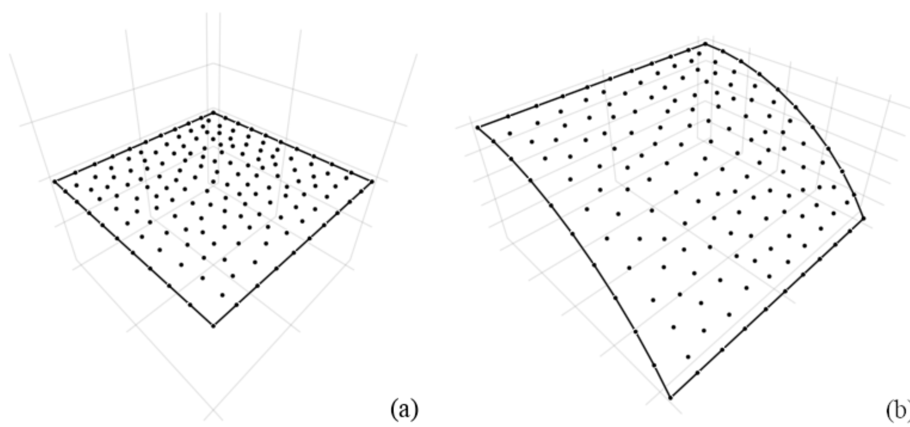

Figure 1: Meshfree discretization for patch test

References

[1] H. Dah-wei, A method for establishing generalized variational principle 6 (6) 501–509. doi:10.1007/BF01876390.
URL http://link.springer.com/10.1007/BF01876390