A principle meshfree formulation for thin shells with naturally consistent enforcement of essential boundary conditions

1. Introduction

Thin shell is one of the most frequently used structure in engineering practice, where the thickness of this kind structure is often much smaller than its radius. With the Kirchhoff-Love hypothesis [1–3], the transverse shear deformation is eliminated in thin shell analysis, such that at least C1 continuous shape functions are required within Galerkin methods. In static and dynamic simulation of structure, the conventional finite element methods [1,2] are one of the most popular approximation scheme, however the construction of C1 continuity is still a big challenge for cell-based finite element methods. In last three decades, the meshfree methods [1–3] equipped high order smoothed shape functions have attracted significant research attention, while the meshfree shape functions are established based upon a set of scattered nodes and the high order continuity of shape functions is easily fulfilled even with low order basis function. For thin shell analysis, this high order meshfree approximations can also alleviate the membrane locking caused by the mismatched approximation order of membrane strain and bending strain [1]. Moreover, in general, the nodal-based meshfree approximations can release the burden of mesh distortion and have the flexibility of local refinement. Due to these advantages, a wide variety meshfree methods are proposed and have been applied to many scientific or engineering fields. Among of them, moving least squares (MLS) and reproducing kernel (RK) meshfree approximations built their shape functions by enforcing the so-call consistency conditions, where the consistency conditions require that the corresponding approximations should exactly reproduce every functions spanned by basis functions, and this conditions usually serve as a basic requirement for the error convergence of resolved Galerkin solutions [1]. However, the high order smoothed meshfree shape functions accompany with the severely overlapping supports, which leads to a misalignment between numerical integration domains and supports of shape functions. As a result, the meshfree shape functions usually exhibit a piecewise rational nature in each integration domains, and it brings a serious difficulty to the accurate numerical integration in Galerkin weak forms [1].

2. Hu-Washizu's formulation of complementary energy for thin shell

2.1. Kinematics for thin shell

Consider the configuration of a shell $\bar{\Omega}$, as shown in Fig. **??**, which can be easily described by a parametric curvilinear coordinate system $\boldsymbol{\xi} = \{\xi^i\}_{i=1,2,3}$. The mid-surface of the shell is specified by the in-plane coordinates $\boldsymbol{\xi} = \{\xi^\alpha\}_{\alpha=1,2}$, as the thickness direction of shell is by ξ^3, $-\frac{h}{2} \le \xi^3 \le \frac{h}{2}$, h is the thick of shell. In this work, Latin indices take 22 the values from 1 to 3, and Greek indices are evaluated by 1 or 2. For the Kirchhoff hypothesis [**?**], the position $\boldsymbol{x} \in \bar{\Omega}$ are defined by linear functions with respect to ξ^3 :

$$\boldsymbol{x}(\xi^1, \xi^2, \xi^3) = \boldsymbol{r}(\xi^1, \xi^2) + \xi^3 \boldsymbol{a}_3(\xi^1, \xi^2) \tag{1}$$

in which $\boldsymbol{r}$ means the position on the mid-surface of shell, and the $\boldsymbol{a}_3$ is corresponding normal direction. For the mid-surface of shell, the in-plane covariant base vector with respect to ξ^α can be derived by a trivial partial differentiation to $\boldsymbol{r}$:

$$\boldsymbol{a}_\alpha = \frac{\partial \boldsymbol{r}}{\partial \xi^\alpha} = \boldsymbol{r}_{,\alpha}, \alpha = 1, 2 \tag{2}$$

for a clear expression, the subscript comma denotes the partial differentiation operation with respect to in-plane coordinates ξ^α. And the normal vector $\boldsymbol{a}_3$ can be obtained by the normalized cross product of $\boldsymbol{a}_\alpha$'s as follow:

$$\boldsymbol{a}_3 = \frac{\boldsymbol{a}_1 \times \boldsymbol{a}_2}{\|\boldsymbol{a}_1 \times \boldsymbol{a}_2\|} \tag{3}$$

where $\| \bullet \|$ is the Euclidean norm operator.

With the assumption of infinitesimal deformation, the strain components respected to global contravariant base can be sated as:

$$\epsilon_{ij} = \frac{1}{2}(\boldsymbol{x}_{,i} \cdot \boldsymbol{u}_{,j} + \boldsymbol{u}_{,i} \cdot \boldsymbol{x}_{,j}) \tag{4}$$

where $\boldsymbol{u}$ is the displacement for shell deformation. To fulfillment with Kirchhoff hypothesis, the displacement is assumed to be the following form:

$$\boldsymbol{u}(\xi^1, \xi^2, \xi^3) = \boldsymbol{v}(\xi^1, \xi^2) + \boldsymbol{\theta}(\xi^1, \xi^2)\xi^3 \tag{5}$$

in which the quadratic and higher order terms are neglected. $\boldsymbol{v}$, $\boldsymbol{\theta}$ respect the displacement and rotation in mid-surface.

Subsequently, plugging Eqs. (1) and (5) into Eq. (4) and neglecting quadratic terms, the strain components can be rephrased as follows:

$$\begin{aligned}\epsilon_{\alpha\beta} &= \frac{1}{2}(\boldsymbol{a}_\alpha \cdot \boldsymbol{v}_{,\beta} + \boldsymbol{v}_{,\alpha} \cdot \boldsymbol{a}_\beta) \\ &\quad + \frac{1}{2}(\boldsymbol{a}_{3,\alpha} \cdot \boldsymbol{v}_{,\beta} + \boldsymbol{v}_{,\alpha} \cdot \boldsymbol{a}_{3,\beta} + \boldsymbol{a}_\alpha \cdot \boldsymbol{\theta}_{,\beta} + \boldsymbol{\theta}_{,\alpha} \cdot \boldsymbol{a}_\beta)\xi^3 \\ &= \varepsilon_{\alpha\beta} + \kappa_{\alpha\beta}\xi^3\end{aligned} \tag{6a}$$

$$\epsilon_{\alpha 3} = \frac{1}{2}(\boldsymbol{a}_\alpha \cdot \boldsymbol{\theta} + \boldsymbol{v}_{,\alpha} \cdot \boldsymbol{a}_3) + \frac{1}{2}(\boldsymbol{a}_3 \cdot \boldsymbol{\theta})_{,\alpha}\xi^3 \tag{6b}$$

$$\epsilon_{33} = \boldsymbol{a}_3 \cdot \boldsymbol{\theta} \tag{6c}$$

where $\varepsilon_{\alpha\beta}$, $\kappa_{\alpha\beta}$ are membrane and bending strains respectively:

$$\varepsilon_{\alpha\beta} = \frac{1}{2}(\boldsymbol{a}_\alpha \cdot \boldsymbol{v}_{,\beta} + \boldsymbol{v}_{,\alpha} \cdot \boldsymbol{a}_\beta) \tag{7}$$

$$\kappa_{\alpha\beta} = \frac{1}{2}(\boldsymbol{a}_{3,\alpha} \cdot \boldsymbol{v}_{,\beta} + \boldsymbol{v}_{,\alpha} \cdot \boldsymbol{a}_{3,\beta} + \boldsymbol{a}_\alpha \cdot \boldsymbol{\theta}_{,\beta} + \boldsymbol{\theta}_{,\alpha} \cdot \boldsymbol{a}_\beta) \tag{8}$$

In accordance with Kirchhoff hypothesis, the thickness of shell will not change and the deformation related with direction of ξ^3 will be vanished, i.e. $\epsilon_{3i} = 0$. Thus, the rotation $\boldsymbol{\theta}$ can be rewritten as:

$$\epsilon_{3i} = 0 \Rightarrow \begin{cases} \boldsymbol{\theta} \cdot \boldsymbol{a}_\alpha + \boldsymbol{v}_{,\alpha} \cdot \boldsymbol{a}_3 = 0 \\ \boldsymbol{\theta} \cdot \boldsymbol{a}_3 = 0 \end{cases} \Rightarrow \boldsymbol{\theta} = -\boldsymbol{v}_{,\alpha} \cdot \boldsymbol{a}_3 \boldsymbol{a}^\alpha \tag{9}$$

where $\boldsymbol{a}^\alpha$'s are the in-plane contravariant base vectors, $\boldsymbol{a}^\alpha \cdot \boldsymbol{a}_\beta = \delta^\alpha_\beta$, δ is the Kronecker delta function. Substituting Eq. (9) into Eq. (8) leads to:

$$\kappa_{\alpha\beta} = (\Gamma^\gamma_{\alpha\beta} \boldsymbol{v}_{,\gamma} - \boldsymbol{v}_{,\alpha\beta}) \cdot \boldsymbol{a}_3 = -\boldsymbol{v}_{,\alpha}|_\beta \cdot \boldsymbol{a}_3 \tag{10}$$

in which $\Gamma^\gamma_{\alpha\beta} = \boldsymbol{a}_{\alpha,\beta} \cdot \boldsymbol{a}^\gamma$ is namely Christoffel symbol of the second kind.

$$\varepsilon_{\alpha\beta} = \frac{1}{2}(\boldsymbol{a}_\alpha \cdot \boldsymbol{v}_{,\beta} + \boldsymbol{a}_\beta \cdot \boldsymbol{v}_{,\alpha}) \tag{11}$$

$$\theta_{\boldsymbol{n}} = \boldsymbol{a}_3 \cdot \boldsymbol{v}_{,\alpha} n^\alpha \tag{12}$$

$$\kappa_{\alpha\beta} = (\Gamma^\gamma_{\alpha\beta} \boldsymbol{v}_{,\gamma} - \boldsymbol{v}_{,\alpha\beta}) \cdot \boldsymbol{a}_3 = -\boldsymbol{v}_{,\alpha}|_\beta \cdot \boldsymbol{a}_3 \tag{13}$$

$$\boldsymbol{t} = \boldsymbol{t}_N + \boldsymbol{t}_M \tag{14}$$

$$\boldsymbol{t}_N = \boldsymbol{a}_\alpha N^{\alpha\beta} n_\beta \tag{15}$$

$$\boldsymbol{t}_M = (\boldsymbol{a}_3 M^{\alpha\beta})|_\beta n_\alpha + (\boldsymbol{a}_3 M^{\alpha\beta} s_\alpha n_\beta)_{,\gamma} s^\gamma \tag{16}$$

$$M_{\boldsymbol{nn}} = M^{\alpha\beta} n_\alpha n_\beta \tag{17}$$

$$\boldsymbol{b} = \boldsymbol{b}_N + \boldsymbol{b}_M \tag{18}$$

$$\boldsymbol{b}_N = (\boldsymbol{a}_\alpha N^{\alpha\beta})|_\beta \tag{19}$$

$$\boldsymbol{b}_M = (\boldsymbol{a}_3 M^{\alpha\beta})_{,\alpha}|_\beta \tag{20}$$

$$P = -[[M^{\alpha\beta} s_\alpha n_\beta]] \tag{21}$$

2.2. Galerkin weak form for Hu-Washizu principle of complementary energy

In accordance with the Hu-Washizu variational principle of complementary energy [1], the corresponding complementary functional, denoted by Π, is listed as follow:

$$
\begin{aligned}
&\Pi(\boldsymbol{v}, \varepsilon_{\alpha\beta}, \kappa_{\alpha\beta}, N^{\alpha\beta}, M^{\alpha\beta}) \\
&= \int_\Omega \frac{h}{2}\varepsilon_{\alpha\beta}C^{\alpha\beta\gamma\eta}\varepsilon_{\gamma\eta}d\Omega + \int_\Omega \frac{h^3}{24}\kappa_{\alpha\beta}C^{\alpha\beta\gamma\eta}\kappa_{\gamma\eta}d\Omega \\
&+ \int_\Omega \varepsilon_{\alpha\beta}(N^{\alpha\beta} - hC^{\alpha\beta\gamma\eta}\varepsilon_{\gamma\eta})d\Omega + \int_\Omega \kappa_{\alpha\beta}(M^{\alpha\beta} - \frac{h^3}{12}C^{\alpha\beta\gamma\eta}\kappa_{\gamma\eta})d\Omega \\
&- \int_{\Gamma_v} \boldsymbol{t}\cdot\bar{\boldsymbol{v}}d\Gamma + \int_{\Gamma_\theta} M_{\boldsymbol{nn}}\bar{\theta}_{\boldsymbol{n}}d\Gamma - (P\boldsymbol{a}_3\cdot\bar{\boldsymbol{v}})_{\boldsymbol{x}\in C_w} \\
&+ \int_{\Gamma_M} \theta_{\boldsymbol{n}}(M_{\boldsymbol{nn}} - \bar{M}_{\boldsymbol{nn}})d\Gamma - \int_{\Gamma_t} \boldsymbol{v}\cdot(\boldsymbol{t}-\bar{\boldsymbol{t}})d\Gamma - \boldsymbol{v}\cdot\boldsymbol{a}_3(P-\bar{P})_{\boldsymbol{x}\in C_P} \\
&- \int_\Omega \boldsymbol{v}\cdot(\boldsymbol{b}-\bar{\boldsymbol{b}})d\Omega
\end{aligned} \tag{22}
$$

Introducing a standard variational argument to Eq. (22), $\delta\Pi = 0$, and considering the arbitrariness of virtual variables, $\delta\boldsymbol{v}$, $\delta\varepsilon_{\alpha\beta}$, $\delta\kappa_{\alpha\beta}$, $N^{\alpha\beta}$, $M^{\alpha\beta}$ lead to the following weak form:

$$
-\int_\Omega h\delta\varepsilon_{\alpha\beta}C^{\alpha\beta\gamma\eta}\varepsilon_{\gamma\eta}d\Omega + \int_\Omega \delta\varepsilon_{\alpha\beta}N^{\alpha\beta}d\Omega = 0 \tag{23a}
$$

$$
-\int_\Omega \frac{h^3}{12}\delta\kappa_{\alpha\beta}C^{\alpha\beta\gamma\eta}\kappa_{\gamma\eta}d\Omega + \int_\Omega \delta\kappa_{\alpha\beta}M^{\alpha\beta}d\Omega = 0 \tag{23b}
$$

$$
\begin{aligned}
&\int_\Omega \delta N^{\alpha\beta}\varepsilon_{\alpha\beta}d\Omega - \int_\Gamma \delta\boldsymbol{t}_N\cdot\boldsymbol{v}d\Gamma + \int_\Omega \delta\boldsymbol{b}_N\cdot\boldsymbol{v}d\Omega \\
&\qquad + \int_{\Gamma_v}\delta\boldsymbol{t}_N\cdot\boldsymbol{v}d\Gamma = \int_{\Gamma_v}\delta\boldsymbol{t}_N\cdot\bar{\boldsymbol{v}}d\Gamma
\end{aligned} \tag{23c}
$$

$$
\begin{aligned}
&\int_\Omega \delta M^{\alpha\beta}\kappa_{\alpha\beta}d\Omega - \int_\Gamma \delta M_{\boldsymbol{nn}}\theta_{\boldsymbol{n}}d\Gamma + \int_\Gamma \delta\boldsymbol{t}_M\cdot\boldsymbol{v}d\Gamma + (\delta P\boldsymbol{a}_3\cdot\boldsymbol{v})_{\boldsymbol{x}\in C} + \int_\Omega \delta\boldsymbol{b}_M\cdot\boldsymbol{v}d\Omega \\
&+ \int_{\Gamma_\theta}\delta M_{\boldsymbol{nn}}\theta_{\boldsymbol{n}}d\Gamma - \int_{\Gamma_v}\delta\boldsymbol{t}_M\cdot\boldsymbol{v}d\Gamma - (\delta P\boldsymbol{a}_3\cdot\boldsymbol{v})_{\boldsymbol{x}\in C_v} \\
&= \int_{\Gamma_\theta}\delta M_{\boldsymbol{nn}}\bar{\theta}_{\boldsymbol{n}}d\Gamma - \int_{\Gamma_v}\delta\boldsymbol{t}_M\cdot\bar{\boldsymbol{v}}d\Gamma - (\delta P\boldsymbol{a}_3\cdot\bar{\boldsymbol{v}})_{\boldsymbol{x}\in C_v}
\end{aligned} \tag{23d}
$$

$$
\begin{aligned}
&\int_{\Gamma} \delta\theta_{\boldsymbol{n}} M_{\boldsymbol{nn}} d\Gamma - \int_{\Gamma} \delta\boldsymbol{v} \cdot \boldsymbol{t} d\Gamma - (\delta\boldsymbol{v} \cdot \boldsymbol{a}_3 P)_{\boldsymbol{x}\in C} + \int_{\Omega} \delta\boldsymbol{v} \cdot \boldsymbol{b} d\Omega \\
&- \int_{\Gamma_\theta} \delta\theta_{\boldsymbol{n}} M_{\boldsymbol{nn}} d\Gamma + \int_{\Gamma_v} \delta\boldsymbol{v} \cdot \boldsymbol{t} d\Gamma + (\delta\boldsymbol{v} \cdot \boldsymbol{a}_3 P)_{\boldsymbol{x}\in C_v} = - \int_{\Gamma_t} \delta\boldsymbol{v} \cdot \bar{\boldsymbol{t}} d\Gamma - \int_{\Omega} \delta\boldsymbol{v} \cdot \bar{\boldsymbol{b}} d\Omega
\end{aligned}
\tag{23e}
$$

where the geometric relationships of Eq. () is used herein.

3. Mixed meshfree formulation for modified Hellinger-Reissner weak form

3.1. Reproducing kernel approximation for displacement

In this study, the displacement is approximated by traditional reproducing kernel approximation. As shown in Fig, the mid-surface of the shell Ω is discretized by a set of meshfree nodes $\{\boldsymbol{\xi}_I\}_{I=1}^{n_p}$ in parametric configuration, where n_p is the total number of meshfree nodes. The approximated displacement namely $\boldsymbol{v}^h$ can be expressed by:

$$\boldsymbol{v}(\boldsymbol{\xi}) = \sum_{I=1}^{n_p} \Psi_I(\boldsymbol{\xi}) \boldsymbol{d}_I \tag{24}$$

in which Ψ_I and $\boldsymbol{d}_I$ is the shape function and nodal coefficient tensor related by node $\boldsymbol{\xi}_I$. According to reproducing kernel approximation [**?**], the shape function takes the following form:

$$\Psi_I(\boldsymbol{\xi}) = \boldsymbol{p}^T(\boldsymbol{\xi}) \boldsymbol{c}(\boldsymbol{\xi}) \phi(\boldsymbol{\xi}_I - \boldsymbol{\xi}) \tag{25}$$

where $\boldsymbol{p}$ is the basis function vector, and in this study, the following quadratic basis function is considered:

$$\boldsymbol{p} = \{1,\ \xi^1,\ \xi^2,\ (\xi^1)^2, \xi^1\xi^2, (\xi^2)^2\}^T \tag{26}$$

The kernel function denoted byϕ controls the support and smoothness of meshfree shape functions. The quantic B-spline function with square support is used herein as the kernel function:

$$\phi(\boldsymbol{\xi}_I - \boldsymbol{\xi}) = \phi(\hat{s}_1)\phi(\hat{s}_2), \quad \hat{s}_\alpha = \frac{|\xi^\alpha_I - \xi^\alpha|}{s_{\alpha I}} \tag{27}$$

with

$$\phi(\hat{s}_\alpha) = \frac{1}{5!} \begin{cases} (3-3\hat{s}_\alpha)^5 - 6(2-3\hat{s}_\alpha)^5 + 15(1-3\hat{s}_\alpha)^5 & \hat{s}_\alpha \leq \frac{1}{3} \\ (3-3\hat{s}_\alpha)^5 - 6(2-3\hat{s}_\alpha)^5 & \frac{1}{3} < \hat{s}_\alpha \leq \frac{2}{3} \\ (3-3\hat{s}_\alpha)^5 & \frac{2}{3} < \hat{s}_\alpha \leq 1 \\ 0 & \hat{s}_\alpha > 1 \end{cases} \tag{28}$$

and $\hat{s}_{\alpha I}$ means the characterized size of support for meshfree shape function Ψ_I.

The unknown vector $\boldsymbol{c}$ in shape function are determined by the fulfillment of the so-call consistency condition:

$$\sum_{I=1}^{n_p} \Psi_I(\boldsymbol{\xi}) \boldsymbol{p}(\boldsymbol{\xi}_I) = \boldsymbol{p}(\boldsymbol{\xi}) \tag{29}$$

or equivalently

$$\sum_{I=1}^{n_p} \Psi_I(\boldsymbol{\xi}) \boldsymbol{p}(\boldsymbol{\xi}_I - \boldsymbol{\xi}) = \boldsymbol{p}(\boldsymbol{0}) \tag{30}$$

Substituting Eq. (24) into (30), yields:

$$\boldsymbol{A}(\boldsymbol{\xi})\boldsymbol{c}(\boldsymbol{\xi}) = \boldsymbol{p}(\boldsymbol{0}) \quad \Rightarrow \quad \boldsymbol{c}(\boldsymbol{\xi}) = \boldsymbol{A}^{-1}(\boldsymbol{\xi})\boldsymbol{p}(\boldsymbol{0}) \tag{31}$$

where $\boldsymbol{A}$ is the moment matrix:

$$\boldsymbol{A}(\boldsymbol{\xi}) = \sum_{I=1}^{n_p} \phi(\boldsymbol{\xi}_I - \boldsymbol{\xi})\boldsymbol{p}(\boldsymbol{\xi}_I - \boldsymbol{\xi})\boldsymbol{p}^T(\boldsymbol{\xi}_I - \boldsymbol{\xi}) \tag{32}$$

Taking Eq. (31) back into Eq. (24), the expression of meshfree shape function can be given by:

$$\Psi_I(\boldsymbol{\xi}) = \boldsymbol{p}^T(\boldsymbol{\xi}_I - \boldsymbol{\xi})\boldsymbol{A}^{-1}(\boldsymbol{\xi})\boldsymbol{p}(\boldsymbol{0})\phi(\boldsymbol{\xi}_I - \boldsymbol{\xi}) \tag{33}$$

3.2. Reproducing kernel gradient smoothing approximation for effective stress and strain

In Galerkin meshfree formulation, the mid-plane of thin shell Ω is split by a set of integration cells Ω_C's, $\cup_{C=1}^{n_e}\Omega_C \approx \Omega$. With the inspiration of reproducing kernel smoothing framework, the Cartesian and covariant derivatives of displacement, $\boldsymbol{v}_{,\alpha}$ and $-\boldsymbol{v}_{,\alpha}|_\beta$, in strains $\varepsilon_{\alpha\beta}$, $\kappa_{\alpha\beta}$ are approximated by $(p-1)$-th order polynomials in each integration cells. In integration cell Ω_C, the approximated derivatives and strains denoted by $\boldsymbol{v}^h_{,\alpha}$, $\varepsilon^h_{\alpha\beta}$ and $-\boldsymbol{v}^h_{,\alpha}|_\beta$, $\kappa^h_{\alpha\beta}$ can be expressed by:

$$\boldsymbol{v}^h_{,\alpha}(\boldsymbol{\xi}) = \boldsymbol{q}^T(\boldsymbol{\xi})\boldsymbol{d}^\varepsilon_\alpha, \quad \varepsilon^h_{\alpha\beta}(\boldsymbol{\xi}) = \boldsymbol{q}^T(\boldsymbol{\xi})\frac{1}{2}(\boldsymbol{a}_\alpha \cdot \boldsymbol{d}^\varepsilon_\beta + \boldsymbol{a}_\beta \cdot \boldsymbol{d}^\varepsilon_\alpha) \tag{34}$$

$$-\boldsymbol{v}^h_{,\alpha}|_\beta(\boldsymbol{\xi}) = \boldsymbol{q}^T(\boldsymbol{\xi})\boldsymbol{d}^\kappa_{\alpha\beta}, \quad \kappa^h_{\alpha\beta}(\boldsymbol{\xi}) = \boldsymbol{q}^T(\boldsymbol{\xi})\boldsymbol{a}_3 \cdot \boldsymbol{d}^\kappa_{\alpha\beta} \tag{35}$$

where $\boldsymbol{q}$ is the $(p-1)$th order polynomial vector and has the following form:

$$\boldsymbol{q} = \{1,\ \xi^1,\ \xi^2,\ \dots, (\xi^2)^{p-1}\}^T \tag{36}$$

and the $\boldsymbol{d}^\varepsilon_\alpha$, $\boldsymbol{d}^\kappa_{\alpha\beta}$ are the corresponding coefficient vector tensors. For the conciseness, the mixed usage of tensor and vector is introduced in this study, for example, the component of coefficient tensor vector $\boldsymbol{d}^\varepsilon_{\alpha I}$, $\boldsymbol{d}^\varepsilon_\alpha = \{\boldsymbol{d}^\varepsilon_{\alpha I}\}$, is a three dimensional tensor, $\dim \boldsymbol{d}^\varepsilon_{\alpha I} = \dim \boldsymbol{v}$.

In order to meet the integration constraint of thin shell problem, the approximated stresses $N^{\alpha\beta h}$, $M^{\alpha\beta h}$ are assumed to be a similar form with strains, yields:

$$N^{\alpha\beta h}(\boldsymbol{\xi}) = \boldsymbol{q}^T(\boldsymbol{\xi})\boldsymbol{a}^\alpha \cdot \boldsymbol{d}^N_\beta, \quad \boldsymbol{a}_\alpha N^{\alpha\beta h}(\boldsymbol{\xi}) = \boldsymbol{q}^T(\boldsymbol{\xi})\boldsymbol{d}^N_\beta \tag{37}$$

$$M^{\alpha\beta h}(\boldsymbol{\xi}) = \boldsymbol{q}^T(\boldsymbol{\xi})\boldsymbol{a}_3 \cdot \boldsymbol{d}^M_{\alpha\beta}, \quad \boldsymbol{a}_3 M^{\alpha\beta h}(\boldsymbol{\xi}) = \boldsymbol{q}^T(\boldsymbol{\xi})\boldsymbol{d}^M_{\alpha\beta} \tag{38}$$

substituting the approximations of Eqs. (24), (34), (35), (37), (38) into Eqs. (23c), (23d) can express $\boldsymbol{d}^\varepsilon_\beta$ and $\boldsymbol{d}^\kappa_{\alpha\beta}$ by $\boldsymbol{d}$ as:

$$\boldsymbol{d}^\varepsilon_\beta = \boldsymbol{G}^{-1}\left(\sum_{I=1}^{n_p}(\tilde{\boldsymbol{g}}_{\beta I} - \bar{\boldsymbol{g}}_{\beta I})\boldsymbol{d}_I + \hat{\boldsymbol{g}}_\beta\right) \tag{39}$$

$$\boldsymbol{d}^{\kappa}_{\alpha\beta} = \boldsymbol{G}^{-1}\left(\sum_{I=1}^{n_p}(\tilde{\boldsymbol{g}}_{\alpha\beta I} - \bar{\boldsymbol{g}}_{\alpha\beta I})\boldsymbol{d}_I + \hat{\boldsymbol{g}}_{\alpha\beta}\right) \tag{40}$$

with

$$\boldsymbol{G} = \int_{\Omega_C} \boldsymbol{q}^T\boldsymbol{q}d\Omega \tag{41}$$

$$\tilde{\boldsymbol{g}}_{\beta I} = \int_{\Gamma_C} \Psi_I \boldsymbol{q} n_\beta d\Gamma - \int_{\Omega_C} \Psi_I \boldsymbol{q}^*|_\beta d\Omega \tag{42a}$$

$$\bar{\boldsymbol{g}}_{\beta I} = \int_{\Gamma_C\cap\Gamma_v} \Psi_I \boldsymbol{q} n_\beta d\Gamma \tag{42b}$$

$$\hat{\boldsymbol{g}}_{\beta} = \int_{\Gamma_C\cap\Gamma_v} \boldsymbol{q} n_\beta \bar{\boldsymbol{v}} d\Gamma \tag{42c}$$

$$\begin{aligned}\tilde{\boldsymbol{g}}_{\alpha\beta I} = &\int_{\Gamma_C} \Psi_{I,\gamma} n^\gamma \boldsymbol{q} n_\alpha n_\beta d\Gamma - \int_{\Gamma_C} \Psi_I(\boldsymbol{q}^{**}|_\beta n_\alpha + (\boldsymbol{q} s_\alpha n_\beta)_{,\gamma} s^\gamma)d\Gamma \\ &+ [[\Psi_I \boldsymbol{q} s_\alpha n_\beta]]_{\boldsymbol{x}\in C_C} - \int_{\Omega_C} \Psi \boldsymbol{q}^{**}_{,\alpha}|_\beta d\Omega\end{aligned} \tag{43a}$$

$$\begin{aligned}\bar{\boldsymbol{g}}_{\alpha\beta I} = &\int_{\Gamma_C\cap\Gamma_\theta} \Psi_{I,\gamma} n^\gamma \boldsymbol{q} n_\alpha n_\beta d\Gamma - \int_{\Gamma_C\cap\Gamma_v} \Psi_I(\boldsymbol{q}^{**}|_\beta n_\alpha + (\boldsymbol{q} s_\alpha n_\beta)_{,\gamma} s^\gamma)d\Gamma \\ &+ [[\Psi_I \boldsymbol{q} s_\alpha n_\beta]]_{\boldsymbol{x}\in C_C\cap C_v}\end{aligned} \tag{43b}$$

$$\begin{aligned}\hat{\boldsymbol{g}}_{\alpha\beta} = &\int_{\Gamma_C\cap\Gamma_\theta} \boldsymbol{q} n_\alpha n_\beta \boldsymbol{a}_3 \bar{\theta}_{\boldsymbol{n}} d\Gamma - \int_{\Gamma_C\cap\Gamma_v} (\boldsymbol{q}^{**}|_\beta n_\alpha + (\boldsymbol{q} s_\alpha n_\beta)_{,\gamma} s^\gamma)\bar{\boldsymbol{v}} d\Gamma \\ &+ [[\boldsymbol{q} s_\alpha n_\beta \bar{\boldsymbol{v}}]]_{\boldsymbol{x}\in C_C\cap C_v}\end{aligned} \tag{43c}$$

plugging Eqs. (39) and (40) back into Eqs. (34) and (35) respectively gives the final expression of $\boldsymbol{v}^h_{,\alpha}$, $\varepsilon^h_{\alpha\beta}$ and $-\boldsymbol{v}^h_{,\alpha\beta}$, $\boldsymbol{\kappa}^h_{\alpha\beta}$ as:

$$\boldsymbol{v}^h_{,\alpha} = \sum_{I=1}^{n_p}(\tilde{\Psi}_{I,\alpha} - \bar{\Psi}_{I,\alpha})\boldsymbol{d}_I + \boldsymbol{q}^T\boldsymbol{G}^{-1}\hat{\boldsymbol{g}}_\alpha \tag{44a}$$

$$\begin{aligned}\varepsilon^h_{\alpha\beta} &= \sum_{I=1}^{n_p}\frac{1}{2}(\boldsymbol{a}_\alpha\tilde{\Psi}_{I,\beta} + \boldsymbol{a}_\beta\tilde{\Psi}_{I,\alpha})\cdot\boldsymbol{d}_I - \sum_{I=1}^{n_p}\frac{1}{2}(\boldsymbol{a}_\alpha\bar{\Psi}_{I,\beta} + \boldsymbol{a}_\beta\bar{\Psi}_{I,\alpha})\cdot\boldsymbol{d}_I \\ &+ \boldsymbol{q}^T\boldsymbol{G}^{-1}\frac{1}{2}(\boldsymbol{a}_\alpha\cdot\hat{\boldsymbol{g}}_\beta + \boldsymbol{a}_\beta\cdot\hat{\boldsymbol{g}}_\alpha) \\ &= \tilde{\varepsilon}^h_{\alpha\beta} - \bar{\varepsilon}^h_{\alpha\beta} + \hat{\varepsilon}^h_{\alpha\beta}\end{aligned} \tag{44b}$$

$$-\boldsymbol{v}^h_{,\alpha}|_\beta = \sum_{I=1}^{n_p}(\tilde{\Psi}_{I,\alpha\beta} - \bar{\Psi}_{I,\alpha\beta})\boldsymbol{d}_I + \boldsymbol{q}^T\boldsymbol{G}^{-1}\hat{\boldsymbol{g}}_{\alpha\beta} \tag{45a}$$

$$\begin{aligned}\kappa^h_{\alpha\beta} &= \sum_{I=1}^{n_p}\tilde{\Psi}_{I,\alpha\beta}\boldsymbol{a}_3\cdot\boldsymbol{d}_I - \sum_{I=1}^{n_p}\bar{\Psi}_{I,\alpha\beta}\boldsymbol{a}_3\cdot\boldsymbol{d}_I + \boldsymbol{q}^T\boldsymbol{G}^{-1}\boldsymbol{a}_3\cdot\hat{\boldsymbol{g}}_{\alpha\beta} \\ &= \tilde{\kappa}^h_{\alpha\beta} - \bar{\kappa}^h_{\alpha\beta} + \hat{\kappa}^h_{\alpha\beta}\end{aligned} \tag{45b}$$

with

$$\tilde{\Psi}_{I,\alpha}(\boldsymbol{\xi}) = \boldsymbol{q}^T(\boldsymbol{\xi})\boldsymbol{G}^{-1}\tilde{\boldsymbol{g}}_{\alpha I}, \quad \bar{\Psi}_{I,\alpha}(\boldsymbol{\xi}) = \boldsymbol{q}^T(\boldsymbol{\xi})\boldsymbol{G}^{-1}\tilde{\boldsymbol{g}}_{\alpha I} \tag{46}$$

$$\begin{cases} \tilde{\varepsilon}^h_{\alpha\beta} = \sum_{I=1}^{n_p} \frac{1}{2}(\boldsymbol{a}_\alpha \tilde{\Psi}_{I,\beta} + \boldsymbol{a}_\beta \tilde{\Psi}_{I,\alpha}) \cdot \boldsymbol{d}_I \\ \bar{\varepsilon}^h_{\alpha\beta} = \sum_{I=1}^{n_p} \frac{1}{2}(\boldsymbol{a}_\alpha \bar{\Psi}_{I,\beta} + \boldsymbol{a}_\beta \bar{\Psi}_{I,\alpha}) \cdot \boldsymbol{d}_I \\ \hat{\varepsilon}^h_{\alpha\beta} = \boldsymbol{q}^T\boldsymbol{G}^{-1}\frac{1}{2}(\boldsymbol{a}_\alpha \cdot \hat{\boldsymbol{g}}_\beta + \boldsymbol{a}_\beta \cdot \hat{\boldsymbol{g}}_\alpha) \end{cases} \tag{47}$$

$$\tilde{\Psi}_{I,\alpha\beta}(\boldsymbol{\xi}) = \boldsymbol{q}^T(\boldsymbol{\xi})\boldsymbol{G}^{-1}\tilde{\boldsymbol{g}}_{\alpha\beta I}, \quad \bar{\Psi}_{I,\alpha\beta}(\boldsymbol{\xi}) = \boldsymbol{q}^T(\boldsymbol{\xi})\boldsymbol{G}^{-1}\tilde{\boldsymbol{g}}_{\alpha\beta I} \tag{48}$$

$$\begin{cases} \tilde{\kappa}^h_{\alpha\beta} = \sum_{I=1}^{n_p} \tilde{\Psi}_{I,\alpha\beta}\boldsymbol{a}_3 \cdot \boldsymbol{d}_I \\ \bar{\kappa}^h_{\alpha\beta} = \sum_{I=1}^{n_p} \bar{\Psi}_{I,\alpha\beta}\boldsymbol{a}_3 \cdot \boldsymbol{d}_I \\ \hat{\kappa}^h_{\alpha\beta} = \boldsymbol{q}^T\boldsymbol{G}^{-1}\boldsymbol{a}_3 \cdot \hat{\boldsymbol{g}}_{\alpha\beta} \end{cases} \tag{49}$$

Furthermore, taking Eqs. (34) and (35) into Eqs.(23a) and (23b) can obtain the approximated effective stresses $N^{\alpha\beta h}$, $M^{\alpha\beta h}$ and their coefficients $\boldsymbol{d}^N_\beta$, $\boldsymbol{d}^M_{\alpha\beta}$ as:

$$\begin{aligned} &\frac{1}{2}(\delta\boldsymbol{d}^\varepsilon_\alpha \cdot \boldsymbol{a}_\beta + \delta\boldsymbol{d}^\varepsilon_\beta \ \cdot \boldsymbol{a}_\alpha)hC^{\alpha\beta\gamma\eta}\frac{1}{2}(\boldsymbol{a}_\gamma \cdot \boldsymbol{d}^\varepsilon_\eta + \boldsymbol{a}_\gamma \cdot \boldsymbol{d}^\varepsilon_\eta)\boldsymbol{G} \\ =&\frac{1}{2}(\delta\boldsymbol{d}^\varepsilon_\alpha \cdot \boldsymbol{d}^N_\beta + \delta\boldsymbol{d}^\varepsilon_\beta \cdot \boldsymbol{d}^N_\alpha)\boldsymbol{G} \\ \Rightarrow \ \boldsymbol{d}^\beta_N =&\boldsymbol{a}_\beta hC^{\alpha\beta\gamma\eta}\frac{1}{2}(\boldsymbol{a}_\gamma \cdot \boldsymbol{d}^\varepsilon_\eta + \boldsymbol{a}_\eta \cdot \boldsymbol{d}^\varepsilon_\gamma) \end{aligned} \tag{50}$$

$$\begin{aligned} &\delta\boldsymbol{d}^\kappa_{\alpha\beta} \cdot \boldsymbol{a}_3\frac{h^3}{12}C^{\alpha\beta\gamma\eta}\boldsymbol{a}_3 \cdot \boldsymbol{d}^\kappa_{\gamma\eta}\boldsymbol{G} = \delta\boldsymbol{d}^\kappa_{\alpha\beta} \cdot \boldsymbol{d}^M_{\alpha\beta}\boldsymbol{G} \\ \Rightarrow \ &\boldsymbol{d}^{\alpha\beta}_M = \boldsymbol{a}_3\frac{h^3}{12}C^{\alpha\beta\gamma\eta}\boldsymbol{a}_3 \cdot \boldsymbol{d}^\kappa_{\gamma\eta} \end{aligned} \tag{51}$$

$$N^{\alpha\beta h} = hC^{\alpha\beta\gamma\eta}(\tilde{\varepsilon}^h_{\gamma\eta} - \bar{\varepsilon}^h_{\gamma\eta} + \hat{\varepsilon}^h_{\gamma\eta}) = \tilde{N}^{\alpha\beta h} - \bar{N}^{\alpha\beta h} + \hat{N}^{\alpha\beta h} \tag{52}$$

$$M^{\alpha\beta h} = \frac{h^3}{12}C^{\alpha\beta\gamma\eta}(\tilde{\kappa}^h_{\gamma\eta} - \bar{\kappa}^h_{\gamma\eta} + \hat{\kappa}^h_{\gamma\eta}) = \tilde{M}^{\alpha\beta h} - \bar{M}^{\alpha\beta h} + \hat{M}^{\alpha\beta h} \tag{53}$$

with

$$\tilde{N}^{\alpha\beta h} = hC^{\alpha\beta\gamma\eta}\tilde{\varepsilon}^h_{\gamma\eta}, \quad \bar{N}^{\alpha\beta h} = hC^{\alpha\beta\gamma\eta}\bar{\varepsilon}^h_{\gamma\eta}, \quad \hat{N}^{\alpha\beta h} = hC^{\alpha\beta\gamma\eta}\hat{\varepsilon}^h_{\gamma\eta} \tag{54}$$

$$\tilde{M}^{\alpha\beta h} = \frac{h^3}{12}C^{\alpha\beta\gamma\eta}\tilde{\kappa}^h_{\gamma\eta}, \quad \bar{M}^{\alpha\beta h} = \frac{h^3}{12}C^{\alpha\beta\gamma\eta}\bar{\kappa}^h_{\gamma\eta}, \quad \hat{M}^{\alpha\beta h} = \frac{h^3}{12}C^{\alpha\beta\gamma\eta}\hat{\kappa}^h_{\gamma\eta} \tag{55}$$

It is noted that, referring to reproducing kernel gradient smoothing framework [**?**], $\tilde{\Psi}_{I,\alpha}$, $\tilde{\Psi}_{I,\alpha\beta}$ are actually the first and second order smoothed gradients in curvilinear coordinates. $\tilde{\boldsymbol{g}}_{\alpha I}$ and $\tilde{\boldsymbol{g}}_{\alpha\beta I}$ are the right hand side integration constraints for first and second order gradients, then this formulation can meet the variational consistency for the p-th order polynomials. It should be known that, in curved model, the variational consistency for non-polynomial functions, like trigonometric functions, should be required for the polynomial solution. Even with p-th order variational consistency, the proposed formulation can not exactly reproduce the solution spanned by basis functions, however the accuracy of reproducing kernel smoothed gradients is still better that traditonal meshfree formulation, this will be evidenced by numerical examples in further section.

4. Naturally variational enforcement for essential boundary conditions

4.1. Discrete equilibrium equations

With the approximated effective stresses and strains, the last equation of weak form becomes:

$$-\sum_{C=1}^{n_e}(\tilde{\boldsymbol{g}}^T_{\alpha I}-\bar{\boldsymbol{g}}^T_{\alpha I})\boldsymbol{d}^\alpha_N-\sum_{C=1}^{n_e}(\tilde{\boldsymbol{g}}^T_{\alpha\beta I}-\bar{\boldsymbol{g}}^T_{\alpha\beta I})\boldsymbol{d}^{\alpha\beta}_M=\boldsymbol{f}_I \tag{56}$$

where $\boldsymbol{f}_I$'s are the components of the traditional force vector:

$$\boldsymbol{f}_I=\int_{\Gamma_t}\Psi_I\bar{\boldsymbol{t}}d\Gamma-\int_{\Gamma_M}\Psi_{I,\gamma}n^\gamma\bar{M}_{\boldsymbol{nn}}d\Gamma+[[\Psi_I\boldsymbol{a}_3\bar{P}]]_{\boldsymbol{x}\in C_P}+\int_\Omega\Psi_I\bar{\boldsymbol{b}}d\Omega \tag{57}$$

and further substituting coefficients $\boldsymbol{d}^\alpha_N$, $\boldsymbol{d}^{\alpha\beta}_M$ into Eq. (56) gives the final discrete equilibrium equations:

$$\begin{aligned}
&-\sum_{C=1}^{n_c}(\tilde{\boldsymbol{g}}^T_{\alpha I}-\bar{\boldsymbol{g}}^T_{\alpha I})\boldsymbol{d}^\alpha_N-\sum_{C=1}^{n_c}(\tilde{\boldsymbol{g}}^T_{\alpha\beta I}-\bar{\boldsymbol{g}}^T_{\alpha\beta I})\boldsymbol{d}^{\alpha\beta}_M\\
&=\sum_{C=1}^{n_e}\sum_{J=1}^{n_p}\begin{pmatrix}
\boldsymbol{a}_\alpha\tilde{\boldsymbol{g}}^T_{\beta I}hC^{\alpha\beta\gamma\eta}\boldsymbol{a}_\gamma\tilde{\boldsymbol{g}}_{\eta J}+\tilde{\boldsymbol{g}}^T_{\alpha\beta I}\boldsymbol{a}_3\dfrac{h^3}{12}C^{\alpha\beta\gamma\eta}\boldsymbol{a}_3\tilde{\boldsymbol{g}}_{\gamma\eta}\\
-\boldsymbol{a}_\alpha\bar{\boldsymbol{g}}^T_{\beta I}hC^{\alpha\beta\gamma\eta}\boldsymbol{a}_\gamma\tilde{\boldsymbol{g}}_{\eta J}-\boldsymbol{a}_\alpha\tilde{\boldsymbol{g}}^T_{\beta I}hC^{\alpha\beta\gamma\eta}\boldsymbol{a}_\gamma\bar{\boldsymbol{g}}_{\eta J}\\
-\bar{\boldsymbol{g}}^T_{\alpha\beta I}\boldsymbol{a}_3\dfrac{h^3}{12}C^{\alpha\beta\gamma\eta}\boldsymbol{a}_3\tilde{\boldsymbol{g}}_{\gamma\eta J}-\tilde{\boldsymbol{g}}^T_{\alpha\beta I}\boldsymbol{a}_3\dfrac{h^3}{12}C^{\alpha\beta\gamma\eta}\boldsymbol{a}_3\bar{\boldsymbol{g}}_{\gamma\eta J}\\
+\boldsymbol{a}_\alpha\tilde{\boldsymbol{g}}^T_{\beta I}hC^{\alpha\beta\gamma\eta}\boldsymbol{a}_\gamma\hat{\boldsymbol{g}}_{\eta J}-\boldsymbol{a}_\alpha\bar{\boldsymbol{g}}^T_{\beta I}hC^{\alpha\beta\gamma\eta}\boldsymbol{a}_\gamma\hat{\boldsymbol{g}}_{\eta J}\\
+\tilde{\boldsymbol{g}}^T_{\alpha\beta I}\boldsymbol{a}_3\dfrac{h^3}{12}C^{\alpha\beta\gamma\eta}\boldsymbol{a}_3\hat{\boldsymbol{g}}_{\gamma\eta J}-\bar{\boldsymbol{g}}^T_{\alpha\beta I}\boldsymbol{a}_3\dfrac{h^3}{12}C^{\alpha\beta\gamma\eta}\boldsymbol{a}_3\hat{\boldsymbol{g}}_{\gamma\eta J}\\
+\boldsymbol{a}_\alpha\bar{\boldsymbol{g}}^T_{\beta I}hC^{\alpha\beta\gamma\eta}\boldsymbol{a}_\gamma\bar{\boldsymbol{g}}_{\eta J}+\bar{\boldsymbol{g}}^T_{\alpha\beta I}\boldsymbol{a}_3\dfrac{h^3}{12}C^{\alpha\beta\gamma\eta}\boldsymbol{a}_3\bar{\boldsymbol{g}}_{\gamma\eta J}
\end{pmatrix}\\
&=\sum_{J=1}^{n_p}(\boldsymbol{K}_{IJ}+\tilde{\boldsymbol{K}}_{IJ}+\bar{\boldsymbol{K}}_{IJ})\cdot\boldsymbol{d}_J-\tilde{\boldsymbol{f}}_I-\bar{\boldsymbol{f}}_I
\end{aligned} \tag{58}$$

where

$$\boldsymbol{K}_{IJ}=\int_\Omega\tilde{\boldsymbol{\varepsilon}}_{\alpha\beta I}\tilde{\boldsymbol{N}}^{\alpha\beta}_J d\Omega+\int_\Omega\tilde{\boldsymbol{\kappa}}_{\alpha\beta I}\tilde{\boldsymbol{M}}^{\alpha\beta}_J d\Omega \tag{59}$$

$$\begin{aligned}
\tilde{\boldsymbol{K}}_{IJ}=&-\int_{\Gamma_v}(\Psi_I\tilde{\boldsymbol{t}}_J+\tilde{\boldsymbol{t}}_I\Psi_J)d\Gamma\\
&+\int_{\Gamma_\theta}(\Psi_{I,\gamma}n^\gamma\boldsymbol{a}_3\tilde{M}_{\boldsymbol{nn}J}+\boldsymbol{a}_3\tilde{M}_{\boldsymbol{nn}I}\Psi_{I,\gamma}n^\gamma)d\Gamma\\
&+([[\Psi_I\boldsymbol{a}_3P_J]]+[[P_I\boldsymbol{a}_3\Psi_J]])_{\boldsymbol{x}\in C_v}
\end{aligned} \tag{60a}$$

$$\tilde{\boldsymbol{f}}_I=-\int_{\Gamma_v}\tilde{\boldsymbol{t}}_I\cdot\bar{\boldsymbol{v}}d\Gamma+\int_{\Gamma_\theta}\tilde{M}_{\boldsymbol{nn}}\bar{\theta}_{\boldsymbol{n}}d\Gamma+[[\tilde{P}_I\boldsymbol{a}_3\cdot\bar{\boldsymbol{v}}]]_{\boldsymbol{x}\in C_v} \tag{60b}$$

$$\bar{\boldsymbol{K}}_{IJ} = -\int_{\Gamma_v} \bar{\boldsymbol{t}}_I \Psi_J d\Gamma + \int_{\Gamma_\theta} \boldsymbol{a}_3 \bar{M}_{\boldsymbol{nn}I} \Psi_{J,\gamma} n^\gamma d\Gamma + [[\bar{P}_I \boldsymbol{a}_3 \Psi_J]]_{\boldsymbol{x}\in C_v} \tag{61a}$$

$$\bar{\boldsymbol{f}}_I = -\int_{\Gamma_v} \bar{\boldsymbol{t}}_I \cdot \bar{\boldsymbol{v}} d\Gamma + \int_{\Gamma_\theta} \bar{M}_{\boldsymbol{nn}} \bar{\theta}_{\boldsymbol{n}} d\Gamma + [[\bar{P}_I \boldsymbol{a}_3 \cdot \bar{\boldsymbol{v}}]]_{\boldsymbol{x}\in C_v} \tag{61b}$$

The detailed derivations of Eqs (59)-(61) are listed in the Appendix. As shown in these equations, the Eq. (59) is the conventional stiffness matrix evaluated by smoothed gradients $\tilde{\Psi}_{I,\alpha}$, $\tilde{\Psi}_{I,\alpha}|_\beta$, and the Eqs. (60) and (61) contribute for the enforcement of essential boundary.

4.2. Comparison with Nitsche's method

The Nitsche's method for enforcing essential boundary can be regarded as a combination of Lagrangian multiplier method and penalty method, in which the Lagrangian multiplier is represented by the approximated displacement. The corresponding total potential energy functional Π_P is given by:

$$\begin{aligned}
\Pi_P(\boldsymbol{v}) = & \int_\Omega \frac{1}{2}\varepsilon_{\alpha\beta} N^{\alpha\beta} d + \int_\Omega \frac{1}{2}\kappa_{\alpha\beta} M^{\alpha\beta} d\Omega \\
& - \int_{\Gamma_t} \boldsymbol{v}\cdot\bar{\boldsymbol{t}} d\Gamma + \int_{\Gamma_M} \boldsymbol{v}_{,\gamma} n^\gamma \boldsymbol{a}_3 M_{\boldsymbol{nn}} d\Gamma + (\boldsymbol{v}\cdot\boldsymbol{a}_3 P)_{\boldsymbol{x}\in C_P} - \int_\Omega \boldsymbol{v}\cdot\bar{\boldsymbol{b}} d\Omega \\
& \underbrace{- \int_{\Gamma_v} \boldsymbol{t}\cdot(\boldsymbol{v}-\bar{\boldsymbol{v}}) d\Gamma + \int_{\Gamma_\theta} M_{\boldsymbol{nn}}(\theta_{\boldsymbol{n}} - \bar{\theta}_{\boldsymbol{n}}) d\Gamma + (P\boldsymbol{a}_3\cdot(\boldsymbol{v}-\bar{\boldsymbol{v}}))_{\boldsymbol{x}\in C_v}}_{\text{consistent term}} \\
& \underbrace{+ \frac{\alpha_v}{2}\int_{\Gamma_v} \boldsymbol{v}\cdot\boldsymbol{v} d\Gamma + \frac{\alpha_\theta}{2}\int_{\Gamma_\theta} \theta_{\boldsymbol{n}}^2 d\Gamma + \frac{\alpha_C}{2}(\boldsymbol{v}\cdot\boldsymbol{v})_{\boldsymbol{x}\in C_v}}_{\text{stabilized term}}
\end{aligned} \tag{62}$$

where the consistent term rephrased from Lagrangian multiplier method contributes to enforce the essential boundary and meet the variational consistency condition. However the consistent term can not always ensure the coercivity of stiffness, so the penalty method is introduced to be regarded as a stabilized term. With a standard variational argument, the corresponding weak form can be stated as:

$$
\begin{aligned}
\delta\Pi_P(\boldsymbol{v}) = & \int_\Omega \delta\varepsilon_{\alpha\beta} N^{\alpha\beta} d + \int_\Omega \delta\kappa_{\alpha\beta} M^{\alpha\beta} d\Omega \\
& - \int_{\Gamma_t} \delta\boldsymbol{v} \cdot \bar{\boldsymbol{t}} d\Gamma + \int_{\Gamma_M} \delta\boldsymbol{v}_{,\gamma} n^\gamma \boldsymbol{a}_3 M_{\boldsymbol{nn}} d\Gamma + (\delta\boldsymbol{v} \cdot \boldsymbol{a}_3 P)_{\boldsymbol{x}\in C_P} - \int_\Omega \delta\boldsymbol{v} \cdot \bar{\boldsymbol{b}} d\Omega \\
& - \int_{\Gamma_v} \delta\boldsymbol{v} \cdot \boldsymbol{t} d\Gamma + \int_{\Gamma_\theta} \delta\theta_{\boldsymbol{n}} M_{\boldsymbol{nn}} d\Gamma + (\boldsymbol{v} \cdot \boldsymbol{a}_3 P)_{\boldsymbol{x}\in C_v} \\
& - \int_{\Gamma_v} \delta\boldsymbol{t} \cdot (\boldsymbol{v} - \bar{\boldsymbol{v}}) d\Gamma + \int_{\Gamma_\theta} \delta M_{\boldsymbol{nn}} (\theta_{\boldsymbol{n}} - \bar{\theta}_{\boldsymbol{n}}) d\Gamma + (\delta P \boldsymbol{a}_3 \cdot (\boldsymbol{v} - \bar{\boldsymbol{v}}))_{\boldsymbol{x}\in C_v} \\
& + \alpha_v \int_{\Gamma_v} \delta\boldsymbol{v} \cdot \boldsymbol{v} d\Gamma + \alpha_\theta \int_{\Gamma_\theta} \delta\theta_{\boldsymbol{n}} \theta_{\boldsymbol{n}} d\Gamma + \alpha_C (\delta\boldsymbol{v} \cdot \boldsymbol{v})_{\boldsymbol{x}\in C_v} \\
& = 0
\end{aligned}
\tag{63}
$$

in which α_v, α_θ and α_C are experimental artificial parameters. Further invoking the conventional reproducing kernel approximation of Eq. (24) leads to the following discrete equilibrium equations:

$$
\sum_{J=1}^{n_p} (\boldsymbol{K}_{IJ} + \boldsymbol{K}^c_{IJ} + \boldsymbol{K}^s_{IJ}) \boldsymbol{d}_J = \boldsymbol{f}_I + \boldsymbol{f}^c + \boldsymbol{f}^s \tag{64}
$$

where the stiffness $\boldsymbol{K}_{IJ}$ is identical with Eq. (59). $\boldsymbol{K}^c_{IJ}$ and $\boldsymbol{K}^s_{IJ}$ are the stiffness matrix for consistent and stabilized terms respectively, and have the following forms:

$$
\begin{aligned}
\boldsymbol{K}^c_{IJ} = & - \int_{\Gamma_v} \Big((\boldsymbol{\mathcal{T}}^\alpha \Psi_{I,\alpha} + \boldsymbol{\mathcal{V}}^{\alpha\beta} \Psi_{I,\alpha}|_\beta) \Psi_J + \Psi_I (\boldsymbol{\mathcal{T}}^\alpha \Psi_{J,\alpha} + \boldsymbol{\mathcal{V}}^{\alpha\beta} \Psi_{J,\alpha}|_\beta) \Big) d\Gamma \\
& + \int_{\Gamma_M} (\mathcal{M}^{\alpha\beta} \Psi_{I,\alpha}|_\beta \boldsymbol{a}_3 \Psi_{J,\gamma} n^\gamma + \Psi_{I,\gamma} n^\gamma \boldsymbol{a}_3 \mathcal{M}^{\alpha\beta} \Psi_{I,\alpha}|_\beta) d\Gamma
\end{aligned}
\tag{65a}
$$

5. Numerical examples

	Linear patch test		Quadratic patch test	
	L_2-Error	H_e-Error	L_2-Error	H_e-Error
GI-Penalty				
GI-Nitsche				
RKGSI-Penalty				
RKGSI-Nitsche				
RKGSI-HR				

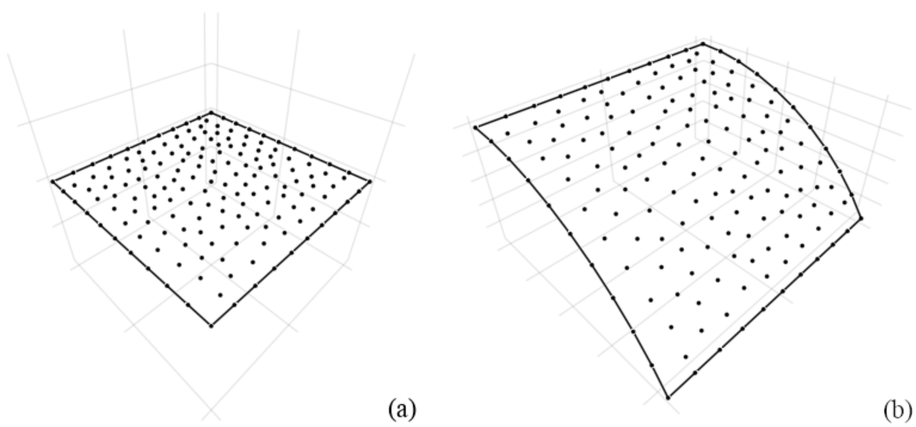

Figure 1: Meshfree discretization for patch test

6. Conclusion

Appendix A. Covariant derivatives

This Appendix lists the covariant derivatives needed for the method developed in this study.

$$v_\alpha|_\beta \tag{A.1}$$

Appendix B. Derivations for stiffness metrics and force vectors

This Appendix details the derivations of stiffness

References

[1] H. Dah-wei, A method for establishing generalized variational principle 6 (6) 501–509. doi:10.1007/BF01876390.
URL http://link.springer.com/10.1007/BF01876390