

1 An quasi-consistent efficient meshfree thin shell
2 formulation naturally accommodating essential
3 boundary conditions

4 Junchao Wu^{a,*}, Yangtao Xu^b, Bin Xu^a, Syed Humayun Basha^c

^aKey Laboratory for Intelligent Infrastructure and Monitoring of Fujian Province, College
of Civil Engineering, Huaqiao University, Xiamen, Fujian, 361021, China

^bCollege of Civil Engineering, Huaqiao University, Xiamen, Fujian, 361021, China

^cKey Laboratory for Structural Engineering and Disaster Prevention of Fujian Province,
College of Civil Engineering, Huaqiao University, Xiamen, Fujian, 361021, China

5 **Abstract**

This research proposed an efficient and quasi-consistent meshfree thin shell formulation with natural enforcement of essential boundary conditions. Within the framework of the Hu-Washizu variational principle, a mixed formulation of displacements, strains and stresses is employed in this approach, where the displacements are discretized using meshfree shape functions, and the strains and stresses are expressed using smoothed gradients, covariant smoothed gradients and covariant bases. The smoothed gradients satisfy the first and second order integration constraint and have quasi-consistent consistency. Owing to Hu-Washizu variational principle, the essential boundary conditions automatically arise in its weak form. As a result, the suggested technique's enforcement of essential boundary conditions resembles that of the traditional Nitsche's method. Contrary to Nitsche's method, the costly higher order derivatives of conventional meshfree shape functions were replaced by the smoothed gradients with fast computation, which improve the efficiency. Meanwhile, the proposed formulation features a naturally stabilized term without adding any artificial stabilization factors, which eliminates the stabilization parameter-dependent issue in the Nitsche's method. The efficacy of the proposed Hu-Washizu meshfree thin shell formulation is illustrated by a set of classical standard thin shell problems.

6 *Keywords:* Meshfree, Thin shell, Hu-Washizu variational principle,
7 Reproducing kernel gradient smoothing, Essential boundary condition

*Corresponding author
Email address: jcwu@hqu.edu.cn (Junchao Wu)

8 1. Introduction

9 Thin shell structure follows the Kirchhoff hypothesis that neglects the shear
10 deformation [1], which requires the approximation should have at least C^1 con-
11 tinuity in Galerkin formulations. The traditional finite element methods usually
12 only has C^0 continuous shape functions, and it more prefers Mindlin thick shear
13 theory, hybrid and mixed models in simulation of shell structure [2]. In last three
14 decades, the meshfree methods [3, 4] equipped high order smoothed shape
15 functions have attracted significant research attention, while the meshfree shape
16 functions are established based upon a set of scattered nodes and the high order
17 continuity of shape functions is easily fulfilled even with low order basis func-
18 tion. For thin shell analysis, this high order meshfree approximations can also
19 alleviate the membrane locking caused by the mismatched approximation order
20 of membrane strain and bending strain [5]. Moreover, in general, the nodal-
21 based meshfree approximations can release the burden of mesh distortion and
22 have the flexibility of local refinement. Due to these advantages, a wide variety
23 meshfree methods are proposed and have been applied to many scientific or
24 engineering fields. However, the high order smoothed meshfree shape functions
25 accompany with the enlarged and overlapping supports, which may also leads
26 to many issues for shape functions. One is the loss of Kronecker delta property
27 [6], which leads to that the essential boundary conditions cannot be enforced
28 directly like finite element methods. Another issue is that the variational con-
29 sistency or said integration constraint cannot be satisfied, which is caused by
30 the misalignment between numerical integration domains and supports of shape
31 functions, and the shape functions exhibit a piecewise rational nature in each
32 integration domains. Variational consistency is of importance to the solution
33 accuracy in Galerkin formulations [7].

34 To directly enforce the essential boundary for Galerkin meshfree methods,
35 several approaches have been proposed for the recovery of shape functions' Kro-
36 necker property. For examples, interpolation element-free method [7], mixed
37 transformation method [8], boundary singular kernel method [8] etc. However,
38 these methods are not based on a variational setting, and cannot guarantee
39 the variational consistency, enforcing accuracy may be worse on where there
40 is no meshfree node. In contrast, enforcing the essential boundary conditions
41 by a variational approach are more preferred for Galerkin meshfree methods.
42 Belytschko et al. [3, 5] firstly introduced the variational consistent Lagrange
43 multiplier method to Galerkin meshfree method, in which the extra degrees of
44 freedom should be employed for discretion of Lagrange multiplier. And this
45 method has been extended to geometrically nonlinear thin shells by Ivannikov
46 et al. [9]. To eliminate the extra degrees of freedom, Lu et al. [9] represented
47 the Lagrange multiplier by corresponding tractions and proposed the modified
48 variational essential boundary enforcement method. However, the coercivity
49 of this approach is not always ensured and potentially reduces the accuracy.
50 Zhu and Atluri [10] pioneered the penalty method for meshfree method, mak-
51 ing it straightforward approach for enforcing essential boundary conditions via
52 Galerkin weak form. However, penalty method suffers from a lack of varia-

53 tional consistency, and requires the experimental artificial parameters, whose
54 optimal value is hard to be determined. Fernández-Méndez and Huerta [6] used
55 the Nitsche’s method in meshfree formulation for imposing essential boundary
56 conditions. This method can be viewed as a hybrid of modified variational
57 method and penalty method, since its consistent term that ensure variational
58 consistency generated by modified variational method, and the penalty method
59 is employed as stabilized term to recovery the coercivity. Skatulla and Sansour
60 [11] further extended Nitsche’s method for thin shell analysis and proposed an
61 iteration algorithm to determine artificial parameters at each integration points.

62 To address the issue of numerical integration, a serial of consistent integra-
63 tion scheme has been developed for Galerkin meshfree methods. For instance,
64 stabilized conforming nodal integration [12], variational consistent integration
65 [13], quadratic consistent integration [?], reproducing kernel gradient smooth-
66 ing integration [?], consistent projection integration [14] etc. The most con-
67 sistent integration scheme is established by assumed strain approach, while the
68 costly higher order derivatives of traditional meshfree shape functions are re-
69 placed by smoothed gradient, and show a high efficiency. Moreover, in order to
70 achieve the global variational consistency, a consistent essential boundary condi-
71 tion enforcement should cooperate with the consistent integration scheme. The
72 pair of consistent integration scheme and Nitsche’s method for the treatment of
73 essential boundary conditions shows a good performance, since it no needs the
74 extra degrees of freedom and can fulfilled the coercivity. However, in Nitsche’s
75 method, the artificial parameters still exist in stabilized term and the costly
76 higher order derivatives should be recalled, especially for thin plate and thin
77 shell problems [15]. Recently, Wu et al [? 16] proposed a efficient and stabilized
78 essential boundary condition enforcement based upon the Hellinger-Reissner
79 (HR) variational principle, where the reproducing kernel gradient smoothing
80 integration is recast by a mixed formulation in Hellinger-Reissner weak form.
81 The terms for enforcing essential boundary conditions is mostly identical with
82 Nitsche’s method, both have consistent term and stabilized term. Nevertheless,
83 the stabilized term of this method naturally exist in Hellinger-Reissner weak
84 form and no longer needs the artificial parameters, even for essential boundary
85 enforcement, total of the higher order derivatives are represented by smoothed
86 gradients and their derivatives.

87 In this study, an efficient and stabilized variational consistent meshfree
88 method with naturally enforcing the essential boundary conditions is devel-
89 oped for thin shell structure. Follow the ideas of Hellinger-Reissner principle
90 base consistent meshfree method, the Hu-Washizu variational principle of com-
91plementary energy [17] with variables of displacement, strains and stresses is
92 employed, where the displacement is approximated by conventional meshfree
93 shape functions, and the strains and stresses are expressed by the smoothed
94 gradients or covariant smoothed gradients with covariant bases. It should be
95 noted that the smoothed gradients inherently embed the first two order inte-
96gration constraints, however, due to the non-polynomial property of stresses,
97 the fulfillment of these integration constraint only can get a quasi-satisfaction
98 of variational consistency. All of the essential boundary conditions about dis-

99 placements and rotations are considered in Hu-Washizu weak form, and present
 100 a Nitsche-like formalism but without any artificial parameters. Comparing with
 101 Nitsche’s method, the costly higher order derivatives are replaced by conven-
 102 tional reproducing smoothed gradients and its direct derivatives. Taking the
 103 advantages of reproducing kernel gradient smoothing framework, the smoothed
 104 gradients shows a better performance on efficiency than conventional derivatives
 105 of shape functions, which improves the computational efficiency of meshfree for-
 106 mulation.

107 The remainder of this paper is organized as follows. Section 2 briefly de-
 108 scribes the kinematics of thin shell structure and the corresponding Hu-Washizu
 109 principle weak form. Subsequently, the mixed formulation regarding the dis-
 110 placements, strains and stresses in accordance with Hu-Washizu weak form is
 111 presented in Section 3. Section 4 derives the discrete equilibrium equations
 112 with the naturally accommodation of essential, and compares them with those
 113 of Nitsche’s method. The efficacy of the proposed Hu-Washizu meshfree thin
 114 shell formulation is validated by numerical results in Section 5. Concluding
 115 remarks are finally drawn in Section 6.

116 2. Hu-Washizu's formulation of complementary energy for thin shell

117 2.1. Kinematics for thin shell

118 Consider the configuration of a shell $\bar{\Omega}$, as shown in Fig. ??, which can be
 119 easily described by a parametric curvilinear coordinate system $\boldsymbol{\xi} = \{\xi^i\}_{i=1,2,3}$.
 120 The mid-surface of the shell denoted by Ω is specified by the in-plane coordinates
 121 $\boldsymbol{\xi} = \{\xi^\alpha\}_{\alpha=1,2}$, as the thickness direction of shell is by ξ^3 , $-\frac{h}{2} \leq \xi^3 \leq \frac{h}{2}$, h is
 122 the thickness of shell. In this work, Latin indices take the values from 1 to 3,
 123 and Greek indices are evaluated by 1 or 2. For the Kirchhoff hypothesis [5], the
 124 position $\mathbf{x} \in \bar{\Omega}$ are defined by linear functions with respect to ξ^3 :

$$\mathbf{x}(\xi^1, \xi^2, \xi^3) = \mathbf{r}(\xi^1, \xi^2) + \xi^3 \mathbf{a}_3(\xi^1, \xi^2) \quad (1)$$

125 in which \mathbf{r} means the position on the mid-surface of shell, and the \mathbf{a}_3 is corre-
 126 sponding normal direction. For the mid-surface of shell, the in-plane covariant
 127 base vector with respect to ξ^α can be derived by a trivial partial differentiation
 128 to \mathbf{r} :

$$\mathbf{a}_\alpha = \frac{\partial \mathbf{r}}{\partial \xi^\alpha} = \mathbf{r}_{,\alpha}, \alpha = 1, 2 \quad (2)$$

129 for a clear expression, the subscript comma denotes the partial differentiation
 130 operation with respect to in-plane coordinates ξ^α . And the normal vector \mathbf{a}_3
 131 can be obtained by the normalized cross product of \mathbf{a}_α 's as follow:

$$\mathbf{a}_3 = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{\|\mathbf{a}_1 \times \mathbf{a}_2\|} \quad (3)$$

132 where $\|\bullet\|$ is the Euclidean norm operator.

133 With the assumption of infinitesimal deformation, the strain components
 134 respected to global contravariant base can be sated as:

$$\epsilon_{ij} = \frac{1}{2}(\mathbf{x}_{,i} \cdot \mathbf{u}_{,j} + \mathbf{u}_{,i} \cdot \mathbf{x}_{,j}) \quad (4)$$

135 where \mathbf{u} is the displacement for shell deformation. To fulfillment with Kirchhoff
 136 hypothesis, the displacement is assumed to be the following form:

$$\mathbf{u}(\xi^1, \xi^2, \xi^3) = \mathbf{v}(\xi^1, \xi^2) + \boldsymbol{\theta}(\xi^1, \xi^2) \xi^3 \quad (5)$$

137 in which the quadratic and higher order terms are neglected. \mathbf{v} , $\boldsymbol{\theta}$ respect the
 138 displacement and rotation in mid-surface.

139 Subsequently, plugging Eqs. (1) and (5) into Eq. (4) and neglecting quadratic
 140 terms, the strain components can be rephrased as follows:

$$\begin{aligned} \epsilon_{\alpha\beta} &= \frac{1}{2}(\mathbf{a}_\alpha \cdot \mathbf{v}_{,\beta} + \mathbf{v}_{,\alpha} \cdot \mathbf{a}_\beta) \\ &+ \frac{1}{2}(\mathbf{a}_{3,\alpha} \cdot \mathbf{v}_{,\beta} + \mathbf{v}_{,\alpha} \cdot \mathbf{a}_{3,\beta} + \mathbf{a}_\alpha \cdot \boldsymbol{\theta}_{,\beta} + \boldsymbol{\theta}_{,\alpha} \cdot \mathbf{a}_\beta) \xi^3 \\ &= \epsilon_{\alpha\beta} + \kappa_{\alpha\beta} \xi^3 \end{aligned} \quad (6a)$$

$$\epsilon_{\alpha 3} = \frac{1}{2}(\mathbf{a}_\alpha \cdot \boldsymbol{\theta} + \mathbf{v}_{,\alpha} \cdot \mathbf{a}_3) + \frac{1}{2}(\mathbf{a}_3 \cdot \boldsymbol{\theta})_{,\alpha} \xi^3 \quad (6b)$$

$$\epsilon_{33} = \mathbf{a}_3 \cdot \boldsymbol{\theta} \quad (6c)$$

141 where $\varepsilon_{\alpha\beta}$, $\kappa_{\alpha\beta}$ are membrane and bending strains respectively:

$$\varepsilon_{\alpha\beta} = \frac{1}{2}(\mathbf{a}_\alpha \cdot \mathbf{v}_{,\beta} + \mathbf{v}_{,\alpha} \cdot \mathbf{a}_\beta) \quad (7)$$

142

$$\kappa_{\alpha\beta} = \frac{1}{2}(\mathbf{a}_{3,\alpha} \cdot \mathbf{v}_{,\beta} + \mathbf{v}_{,\alpha} \cdot \mathbf{a}_{3,\beta} + \mathbf{a}_\alpha \cdot \boldsymbol{\theta}_{,\beta} + \boldsymbol{\theta}_{,\alpha} \cdot \mathbf{a}_\beta) \quad (8)$$

143 In accordance with Kirchhoff hypothesis, the thickness of shell will not
144 change and the deformation related with direction of ξ^3 will be vanished, i.e.
145 $\epsilon_{3i} = 0$. Thus, the rotation $\boldsymbol{\theta}$ can be rewritten as:

$$\epsilon_{3i} = 0 \Rightarrow \begin{cases} \boldsymbol{\theta} \cdot \mathbf{a}_\alpha + \mathbf{v}_{,\alpha} \cdot \mathbf{a}_3 = 0 \\ \boldsymbol{\theta} \cdot \mathbf{a}_3 = 0 \end{cases} \Rightarrow \boldsymbol{\theta} = -\mathbf{v}_{,\alpha} \cdot \mathbf{a}_3 \mathbf{a}^\alpha \quad (9)$$

146 where \mathbf{a}^α 's are the in-plane contravariant base vectors, $\mathbf{a}^\alpha \cdot \mathbf{a}_\beta = \delta_\beta^\alpha$, δ is the
147 Kronecker delta function. Substituting Eq. (9) into Eq. (8) leads to:

$$\kappa_{\alpha\beta} = (\Gamma_{\alpha\beta}^\gamma \mathbf{v}_{,\gamma} - \mathbf{v}_{,\alpha\beta}) \cdot \mathbf{a}_3 = -\mathbf{v}_{,\alpha}|_\beta \cdot \mathbf{a}_3 \quad (10)$$

148 in which $\Gamma_{\alpha\beta}^\gamma = \mathbf{a}_{\alpha,\beta} \cdot \mathbf{a}^\gamma$ is namely Christoffel symbol of the second kind. And
149 $\mathbf{v}_{,\alpha}|_\beta$ is the in-plane covariant derivative of $\mathbf{v}_{,\alpha}$, i.e. $\mathbf{v}_{,\alpha}|_\beta = \Gamma_{\alpha\beta}^\gamma \mathbf{v}_{,\gamma} - \mathbf{v}_{,\alpha\beta}$.

150 2.2. Galerkin weak form for Hu-Washizu principle of complementary energy

151 In this study, the Hu-Washizu variational principle of complementary energy
152 [17] is used herein for development of this method, the corresponding comple-
153 mentary functional, denoted by Π_C , is listed as follow:

$$\begin{aligned} \Pi_C(\varepsilon_{\alpha\beta}, \kappa_{\alpha\beta}, N^{\alpha\beta}, M^{\alpha\beta}) &= \int_\Omega \frac{h}{2} \varepsilon_{\alpha\beta} C^{\alpha\beta\gamma\eta} \varepsilon_{\gamma\eta} d\Omega + \int_\Omega \frac{h^3}{24} \kappa_{\alpha\beta} C^{\alpha\beta\gamma\eta} \kappa_{\gamma\eta} d\Omega \\ &+ \int_\Omega \varepsilon_{\alpha\beta} (N^{\alpha\beta} - h C^{\alpha\beta\gamma\eta} \varepsilon_{\gamma\eta}) d\Omega + \int_\Omega \kappa_{\alpha\beta} (M^{\alpha\beta} - \frac{h^3}{12} C^{\alpha\beta\gamma\eta} \kappa_{\gamma\eta}) d\Omega \\ &- \int_{\Gamma_v} \mathbf{T} \cdot \bar{\mathbf{v}} d\Gamma + \int_{\Gamma_\theta} M_{\mathbf{n}\mathbf{n}} \bar{\boldsymbol{\theta}}_{\mathbf{n}} d\Gamma - (P \mathbf{a}_3 \cdot \bar{\mathbf{v}})_{\mathbf{x} \in C_w} \end{aligned} \quad (11)$$

154 where $C^{\alpha\beta\gamma\eta}$'s are the components of fourth order elasticity tensor with re-
155 spect to covariant base and plane stress assumption, and it can be expressed
156 by Young's modulus E , Poisson rate ν and the in-plane contravariant metric
157 coefficients $a^{\alpha\beta}$'s, $a^{\alpha\beta} = \mathbf{a}^\alpha \cdot \mathbf{a}^\beta$, as follow:

$$C^{\alpha\beta\gamma\eta} = \frac{E}{2(1+\nu)} (a^{\alpha\gamma} a^{\beta\eta} + a^{\alpha\eta} a^{\beta\gamma} + \frac{2\nu}{1-\nu} a^{\alpha\beta} a^{\gamma\eta}) \quad (12)$$

158 and $N^{\alpha\beta}$, $M^{\alpha\beta}$ are the components of membrane and bending stresses given by:

$$N^{\alpha\beta} = h C^{\alpha\beta\gamma\eta} \varepsilon_{\gamma\eta}, \quad M^{\alpha\beta} = \frac{h^3}{12} C^{\alpha\beta\gamma\eta} \kappa_{\gamma\eta} \quad (13)$$

Essential boundaries on the edges and corners denoted by Γ_v , Γ_θ and C_v are naturally existed in complementary energy functional, $\bar{\mathbf{v}}$, $\bar{\theta}_{\mathbf{n}}$ are the corresponding prescribed displacement and normal rotation. \mathbf{T} , $M_{\mathbf{nn}}$ and P can be determined by Euler-Lagrange equations of shell problem [15] as follows:

$$\mathbf{T} = \mathbf{T}_N + \mathbf{T}_M \rightarrow \begin{cases} \mathbf{T}_N = \mathbf{a}_\alpha N^{\alpha\beta} n_\beta \\ \mathbf{T}_M = (\mathbf{a}_3 M^{\alpha\beta} s_\alpha n_\beta)_{,\gamma} s^\gamma + (\mathbf{a}_3 M^{\alpha\beta})|_\beta n_\alpha \end{cases} \quad (14)$$

$$M_{\mathbf{nn}} = M^{\alpha\beta} n_\alpha n_\beta \quad (15)$$

$$P = -[[M^{\alpha\beta} s_\alpha n_\beta]] \quad (16)$$

where $\mathbf{n} = n^\alpha \mathbf{a}_\alpha = n_\alpha \mathbf{a}^\alpha$ and $\mathbf{s} = s^\alpha \mathbf{a}_\alpha = s_\alpha \mathbf{a}^\alpha$ are the outward normal and tangent directions on boundaries. $[[f]]$ is the jump operator defined by:

$$[[f]]_{\mathbf{x}=\mathbf{x}_c} = \lim_{\epsilon \rightarrow 0^+} (f(\mathbf{x}_c + \epsilon) - f(\mathbf{x}_c - \epsilon)), \mathbf{x}_c \in \Gamma \quad (17)$$

where f is an arbitrary function on Γ .

Moreover, the natural boundary conditions should be applied by Lagrangian multiplier method with displacement \mathbf{v} regarded as multiplier. Thus then the new complementary energy functional namely Π is given by:

$$\begin{aligned} \Pi(\mathbf{v}, \varepsilon_{\alpha\beta}, \kappa_{\alpha\beta}, N^{\alpha\beta}, M^{\alpha\beta}) \\ = \Pi_C(\varepsilon_{\alpha\beta}, \kappa_{\alpha\beta}, N^{\alpha\beta}, M^{\alpha\beta}) + \int_{\Gamma_M} \theta_{\mathbf{n}} (M_{\mathbf{nn}} - \bar{M}_{\mathbf{nn}}) d\Gamma \\ - \int_{\Gamma_T} \mathbf{v} \cdot (\mathbf{T} - \bar{\mathbf{T}}) d\Gamma - \mathbf{v} \cdot \mathbf{a}_3 (P - \bar{P})_{\mathbf{x} \in C_P} - \int_{\Omega} \mathbf{v} \cdot (\mathbf{b} - \bar{\mathbf{b}}) d\Omega \end{aligned} \quad (18)$$

where $\bar{\mathbf{T}}$, $\bar{M}_{\mathbf{nn}}$ and \bar{P} are the corresponding prescribed traction, bending moment and concentrated force on edges Γ_T , Γ_M and corner C_P respectively. All the boundaries meet the following geometric relationships:

$$\begin{cases} \Gamma = \Gamma_v \cup \Gamma_T \cup \Gamma_\theta \cup \Gamma_M, & C = C_v \cup C_P, \\ \Gamma_v \cap \Gamma_T = \Gamma_\theta \cap \Gamma_M = C_v \cap C_P = \emptyset \end{cases} \quad (19)$$

and $\bar{\mathbf{b}}$ stands for the prescribed body force in Ω , \mathbf{b} also can be given based upon Euler-Lagrange equations [15] as:

$$\mathbf{b} = \mathbf{b}_N + \mathbf{b}_M \rightarrow \begin{cases} \mathbf{b}_N = (\mathbf{a}_\alpha N^{\alpha\beta})|_\beta \\ \mathbf{b}_M = (\mathbf{a}_3 M^{\alpha\beta})|_{\alpha\beta} \end{cases} \quad (20)$$

Introducing a standard variational argument to Eq. (18), $\delta\Pi = 0$, and considering the arbitrariness of virtual variables, $\delta\mathbf{v}$, $\delta\varepsilon_{\alpha\beta}$, $\delta\kappa_{\alpha\beta}$, $N^{\alpha\beta}$, $M^{\alpha\beta}$ lead to the following weak form:

$$- \int_{\Omega} h \delta\varepsilon_{\alpha\beta} C^{\alpha\beta\gamma\eta} \varepsilon_{\gamma\eta} d\Omega + \int_{\Omega} \delta\varepsilon_{\alpha\beta} N^{\alpha\beta} d\Omega = 0 \quad (21a)$$

179

$$-\int_{\Omega} \frac{h^3}{12} \delta \kappa_{\alpha\beta} C^{\alpha\beta\gamma\eta} \kappa_{\gamma\eta} d\Omega + \int_{\Omega} \delta \kappa_{\alpha\beta} M^{\alpha\beta} d\Omega = 0 \quad (21b)$$

180

$$\begin{aligned} \int_{\Omega} \delta N^{\alpha\beta} \varepsilon_{\alpha\beta} d\Omega - \int_{\Gamma} \delta \mathbf{T}_N \cdot \mathbf{v} d\Gamma + \int_{\Omega} \delta \mathbf{b}_N \cdot \mathbf{v} d\Omega \\ + \int_{\Gamma_v} \delta \mathbf{T}_N \cdot \mathbf{v} d\Gamma = \int_{\Gamma_v} \delta \mathbf{T}_N \cdot \bar{\mathbf{v}} d\Gamma \end{aligned} \quad (21c)$$

181

$$\begin{aligned} \int_{\Omega} \delta M^{\alpha\beta} \kappa_{\alpha\beta} d\Omega - \int_{\Gamma} \delta M_{\mathbf{n}\mathbf{n}} \theta_{\mathbf{n}} d\Gamma + \int_{\Gamma} \delta \mathbf{T}_M \cdot \mathbf{v} d\Gamma + (\delta P \mathbf{a}_3 \cdot \mathbf{v})_{\mathbf{x} \in C} + \int_{\Omega} \delta \mathbf{b}_M \cdot \mathbf{v} d\Omega \\ + \int_{\Gamma_{\theta}} \delta M_{\mathbf{n}\mathbf{n}} \theta_{\mathbf{n}} d\Gamma - \int_{\Gamma_v} \delta \mathbf{T}_M \cdot \mathbf{v} d\Gamma - (\delta P \mathbf{a}_3 \cdot \mathbf{v})_{\mathbf{x} \in C_v} \\ = \int_{\Gamma_{\theta}} \delta M_{\mathbf{n}\mathbf{n}} \bar{\theta}_{\mathbf{n}} d\Gamma - \int_{\Gamma_v} \delta \mathbf{T}_M \cdot \bar{\mathbf{v}} d\Gamma - (\delta P \mathbf{a}_3 \cdot \bar{\mathbf{v}})_{\mathbf{x} \in C_v} \end{aligned} \quad (21d)$$

182

$$\begin{aligned} \int_{\Gamma} \delta \theta_{\mathbf{n}} M_{\mathbf{n}\mathbf{n}} d\Gamma - \int_{\Gamma} \delta \mathbf{v} \cdot \mathbf{T} d\Gamma - (\delta \mathbf{v} \cdot \mathbf{a}_3 P)_{\mathbf{x} \in C} + \int_{\Omega} \delta \mathbf{v} \cdot \mathbf{b} d\Omega \\ - \int_{\Gamma_{\theta}} \delta \theta_{\mathbf{n}} M_{\mathbf{n}\mathbf{n}} d\Gamma + \int_{\Gamma_v} \delta \mathbf{v} \cdot \mathbf{T} d\Gamma + (\delta \mathbf{v} \cdot \mathbf{a}_3 P)_{\mathbf{x} \in C_v} = - \int_{\Gamma_T} \delta \mathbf{v} \cdot \bar{\mathbf{t}} d\Gamma - \int_{\Omega} \delta \mathbf{v} \cdot \bar{\mathbf{b}} d\Omega \end{aligned} \quad (21e)$$

183 where the geometric relationships of Eq. (19) is used herein.

184 3. Mixed meshfree formulation for modified Hellinger-Reissner weak 185 form

186 3.1. Reproducing kernel approximation for displacement

187 In this study, the displacement is approximated by traditional reproducing
188 kernel approximation. As shown in Fig, the mid-surface of the shell Ω is dis-
189 cretized by a set of meshfree nodes $\{\boldsymbol{\xi}_I\}_{I=1}^{n_p}$ in parametric configuration, where
190 n_p is the total number of meshfree nodes. The approximated displacement
191 namely \mathbf{v}^h can be expressed by:

$$\mathbf{v}(\boldsymbol{\xi}) = \sum_{I=1}^{n_p} \Psi_I(\boldsymbol{\xi}) \mathbf{d}_I \quad (22)$$

192 in which Ψ_I and \mathbf{d}_I is the shape function and nodal coefficient tensor related by
193 node $\boldsymbol{\xi}_I$. According to reproducing kernel approximation [4], the shape function
194 takes the following form:

$$\Psi_I(\boldsymbol{\xi}) = \mathbf{p}^T(\boldsymbol{\xi}) \mathbf{c}(\boldsymbol{\xi}) \phi(\boldsymbol{\xi}_I - \boldsymbol{\xi}) \quad (23)$$

195 where \mathbf{p} is the basis function vector, and in this study, the following quadratic
196 basis function is considered:

$$\mathbf{p} = \{1, \xi^1, \xi^2, (\xi^1)^2, \xi^1 \xi^2, (\xi^2)^2\}^T \quad (24)$$

197 The kernel function denoted by ϕ controls the support and smoothness of
198 meshfree shape functions. The quantic B-spline function with square support is
199 used herein as the kernel function:

$$\phi(\boldsymbol{\xi}_I - \boldsymbol{\xi}) = \phi(\hat{s}_1) \phi(\hat{s}_2), \quad \hat{s}_\alpha = \frac{|\xi_I^\alpha - \xi^\alpha|}{s_{\alpha I}} \quad (25)$$

200 with

$$\phi(\hat{s}_\alpha) = \frac{1}{5!} \begin{cases} (3 - 3\hat{s}_\alpha)^5 - 6(2 - 3\hat{s}_\alpha)^5 + 15(1 - 3\hat{s}_\alpha)^5 & \hat{s}_\alpha \leq \frac{1}{3} \\ (3 - 3\hat{s}_\alpha)^5 - 6(2 - 3\hat{s}_\alpha)^5 & \frac{1}{3} < \hat{s}_\alpha \leq \frac{2}{3} \\ (3 - 3\hat{s}_\alpha)^5 & \frac{2}{3} < \hat{s}_\alpha \leq 1 \\ 0 & \hat{s}_\alpha > 1 \end{cases} \quad (26)$$

201 and $\hat{s}_{\alpha I}$ means the characterized size of support for meshfree shape function Ψ_I .

202 The unknown vector \mathbf{c} in shape function are determined by the fulfillment
203 of the so-call consistency condition:

$$\sum_{I=1}^{n_p} \Psi_I(\boldsymbol{\xi}) \mathbf{p}(\boldsymbol{\xi}_I) = \mathbf{p}(\boldsymbol{\xi}) \quad (27)$$

204 or equivalently

$$\sum_{I=1}^{n_p} \Psi_I(\boldsymbol{\xi}) \mathbf{p}(\boldsymbol{\xi}_I - \boldsymbol{\xi}) = \mathbf{p}(\mathbf{0}) \quad (28)$$

Substituting Eq. (22) into (28), yields:

$$\mathbf{A}(\boldsymbol{\xi})\mathbf{c}(\boldsymbol{\xi}) = \mathbf{p}(\mathbf{0}) \quad \Rightarrow \quad \mathbf{c}(\boldsymbol{\xi}) = \mathbf{A}^{-1}(\boldsymbol{\xi})\mathbf{p}(\mathbf{0}) \quad (29)$$

where \mathbf{A} is the moment matrix:

$$\mathbf{A}(\boldsymbol{\xi}) = \sum_{I=1}^{n_p} \phi(\boldsymbol{\xi}_I - \boldsymbol{\xi}) \mathbf{p}(\boldsymbol{\xi}_I - \boldsymbol{\xi}) \mathbf{p}^T(\boldsymbol{\xi}_I - \boldsymbol{\xi}) \quad (30)$$

Taking Eq. (29) back into Eq. (22), the expression of meshfree shape function can be given by:

$$\Psi_I(\boldsymbol{\xi}) = \mathbf{p}^T(\boldsymbol{\xi}_I - \boldsymbol{\xi}) \mathbf{A}^{-1}(\boldsymbol{\xi}) \mathbf{p}(\mathbf{0}) \phi(\boldsymbol{\xi}_I - \boldsymbol{\xi}) \quad (31)$$

3.2. Reproducing kernel gradient smoothing approximation for effective stress and strain

In Galerkin meshfree formulation, the mid-plane of thin shell Ω is split by a set of integration cells Ω_C 's, $\cup_{C=1}^{n_e} \Omega_C \approx \Omega$. With the inspiration of reproducing kernel smoothing framework, the Cartesian and covariant derivatives of displacement, $\mathbf{v}_{,\alpha}$ and $-\mathbf{v}_{,\alpha}|_{\beta}$, in strains $\varepsilon_{\alpha\beta}$, $\kappa_{\alpha\beta}$ are approximated by $(p-1)$ -th order polynomials in each integration cells. In integration cell Ω_C , the approximated derivatives and strains denoted by $\mathbf{v}_{,\alpha}^h$, $\varepsilon_{\alpha\beta}^h$ and $-\mathbf{v}_{,\alpha}|_{\beta}^h$, $\kappa_{\alpha\beta}^h$ can be expressed by:

$$\mathbf{v}_{,\alpha}^h(\boldsymbol{\xi}) = \mathbf{q}^T(\boldsymbol{\xi}) \mathbf{d}_{\alpha}^{\varepsilon}, \quad \varepsilon_{\alpha\beta}^h(\boldsymbol{\xi}) = \mathbf{q}^T(\boldsymbol{\xi}) \frac{1}{2} (\mathbf{a}_{\alpha} \cdot \mathbf{d}_{\beta}^{\varepsilon} + \mathbf{a}_{\beta} \cdot \mathbf{d}_{\alpha}^{\varepsilon}) \quad (32)$$

$$-\mathbf{v}_{,\alpha}|_{\beta}^h(\boldsymbol{\xi}) = \mathbf{q}^T(\boldsymbol{\xi}) \mathbf{d}_{\alpha\beta}^{\kappa}, \quad \kappa_{\alpha\beta}^h(\boldsymbol{\xi}) = \mathbf{q}^T(\boldsymbol{\xi}) \mathbf{a}_3 \cdot \mathbf{d}_{\alpha\beta}^{\kappa} \quad (33)$$

where \mathbf{q} is the linear polynomial vector and has the following form:

$$\mathbf{q} = \{1, \xi^1, \xi^2\}^T \quad (34)$$

and the $\mathbf{d}_{\alpha}^{\varepsilon}$, $\mathbf{d}_{\alpha\beta}^{\kappa}$ are the corresponding coefficient vector tensors. For the conciseness, the mixed usage of tensor and vector is introduced in this study, for example, the component of coefficient tensor vector $\mathbf{d}_{\alpha I}^{\varepsilon}$, $\mathbf{d}_{\alpha}^{\varepsilon} = \{\mathbf{d}_{\alpha I}^{\varepsilon}\}$, is a three dimensional tensor, $\dim \mathbf{d}_{\alpha I}^{\varepsilon} = \dim \mathbf{v}$.

In order to meet the integration constraint of thin shell problem, the approximated stresses $N^{\alpha\beta h}$, $M^{\alpha\beta h}$ are assumed to be a similar form with strains, yields:

$$N^{\alpha\beta h}(\boldsymbol{\xi}) = \mathbf{q}^T(\boldsymbol{\xi}) \mathbf{a}^{\alpha} \cdot \mathbf{d}_N^{\beta}, \quad \mathbf{a}_{\alpha} N^{\alpha\beta h}(\boldsymbol{\xi}) = \mathbf{q}^T(\boldsymbol{\xi}) \mathbf{d}_N^{\beta} \quad (35)$$

$$M^{\alpha\beta h}(\boldsymbol{\xi}) = \mathbf{q}^T(\boldsymbol{\xi}) \mathbf{a}_3 \cdot \mathbf{d}_M^{\alpha\beta}, \quad \mathbf{a}_3 M^{\alpha\beta h}(\boldsymbol{\xi}) = \mathbf{q}^T(\boldsymbol{\xi}) \mathbf{d}_M^{\alpha\beta} \quad (36)$$

substituting the approximations of Eqs. (22), (32), (33), (35), (36) into Eqs. (21c), (21d) can express $\mathbf{d}_{\beta}^{\varepsilon}$ and $\mathbf{d}_{\alpha\beta}^{\kappa}$ by \mathbf{d} as:

$$\mathbf{d}_{\beta}^{\varepsilon} = \mathbf{G}^{-1} \left(\sum_{I=1}^{n_p} (\tilde{\mathbf{g}}_{\beta I} - \bar{\mathbf{g}}_{\beta I}) \mathbf{d}_I + \hat{\mathbf{g}}_{\beta} \right) \quad (37)$$

230

$$\mathbf{d}_{\alpha\beta}^\kappa = \mathbf{G}^{-1} \left(\sum_{I=1}^{n_p} (\tilde{\mathbf{g}}_{\alpha\beta I} - \bar{\mathbf{g}}_{\alpha\beta I}) \mathbf{d}_I + \hat{\mathbf{g}}_{\alpha\beta} \right) \quad (38)$$

231 with

$$\mathbf{G} = \int_{\Omega_C} \mathbf{q}^T \mathbf{q} d\Omega \quad (39)$$

232

$$\tilde{\mathbf{g}}_{\beta I} = \int_{\Gamma_C} \Psi_I \mathbf{q} n_\beta d\Gamma - \int_{\Omega_C} \Psi_I \mathbf{q}_{|\beta} d\Omega \quad (40a)$$

$$\bar{\mathbf{g}}_{\beta I} = \int_{\Gamma_C \cap \Gamma_v} \Psi_I \mathbf{q} n_\beta d\Gamma \quad (40b)$$

$$\hat{\mathbf{g}}_\beta = \int_{\Gamma_C \cap \Gamma_v} \mathbf{q} n_\beta \bar{\mathbf{v}} d\Gamma \quad (40c)$$

233

$$\begin{aligned} \tilde{\mathbf{g}}_{\alpha\beta I} &= \int_{\Gamma_C} \Psi_{I,\gamma} n^\gamma \mathbf{q} n_\alpha n_\beta d\Gamma - \int_{\Gamma_C} \Psi_I (\mathbf{q}_{|\beta} n_\alpha + (\mathbf{q} s_\alpha n_\beta)_{,\gamma} s^\gamma) d\Gamma \\ &\quad + [[\Psi_I \mathbf{q} s_\alpha n_\beta]]_{\mathbf{x} \in C_C} - \int_{\Omega_C} \Psi \mathbf{q}_{,\alpha|\beta} d\Omega \end{aligned} \quad (41a)$$

$$\begin{aligned} \bar{\mathbf{g}}_{\alpha\beta I} &= \int_{\Gamma_C \cap \Gamma_\theta} \Psi_{I,\gamma} n^\gamma \mathbf{q} n_\alpha n_\beta d\Gamma - \int_{\Gamma_C \cap \Gamma_v} \Psi_I (\mathbf{q}_{|\beta} n_\alpha + (\mathbf{q} s_\alpha n_\beta)_{,\gamma} s^\gamma) d\Gamma \\ &\quad + [[\Psi_I \mathbf{q} s_\alpha n_\beta]]_{\mathbf{x} \in C_C \cap C_v} \end{aligned} \quad (41b)$$

$$\begin{aligned} \hat{\mathbf{g}}_{\alpha\beta} &= \int_{\Gamma_C \cap \Gamma_\theta} \mathbf{q} n_\alpha n_\beta \mathbf{a}_3 \bar{\boldsymbol{\theta}}_\alpha d\Gamma - \int_{\Gamma_C \cap \Gamma_v} (\mathbf{q}_{|\beta} n_\alpha + (\mathbf{q} s_\alpha n_\beta)_{,\gamma} s^\gamma) \bar{\mathbf{v}} d\Gamma \\ &\quad + [[\mathbf{q} s_\alpha n_\beta \bar{\mathbf{v}}]]_{\mathbf{x} \in C_C \cap C_v} \end{aligned} \quad (41c)$$

234 where evaluations of $\mathbf{q}_{|\beta}$, $\mathbf{q}_{,|\beta}$ are detail in Appendix ?? . Further plugging Eqs.
 235 (37) and (38) back into Eqs. (32) and (33) respectively gives the final expression
 236 of $\mathbf{v}_{,\alpha}^h$, $\varepsilon_{\alpha\beta}^h$ and $-\mathbf{v}_{,\alpha\beta}^h$, $\boldsymbol{\kappa}_{\alpha\beta}^h$ as:

$$\mathbf{v}_{,\alpha}^h = \sum_{I=1}^{n_p} (\tilde{\Psi}_{I,\alpha} - \bar{\Psi}_{I,\alpha}) \mathbf{d}_I + \mathbf{q}^T \mathbf{G}^{-1} \hat{\mathbf{g}}_\alpha \quad (42a)$$

237

$$\begin{aligned} \varepsilon_{\alpha\beta}^h &= \sum_{I=1}^{n_p} \frac{1}{2} (\mathbf{a}_\alpha \tilde{\Psi}_{I,\beta} + \mathbf{a}_\beta \tilde{\Psi}_{I,\alpha}) \cdot \mathbf{d}_I - \sum_{I=1}^{n_p} \frac{1}{2} (\mathbf{a}_\alpha \bar{\Psi}_{I,\beta} + \mathbf{a}_\beta \bar{\Psi}_{I,\alpha}) \cdot \mathbf{d}_I \\ &\quad + \mathbf{q}^T \mathbf{G}^{-1} \frac{1}{2} (\mathbf{a}_\alpha \cdot \hat{\mathbf{g}}_\beta + \mathbf{a}_\beta \cdot \hat{\mathbf{g}}_\alpha) \\ &= \hat{\varepsilon}_{\alpha\beta}^h - \bar{\varepsilon}_{\alpha\beta}^h + \hat{\varepsilon}_{\alpha\beta}^h \end{aligned} \quad (42b)$$

238

$$-\mathbf{v}_{,\alpha}^h|_\beta = \sum_{I=1}^{n_p} (\tilde{\Psi}_{I,\alpha\beta} - \bar{\Psi}_{I,\alpha\beta}) \mathbf{d}_I + \mathbf{q}^T \mathbf{G}^{-1} \hat{\mathbf{g}}_{\alpha\beta} \quad (43a)$$

239

$$\begin{aligned}\kappa_{\alpha\beta}^h &= \sum_{I=1}^{n_p} \tilde{\Psi}_{I,\alpha\beta} \mathbf{a}_3 \cdot \mathbf{d}_I - \sum_{I=1}^{n_p} \bar{\Psi}_{I,\alpha\beta} \mathbf{a}_3 \cdot \mathbf{d}_I + \mathbf{q}^T \mathbf{G}^{-1} \mathbf{a}_3 \cdot \hat{\mathbf{g}}_{\alpha\beta} \\ &= \tilde{\kappa}_{\alpha\beta}^h - \bar{\kappa}_{\alpha\beta}^h + \hat{\kappa}_{\alpha\beta}^h\end{aligned}\quad (43b)$$

240 with

$$\tilde{\Psi}_{I,\alpha}(\boldsymbol{\xi}) = \mathbf{q}^T(\boldsymbol{\xi}) \mathbf{G}^{-1} \tilde{\mathbf{g}}_{\alpha I}, \quad \bar{\Psi}_{I,\alpha}(\boldsymbol{\xi}) = \mathbf{q}^T(\boldsymbol{\xi}) \mathbf{G}^{-1} \tilde{\mathbf{g}}_{\alpha I} \quad (44)$$

241

$$\begin{cases} \tilde{\varepsilon}_{\alpha\beta}^h = \sum_{I=1}^{n_p} \frac{1}{2} (\mathbf{a}_\alpha \tilde{\Psi}_{I,\beta} + \mathbf{a}_\beta \tilde{\Psi}_{I,\alpha}) \cdot \mathbf{d}_I \\ \bar{\varepsilon}_{\alpha\beta}^h = \sum_{I=1}^{n_p} \frac{1}{2} (\mathbf{a}_\alpha \bar{\Psi}_{I,\beta} + \mathbf{a}_\beta \bar{\Psi}_{I,\alpha}) \cdot \mathbf{d}_I \\ \hat{\varepsilon}_{\alpha\beta}^h = \mathbf{q}^T \mathbf{G}^{-1} \frac{1}{2} (\mathbf{a}_\alpha \cdot \hat{\mathbf{g}}_\beta + \mathbf{a}_\beta \cdot \hat{\mathbf{g}}_\alpha) \end{cases} \quad (45)$$

242

$$\tilde{\Psi}_{I,\alpha\beta}(\boldsymbol{\xi}) = \mathbf{q}^T(\boldsymbol{\xi}) \mathbf{G}^{-1} \tilde{\mathbf{g}}_{\alpha\beta I}, \quad \bar{\Psi}_{I,\alpha\beta}(\boldsymbol{\xi}) = \mathbf{q}^T(\boldsymbol{\xi}) \mathbf{G}^{-1} \tilde{\mathbf{g}}_{\alpha\beta I} \quad (46)$$

243

$$\begin{cases} \tilde{\kappa}_{\alpha\beta}^h = \sum_{I=1}^{n_p} \tilde{\Psi}_{I,\alpha\beta} \mathbf{a}_3 \cdot \mathbf{d}_I \\ \bar{\kappa}_{\alpha\beta}^h = \sum_{I=1}^{n_p} \bar{\Psi}_{I,\alpha\beta} \mathbf{a}_3 \cdot \mathbf{d}_I \\ \hat{\kappa}_{\alpha\beta}^h = \mathbf{q}^T \mathbf{G}^{-1} \mathbf{a}_3 \cdot \hat{\mathbf{g}}_{\alpha\beta} \end{cases} \quad (47)$$

244 Furthermore, taking Eqs. (32) and (33) into Eqs.(21a) and (21b) can obtain
 245 the approximated effective stresses $N^{\alpha\beta h}$, $M^{\alpha\beta h}$ and their coefficients \mathbf{d}_N^β , $\mathbf{d}_M^{\alpha\beta}$
 246 as:

$$\delta \mathbf{d}_\alpha^\varepsilon \cdot \mathbf{G}_N^{\alpha\eta} \cdot \mathbf{d}_\eta^\varepsilon = \delta \mathbf{d}_\alpha^\varepsilon \cdot \mathbf{d}_N^\alpha \mathbf{G} \Rightarrow \mathbf{d}_N^\alpha = \mathbf{G}^{-1} \mathbf{G}_N^{\alpha\eta} \cdot \mathbf{d}_\eta^\varepsilon \quad (48)$$

247

$$\delta \mathbf{d}_{\alpha\beta}^\kappa : \mathbf{G}_M^{\alpha\beta\gamma\eta} : \mathbf{d}_{\gamma\eta}^\kappa \mathbf{G} = \delta \mathbf{d}_{\alpha\beta}^\kappa \cdot \mathbf{d}_M^{\alpha\beta} \mathbf{G} \Rightarrow \mathbf{d}_M^{\alpha\beta} = \mathbf{G}^{-1} \mathbf{G}_M^{\alpha\beta} \cdot \mathbf{d}_{\gamma\eta}^\kappa \quad (49)$$

248 with

$$\mathbf{G}_N^{\alpha\eta} = \int_{\Omega_C} \mathbf{a}_\beta h C^{\alpha\beta\gamma\eta} \mathbf{a}_\gamma \mathbf{q} \mathbf{q}^T d\Omega \quad (50)$$

249

$$\mathbf{G}_M^{\alpha\beta\gamma\eta} = \int_{\Omega_C} \mathbf{a}_3 \frac{h^3}{12} C^{\alpha\beta\gamma\eta} \mathbf{a}_3 \mathbf{q} \mathbf{q}^T d\Omega \quad (51)$$

250 Finally, taking Eqs. (48) and (49) back to Eqs. (35), (36) can express the
 251 components of stresses as follows:

$$N^{\alpha\beta h} = \mathbf{q}^T \mathbf{G}^{-1} \frac{1}{2} (\mathbf{a}^\alpha \cdot \mathbf{G}_N^{\beta\gamma} + \mathbf{a}^\beta \cdot \mathbf{G}_N^{\alpha\gamma}) \cdot \mathbf{d}_\gamma^\varepsilon \quad (52)$$

252

$$M^{\alpha\beta h} = \mathbf{q}^T \mathbf{G}^{-1} \mathbf{a}_3 \cdot \mathbf{G}_M^{\alpha\beta\gamma\eta} \cdot \mathbf{d}_{\gamma\eta}^\kappa \quad (53)$$

253 It is noted that, referring to reproducing kernel gradient smoothing frame-
 254 work [?], $\tilde{\Psi}_{I,\alpha}$, $\tilde{\Psi}_{I,\alpha\beta}$ are actually the first and second order smoothed gradients

255 in curvilinear coordinates. $\tilde{\mathbf{g}}_{\alpha I}$ and $\tilde{\mathbf{g}}_{\alpha\beta I}$ are the right hand side integration con-
 256 straints for first and second order gradients, then this formulation can meet the
 257 variational consistency for the p -th order polynomials. It should be known that,
 258 in curved model, the variational consistency for non-polynomial functions, like
 259 trigonometric functions, should be required for the polynomial solution. Even
 260 with p -th order variational consistency, the proposed formulation can not ex-
 261 actly reproduce the solution spanned by basis functions, however the accuracy
 262 of reproducing kernel smoothed gradients is still better than traditional meshfree
 263 formulation, this will be evidenced by numerical examples in further section.

$$\begin{aligned}
 N^{\alpha\beta h} &= \mathbf{q}^T \mathbf{G}^{-1} \mathbf{a}_\alpha \cdot \int_{\Omega_C} \mathbf{a}_\zeta h C^{\zeta\beta\gamma\eta} \mathbf{a}_\gamma \mathbf{q} \mathbf{q}^T d\Omega \cdot \mathbf{d}_\eta^\varepsilon \\
 &= \mathbf{q}^T \mathbf{G}^{-1} \int_{\Omega_C} \mathbf{a}_\alpha h C^{\alpha\beta\gamma\eta} \mathbf{a}_\gamma \mathbf{q} \mathbf{q}^T d\Omega \cdot \mathbf{d}_\eta^\varepsilon \\
 &= \mathbf{q}^T \mathbf{G}^{-1} \int_{\Omega_C} \mathbf{a}_\alpha \mathbf{q} h C^{\alpha\beta\gamma\eta} \varepsilon_{\alpha\beta}^h d\Omega \\
 &= \mathbf{q}^T \mathbf{G}^{-1} \int_{\Omega_C} \mathbf{q} \mathbf{a}_\alpha (\tilde{N}^{\alpha\beta h} - \bar{N}^{\alpha\beta h} + \hat{N}^{\alpha\beta h}) d\Omega
 \end{aligned} \tag{54}$$

264 **4. Naturally variational enforcement for essential boundary condi-**
 265 **tions**

266 *4.1. Discrete equilibrium equations*

267 With the approximated effective stresses and strains, the last equation of
 268 weak form becomes:

$$\begin{aligned}
 & \int_{\Gamma_C} \delta \theta_n M_{nn} d\Gamma - \int_{\Gamma_C} \delta \mathbf{v} \cdot \mathbf{T} d\Gamma - (\delta \mathbf{v} \cdot \mathbf{a}_3 P)_{\mathbf{x} \in C_C} + \int_{\Omega_C} \delta \mathbf{v} \cdot \mathbf{b} d\Omega \\
 &= - \int_{\Gamma_C} \delta \mathbf{v} \cdot \mathbf{T}_N d\Gamma + \int_{\Omega_C} \delta \mathbf{v} \cdot \mathbf{b}_N d\Omega \\
 &+ \int_{\Gamma_C} \delta \theta_n M_{nn} d\Gamma - \int_{\Gamma_C} \delta \mathbf{v} \cdot \mathbf{T}_M d\Gamma - (\delta \mathbf{v} \cdot \mathbf{a}_3 P)_{\mathbf{x} \in C_C} + \int_{\Omega_C} \delta \mathbf{v} \cdot \mathbf{b}_M d\Omega \\
 &= - \sum_{I=1}^{n_p} \delta \mathbf{d}_I \cdot (\tilde{\mathbf{g}}_{\alpha I}^T \mathbf{d}_N^\alpha + \tilde{\mathbf{g}}_{\alpha \beta}^T \mathbf{d}_M^{\alpha \beta}) \\
 &= - \sum_{I=1}^{n_p} \delta \mathbf{d}_I \cdot ((\mathbf{G}^{-1} \tilde{\mathbf{g}}_{\alpha I})^T \mathbf{G}_N^{\alpha \eta} \cdot \mathbf{d}_\eta^\varepsilon) \\
 &= - \sum_{I=1}^{n_p} \delta \mathbf{d}_I \cdot ((\mathbf{G}^{-1} \tilde{\mathbf{g}}_{\alpha I})^T \int_{\Omega_C} \mathbf{a}_\beta \mathbf{q} h C^{\alpha \beta \gamma \eta} \mathbf{q}^T \mathbf{a}_\gamma \cdot \mathbf{d}_\eta^\varepsilon d\Omega) \\
 &= \int_{\Omega_C} \delta \tilde{\varepsilon}_{\alpha \beta} (\tilde{N}^{\alpha \beta} - \bar{N}^{\alpha \beta} + \hat{N}^{\alpha \beta}) d\Omega
 \end{aligned} \tag{55}$$

269

$$- \sum_{C=1}^{n_e} \sum_{I=1}^{n_p} \delta \mathbf{d}_I \cdot \left((\tilde{\mathbf{g}}_{\alpha I}^T - \bar{\mathbf{g}}_{\alpha I}^T) \mathbf{d}_N^\alpha + (\tilde{\mathbf{g}}_{\alpha \beta I}^T - \bar{\mathbf{g}}_{\alpha \beta I}^T) \mathbf{d}_M^{\alpha \beta} \right) = - \sum_{I=1}^{n_p} \delta \mathbf{d}_I \cdot \mathbf{f}_I \tag{56}$$

270 where \mathbf{f}_I 's are the components of the traditional force vector:

$$\mathbf{f}_I = \int_{\Gamma_t} \Psi_I \bar{\mathbf{t}} d\Gamma - \int_{\Gamma_M} \Psi_{I,\gamma} n^\gamma \bar{M}_{nn} d\Gamma + [[\Psi_I \mathbf{a}_3 \bar{P}]]_{\mathbf{x} \in C_P} + \int_{\Omega} \Psi_I \bar{\mathbf{b}} d\Omega \tag{57}$$

$$\begin{aligned}
 \sum_{I=1}^{n_p} \delta \mathbf{d}_I \cdot \tilde{\mathbf{g}}_{\alpha I}^T \mathbf{d}_N^\alpha &= \sum_{I=1}^{n_p} \delta \mathbf{d}_I \cdot (\mathbf{G}^{-1} \tilde{\mathbf{g}}_{\alpha I})^T \mathbf{G} \mathbf{d}_N^\alpha \\
 &= \int_{\Omega_C} \sum_{I=1}^{n_p} \delta \mathbf{d}_I \cdot (\mathbf{q}^T \mathbf{G}^{-1} \tilde{\mathbf{g}}_{\alpha I})^T \mathbf{q}^T \mathbf{d}_N^\alpha d\Omega \\
 &= \int_{\Omega_C} \sum_{I=1}^{n_p} \delta \mathbf{d}_I \cdot \mathbf{a}_\beta (\mathbf{q}^T \mathbf{G}^{-1} \tilde{\mathbf{g}}_{\alpha I})^T N^{\alpha \beta h} d\Omega \\
 &= \int_{\Omega_C} \delta \tilde{\varepsilon}_{\alpha \beta}^h N^{\alpha \beta h} d\Omega
 \end{aligned} \tag{58}$$

271

$$\begin{aligned}
& \sum_{I=1}^{n_p} \delta \mathbf{d}_I \cdot \left((\tilde{\mathbf{g}}_{\alpha I}^T - \bar{\mathbf{g}}_{\alpha I}^T) \mathbf{d}_N^\alpha + (\tilde{\mathbf{g}}_{\alpha \beta I}^T - \bar{\mathbf{g}}_{\alpha \beta I}^T) \mathbf{d}_M^{\alpha \beta} \right) \\
&= \int_{\Omega_C} \left((\delta \tilde{\varepsilon}_{\alpha \beta} - \delta \bar{\varepsilon}_{\alpha \beta}) N^{\alpha \beta h} + (\delta \tilde{\kappa}_{\alpha \beta} - \delta \bar{\kappa}_{\alpha \beta}) M^{\alpha \beta h} \right) d\Omega
\end{aligned} \tag{59}$$

272

$$\begin{aligned}
& \int_{\Omega_C} \delta \tilde{\varepsilon}_{\alpha \beta} N^{\alpha \beta h} d\Omega \\
&= \int_{\Omega_C} \delta \tilde{\varepsilon}_{\alpha \beta} N^{\alpha \beta h} d\Omega
\end{aligned} \tag{60}$$

273 and further substituting coefficients \mathbf{d}_N^α , $\mathbf{d}_M^{\alpha \beta}$ into Eq. (56) gives the final discrete
 274 equilibrium equations:

$$\begin{aligned}
& - \sum_{C=1}^{n_c} (\tilde{\mathbf{g}}_{\alpha I}^T - \bar{\mathbf{g}}_{\alpha I}^T) \mathbf{d}_N^\alpha - \sum_{C=1}^{n_c} (\tilde{\mathbf{g}}_{\alpha \beta I}^T - \bar{\mathbf{g}}_{\alpha \beta I}^T) \mathbf{d}_M^{\alpha \beta} \\
&= \sum_{C=1}^{n_e} \sum_{J=1}^{n_p} \left(\begin{aligned}
& \mathbf{a}_\alpha \tilde{\mathbf{g}}_{\beta I}^T h C^{\alpha \beta \gamma \eta} \mathbf{a}_\gamma \tilde{\mathbf{g}}_{\eta J} + \tilde{\mathbf{g}}_{\alpha \beta I}^T \mathbf{a}_3 \frac{h^3}{12} C^{\alpha \beta \gamma \eta} \mathbf{a}_3 \tilde{\mathbf{g}}_{\gamma \eta} \\
& - \mathbf{a}_\alpha \bar{\mathbf{g}}_{\beta I}^T h C^{\alpha \beta \gamma \eta} \mathbf{a}_\gamma \tilde{\mathbf{g}}_{\eta J} - \mathbf{a}_\alpha \tilde{\mathbf{g}}_{\beta I}^T h C^{\alpha \beta \gamma \eta} \mathbf{a}_\gamma \bar{\mathbf{g}}_{\eta J} \\
& - \bar{\mathbf{g}}_{\alpha \beta I}^T \mathbf{a}_3 \frac{h^3}{12} C^{\alpha \beta \gamma \eta} \mathbf{a}_3 \tilde{\mathbf{g}}_{\gamma \eta J} - \tilde{\mathbf{g}}_{\alpha \beta I}^T \mathbf{a}_3 \frac{h^3}{12} C^{\alpha \beta \gamma \eta} \mathbf{a}_3 \bar{\mathbf{g}}_{\gamma \eta J} \\
& + \mathbf{a}_\alpha \tilde{\mathbf{g}}_{\beta I}^T h C^{\alpha \beta \gamma \eta} \mathbf{a}_\gamma \hat{\mathbf{g}}_{\eta J} - \mathbf{a}_\alpha \bar{\mathbf{g}}_{\beta I}^T h C^{\alpha \beta \gamma \eta} \mathbf{a}_\gamma \hat{\mathbf{g}}_{\eta J} \\
& + \tilde{\mathbf{g}}_{\alpha \beta I}^T \mathbf{a}_3 \frac{h^3}{12} C^{\alpha \beta \gamma \eta} \mathbf{a}_3 \hat{\mathbf{g}}_{\gamma \eta J} - \bar{\mathbf{g}}_{\alpha \beta I}^T \mathbf{a}_3 \frac{h^3}{12} C^{\alpha \beta \gamma \eta} \mathbf{a}_3 \hat{\mathbf{g}}_{\gamma \eta J} \\
& + \mathbf{a}_\alpha \bar{\mathbf{g}}_{\beta I}^T h C^{\alpha \beta \gamma \eta} \mathbf{a}_\gamma \bar{\mathbf{g}}_{\eta J} + \bar{\mathbf{g}}_{\alpha \beta I}^T \mathbf{a}_3 \frac{h^3}{12} C^{\alpha \beta \gamma \eta} \mathbf{a}_3 \bar{\mathbf{g}}_{\gamma \eta J}
\end{aligned} \right) \\
&= \sum_{J=1}^{n_p} (\mathbf{K}_{IJ} + \tilde{\mathbf{K}}_{IJ} + \bar{\mathbf{K}}_{IJ}) \cdot \mathbf{d}_J - \tilde{\mathbf{f}}_I - \bar{\mathbf{f}}_I
\end{aligned} \tag{61}$$

275 where

$$\mathbf{K}_{IJ} = \int_{\Omega} \tilde{\varepsilon}_{\alpha \beta I} \tilde{\mathbf{N}}_J^{\alpha \beta} d\Omega + \int_{\Omega} \tilde{\kappa}_{\alpha \beta I} \tilde{\mathbf{M}}_J^{\alpha \beta} d\Omega \tag{62}$$

276

$$\begin{aligned}
\tilde{\mathbf{K}}_{IJ} = & - \int_{\Gamma_v} (\Psi_I \tilde{\mathbf{t}}_J + \tilde{\mathbf{t}}_I \Psi_J) d\Gamma \\
& + \int_{\Gamma_\theta} (\Psi_{I,\gamma} n^\gamma \mathbf{a}_3 \tilde{M}_{nnJ} + \mathbf{a}_3 \tilde{M}_{nnI} \Psi_{I,\gamma} n^\gamma) d\Gamma \\
& + ([[\Psi_I \mathbf{a}_3 P_J]] + [[P_I \mathbf{a}_3 \Psi_J]])_{\mathbf{x} \in C_v}
\end{aligned} \tag{63a}$$

$$\tilde{\mathbf{f}}_I = - \int_{\Gamma_v} \tilde{\mathbf{t}}_I \cdot \bar{\mathbf{v}} d\Gamma + \int_{\Gamma_\theta} \tilde{M}_{nn} \bar{\theta}_n d\Gamma + [[\tilde{P}_I \mathbf{a}_3 \cdot \bar{\mathbf{v}}]]_{\mathbf{x} \in C_v} \tag{63b}$$

$$\bar{\mathbf{K}}_{IJ} = - \int_{\Gamma_v} \bar{\mathbf{t}}_I \Psi_J d\Gamma + \int_{\Gamma_\theta} \mathbf{a}_3 \bar{M}_{nnI} \Psi_{J,\gamma} n^\gamma d\Gamma + [[\bar{P}_I \mathbf{a}_3 \Psi_J]]_{\mathbf{x} \in C_v} \quad (64a)$$

$$\bar{\mathbf{f}}_I = - \int_{\Gamma_v} \bar{\mathbf{t}}_I \cdot \bar{\mathbf{v}} d\Gamma + \int_{\Gamma_\theta} \bar{M}_{nn} \bar{\theta}_n d\Gamma + [[\bar{P}_I \mathbf{a}_3 \cdot \bar{\mathbf{v}}]]_{\mathbf{x} \in C_v} \quad (64b)$$

278 The detailed derivations of Eqs (62)-(64) are listed in the Appendix. As
 279 shown in these equations, the Eq. (62) is the conventional stiffness matrix
 280 evaluated by smoothed gradients $\tilde{\Psi}_{I,\alpha}$, $\tilde{\Psi}_{I,\alpha|\beta}$, and the Eqs. (63) and (64)
 281 contribute for the enforcement of essential boundary.

282 4.2. Comparison with Nitsche's method

283 The Nitsche's method for enforcing essential boundary can be regarded as a
 284 combination of Lagrangian multiplier method and penalty method, in which the
 285 Lagrangian multiplier is represented by the approximated displacement. The
 286 corresponding total potential energy functional Π_P is given by:

$$\begin{aligned} \Pi_P(\mathbf{v}) = & \int_{\Omega} \frac{1}{2} \varepsilon_{\alpha\beta} N^{\alpha\beta} d\Omega + \int_{\Omega} \frac{1}{2} \kappa_{\alpha\beta} M^{\alpha\beta} d\Omega \\ & - \int_{\Gamma_t} \mathbf{v} \cdot \bar{\mathbf{t}} d\Gamma + \int_{\Gamma_M} \mathbf{v}_{,\gamma} n^\gamma \mathbf{a}_3 M_{nn} d\Gamma + (\mathbf{v} \cdot \mathbf{a}_3 P)_{\mathbf{x} \in C_P} - \int_{\Omega} \mathbf{v} \cdot \bar{\mathbf{b}} d\Omega \\ & - \underbrace{\int_{\Gamma_v} \mathbf{t} \cdot (\mathbf{v} - \bar{\mathbf{v}}) d\Gamma + \int_{\Gamma_\theta} M_{nn} (\theta_n - \bar{\theta}_n) d\Gamma + (P \mathbf{a}_3 \cdot (\mathbf{v} - \bar{\mathbf{v}}))_{\mathbf{x} \in C_v}}_{\text{consistent term}} \quad (65) \\ & + \underbrace{\frac{\alpha_v}{2} \int_{\Gamma_v} \mathbf{v} \cdot \mathbf{v} d\Gamma + \frac{\alpha_\theta}{2} \int_{\Gamma_\theta} \theta_n^2 d\Gamma + \frac{\alpha_C}{2} (\mathbf{v} \cdot \mathbf{v})_{\mathbf{x} \in C_v}}_{\text{stabilized term}} \end{aligned}$$

287 where the consistent term rephrased from Lagrangian multiplier method con-
 288 tributes to enforce the essential boundary and meet the variational consistency
 289 condition. However the consistent term can not always ensure the coercivity
 290 of stiffness, so the penalty method is introduced to be regarded as a stabilized
 291 term. With a standard variational argument, the corresponding weak form can

292 be stated as:

$$\begin{aligned}
\delta\Pi_P(\mathbf{v}) &= \int_{\Omega} \delta\varepsilon_{\alpha\beta} N^{\alpha\beta} d\Omega + \int_{\Omega} \delta\kappa_{\alpha\beta} M^{\alpha\beta} d\Omega \\
&\quad - \int_{\Gamma_t} \delta\mathbf{v} \cdot \bar{\mathbf{t}} d\Gamma + \int_{\Gamma_M} \delta\mathbf{v}_{,\gamma} n^{\gamma} \mathbf{a}_3 M_{nn} d\Gamma + (\delta\mathbf{v} \cdot \mathbf{a}_3 P)_{\mathbf{x} \in C_P} - \int_{\Omega} \delta\mathbf{v} \cdot \bar{\mathbf{b}} d\Omega \\
&\quad - \int_{\Gamma_v} \delta\mathbf{v} \cdot \mathbf{t} d\Gamma + \int_{\Gamma_{\theta}} \delta\theta_{\mathbf{n}} M_{nn} d\Gamma + (\mathbf{v} \cdot \mathbf{a}_3 P)_{\mathbf{x} \in C_v} \\
&\quad - \int_{\Gamma_v} \delta\mathbf{t} \cdot (\mathbf{v} - \bar{\mathbf{v}}) d\Gamma + \int_{\Gamma_{\theta}} \delta M_{nn} (\theta_{\mathbf{n}} - \bar{\theta}_{\mathbf{n}}) d\Gamma + (\delta P \mathbf{a}_3 \cdot (\mathbf{v} - \bar{\mathbf{v}}))_{\mathbf{x} \in C_v} \\
&\quad + \alpha_v \int_{\Gamma_v} \delta\mathbf{v} \cdot \mathbf{v} d\Gamma + \alpha_{\theta} \int_{\Gamma_{\theta}} \delta\theta_{\mathbf{n}} \theta_{\mathbf{n}} d\Gamma + \alpha_C (\delta\mathbf{v} \cdot \mathbf{v})_{\mathbf{x} \in C_v} \\
&= 0
\end{aligned} \tag{66}$$

293 in which α_v , α_{θ} and α_C are experimental artificial parameters. Further invoking
294 the conventional reproducing kernel approximation of Eq. (22) leads to the
295 following discrete equilibrium equations:

$$\sum_{J=1}^{n_p} (\mathbf{K}_{IJ} + \mathbf{K}_{IJ}^c + \mathbf{K}_{IJ}^s) \mathbf{d}_J = \mathbf{f}_I + \mathbf{f}^c + \mathbf{f}^s \tag{67}$$

296 where the stiffness \mathbf{K}_{IJ} is identical with Eq. (62). \mathbf{K}_{IJ}^c and \mathbf{K}_{IJ}^s are the stiffness
297 matrix for consistent and stabilized terms respectively, and have the following
298 forms:

$$\begin{aligned}
\mathbf{K}_{IJ}^c &= - \int_{\Gamma_v} \left((\mathcal{T}^{\alpha} \Psi_{I,\alpha} + \mathcal{V}^{\alpha\beta} \Psi_{I,\alpha|\beta}) \Psi_J + \Psi_I (\mathcal{T}^{\alpha} \Psi_{J,\alpha} + \mathcal{V}^{\alpha\beta} \Psi_{J,\alpha|\beta}) \right) d\Gamma \\
&\quad + \int_{\Gamma_M} (\mathcal{M}^{\alpha\beta} \Psi_{I,\alpha|\beta} \mathbf{a}_3 \Psi_{J,\gamma} n^{\gamma} + \Psi_{I,\gamma} n^{\gamma} \mathbf{a}_3 \mathcal{M}^{\alpha\beta} \Psi_{J,\alpha|\beta}) d\Gamma
\end{aligned} \tag{68a}$$

299 5. Numerical examples

300 In this section, several examples are carried out to verify proposed method,
 301 which employs the consistent reproducing kernel gradient smoothing integration
 302 scheme (RKGSI) and the non-consistent Gauss integration scheme (GI) with
 303 penalty method, Nitsche's method and the proposed Hu-Washizu formulation
 304 (HW) to enforce the essential boundary conditions. A normalized support size of
 305 2.5 is used for all methods to ensure the requirement of quadratic base meshfree
 306 approximation. To eliminate the influence of integration, the Gauss integration
 307 scheme use 6 Gauss points for domain integration and 3 points for boundary
 308 integration, and such that the number of integration points are identical between
 309 Gauss scheme and RKGSI scheme. The error estimates of displacement namely
 310 L_2 -Error and energy namely H_e -Error is used here:

$$\begin{aligned}
 L_2\text{-Error} &= \frac{\sqrt{\int_{\Omega} (\mathbf{v} - \mathbf{v}^h) \cdot (\mathbf{v} - \mathbf{v}^h) d\Omega}}{\sqrt{\mathbf{v} \cdot \mathbf{v}}} \\
 H_e\text{-Error} &= \frac{\sqrt{\int_{\Omega} \left((\varepsilon_{\alpha\beta} - \varepsilon_{\alpha\beta}^h)(N^{\alpha\beta} - N^{\alpha\beta h}) + \int_{\Omega} (\kappa_{\alpha\beta} - \kappa_{\alpha\beta}^h)(M^{\alpha\beta} - M^{\alpha\beta h}) \right) d\Omega}}{\sqrt{\int_{\Omega} (\varepsilon_{\alpha\beta} N^{\alpha\beta} + \kappa_{\alpha\beta} M^{\alpha\beta}) d\Omega}}
 \end{aligned} \tag{69}$$

311 5.1. Patch tests

312 The linear and quadratic patch tests for flat and curved thin shell are firstly
 313 study to verify the variational consistency of the proposed method. As shown in
 314 Fig. 1, the flat and curved model is depicted by an identical parametric domain
 315 $\Omega = (0, 1) \otimes (0, 1)$, where the cylindrical coordinate system with radius $R = 1$ is
 316 employed to describe the curved model, and the whole domain Ω is discretized
 317 by 165 meshfree nodes. All the boundaries are enforced as essential boundary
 318 conditions with the following manufactured exact solution:

$$\mathbf{v} = \begin{Bmatrix} (\xi^1 + 2\xi^2)^n \\ (3\xi^1 + 4\xi^2)^n \\ (5\xi^1 + 6\xi^2)^n \end{Bmatrix}, \quad n = \begin{cases} 1 & \text{Linear patch test} \\ 2 & \text{Quadratic patch test} \end{cases} \tag{70}$$

Figure 1: Meshfree discretization for patch test

319 Table 1 lists the L_2 - and H_e -Error results of patch test with flat model,
 320 where the RKGSI with variational consistent essential boundary enforcement,
 321 i.e. RKGSI-Nitsche and RKGSI-HW, can pass the linear and quadratic path
 322 test. Due to the loss of variational consistency condition, even with Nitsche's
 323 method, Gauss meshfree formulations show noticeable errors. Table 2 shows the
 324 results for curved model, which indicated that all the mehtods cannot pass the
 325 patch test, which mainly because the proposed smoothed gradient of Eqs. (35),

(36) is unable to exactly reproduce the non-polynomial membrane and bending stress. However, the RKGSI-HW and RKGSI-Nitsche also performance better accuracy than other methods due to the fulfillment of first two order variational consistency. Meanwhile, the bending moment contours of M^{12} are listed in Fig. 3, which further verify that the proposed method obtain a satisfactory result comparing with exact solution, the conventional Gauss meshree formulations show observable errors.

Table 1: Results of patch test for flat model

	Linear patch test		Quadratic patch test	
	L_2 -Error	H_e -Error	L_2 -Error	H_e -Error
GI-Penalty	$4.45E-4$	$1.35E-2$	$2.01E-3$	$1.63E-2$
GI-Nitsche	$4.51E-4$	$1.42E-2$	$1.22E-3$	$1.68E-2$
RKGSI-Penalty	$3.64E-9$	$6.77E-8$	$4.54E-9$	$6.57E-8$
RKGSI-Nitsche	$3.31E-12$	$1.34E-11$	$5.98E-12$	$1.21E-11$
RKGSI-HR	$6.67E-13$	$1.50E-11$	$1.07E-12$	$1.26E-11$

Table 2: Results of patch test for cylindrical model.

	Linear patch test		Quadratic patch test	
	L_2 -Error	H_e -Error	L_2 -Error	H_e -Error
GI-Penalty	$3.79E-4$	$1.30E-2$	$1.74E-3$	$1.37E-2$
GI-Nitsche	$4.04E-4$	$1.42E-2$	$1.15E-3$	$1.49E-2$
RKGSI-Penalty	$1.47E-4$	$5.39E-3$	$2.26E-4$	$2.09E-3$
RKGSI-Nitsche	$2.41E-6$	$7.37E-5$	$2.47E-6$	$2.89E-5$
RKGSI-HR	$4.28E-6$	$1.30E-4$	$9.69E-6$	$2.41E-4$

Figure 2: Contour plots of M^{12} for curved shell patch test.

5.2. Scordelis-Lo roof

This example consider the classical Scordelis-Lo roof problem, as shown in Fig., the cylindrical roof has the radius $R = 25$, length $L = 50$, thickness $h = 0.25$, Young's modulus $E = 4.32 \times 10^8$ and Poisson rate $\nu = 0.0$. An uniform body force of $b_z = -90$ are enforced in whole roof and the curved edges are enforced by $v_x = v_z = 0$, and the straight edges are free.

Due to the symmetry, only a quadrant of the model is considered for meshfree analysis, which is discretized by the 5×8 , 11×16 , 17×24 and 23×32 meshfree nodes.

Figure 3: Description of Scordelis-Lo roof problem.

342 6. Conclusion

343 An efficient and quasi-consistent meshfree thin shell formulation was pre-
344 sented to naturally enforce the essential boundary conditions. In this approach,
345 the mixed formulation with Hu-Washizu principle weak form is employed, where
346 the displacement is discretized by traditional meshfree shape functions, the
347 strains and stresses can be expressed by reproducing kernel smoothed gradients
348 and covariant smoothed gradients. The smoothed gradient naturally embed
349 the first two order integration constraint, and has a quasi variational consis-
350 tency for curved models in each integration cells. Owing to the Hu-Washizu
351 variational principle, the essential boundary condition enforcement has a sim-
352 ilar form with conventional Nitsche's method, both have the consistent term
353 and stabilized term. Compared with Nitsche's method, the costly high order
354 derivatives in Nitsche's consistent term have been replaced by smoothed gra-
355 dients, which shows great computational speed due to the reproducing kernel
356 gradient smoothing framework. Meanwhile, the stabilized term is naturally ex-
357 isted in Hu-Washizu weak form, and the artificial parameter needed in Nitsche's
358 stabilized term has been vanished, which can automatically maintain the coer-
359 civity for stiffness matrix. Numerical results demonstrated that the proposed
360 Hu-Washizu quasi-consistent meshfree thin shell formulation show great perfor-
361 mance in terms of accuracy, efficiency and stability

362 Appendix A. Green's theorems for in-plane vector

363 This Appendix discuss two kinds of Green's theorems used for the devel-
 364 opment of the method. For an arbitrary vector v^α and a scalar function f ,
 365 with the Green's theorem for in-plane vector, the first Green's theorem is list
 366 as follow [15]:

$$\begin{aligned} \int_{\Omega} f_{,\alpha} v^\alpha d\Omega &= \int_{\Gamma} f v^\alpha n_\alpha d\Gamma - \int_{\Omega} f (v_{,\alpha}^\alpha + \Gamma_{\beta\alpha}^\beta v^\alpha) d\Omega \\ &= \int_{\Gamma} f v^\alpha n_\alpha d\Gamma - \int_{\Omega} f v^\alpha|_\alpha d\Omega \end{aligned} \quad (\text{A.1})$$

367 where $\Gamma_{\alpha\beta}^\gamma = \mathbf{a}_{\alpha,\beta} \cdot \mathbf{a}^\gamma$ denotes the Christoffel symbol of the second kind. $v^\alpha|_\alpha$
 368 can be regarded as the in-plane covariant derivative of the vector v^α :

$$v^\alpha|_\alpha = v_{,\alpha}^\alpha + \Gamma_{\beta\alpha}^\beta v^\alpha \quad (\text{A.2})$$

369 The second Green's theorem is established with a mixed form of second order
 370 derivative, let $A^{\alpha\beta}$ be an arbitrary symmetric second order tensor, the Green's
 371 theorem yields [15]:

$$\begin{aligned} \int_{\Omega} f_{,\alpha}|_\beta A^{\alpha\beta} d\Omega &= \int_{\Gamma} f_{,\gamma} n^\gamma A^{\alpha\beta} n_\alpha n_\beta d\Gamma - \int_{\Gamma} f (A^{\alpha\beta} s_\alpha n_\beta)_{,\gamma} s^\gamma d\Gamma + [[f A^{\alpha\beta} s_\alpha n_\beta]]_{\mathbf{x} \in C} \\ &\quad - \int_{\Gamma} f (A_{,\beta}^{\alpha\beta} n_\alpha + \Gamma_{\alpha\beta}^\gamma A^{\alpha\beta} n_\gamma + \Gamma_{\gamma\beta}^\gamma A^{\alpha\beta} n_\alpha) d\Gamma \\ &\quad + \int_{\Omega} f \left(\Gamma_{\alpha\beta,\gamma}^\gamma A^{\alpha\beta} + \Gamma_{\alpha\beta}^\gamma A_{,\gamma}^{\alpha\beta} + \Gamma_{\eta\gamma}^\eta \Gamma_{\alpha\beta}^\gamma A^{\alpha\beta} \right. \\ &\quad \left. + A_{,\alpha\beta}^{\alpha\beta} + \Gamma_{\gamma\beta,\alpha}^\gamma A^{\alpha\beta} + 2\Gamma_{\gamma\alpha}^\gamma A_{,\beta}^{\alpha\beta} + \Gamma_{\gamma\alpha}^\gamma \Gamma_{\eta\beta}^\eta A^{\alpha\beta} \right) d\Omega \\ &= \int_{\Gamma} f_{,\gamma} n^\gamma A^{\alpha\beta} n_\alpha n_\beta d\Gamma - \int_{\Gamma} f (A^{\alpha\beta} s_\alpha n_\beta)_{,\gamma} s^\gamma d\Gamma + [[f A^{\alpha\beta} s_\alpha n_\beta]]_{\mathbf{x} \in C} \\ &\quad - \int_{\Gamma} f A^{\alpha\beta}|_\beta n_\alpha d\Gamma + \int_{\Omega} f A^{\alpha\beta}|_{\alpha\beta} d\Omega \end{aligned} \quad (\text{A.3})$$

372 with

$$A^{\alpha\beta}|_\beta = A_{,\beta}^{\alpha\beta} + \Gamma_{\beta\gamma}^\alpha A^{\beta\gamma} + \Gamma_{\gamma\beta}^\gamma A^{\alpha\beta} \quad (\text{A.4})$$

373

$$\begin{aligned} A^{\alpha\beta}|_{\alpha\beta} &= \Gamma_{\alpha\beta,\gamma}^\gamma A^{\alpha\beta} + \Gamma_{\alpha\beta}^\gamma A_{,\gamma}^{\alpha\beta} + \Gamma_{\eta\gamma}^\eta \Gamma_{\alpha\beta}^\gamma A^{\alpha\beta} \\ &\quad + A_{,\alpha\beta}^{\alpha\beta} + \Gamma_{\gamma\beta,\alpha}^\gamma A^{\alpha\beta} + 2\Gamma_{\gamma\alpha}^\gamma A_{,\beta}^{\alpha\beta} + \Gamma_{\gamma\alpha}^\gamma \Gamma_{\eta\beta}^\eta A^{\alpha\beta} \end{aligned} \quad (\text{A.5})$$

374 For the sake of brevity, the notion of covariant derivative is extended to
 375 scalar function as:

$$f|_\alpha = f_{,\alpha} + \Gamma_{\beta\alpha}^\beta f \quad (\text{A.6})$$

376

$$f|_\beta n_\alpha = f_{,\beta} n_\alpha + \Gamma_{\alpha\beta}^\gamma f n_\gamma + \Gamma_{\gamma\beta}^\gamma f n_\alpha \quad (\text{A.7})$$

377

$$\begin{aligned} f|_{\alpha\beta} &= \Gamma_{\alpha\beta,\gamma}^\gamma f + \Gamma_{\alpha\beta}^\gamma f_{,\gamma} + \Gamma_{\eta\gamma}^\eta \Gamma_{\alpha\beta}^\gamma f \\ &\quad + f_{,\alpha\beta} + \Gamma_{\gamma\beta,\alpha}^\gamma f + 2\Gamma_{\gamma\alpha}^\gamma f_{,\beta} + \Gamma_{\gamma\alpha}^\gamma \Gamma_{\eta\beta}^\eta f \end{aligned} \quad (\text{A.8})$$

³⁷⁸ **Appendix B. Derivations for stiffness metrics and force vectors**

³⁷⁹ This Appendix details the derivations of stiffness

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