

A principle meshfree formulation for thin shells with naturally consistent enforcement of essential boundary conditions

Junchao Wu^{a,*}, Yangtao Xu^b, Bin Xu^a, Syed Humayun Basha^c

^aKey Laboratory for Intelligent Infrastructure and Monitoring of Fujian Province, College of Civil Engineering, Huaqiao University, Xiamen, Fujian, 361021, China

^bCollege of Civil Engineering, Huaqiao University, Xiamen, Fujian, 361021, China

^cKey Laboratory for Structural Engineering and Disaster Prevention of Fujian Province, College of Civil Engineering, Huaqiao University, Xiamen, Fujian, 361021, China

Abstract

Thin shell problems ignore the shear deformations and this leads to a requirement of C1 continuous approximations. Meshfree methods equipped with high order smoothed shape functions is suitable for thin shell analysis, since the high order shape function can also suppress the membrane locking in thin shell problems. However, meshfree shape function always perform a natural rational property, this is a big challenge to meet integration consistency for traditional Gauss integration rule within Galerkin weak form, while integration consistency serves a key role in accuracy of Galerkin meshfree methods. In this work, we proposed a reproducing kernel gradient smoothing integration (RKGSI) algorithm for thin shell problems, while the first and second order smoothed gradients are constructed based upon reproducing kernel smoothing gradient framework, with the aid of this framework, the integration consistency becomes a natural property by a replacement between smoothed gradients and traditional gradients of shape functions in Galerkin weak form. The order of basis functions used in smoothed gradient is determined by ensuring the optimal order of error convergence respected to energy norm. The traditional costly second order gradients are totally eliminated in RKGSI formulation. To further increase the efficiency of proposed method, a set of integration schemes are developed for consistent assembly of stiffness matrix, force vector and smoothed gradients, where the number of integration points, which accompanied with calculation of traditional shape functions and their first order gradients, are minimized by a global point of view. It is evident that the smoothed gradients meets the reproducing consistency of gradients that can ensure the optimal convergence property. The numerical examples demonstrate the efficacy and efficiency of proposed method, while the RKGSI performs a comparable result in energy error with interpolation by meshfree approximations.

Keywords: Meshfree, Thin shell, Hu-Washizu variational principle, Reproducing kernel gradient smoothing, Essential boundary condition

8 1. Introduction

9 Thin shell is one of the most frequently used structure in engineering prac-
10 tice, where the thickness of this kind structure is often much smaller than its
11 radius. With the Kirchhoff-Love hypothesis [1–3], the transverse shear defor-
12 mation is eliminated in thin shell analysis, such that at least C1 continuous
13 shape functions are required within Galerkin methods. In static and dynamic
14 simulation of structure, the conventional finite element methods [1,2] are one
15 of the most popular approximation scheme, however the construction of C1
16 continuity is still a big challenge for cell-based finite element methods. In last
17 three decades, the meshfree methods [1–3] equipped high order smoothed shape
18 functions have attracted significant research attention, while the meshfree shape
19 functions are established based upon a set of scattered nodes and the high or-
20 der continuity of shape functions is easily fulfilled even with low order basis
21 function. For thin shell analysis, this high order meshfree approximations can
22 also alleviate the membrane locking caused by the mismatched approximation
23 order of membrane strain and bending strain [1]. Moreover, in general, the
24 nodal-based meshfree approximations can release the burden of mesh distortion
25 and have the flexibility of local refinement. Due to these advantages, a wide
26 variety meshfree methods are proposed and have been applied to many scien-
27 tific or engineering fields. Among of them, moving least squares (MLS) and
28 reproducing kernel (RK) meshfree approximations built their shape functions
29 by enforcing the so-call consistency conditions, where the consistency condi-
30 tions require that the corresponding approximations should exactly reproduce
31 every functions spanned by basis functions, and this conditions usually serve
32 as a basic requirement for the error convergence of resolved Galerkin solutions
33 [1]. However, the high order smoothed meshfree shape functions accompany
34 with the severely overlapping supports, which leads to a misalignment between
35 numerical integration domains and supports of shape functions. As a result,
36 the meshfree shape functions usually exhibit a piecewise rational nature in each
37 integration domains, and it brings a serious difficulty to the accurate numerical
38 integration in Galerkin weak forms [1].

2. Hu-Washizu's formulation of complementary energy for thin shell

2.1. Kinematics for thin shell

Consider the configuration of a shell $\bar{\Omega}$, as shown in Fig. ??, which can be easily described by a parametric curvilinear coordinate system $\boldsymbol{\xi} = \{\xi^i\}_{i=1,2,3}$. The mid-surface of the shell is specified by the in-plane coordinates $\boldsymbol{\xi} = \{\xi^\alpha\}_{\alpha=1,2}$, as the thickness direction of shell is by ξ^3 , $-\frac{h}{2} \leq \xi^3 \leq \frac{h}{2}$, h is the thickness of shell. In this work, Latin indices take the values from 1 to 3, and Greek indices are evaluated by 1 or 2. For the Kirchhoff hypothesis [?], the position $\mathbf{x} \in \bar{\Omega}$ are defined by linear functions with respect to ξ^3 :

$$\mathbf{x}(\xi^1, \xi^2, \xi^3) = \mathbf{r}(\xi^1, \xi^2) + \xi^3 \mathbf{a}_3(\xi^1, \xi^2) \quad (1)$$

in which \mathbf{r} means the position on the mid-surface of shell, and the \mathbf{a}_3 is corresponding normal direction. For the mid-surface of shell, the in-plane covariant base vector with respect to ξ^α can be derived by a trivial partial differentiation to \mathbf{r} :

$$\mathbf{a}_\alpha = \frac{\partial \mathbf{r}}{\partial \xi^\alpha} = \mathbf{r}_{,\alpha}, \alpha = 1, 2 \quad (2)$$

for a clear expression, the subscript comma denotes the partial differentiation operation with respect to in-plane coordinates ξ^α . And the normal vector \mathbf{a}_3 can be obtained by the normalized cross product of \mathbf{a}_α 's as follow:

$$\mathbf{a}_3 = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{\|\mathbf{a}_1 \times \mathbf{a}_2\|} \quad (3)$$

where $\|\bullet\|$ is the Euclidean norm operator.

With the assumption of infinitesimal deformation, the strain components respected to global contravariant base can be sated as:

$$\epsilon_{ij} = \frac{1}{2}(\mathbf{x}_{,i} \cdot \mathbf{u}_{,j} + \mathbf{u}_{,i} \cdot \mathbf{x}_{,j}) \quad (4)$$

where \mathbf{u} is the displacement for shell deformation. To fulfillment with Kirchhoff hypothesis, the displacement is assumed to be the following form:

$$\mathbf{u}(\xi^1, \xi^2, \xi^3) = \mathbf{v}(\xi^1, \xi^2) + \boldsymbol{\theta}(\xi^1, \xi^2) \xi^3 \quad (5)$$

in which the quadratic and higher order terms are neglected. \mathbf{v} , $\boldsymbol{\theta}$ respect the displacement and rotation in mid-surface.

Subsequently, plugging Eqs. (1) and (5) into Eq. (4) and neglecting quadratic terms, the strain components can be rephrased as follows:

$$\begin{aligned} \epsilon_{\alpha\beta} &= \frac{1}{2}(\mathbf{a}_\alpha \cdot \mathbf{v}_{,\beta} + \mathbf{v}_{,\alpha} \cdot \mathbf{a}_\beta) \\ &+ \frac{1}{2}(\mathbf{a}_{3,\alpha} \cdot \mathbf{v}_{,\beta} + \mathbf{v}_{,\alpha} \cdot \mathbf{a}_{3,\beta} + \mathbf{a}_\alpha \cdot \boldsymbol{\theta}_{,\beta} + \boldsymbol{\theta}_{,\alpha} \cdot \mathbf{a}_\beta) \xi^3 \\ &= \varepsilon_{\alpha\beta} + \kappa_{\alpha\beta} \xi^3 \end{aligned} \quad (6a)$$

$$\epsilon_{\alpha 3} = \frac{1}{2}(\mathbf{a}_\alpha \cdot \boldsymbol{\theta} + \mathbf{v}_{,\alpha} \cdot \mathbf{a}_3) + \frac{1}{2}(\mathbf{a}_3 \cdot \boldsymbol{\theta})_{,\alpha} \xi^3 \quad (6b)$$

$$\epsilon_{33} = \mathbf{a}_3 \cdot \boldsymbol{\theta} \quad (6c)$$

64 where $\varepsilon_{\alpha\beta}$, $\kappa_{\alpha\beta}$ are membrane and bending strains respectively:

$$\varepsilon_{\alpha\beta} = \frac{1}{2}(\mathbf{a}_\alpha \cdot \mathbf{v}_{,\beta} + \mathbf{v}_{,\alpha} \cdot \mathbf{a}_\beta) \quad (7)$$

65

$$\kappa_{\alpha\beta} = \frac{1}{2}(\mathbf{a}_{3,\alpha} \cdot \mathbf{v}_{,\beta} + \mathbf{v}_{,\alpha} \cdot \mathbf{a}_{3,\beta} + \mathbf{a}_\alpha \cdot \boldsymbol{\theta}_{,\beta} + \boldsymbol{\theta}_{,\alpha} \cdot \mathbf{a}_\beta) \quad (8)$$

66 In accordance with Kirchhoff hypothesis, the thickness of shell will not
 67 change and the deformation related with direction of ξ^3 will be vanished, i.e.
 68 $\epsilon_{3i} = 0$. Thus, the rotation $\boldsymbol{\theta}$ can be rewritten as:

$$\epsilon_{3i} = 0 \Rightarrow \begin{cases} \boldsymbol{\theta} \cdot \mathbf{a}_\alpha + \mathbf{v}_{,\alpha} \cdot \mathbf{a}_3 = 0 \\ \boldsymbol{\theta} \cdot \mathbf{a}_3 = 0 \end{cases} \Rightarrow \boldsymbol{\theta} = -\mathbf{v}_{,\alpha} \cdot \mathbf{a}_3 \mathbf{a}^\alpha \quad (9)$$

69 where \mathbf{a}^α 's are the in-plane contravariant base vectors, $\mathbf{a}^\alpha \cdot \mathbf{a}_\beta = \delta^\alpha_\beta$, δ is the
 70 Kronecker delta function. Substituting Eq. (9) into Eq. (8) leads to:

$$\kappa_{\alpha\beta} = (\Gamma_{\alpha\beta}^\gamma \mathbf{v}_{,\gamma} - \mathbf{v}_{,\alpha\beta}) \cdot \mathbf{a}_3 = -\mathbf{v}_{,\alpha}|_\beta \cdot \mathbf{a}_3 \quad (10)$$

71 in which $\Gamma_{\alpha\beta}^\gamma = \mathbf{a}_{\alpha,\beta} \cdot \mathbf{a}^\gamma$ is namely Christoffel symbol of the second kind.

72 **3. Mixed meshfree formulation for modified Hellinger-Reissner weak**
 73 **form**

74 *3.1. Reproducing kernel approximation for displacement*

$$\varepsilon_{\alpha\beta} = \frac{1}{2}(\mathbf{a}_\alpha \cdot \mathbf{v}_{,\beta} + \mathbf{a}_\beta \cdot \mathbf{v}_{,\alpha}) \quad (11)$$

$$\theta_{\mathbf{n}} = \mathbf{a}_3 \cdot \mathbf{v}_{,\alpha} n^\alpha \quad (12)$$

$$\kappa_{\alpha\beta} = (\Gamma_{\alpha\beta}^\gamma \mathbf{v}_{,\gamma} - \mathbf{v}_{,\alpha\beta}) \cdot \mathbf{a}_3 = -\mathbf{v}_{,\alpha}|_\beta \cdot \mathbf{a}_3 \quad (13)$$

$$\mathbf{t} = \mathbf{t}_N + \mathbf{t}_M \quad (14)$$

$$\mathbf{t}_N = \mathbf{a}_\alpha N^{\alpha\beta} n_\beta \quad (15)$$

$$\mathbf{t}_M = (\mathbf{a}_3 M^{\alpha\beta})|_\beta n_\alpha + (\mathbf{a}_3 M^{\alpha\beta} s_\alpha n_\beta)_{,\gamma} s^\gamma \quad (16)$$

$$M_{\mathbf{nn}} = M^{\alpha\beta} n_\alpha n_\beta \quad (17)$$

$$\mathbf{b} = \mathbf{b}_N + \mathbf{b}_M \quad (18)$$

$$\mathbf{b}_N = (\mathbf{a}_\alpha N^{\alpha\beta})|_\beta \quad (19)$$

$$\mathbf{b}_M = (\mathbf{a}_3 M^{\alpha\beta})_{,\alpha}|_\beta \quad (20)$$

$$P = -[[M^{\alpha\beta} s_\alpha n_\beta]] \quad (21)$$

75 *3.2. Galerkin weak form for Hu-Washizu principle of complementary energy*

76 In accordance with the Hu-Washizu variational principle of complementary
 77 energy [1], the corresponding complementary functional, denoted by Π , is listed
 78 as follow:

$$\begin{aligned} & \Pi(\mathbf{v}, \varepsilon_{\alpha\beta}, \kappa_{\alpha\beta}, N^{\alpha\beta}, M^{\alpha\beta}) \\ &= \int_\Omega \frac{h}{2} \varepsilon_{\alpha\beta} C^{\alpha\beta\gamma\eta} \varepsilon_{\gamma\eta} d\Omega + \int_\Omega \frac{h^3}{24} \kappa_{\alpha\beta} C^{\alpha\beta\gamma\eta} \kappa_{\gamma\eta} d\Omega \\ &+ \int_\Omega \varepsilon_{\alpha\beta} (N^{\alpha\beta} - h C^{\alpha\beta\gamma\eta} \varepsilon_{\gamma\eta}) d\Omega + \int_\Omega \kappa_{\alpha\beta} (M^{\alpha\beta} - \frac{h^3}{12} C^{\alpha\beta\gamma\eta} \kappa_{\gamma\eta}) d\Omega \\ &- \int_{\Gamma_v} \mathbf{t} \cdot \bar{\mathbf{v}} d\Gamma + \int_{\Gamma_\theta} M_{\mathbf{nn}} \bar{\theta}_{\mathbf{n}} d\Gamma - (P \mathbf{a}_3 \cdot \bar{\mathbf{v}})_{\mathbf{x} \in C_w} \\ &+ \int_{\Gamma_M} \theta_{\mathbf{n}} (M_{\mathbf{nn}} - \bar{M}_{\mathbf{nn}}) d\Gamma - \int_{\Gamma_t} \mathbf{v} \cdot (\mathbf{t} - \bar{\mathbf{t}}) d\Gamma - \mathbf{v} \cdot \mathbf{a}_3 (P - \bar{P})_{\mathbf{x} \in C_P} \\ &- \int_\Omega \mathbf{v} \cdot (\mathbf{b} - \bar{\mathbf{b}}) d\Omega \end{aligned} \quad (22)$$

Introducing a standard variational argument to Eq. (22), $\delta\Pi = 0$, and considering the arbitrariness of virtual variables, $\delta\mathbf{v}$, $\delta\varepsilon_{\alpha\beta}$, $\delta\kappa_{\alpha\beta}$, $N^{\alpha\beta}$, $M^{\alpha\beta}$ lead to the following weak form:

$$-\int_{\Omega} h\delta\varepsilon_{\alpha\beta}C^{\alpha\beta\gamma\eta}\varepsilon_{\gamma\eta}d\Omega + \int_{\Omega} \delta\varepsilon_{\alpha\beta}N^{\alpha\beta}d\Omega = 0 \quad (23a)$$

$$-\int_{\Omega} \frac{h^3}{12}\delta\kappa_{\alpha\beta}C^{\alpha\beta\gamma\eta}\kappa_{\gamma\eta}d\Omega + \int_{\Omega} \delta\kappa_{\alpha\beta}M^{\alpha\beta}d\Omega = 0 \quad (23b)$$

$$\begin{aligned} \int_{\Omega} \delta N^{\alpha\beta}\varepsilon_{\alpha\beta}d\Omega - \int_{\Gamma} \delta\mathbf{t}_N \cdot \mathbf{v}d\Gamma + \int_{\Omega} \delta\mathbf{b}_N \cdot \mathbf{v}d\Omega \\ + \int_{\Gamma_v} \delta\mathbf{t}_N \cdot \mathbf{v}d\Gamma = \int_{\Gamma_v} \delta\mathbf{t}_N \cdot \bar{\mathbf{v}}d\Gamma \end{aligned} \quad (23c)$$

$$\begin{aligned} \int_{\Omega} \delta M^{\alpha\beta}\kappa_{\alpha\beta}d\Omega - \int_{\Gamma} \delta M_{nn}\theta_n d\Gamma + \int_{\Gamma} \delta\mathbf{t}_M \cdot \mathbf{v}d\Gamma + (\delta P\mathbf{a}_3 \cdot \mathbf{v})_{x \in C} + \int_{\Omega} \delta\mathbf{b}_M \cdot \mathbf{v}d\Omega \\ + \int_{\Gamma_{\theta}} \delta M_{nn}\theta_n d\Gamma - \int_{\Gamma_v} \delta\mathbf{t}_M \cdot \mathbf{v}d\Gamma - (\delta P\mathbf{a}_3 \cdot \mathbf{v})_{x \in C_v} \\ = \int_{\Gamma_{\theta}} \delta M_{nn}\bar{\theta}_n d\Gamma - \int_{\Gamma_v} \delta\mathbf{t}_M \cdot \bar{\mathbf{v}}d\Gamma - (\delta P\mathbf{a}_3 \cdot \bar{\mathbf{v}})_{x \in C_v} \end{aligned} \quad (23d)$$

$$\begin{aligned} \int_{\Gamma} \delta\theta_n M_{nn}d\Gamma - \int_{\Gamma} \delta\mathbf{v} \cdot \mathbf{t}d\Gamma - (\delta\mathbf{v} \cdot \mathbf{a}_3 P)_{x \in C} + \int_{\Omega} \delta\mathbf{v} \cdot \mathbf{b}d\Omega \\ - \int_{\Gamma_{\theta}} \delta\theta_n M_{nn}d\Gamma + \int_{\Gamma_v} \delta\mathbf{v} \cdot \mathbf{t}d\Gamma + (\delta\mathbf{v} \cdot \mathbf{a}_3 P)_{x \in C_v} = - \int_{\Gamma_t} \delta\mathbf{v} \cdot \bar{\mathbf{t}}d\Gamma - \int_{\Omega} \delta\mathbf{v} \cdot \bar{\mathbf{b}}d\Omega \end{aligned} \quad (23e)$$

where the geometric relationships of Eq. () is used herein. In this study, the displacement is approximated by traditional reproducing kernel approximation. As shown in Fig, the mid-surface of the shell Ω is discretized by a set of meshfree nodes $\{\boldsymbol{\xi}_I\}_{I=1}^{n_p}$ in parametric configuration, where n_p is the total number of meshfree nodes. The approximated displacement namely \mathbf{v}^h can be expressed by:

$$\mathbf{v}(\boldsymbol{\xi}) = \sum_{I=1}^{n_p} \Psi_I(\boldsymbol{\xi})\mathbf{d}_I \quad (24)$$

in which Ψ_I and \mathbf{d}_I is the shape function and nodal coefficient tensor related by node $\boldsymbol{\xi}_I$. According to reproducing kernel approximation [?], the shape function takes the following form:

$$\Psi_I(\boldsymbol{\xi}) = \mathbf{p}^T(\boldsymbol{\xi})\mathbf{c}(\boldsymbol{\xi})\phi(\boldsymbol{\xi}_I - \boldsymbol{\xi}) \quad (25)$$

where \mathbf{p} is the basis function vector, and in this study, the following quadratic basis function is considered:

$$\mathbf{p} = \{1, \xi^1, \xi^2, (\xi^1)^2, \xi^1 \xi^2, (\xi^2)^2\}^T \quad (26)$$

The kernel function denoted by ϕ controls the support and smoothness of meshfree shape functions. The quantic B-spline function with square support is used herein as the kernel function:

$$\phi(\boldsymbol{\xi}_I - \boldsymbol{\xi}) = \phi(\hat{s}_1)\phi(\hat{s}_2), \quad \hat{s}_\alpha = \frac{|\xi_I^\alpha - \xi^\alpha|}{s_{\alpha I}} \quad (27)$$

with

$$\phi(\hat{s}_\alpha) = \frac{1}{5!} \begin{cases} (3 - 3\hat{s}_\alpha)^5 - 6(2 - 3\hat{s}_\alpha)^5 + 15(1 - 3\hat{s}_\alpha)^5 & \hat{s}_\alpha \leq \frac{1}{3} \\ (3 - 3\hat{s}_\alpha)^5 - 6(2 - 3\hat{s}_\alpha)^5 & \frac{1}{3} < \hat{s}_\alpha \leq \frac{2}{3} \\ (3 - 3\hat{s}_\alpha)^5 & \frac{2}{3} < \hat{s}_\alpha \leq 1 \\ 0 & \hat{s}_\alpha > 1 \end{cases} \quad (28)$$

and $\hat{s}_{\alpha I}$ means the characterized size of support for meshfree shape function Ψ_I .

The unknown vector \mathbf{c} in shape function are determined by the fulfillment of the so-call consistency condition:

$$\sum_{I=1}^{n_p} \Psi_I(\boldsymbol{\xi}) \mathbf{p}(\boldsymbol{\xi}_I) = \mathbf{p}(\boldsymbol{\xi}) \quad (29)$$

or equivalently

$$\sum_{I=1}^{n_p} \Psi_I(\boldsymbol{\xi}) \mathbf{p}(\boldsymbol{\xi}_I - \boldsymbol{\xi}) = \mathbf{p}(\mathbf{0}) \quad (30)$$

Substituting Eq. (24) into (30), yields:

$$\mathbf{A}(\boldsymbol{\xi}) \mathbf{c}(\boldsymbol{\xi}) = \mathbf{p}(\mathbf{0}) \quad \Rightarrow \quad \mathbf{c}(\boldsymbol{\xi}) = \mathbf{A}^{-1}(\boldsymbol{\xi}) \mathbf{p}(\mathbf{0}) \quad (31)$$

where \mathbf{A} is the moment matrix:

$$\mathbf{A}(\boldsymbol{\xi}) = \sum_{I=1}^{n_p} \phi(\boldsymbol{\xi}_I - \boldsymbol{\xi}) \mathbf{p}(\boldsymbol{\xi}_I - \boldsymbol{\xi}) \mathbf{p}^T(\boldsymbol{\xi}_I - \boldsymbol{\xi}) \quad (32)$$

Taking Eq. (31) back into Eq. (24), the expression of meshfree shape function can be given by:

$$\Psi_I(\boldsymbol{\xi}) = \mathbf{p}^T(\boldsymbol{\xi}_I - \boldsymbol{\xi}) \mathbf{A}^{-1}(\boldsymbol{\xi}) \mathbf{p}(\mathbf{0}) \phi(\boldsymbol{\xi}_I - \boldsymbol{\xi}) \quad (33)$$

3.3. Reproducing kernel gradient smoothing approximation for effective stress and strain

In Galerkin meshfree formulation, the mid-plane of thin shell Ω is split by a set of integration cells Ω_C 's, $\cup_{C=1}^{n_e} \Omega_C \approx \Omega$. With the inspiration of reproducing

kernel smoothing framework, the Cartesian and covariant derivatives of displacement, $\mathbf{v}_{,\alpha}$ and $-\mathbf{v}_{,\alpha}|_{\beta}$, in strains $\varepsilon_{\alpha\beta}$, $\kappa_{\alpha\beta}$ are approximated by $(p-1)$ -th order polynomials in each integration cells. In integration cell Ω_C , the approximated derivatives and strains denoted by $\mathbf{v}_{,\alpha}^h$, $\varepsilon_{\alpha\beta}^h$ and $-\mathbf{v}_{,\alpha}^h|_{\beta}$, $\kappa_{\alpha\beta}^h$ can be expressed by:

$$\mathbf{v}_{,\alpha}^h(\boldsymbol{\xi}) = \mathbf{q}^T(\boldsymbol{\xi}) \mathbf{d}_{\alpha}^{\varepsilon}, \quad \varepsilon_{\alpha\beta}^h(\boldsymbol{\xi}) = \mathbf{q}^T(\boldsymbol{\xi}) \frac{1}{2} (\mathbf{a}_{\alpha} \cdot \mathbf{d}_{\beta}^{\varepsilon} + \mathbf{a}_{\beta} \cdot \mathbf{d}_{\alpha}^{\varepsilon}) \quad (34)$$

$$-\mathbf{v}_{,\alpha}^h|_{\beta}(\boldsymbol{\xi}) = \mathbf{q}^T(\boldsymbol{\xi}) \mathbf{d}_{\alpha\beta}^{\kappa}, \quad \kappa_{\alpha\beta}^h(\boldsymbol{\xi}) = \mathbf{q}^T(\boldsymbol{\xi}) \mathbf{a}_3 \cdot \mathbf{d}_{\alpha\beta}^{\kappa} \quad (35)$$

where \mathbf{q} is the $(p-1)$ th order polynomial vector and has the following form:

$$\mathbf{q} = \{1, \xi^1, \xi^2, \dots, (\xi^2)^{p-1}\}^T \quad (36)$$

and the $\mathbf{d}_{\alpha}^{\varepsilon}$, $\mathbf{d}_{\alpha\beta}^{\kappa}$ are the corresponding coefficient vector tensors. For the conciseness, the mixed usage of tensor and vector is introduced in this study, for example, the component of coefficient tensor vector $\mathbf{d}_{\alpha I}^{\varepsilon}$, $\mathbf{d}_{\alpha}^{\varepsilon} = \{\mathbf{d}_{\alpha I}^{\varepsilon}\}$, is a three dimensional tensor, $\dim \mathbf{d}_{\alpha I}^{\varepsilon} = \dim \mathbf{v}$.

In order to meet the integration constraint of thin shell problem, the approximated stresses $N^{\alpha\beta h}$, $M^{\alpha\beta h}$ are assumed to be a similar form with strains, yields:

$$N^{\alpha\beta h}(\boldsymbol{\xi}) = \mathbf{q}^T(\boldsymbol{\xi}) \mathbf{a}^{\alpha} \cdot \mathbf{d}_{\beta}^N, \quad \mathbf{a}_{\alpha} N^{\alpha\beta h}(\boldsymbol{\xi}) = \mathbf{q}^T(\boldsymbol{\xi}) \mathbf{d}_{\beta}^N \quad (37)$$

$$M^{\alpha\beta h}(\boldsymbol{\xi}) = \mathbf{q}^T(\boldsymbol{\xi}) \mathbf{a}_3 \cdot \mathbf{d}_{\alpha\beta}^M, \quad \mathbf{a}_3 M^{\alpha\beta h}(\boldsymbol{\xi}) = \mathbf{q}^T(\boldsymbol{\xi}) \mathbf{d}_{\alpha\beta}^M \quad (38)$$

substituting the approximations of Eqs. (24), (34), (35), (37), (38) into Eqs. (23c), (23d) can express $\mathbf{d}_{\beta}^{\varepsilon}$ and $\mathbf{d}_{\alpha\beta}^{\kappa}$ by \mathbf{d} as:

$$\mathbf{d}_{\beta}^{\varepsilon} = \mathbf{G}^{-1} \left(\sum_{I=1}^{n_p} (\tilde{\mathbf{g}}_{\beta I} - \bar{\mathbf{g}}_{\beta I}) \mathbf{d}_I + \hat{\mathbf{g}}_{\beta} \right) \quad (39)$$

$$\mathbf{d}_{\alpha\beta}^{\kappa} = \mathbf{G}^{-1} \left(\sum_{I=1}^{n_p} (\tilde{\mathbf{g}}_{\alpha\beta I} - \bar{\mathbf{g}}_{\alpha\beta I}) \mathbf{d}_I + \hat{\mathbf{g}}_{\alpha\beta} \right) \quad (40)$$

with

$$\mathbf{G} = \int_{\Omega_C} \mathbf{q}^T \mathbf{q} d\Omega \quad (41)$$

$$\tilde{\mathbf{g}}_{\beta I} = \int_{\Gamma_C} \Psi_I \mathbf{q} n_{\beta} d\Gamma - \int_{\Omega_C} \Psi_I \mathbf{q}^*|_{\beta} d\Omega \quad (42a)$$

$$\bar{\mathbf{g}}_{\beta I} = \int_{\Gamma_C \cap \Gamma_v} \Psi_I \mathbf{q} n_{\beta} d\Gamma \quad (42b)$$

$$\hat{\mathbf{g}}_{\beta} = \int_{\Gamma_C \cap \Gamma_v} \mathbf{q} n_{\beta} \bar{\mathbf{v}} d\Gamma \quad (42c)$$

$$\begin{aligned}\tilde{\mathbf{g}}_{\alpha\beta I} &= \int_{\Gamma_C} \Psi_{I,\gamma} n^\gamma \mathbf{q} n_\alpha n_\beta d\Gamma - \int_{\Gamma_C} \Psi_I (\mathbf{q}^{**}|_\beta n_\alpha + (\mathbf{q} s_\alpha n_\beta)_{,\gamma} s^\gamma) d\Gamma \\ &\quad + [[\Psi_I \mathbf{q} s_\alpha n_\beta]]_{\mathbf{x} \in \mathcal{C}_C} - \int_{\Omega_C} \Psi \mathbf{q}_{,\alpha}^{**}|_\beta d\Omega\end{aligned}\quad (43a)$$

$$\begin{aligned}\bar{\mathbf{g}}_{\alpha\beta I} &= \int_{\Gamma_C \cap \Gamma_\theta} \Psi_{I,\gamma} n^\gamma \mathbf{q} n_\alpha n_\beta d\Gamma - \int_{\Gamma_C \cap \Gamma_v} \Psi_I (\mathbf{q}^{**}|_\beta n_\alpha + (\mathbf{q} s_\alpha n_\beta)_{,\gamma} s^\gamma) d\Gamma \\ &\quad + [[\Psi_I \mathbf{q} s_\alpha n_\beta]]_{\mathbf{x} \in \mathcal{C}_C \cap C_v}\end{aligned}\quad (43b)$$

$$\begin{aligned}\hat{\mathbf{g}}_{\alpha\beta} &= \int_{\Gamma_C \cap \Gamma_\theta} \mathbf{q} n_\alpha n_\beta \mathbf{a}_3 \bar{\theta}_n d\Gamma - \int_{\Gamma_C \cap \Gamma_v} (\mathbf{q}^{**}|_\beta n_\alpha + (\mathbf{q} s_\alpha n_\beta)_{,\gamma} s^\gamma) \bar{\mathbf{v}} d\Gamma \\ &\quad + [[\mathbf{q} s_\alpha n_\beta \bar{\mathbf{v}}]]_{\mathbf{x} \in \mathcal{C}_C \cap C_v}\end{aligned}\quad (43c)$$

134 plugging Eqs. (39) and (40) back into Eqs. (34) and (35) respectively gives the
135 final expression of $\mathbf{v}_{,\alpha}^h$, $\varepsilon_{\alpha\beta}^h$ and $-\mathbf{v}_{,\alpha\beta}^h$, $\kappa_{\alpha\beta}^h$ as:

$$\mathbf{v}_{,\alpha}^h = \sum_{I=1}^{n_p} (\tilde{\Psi}_{I,\alpha} - \bar{\Psi}_{I,\alpha}) \mathbf{d}_I + \mathbf{q}^T \mathbf{G}^{-1} \hat{\mathbf{g}}_\alpha \quad (44a)$$

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$$\begin{aligned}\varepsilon_{\alpha\beta}^h &= \sum_{I=1}^{n_p} \frac{1}{2} (\mathbf{a}_\alpha \tilde{\Psi}_{I,\beta} + \mathbf{a}_\beta \tilde{\Psi}_{I,\alpha}) \cdot \mathbf{d}_I - \sum_{I=1}^{n_p} \frac{1}{2} (\mathbf{a}_\alpha \bar{\Psi}_{I,\beta} + \mathbf{a}_\beta \bar{\Psi}_{I,\alpha}) \cdot \mathbf{d}_I \\ &\quad + \mathbf{q}^T \mathbf{G}^{-1} \frac{1}{2} (\mathbf{a}_\alpha \cdot \hat{\mathbf{g}}_\beta + \mathbf{a}_\beta \cdot \hat{\mathbf{g}}_\alpha) \\ &= \tilde{\varepsilon}_{\alpha\beta}^h - \bar{\varepsilon}_{\alpha\beta}^h + \hat{\varepsilon}_{\alpha\beta}^h\end{aligned}\quad (44b)$$

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$$-\mathbf{v}_{,\alpha\beta}^h = \sum_{I=1}^{n_p} (\tilde{\Psi}_{I,\alpha\beta} - \bar{\Psi}_{I,\alpha\beta}) \mathbf{d}_I + \mathbf{q}^T \mathbf{G}^{-1} \hat{\mathbf{g}}_{\alpha\beta} \quad (45a)$$

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$$\begin{aligned}\kappa_{\alpha\beta}^h &= \sum_{I=1}^{n_p} \tilde{\Psi}_{I,\alpha\beta} \mathbf{a}_3 \cdot \mathbf{d}_I - \sum_{I=1}^{n_p} \bar{\Psi}_{I,\alpha\beta} \mathbf{a}_3 \cdot \mathbf{d}_I + \mathbf{q}^T \mathbf{G}^{-1} \mathbf{a}_3 \cdot \hat{\mathbf{g}}_{\alpha\beta} \\ &= \tilde{\kappa}_{\alpha\beta}^h - \bar{\kappa}_{\alpha\beta}^h + \hat{\kappa}_{\alpha\beta}^h\end{aligned}\quad (45b)$$

139 with

$$\tilde{\Psi}_{I,\alpha}(\boldsymbol{\xi}) = \mathbf{q}^T(\boldsymbol{\xi}) \mathbf{G}^{-1} \tilde{\mathbf{g}}_{\alpha I}, \quad \bar{\Psi}_{I,\alpha}(\boldsymbol{\xi}) = \mathbf{q}^T(\boldsymbol{\xi}) \mathbf{G}^{-1} \bar{\mathbf{g}}_{\alpha I} \quad (46)$$

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$$\begin{cases} \tilde{\varepsilon}_{\alpha\beta}^h = \sum_{I=1}^{n_p} \frac{1}{2} (\mathbf{a}_\alpha \tilde{\Psi}_{I,\beta} + \mathbf{a}_\beta \tilde{\Psi}_{I,\alpha}) \cdot \mathbf{d}_I \\ \bar{\varepsilon}_{\alpha\beta}^h = \sum_{I=1}^{n_p} \frac{1}{2} (\mathbf{a}_\alpha \bar{\Psi}_{I,\beta} + \mathbf{a}_\beta \bar{\Psi}_{I,\alpha}) \cdot \mathbf{d}_I \\ \hat{\varepsilon}_{\alpha\beta}^h = \mathbf{q}^T \mathbf{G}^{-1} \frac{1}{2} (\mathbf{a}_\alpha \cdot \hat{\mathbf{g}}_\beta + \mathbf{a}_\beta \cdot \hat{\mathbf{g}}_\alpha) \end{cases} \quad (47)$$

141

$$\tilde{\Psi}_{I,\alpha\beta}(\boldsymbol{\xi}) = \mathbf{q}^T(\boldsymbol{\xi}) \mathbf{G}^{-1} \tilde{\mathbf{g}}_{\alpha\beta I}, \quad \bar{\Psi}_{I,\alpha\beta}(\boldsymbol{\xi}) = \mathbf{q}^T(\boldsymbol{\xi}) \mathbf{G}^{-1} \bar{\mathbf{g}}_{\alpha\beta I} \quad (48)$$

142

$$\begin{cases} \tilde{\kappa}_{\alpha\beta}^h = \sum_{I=1}^{n_p} \tilde{\Psi}_{I,\alpha\beta} \mathbf{a}_3 \cdot \mathbf{d}_I \\ \bar{\kappa}_{\alpha\beta}^h = \sum_{I=1}^{n_p} \bar{\Psi}_{I,\alpha\beta} \mathbf{a}_3 \cdot \mathbf{d}_I \\ \hat{\kappa}_{\alpha\beta}^h = \mathbf{q}^T \mathbf{G}^{-1} \mathbf{a}_3 \cdot \hat{\mathbf{g}}_{\alpha\beta} \end{cases} \quad (49)$$

143 Furthermore, taking Eqs. (34) and (35) into Eqs.(23a) and (23b) can obtain
 144 the approximated effective stresses $N^{\alpha\beta h}$, $M^{\alpha\beta h}$ and their coefficients \mathbf{d}_{β}^N , $\mathbf{d}_{\alpha\beta}^M$
 145 as:

$$\begin{aligned} & \frac{1}{2}(\delta \mathbf{d}_{\alpha}^{\varepsilon} \cdot \mathbf{a}_{\beta} + \delta \mathbf{d}_{\beta}^{\varepsilon} \cdot \mathbf{a}_{\alpha}) h C^{\alpha\beta\gamma\eta} \frac{1}{2}(\mathbf{a}_{\gamma} \cdot \mathbf{d}_{\eta}^{\varepsilon} + \mathbf{a}_{\eta} \cdot \mathbf{d}_{\gamma}^{\varepsilon}) \mathbf{G} \\ &= \frac{1}{2}(\delta \mathbf{d}_{\alpha}^{\varepsilon} \cdot \mathbf{d}_{\beta}^N + \delta \mathbf{d}_{\beta}^{\varepsilon} \cdot \mathbf{d}_{\alpha}^N) \mathbf{G} \\ \Rightarrow \mathbf{d}_N^{\beta} &= \mathbf{a}_{\beta} h C^{\alpha\beta\gamma\eta} \frac{1}{2}(\mathbf{a}_{\gamma} \cdot \mathbf{d}_{\eta}^{\varepsilon} + \mathbf{a}_{\eta} \cdot \mathbf{d}_{\gamma}^{\varepsilon}) \end{aligned} \quad (50)$$

146

$$\begin{aligned} & \delta \mathbf{d}_{\alpha\beta}^{\kappa} \cdot \mathbf{a}_3 \frac{h^3}{12} C^{\alpha\beta\gamma\eta} \mathbf{a}_3 \cdot \mathbf{d}_{\gamma\eta}^{\kappa} \mathbf{G} = \delta \mathbf{d}_{\alpha\beta}^{\kappa} \cdot \mathbf{d}_{\alpha\beta}^M \mathbf{G} \\ \Rightarrow \mathbf{d}_M^{\alpha\beta} &= \mathbf{a}_3 \frac{h^3}{12} C^{\alpha\beta\gamma\eta} \mathbf{a}_3 \cdot \mathbf{d}_{\gamma\eta}^{\kappa} \end{aligned} \quad (51)$$

147

$$N^{\alpha\beta h} = h C^{\alpha\beta\gamma\eta} (\tilde{\varepsilon}_{\gamma\eta}^h - \bar{\varepsilon}_{\gamma\eta}^h + \hat{\varepsilon}_{\gamma\eta}^h) = \tilde{N}^{\alpha\beta h} - \bar{N}^{\alpha\beta h} + \hat{N}^{\alpha\beta h} \quad (52)$$

148

$$M^{\alpha\beta h} = \frac{h^3}{12} C^{\alpha\beta\gamma\eta} (\tilde{\kappa}_{\gamma\eta}^h - \bar{\kappa}_{\gamma\eta}^h + \hat{\kappa}_{\gamma\eta}^h) = \tilde{M}^{\alpha\beta h} - \bar{M}^{\alpha\beta h} + \hat{M}^{\alpha\beta h} \quad (53)$$

149 with

$$\tilde{N}^{\alpha\beta h} = h C^{\alpha\beta\gamma\eta} \tilde{\varepsilon}_{\gamma\eta}^h, \quad \bar{N}^{\alpha\beta h} = h C^{\alpha\beta\gamma\eta} \bar{\varepsilon}_{\gamma\eta}^h, \quad \hat{N}^{\alpha\beta h} = h C^{\alpha\beta\gamma\eta} \hat{\varepsilon}_{\gamma\eta}^h \quad (54)$$

150

$$\tilde{M}^{\alpha\beta h} = \frac{h^3}{12} C^{\alpha\beta\gamma\eta} \tilde{\kappa}_{\gamma\eta}^h, \quad \bar{M}^{\alpha\beta h} = \frac{h^3}{12} C^{\alpha\beta\gamma\eta} \bar{\kappa}_{\gamma\eta}^h, \quad \hat{M}^{\alpha\beta h} = \frac{h^3}{12} C^{\alpha\beta\gamma\eta} \hat{\kappa}_{\gamma\eta}^h \quad (55)$$

151 It is noted that, referring to reproducing kernel gradient smoothing frame-
 152 work [?], $\tilde{\Psi}_{I,\alpha}$, $\tilde{\Psi}_{I,\alpha\beta}$ are actually the first and second order smoothed gradients
 153 in curvilinear coordinates. $\tilde{\mathbf{g}}_{\alpha I}$ and $\tilde{\mathbf{g}}_{\alpha\beta I}$ are the right hand side integration con-
 154 straints for first and second order gradients, then this formulation can meet the
 155 variational consistency for the p -th order polynomials. It should be known that,
 156 in curved model, the variational consistency for non-polynomial functions, like
 157 trigonometric functions, should be required for the polynomial solution. Even
 158 with p -th order variational consistency, the proposed formulation can not ex-
 159 actly reproduce the solution spanned by basis functions, however the accuracy
 160 of reproducing kernel smoothed gradients is still better than traditional meshfree
 161 formulation, this will be evidenced by numerical examples in further section.

162 **4. Naturally variational enforcement for essential boundary condi-**
 163 **tions**

164 *4.1. Discrete equilibrium equations*

165 With the approximated effective stresses and strains, the last equation of
 166 weak form becomes:

$$-\sum_{C=1}^{n_e}(\tilde{\mathbf{g}}_{\alpha I}^T - \bar{\mathbf{g}}_{\alpha I}^T)\mathbf{d}_N^\alpha - \sum_{C=1}^{n_e}(\tilde{\mathbf{g}}_{\alpha\beta I}^T - \bar{\mathbf{g}}_{\alpha\beta I}^T)\mathbf{d}_M^{\alpha\beta} = \mathbf{f}_I \quad (56)$$

167 where \mathbf{f}_I 's are the components of the traditional force vector:

$$\mathbf{f}_I = \int_{\Gamma_t} \Psi_I \bar{\mathbf{t}} d\Gamma - \int_{\Gamma_M} \Psi_{I,\gamma} n^\gamma \bar{M}_{nn} d\Gamma + [[\Psi_I \mathbf{a}_3 \bar{P}]]_{\mathbf{x} \in C_P} + \int_{\Omega} \Psi_I \bar{\mathbf{b}} d\Omega \quad (57)$$

168 and further substituting coefficients $\mathbf{d}_N^\alpha, \mathbf{d}_M^{\alpha\beta}$ into Eq. (56) gives the final discrete
 169 equilibrium equations:

$$\begin{aligned} & -\sum_{C=1}^{n_e}(\tilde{\mathbf{g}}_{\alpha I}^T - \bar{\mathbf{g}}_{\alpha I}^T)\mathbf{d}_N^\alpha - \sum_{C=1}^{n_e}(\tilde{\mathbf{g}}_{\alpha\beta I}^T - \bar{\mathbf{g}}_{\alpha\beta I}^T)\mathbf{d}_M^{\alpha\beta} \\ & = \sum_{C=1}^{n_e} \sum_{J=1}^{n_p} \begin{pmatrix} \mathbf{a}_\alpha \tilde{\mathbf{g}}_{\beta I}^T h C^{\alpha\beta\gamma\eta} \mathbf{a}_\gamma \tilde{\mathbf{g}}_{\eta J} + \tilde{\mathbf{g}}_{\alpha\beta I}^T \mathbf{a}_3 \frac{h^3}{12} C^{\alpha\beta\gamma\eta} \mathbf{a}_3 \tilde{\mathbf{g}}_{\gamma\eta} \\ -\mathbf{a}_\alpha \bar{\mathbf{g}}_{\beta I}^T h C^{\alpha\beta\gamma\eta} \mathbf{a}_\gamma \tilde{\mathbf{g}}_{\eta J} - \mathbf{a}_\alpha \tilde{\mathbf{g}}_{\beta I}^T h C^{\alpha\beta\gamma\eta} \mathbf{a}_\gamma \bar{\mathbf{g}}_{\eta J} \\ -\bar{\mathbf{g}}_{\alpha\beta I}^T \mathbf{a}_3 \frac{h^3}{12} C^{\alpha\beta\gamma\eta} \mathbf{a}_3 \tilde{\mathbf{g}}_{\gamma\eta J} - \tilde{\mathbf{g}}_{\alpha\beta I}^T \mathbf{a}_3 \frac{h^3}{12} C^{\alpha\beta\gamma\eta} \mathbf{a}_3 \bar{\mathbf{g}}_{\gamma\eta J} \\ +\mathbf{a}_\alpha \tilde{\mathbf{g}}_{\beta I}^T h C^{\alpha\beta\gamma\eta} \mathbf{a}_\gamma \hat{\mathbf{g}}_{\eta J} - \mathbf{a}_\alpha \bar{\mathbf{g}}_{\beta I}^T h C^{\alpha\beta\gamma\eta} \mathbf{a}_\gamma \hat{\mathbf{g}}_{\eta J} \\ +\tilde{\mathbf{g}}_{\alpha\beta I}^T \mathbf{a}_3 \frac{h^3}{12} C^{\alpha\beta\gamma\eta} \mathbf{a}_3 \hat{\mathbf{g}}_{\gamma\eta J} - \bar{\mathbf{g}}_{\alpha\beta I}^T \mathbf{a}_3 \frac{h^3}{12} C^{\alpha\beta\gamma\eta} \mathbf{a}_3 \hat{\mathbf{g}}_{\gamma\eta J} \\ +\mathbf{a}_\alpha \bar{\mathbf{g}}_{\beta I}^T h C^{\alpha\beta\gamma\eta} \mathbf{a}_\gamma \bar{\mathbf{g}}_{\eta J} + \bar{\mathbf{g}}_{\alpha\beta I}^T \mathbf{a}_3 \frac{h^3}{12} C^{\alpha\beta\gamma\eta} \mathbf{a}_3 \bar{\mathbf{g}}_{\gamma\eta J} \end{pmatrix} \\ & = \sum_{J=1}^{n_p} (\mathbf{K}_{IJ} + \tilde{\mathbf{K}}_{IJ} + \bar{\mathbf{K}}_{IJ}) \cdot \mathbf{d}_J - \tilde{\mathbf{f}}_I - \bar{\mathbf{f}}_I \end{aligned} \quad (58)$$

170 where

$$\mathbf{K}_{IJ} = \int_{\Omega} \tilde{\varepsilon}_{\alpha\beta I} \tilde{N}_J^{\alpha\beta} d\Omega + \int_{\Omega} \tilde{\kappa}_{\alpha\beta I} \tilde{M}_J^{\alpha\beta} d\Omega \quad (59)$$

171

$$\begin{aligned} \tilde{\mathbf{K}}_{IJ} &= - \int_{\Gamma_v} (\Psi_I \tilde{\mathbf{t}}_J + \tilde{\mathbf{t}}_I \Psi_J) d\Gamma \\ &+ \int_{\Gamma_\theta} (\Psi_{I,\gamma} n^\gamma \mathbf{a}_3 \tilde{M}_{nnJ} + \mathbf{a}_3 \tilde{M}_{nnI} \Psi_{I,\gamma} n^\gamma) d\Gamma \\ &+ ([[\Psi_I \mathbf{a}_3 P_J]] + [[P_I \mathbf{a}_3 \Psi_J]])_{\mathbf{x} \in C_v} \end{aligned} \quad (60a)$$

$$\tilde{\mathbf{f}}_I = - \int_{\Gamma_v} \tilde{\mathbf{t}}_I \cdot \bar{\mathbf{v}} d\Gamma + \int_{\Gamma_\theta} \tilde{M}_{nn} \bar{\theta}_n d\Gamma + [[\tilde{P}_I \mathbf{a}_3 \cdot \bar{\mathbf{v}}]]_{\mathbf{x} \in C_v} \quad (60b)$$

$$\bar{\mathbf{K}}_{IJ} = - \int_{\Gamma_v} \bar{\mathbf{t}}_I \Psi_J d\Gamma + \int_{\Gamma_\theta} \mathbf{a}_3 \bar{M}_{nnI} \Psi_{J,\gamma} n^\gamma d\Gamma + [[\bar{P}_I \mathbf{a}_3 \Psi_J]]_{\mathbf{x} \in C_v} \quad (61a)$$

$$\bar{\mathbf{f}}_I = - \int_{\Gamma_v} \bar{\mathbf{t}}_I \cdot \bar{\mathbf{v}} d\Gamma + \int_{\Gamma_\theta} \bar{M}_{nn} \bar{\theta}_n d\Gamma + [[\bar{P}_I \mathbf{a}_3 \cdot \bar{\mathbf{v}}]]_{\mathbf{x} \in C_v} \quad (61b)$$

173 The detailed derivations of Eqs (59)-(61) are listed in the Appendix. As
 174 shown in these equations, the Eq. (59) is the conventional stiffness matrix
 175 evaluated by smoothed gradients $\tilde{\Psi}_{I,\alpha}$, $\tilde{\Psi}_{I,\alpha|\beta}$, and the Eqs. (60) and (61)
 176 contribute for the enforcement of essential boundary.

177 4.2. Comparison with Nitsche's method

178 The Nitsche's method for enforcing essential boundary can be regarded as a
 179 combination of Lagrangian multiplier method and penalty method, in which the
 180 Lagrangian multiplier is represented by the approximated displacement. The
 181 corresponding total potential energy functional Π_P is given by:

$$\begin{aligned} \Pi_P(\mathbf{v}) = & \int_{\Omega} \frac{1}{2} \varepsilon_{\alpha\beta} N^{\alpha\beta} d + \int_{\Omega} \frac{1}{2} \kappa_{\alpha\beta} M^{\alpha\beta} d\Omega \\ & - \int_{\Gamma_t} \mathbf{v} \cdot \bar{\mathbf{t}} d\Gamma + \int_{\Gamma_M} \mathbf{v}_{,\gamma} n^\gamma \mathbf{a}_3 M_{nn} d\Gamma + (\mathbf{v} \cdot \mathbf{a}_3 P)_{\mathbf{x} \in C_P} - \int_{\Omega} \mathbf{v} \cdot \bar{\mathbf{b}} d\Omega \\ & - \underbrace{\int_{\Gamma_v} \mathbf{t} \cdot (\mathbf{v} - \bar{\mathbf{v}}) d\Gamma + \int_{\Gamma_\theta} M_{nn} (\theta_n - \bar{\theta}_n) d\Gamma + (P \mathbf{a}_3 \cdot (\mathbf{v} - \bar{\mathbf{v}}))_{\mathbf{x} \in C_v}}_{\text{consistent term}} \quad (62) \\ & + \underbrace{\frac{\alpha_v}{2} \int_{\Gamma_v} \mathbf{v} \cdot \mathbf{v} d\Gamma + \frac{\alpha_\theta}{2} \int_{\Gamma_\theta} \theta_n^2 d\Gamma + \frac{\alpha_C}{2} (\mathbf{v} \cdot \mathbf{v})_{\mathbf{x} \in C_v}}_{\text{stabilized term}} \end{aligned}$$

182 where the consistent term rephrased from Lagrangian multiplier method con-
 183 tributes to enforce the essential boundary and meet the variational consistency
 184 condition. However the consistent term can not always ensure the coercivity
 185 of stiffness, so the penalty method is introduced to be regarded as a stabilized
 186 term. With a standard variational argument, the corresponding weak form can

187 be stated as:

$$\begin{aligned}
\delta\Pi_P(\mathbf{v}) &= \int_{\Omega} \delta\varepsilon_{\alpha\beta} N^{\alpha\beta} d + \int_{\Omega} \delta\kappa_{\alpha\beta} M^{\alpha\beta} d\Omega \\
&\quad - \int_{\Gamma_t} \delta\mathbf{v} \cdot \bar{\mathbf{t}} d\Gamma + \int_{\Gamma_M} \delta\mathbf{v}_{,\gamma} n^{\gamma} \mathbf{a}_3 M_{nn} d\Gamma + (\delta\mathbf{v} \cdot \mathbf{a}_3 P)_{\mathbf{x} \in C_P} - \int_{\Omega} \delta\mathbf{v} \cdot \bar{\mathbf{b}} d\Omega \\
&\quad - \int_{\Gamma_v} \delta\mathbf{v} \cdot \mathbf{t} d\Gamma + \int_{\Gamma_{\theta}} \delta\theta_{\mathbf{n}} M_{nn} d\Gamma + (\mathbf{v} \cdot \mathbf{a}_3 P)_{\mathbf{x} \in C_v} \\
&\quad - \int_{\Gamma_v} \delta\mathbf{t} \cdot (\mathbf{v} - \bar{\mathbf{v}}) d\Gamma + \int_{\Gamma_{\theta}} \delta M_{nn} (\theta_{\mathbf{n}} - \bar{\theta}_{\mathbf{n}}) d\Gamma + (\delta P \mathbf{a}_3 \cdot (\mathbf{v} - \bar{\mathbf{v}}))_{\mathbf{x} \in C_v} \\
&\quad + \alpha_v \int_{\Gamma_v} \delta\mathbf{v} \cdot \mathbf{v} d\Gamma + \alpha_{\theta} \int_{\Gamma_{\theta}} \delta\theta_{\mathbf{n}} \theta_{\mathbf{n}} d\Gamma + \alpha_C (\delta\mathbf{v} \cdot \mathbf{v})_{\mathbf{x} \in C_v} \\
&= 0
\end{aligned} \tag{63}$$

188 in which α_v , α_{θ} and α_C are experimental artificial parameters. Further invoking
189 the conventional reproducing kernel approximation of Eq. (24) leads to the
190 following discrete equilibrium equations:

$$\sum_{J=1}^{n_p} (\mathbf{K}_{IJ} + \mathbf{K}_{IJ}^c + \mathbf{K}_{IJ}^s) \mathbf{d}_J = \mathbf{f}_I + \mathbf{f}^c + \mathbf{f}^s \tag{64}$$

191 where the stiffness \mathbf{K}_{IJ} is identical with Eq. (59). \mathbf{K}_{IJ}^c and \mathbf{K}_{IJ}^s are the stiffness
192 matrix for consistent and stabilized terms respectively, and have the following
193 forms:

$$\begin{aligned}
\mathbf{K}_{IJ}^c &= - \int_{\Gamma_v} \left((\mathcal{T}^{\alpha} \Psi_{I,\alpha} + \mathcal{V}^{\alpha\beta} \Psi_{I,\alpha|\beta}) \Psi_J + \Psi_I (\mathcal{T}^{\alpha} \Psi_{J,\alpha} + \mathcal{V}^{\alpha\beta} \Psi_{J,\alpha|\beta}) \right) d\Gamma \\
&\quad + \int_{\Gamma_M} (\mathcal{M}^{\alpha\beta} \Psi_{I,\alpha|\beta} \mathbf{a}_3 \Psi_{J,\gamma} n^{\gamma} + \Psi_{I,\gamma} n^{\gamma} \mathbf{a}_3 \mathcal{M}^{\alpha\beta} \Psi_{J,\alpha|\beta}) d\Gamma
\end{aligned} \tag{65a}$$

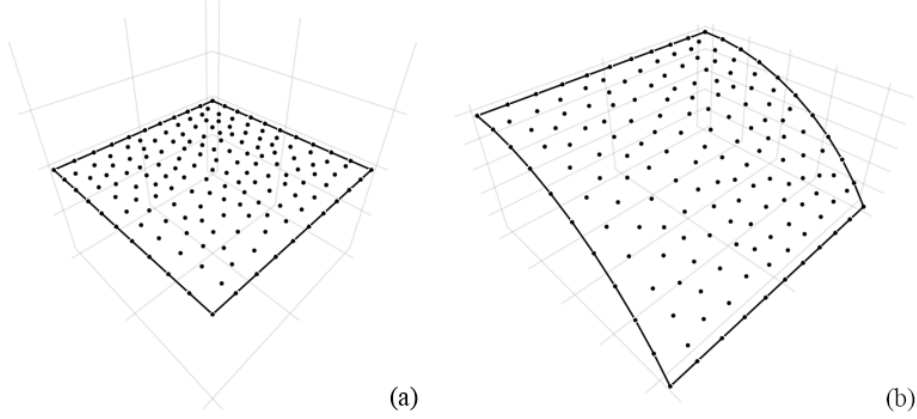


Figure 1: Meshfree discretization for patch test

Table 1: Results of patch test for flat model

	Linear patch test		Quadratic patch test	
	L_2 -Error	H_e -Error	L_2 -Error	H_e -Error
GI-Penalty	$4.45E-4$	$1.35E-2$	$2.01E-3$	$1.63E-2$
GI-Nitsche	$4.51E-4$	$1.42E-2$	$1.22E-3$	$1.68E-2$
RKGSi-Penalty	$3.64E-9$	$6.77E-8$	$4.54E-9$	$6.57E-8$
RKGSi-Nitsche	$3.31E-12$	$1.34E-11$	$5.98E-12$	$1.21E-11$
RKGSi-HR	$6.67E-13$	$1.50E-11$	$1.07E-12$	$1.26E-11$

Table 2: Results of patch test for curved model

	Linear patch test		Quadratic patch test	
	L_2 -Error	H_e -Error	L_2 -Error	H_e -Error
GI-Penalty	$3.79E-4$	$1.30E-2$	$1.74E-3$	$1.37E-2$
GI-Nitsche	$4.04E-4$	$1.42E-2$	$1.15E-3$	$1.49E-2$
RKGSi-Penalty	$3.64E-9$	$6.77E-8$	$4.54E-9$	$6.57E-8$
RKGSi-Nitsche	$3.31E-12$	$1.34E-11$	$5.98E-12$	$1.21E-11$
RKGSi-HR	$4.28E-6$	$1.30E-4$	$9.69E-6$	$1.30E-4$

196 Appendix A. Covariant derivatives

197 This Appendix lists the covariant derivatives needed for the development of
 198 the proposed method. For an arbitrary first order tensor \mathbf{v} presented by in-plane
 199 covariant or contravariant bases as:

$$\mathbf{v} = v_\alpha \mathbf{a}^\alpha + v_3 \mathbf{a}_3 = v^\alpha \mathbf{a}_\alpha + v^3 \mathbf{a}_3 \quad (\text{A.1})$$

200 the partial derivatives of tensor \mathbf{v} with respect to coordinate ξ^α , $\mathbf{v}_{,\alpha}$, can be
 201 evaluated by:

$$\begin{aligned} \mathbf{v}_{,\alpha} &= v_{\beta,\alpha} \mathbf{a}^\beta + v_\beta \mathbf{a}_{,\alpha}^\beta + v_{3,\alpha} \mathbf{a}_3 + v_3 \mathbf{a}_{3,\alpha} \\ &= v_{\beta,\alpha} \mathbf{a}^\beta - \Gamma_{\alpha\gamma}^\beta v_\beta \mathbf{a}^\gamma + v_{3,\alpha} \mathbf{a}_3 - v_3 b_{\alpha\beta} \mathbf{a}^\beta \\ &= v_{\beta,\alpha} \mathbf{a}^\beta - \Gamma_{\alpha\beta}^\gamma v_\gamma \mathbf{a}^\beta + v_{3,\alpha} \mathbf{a}_3 - v_3 b_{\alpha\beta} \mathbf{a}^\beta \\ &= v_\beta |_\alpha \mathbf{a}^\beta + v_{3,\alpha} \mathbf{a}_3 - v_3 b_{\alpha\beta} \mathbf{a}^\beta \end{aligned} \quad (\text{A.2})$$

202 where $\Gamma_{\alpha\beta}^\gamma = \mathbf{a}_{\alpha,\beta} \cdot \mathbf{a}^\gamma$ denotes the Christoffel symbol of the second kind, $b_{\alpha\beta} =$
 203 $\mathbf{a}_{\alpha,\beta} \cdot \mathbf{a}_3 = -\mathbf{a}_\alpha \cdot \mathbf{a}_{3,\beta}$ stands for the curvature tensor. $v_\alpha |_\beta$ can be regarded as
 204 the in-plane covariant derivative of the vector v_α :

$$v_\alpha |_\beta = v_{\alpha,\beta} - \Gamma_{\alpha\beta}^\gamma v_\gamma \quad (\text{A.3})$$

205 Following the same path, the in-plane covariant derivative of v^α is given by:

$$v^\alpha |_\beta = v_{,\beta}^\alpha + \Gamma_{\beta\gamma}^\alpha v^\gamma \quad (\text{A.4})$$

206 **Appendix B. Derivations for stiffness metrics and force vectors**

207 This Appendix details the derivations of stiffness

208 **References**

- 209 [1] H. Dah-wei, A method for establishing generalized variational principle 6 (6)
210 501–509. doi:10.1007/BF01876390.
211 URL <http://link.springer.com/10.1007/BF01876390>