A Hu-Washizu variational consistent meshfree thin shell formulation with naturally accommodating essential boundary conditions

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Abstract

A Hu-Washizu principle based variational consistent meshfree formulation with naturally enforcement of essential boundary conditions is proposed for thin shell analysis. In this approach, a mixed formulation of displacements, strains and stresses within the framework of Hu-Washizu variational principle is employed, where the displacements are discretized by meshfree shape functions, the strains and stresses are expressed as the smoothed gradients and covariant smoothed gradients which meet the first two order integration constraint and have the quasi- varational consistency. Thin shell problems ignore the shear deformations and this leads to a requirement of C1 continuous approximations. Meshfree methods equipped with high order smoothed shape functions is suitable for thin shell analysis, since the high order shape function can also suppress the membrane locking in thin shell problems. However, meshfree shape function always perform a natural rational property, this is a big challenge to meet integration consistency for traditional Gauss integration rule within Galerkin weak form, while integration consistency serves a key role in accuracy of Galerkin meshfree methods. In this work, we proposed a reproducing kernel gradient smoothing integration (RKGSI) algorithm for thin shell problems, while the first and second order smoothed gradients are constructed based upon reproducing kernel smoothing gradient framework, with the aid of this framework, the integration consistency becomes a natural property by a replacement between smoothed gradients and traditional gradients of shape functions in Galerkin weak form. The order of basis functions used in smoothed gradient is determined by ensuring the optimal order of error convergence respected to energy norm. The traditional costly second order gradients are totally eliminated in RKGSI formulation. To further increase the efficiency of proposed method, a set of integration schemes are developed for consistent assembly of stiffness matrix, force vector and smoothed gradients, where the number of integration points, which accompanied with calculation of traditional shape functions and their first order gradients, are minimized by a global point of view. It is evident

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that the smoothed gradients meets the reproducing consistency of gradients that can ensure the optimal convergence property. The numerical examples demonstrate the efficacy and efficiency of proposed method, while the RKGSI performs a comparable result in energy error with interpolation by meshfree approximations.

Keywords: Meshfree, Thin shell, Hu-Washizu variational principle, Reproducing kernel gradient smoothing, Essential boundary condition

1. Introduction

Thin shell structure follows the Kirchhoff hypothesis that neglects the shear deformation [?], which requires the approximation should have at least C^1 continuity in Galerkin formulations. The traditional finite element methods usually only has C^0 continuous shape functions, and it more prefers Mindlin thick shear theory, hybrid and mixed models in simulation of shell structure [?]. In last three decades, the meshfree methods [? ? ?] equipped high order smoothed shape functions have attracted significant research attention, while the meshfree shape functions are established based upon a set of scattered nodes and the high order continuity of shape functions is easily fulfilled even with low order basis function. For thin shell analysis, this high order meshfree approximations can also alleviate the membrane locking caused by the mismatched approximation order of membrane strain and bending strain [?]. Moreover, in general, the nodal-based meshfree approximations can release the burden of mesh distortion and have the flexibility of local refinement. Due to these advantages, a wide variety meshfree methods are proposed and have been applied to many scientific or engineering fields. However, the high order smoothed meshfree shape functions accompany with the enlarged and overlapping supports, which may also leads to many issues for shape functions. One is the loss of Kronecker delta property [?], which leads to that the essential boundary conditions cannot be enforced directly like finite element methods. Another issue is that the variational consistency or said integration constraint cannot be satisfied, which is caused by the misalignment between numerical integration domains and supports of shape functions, and the shape functions exhibit a piecewise rational nature in each integration domains. Variational consistency is of importance to the solution accuracy in Galerkin formulations [?].

To directly enforce the essential boundary for Galerkin meshfree methods, several approaches have been proposed for the recovery of shape functions' Kronecker property. For examples, interpolation element-free method [?], mixed transformation method [?], boundary singular kernel method [?] etc. However, these methods are not based on a variational setting, and cannot guarantee the variational consistency, enforcing accuracy may be worse on where there is no meshfree node. In contrast, enforcing the essential boundary conditions by a variational approach are more preferred for Galerkin meshfree methods. Belytschko et al. [? ?] firstly introduced the variational consistent Lagrange multiplier method to Galerkin meshfree method, in which the extra degrees of freedom should be employed for discretion of Lagrange multiplier. And this method has been extended to geometrically nonlinear thin shells by Ivannikov et al. [?]. To eliminate the extra degrees of freedom, Lu et al. [?] represented the Lagrange multiplier by corresponding tractions and proposed the modified variational essential boundary enforcement method. However, the coercivity of this approach is not always ensured and potentially reduces the accuracy. Zhu and Atluri [?] pioneered the penalty method for meshfree method, making it straightforward approach for enforcing essential boundary conditions via Galerkin weak form. However, penalty method suffers from a lack of variational consistency, and requires the experimental artificial parameters, whose optimal value is hard to be determined. Fernández-Méndez and Huerta [**?**] used the Nitsche's method in meshfree formulation for imposing essential boundary conditions. This method can be viewed as a hybrid of modified variational method and penalty method, since its consistent term that ensure variational consistency generated by modified variational method, and the penalty method is employed as stabilized term to recovery the coercivity. Skatulla and Sansour [**?**] further extended Nitsche's method for thin shell analysis and proposed an iteration algorithm to determine artificial parameters at each integration points.

To address the issue of numerical integration, a serial of consistent integration scheme has been developed for Galerkin meshfree methods. For instance, stabilized conforming nodal integration [**?**], variational consistent integration [**?**], quadratic consistent integration [**?**], reproducing kernel gradient smoothing integration [**?**], consistent projection integration [**?**] etc. The most consistent integration scheme is established by assumed strain approach, while the costly higher order derivatives of traditional meshfree shape functions are replaced by smoothed gradient, and show a high efficiency. Moreover, in order to achieve the global variational consistency, a consistent essential boundary condition enforcement should cooperate with the consistent integration scheme. The pair of consistent integration scheme and Nitsche's method for the treatment of essential boundary conditions shows a good performance, since it no needs the extra degrees of freedom and can fulfilled the coercivity. However,in Nitsche's method, the artificial parameters still exist in stabilized term and the costly higher order derivatives should be recalled, especially for thin plate and thin shell problems [**?**]. Recently, Wu et al [**? ?**] proposed a efficient and stabilized essential boundary condition enforcement based upon the Hellinger-Reissner (HR) variational principle, where the reproducing kernel gradient smoothing integration is recast by a mixed formulation in Hellinger-Reissner weak form. The terms for enforcing essential boundary conditions is mostly identical with Nitsche's method, both have consistent term and stabilized term. Nevertheless, the stabilized term of this method naturally exist in Hellinger-Reissner weak form and no longer needs the artificial parameters, even for essential boundary enforcement, total of the higher order derivatives are represented by smoothed gradients and their derivatives.

In this study, an efficient and stabilized variational consistent meshfree method with naturally enforcing the essential boundary conditions is developed for thin shell structure. Follow the ideas of Hellinger-Reissner principle base consistent meshfree method, the Hu-Washizu variational principle of complementary energy [1] with variables of displacement, strains and stresses is employed, where the displacement is approximated by conventional meshfree shape functions, and the strains and stresses are expressed by the smoothed gradients or covariant smoothed gradients with covariant bases. It should be noted that the smoothed gradients inherently embed the first two order integration constraints, however, due to the non-polynomial property of stresses, the fulfillment of these integration constraint only can get a quasi-satisfaction of variational consistency. All of the essential boundary conditions about displacements and rotations are considered in Hu-Washizu weak form, and present a Nitsche-like formalism but without any artificial parameters. Comparing with Nitsche's method, the costly higher order derivatives are replaced by conventional reproducing smoothed gradients and its direct derivatives. Taking the advantages of reproducing kernel gradient smoothing framework, the smoothed gradients shows a better performance on efficiency than conventional derivatives of shape functions, which improves the computational efficiency of meshfree formulation.

The remainder of this paper is organized as follows. Section 2 briefly describes the kinematics of thin shell structure and the corresponding Hu-Washizu principle weak form. Subsequently, the mixed formulation regarding the displacements, strains and stresses in accordance with Hu-Washizu weak form is presented in Section 3. Section 4 derives the discrete equilibrium equations with the naturally accommodation of essential, and compares them with those of Nitsche's method. The efficacy of the proposed Hu-Washizu meshfree thin shell formulation is validated by numerical results in Section 5. Concluding remarks are finally drawn in Section 6.

2. Hu-Washizu's formulation of complementary energy for thin shell

2.1. Kinematics for thin shell

Consider the configuration of a shell $\bar{\Omega}$, as shown in Fig. **??**, which can be easily described by a parametric curvilinear coordinate system $\boldsymbol{\xi} = \{\xi^i\}_{i=1,2,3}$. The mid-surface of the shell denoted by Ω is specified by the in-plane coordinates $\boldsymbol{\xi} = \{\xi^\alpha\}_{\alpha=1,2}$, as the thickness direction of shell is by ξ^3, $-\frac{h}{2} \leq \xi^3 \leq \frac{h}{2}$, h is the thickness of shell. In this work, Latin indices take the values from 1 to 3, and Greek indices are evaluated by 1 or 2. For the Kirchhoff hypothesis [**?**], the position $\boldsymbol{x} \in \bar{\Omega}$ are defined by linear functions with respect to ξ^3 :

$$\boldsymbol{x}(\xi^1, \xi^2, \xi^3) = \boldsymbol{r}(\xi^1, \xi^2) + \xi^3 \boldsymbol{a}_3(\xi^1, \xi^2) \tag{1}$$

in which $\boldsymbol{r}$ means the position on the mid-surface of shell, and the $\boldsymbol{a}_3$ is corresponding normal direction. For the mid-surface of shell, the in-plane covariant base vector with respect to ξ^α can be derived by a trivial partial differentiation to $\boldsymbol{r}$:

$$\boldsymbol{a}_\alpha = \frac{\partial \boldsymbol{r}}{\partial \xi^\alpha} = \boldsymbol{r}_{,\alpha}, \alpha = 1, 2 \tag{2}$$

for a clear expression, the subscript comma denotes the partial differentiation operation with respect to in-plane coordinates ξ^α. And the normal vector $\boldsymbol{a}_3$ can be obtained by the normalized cross product of $\boldsymbol{a}_\alpha$'s as follow:

$$\boldsymbol{a}_3 = \frac{\boldsymbol{a}_1 \times \boldsymbol{a}_2}{\|\boldsymbol{a}_1 \times \boldsymbol{a}_2\|} \tag{3}$$

where $\| \bullet \|$ is the Euclidean norm operator.

With the assumption of infinitesimal deformation, the strain components respected to global contravariant base can be sated as:

$$\epsilon_{ij} = \frac{1}{2}(\boldsymbol{x}_{,i} \cdot \boldsymbol{u}_{,j} + \boldsymbol{u}_{,i} \cdot \boldsymbol{x}_{,j}) \tag{4}$$

where $\boldsymbol{u}$ is the displacement for shell deformation. To fulfillment with Kirchhoff hypothesis, the displacement is assumed to be the following form:

$$\boldsymbol{u}(\xi^1, \xi^2, \xi^3) = \boldsymbol{v}(\xi^1, \xi^2) + \boldsymbol{\theta}(\xi^1, \xi^2)\xi^3 \tag{5}$$

in which the quadratic and higher order terms are neglected. $\boldsymbol{v}$, $\boldsymbol{\theta}$ respect the displacement and rotation in mid-surface.

Subsequently, plugging Eqs. (1) and (5) into Eq. (4) and neglecting quadratic terms, the strain components can be rephrased as follows:

$$\begin{aligned}
\epsilon_{\alpha\beta} &= \frac{1}{2}(\boldsymbol{a}_\alpha \cdot \boldsymbol{v}_{,\beta} + \boldsymbol{v}_{,\alpha} \cdot \boldsymbol{a}_\beta) \\
&\quad + \frac{1}{2}(\boldsymbol{a}_{3,\alpha} \cdot \boldsymbol{v}_{,\beta} + \boldsymbol{v}_{,\alpha} \cdot \boldsymbol{a}_{3,\beta} + \boldsymbol{a}_\alpha \cdot \boldsymbol{\theta}_{,\beta} + \boldsymbol{\theta}_{,\alpha} \cdot \boldsymbol{a}_\beta)\xi^3 \\
&= \varepsilon_{\alpha\beta} + \kappa_{\alpha\beta}\xi^3
\end{aligned} \tag{6a}$$

$$\epsilon_{\alpha 3} = \frac{1}{2}(\boldsymbol{a}_\alpha \cdot \boldsymbol{\theta} + \boldsymbol{v}_{,\alpha} \cdot \boldsymbol{a}_3) + \frac{1}{2}(\boldsymbol{a}_3 \cdot \boldsymbol{\theta})_{,\alpha}\xi^3 \tag{6b}$$

$$\epsilon_{33} = \boldsymbol{a}_3 \cdot \boldsymbol{\theta} \tag{6c}$$

where $\varepsilon_{\alpha\beta}$, $\kappa_{\alpha\beta}$ are membrane and bending strains respectively:

$$\varepsilon_{\alpha\beta} = \frac{1}{2}(\boldsymbol{a}_\alpha \cdot \boldsymbol{v}_{,\beta} + \boldsymbol{v}_{,\alpha} \cdot \boldsymbol{a}_\beta) \tag{7}$$

$$\kappa_{\alpha\beta} = \frac{1}{2}(\boldsymbol{a}_{3,\alpha} \cdot \boldsymbol{v}_{,\beta} + \boldsymbol{v}_{,\alpha} \cdot \boldsymbol{a}_{3,\beta} + \boldsymbol{a}_\alpha \cdot \boldsymbol{\theta}_{,\beta} + \boldsymbol{\theta}_{,\alpha} \cdot \boldsymbol{a}_\beta) \tag{8}$$

In accordance with Kirchhoff hypothesis, the thickness of shell will not change and the deformation related with direction of ξ^3 will be vanished, i.e. $\epsilon_{3i} = 0$. Thus, the rotation $\boldsymbol{\theta}$ can be rewritten as:

$$\epsilon_{3i} = 0 \Rightarrow \begin{cases} \boldsymbol{\theta} \cdot \boldsymbol{a}_\alpha + \boldsymbol{v}_{,\alpha} \cdot \boldsymbol{a}_3 = 0 \\ \boldsymbol{\theta} \cdot \boldsymbol{a}_3 = 0 \end{cases} \Rightarrow \boldsymbol{\theta} = -\boldsymbol{v}_{,\alpha} \cdot \boldsymbol{a}_3 \boldsymbol{a}^\alpha \tag{9}$$

where $\boldsymbol{a}^\alpha$'s are the in-plane contravariant base vectors, $\boldsymbol{a}^\alpha \cdot \boldsymbol{a}_\beta = \delta^\alpha_\beta$, δ is the Kronecker delta function. Substituting Eq. (9) into Eq. (8) leads to:

$$\kappa_{\alpha\beta} = (\Gamma^\gamma_{\alpha\beta} \boldsymbol{v}_{,\gamma} - \boldsymbol{v}_{,\alpha\beta}) \cdot \boldsymbol{a}_3 = -\boldsymbol{v}_{,\alpha}|_\beta \cdot \boldsymbol{a}_3 \tag{10}$$

in which $\Gamma^\gamma_{\alpha\beta} = \boldsymbol{a}_{\alpha,\beta} \cdot \boldsymbol{a}^\gamma$ is namely Christoffel symbol of the second kind. And $\boldsymbol{v}_{,\alpha}$ is the in-plane covariant derivative of $\boldsymbol{v}_{,\alpha}$, i.e. $\boldsymbol{v}_{,\alpha}|_\beta = \Gamma^\gamma_{\alpha\beta} \boldsymbol{v}_{,\gamma} - \boldsymbol{v}_{,\alpha\beta}$.

2.2. Galerkin weak form for Hu-Washizu principle of complementary energy

In this study, the Hu-Washizu variational principle of complementary energy [1] is used herein for development of this method, the corresponding complementary functional, denoted by Π_C, is listed as follow:

$$\begin{aligned} &\Pi_C(\varepsilon_{\alpha\beta}, \kappa_{\alpha\beta}, N^{\alpha\beta}, M^{\alpha\beta}) \\ &= \int_\Omega \frac{h}{2} \varepsilon_{\alpha\beta} C^{\alpha\beta\gamma\eta} \varepsilon_{\gamma\eta} d\Omega + \int_\Omega \frac{h^3}{24} \kappa_{\alpha\beta} C^{\alpha\beta\gamma\eta} \kappa_{\gamma\eta} d\Omega \\ &+ \int_\Omega \varepsilon_{\alpha\beta} (N^{\alpha\beta} - h C^{\alpha\beta\gamma\eta} \varepsilon_{\gamma\eta}) d\Omega + \int_\Omega \kappa_{\alpha\beta} (M^{\alpha\beta} - \frac{h^3}{12} C^{\alpha\beta\gamma\eta} \kappa_{\gamma\eta}) d\Omega \\ &- \int_{\Gamma_v} \boldsymbol{T} \cdot \bar{\boldsymbol{v}} d\Gamma + \int_{\Gamma_\theta} M_{\boldsymbol{nn}} \bar{\theta}_{\boldsymbol{n}} d\Gamma - (P \boldsymbol{a}_3 \cdot \bar{\boldsymbol{v}})_{\boldsymbol{x} \in C_w} \end{aligned} \tag{11}$$

where $C^{\alpha\beta\gamma\eta}$'s are the components of fourth order elasticity tensor with respect to covariant base and plane stress assumption, and it can be expressed by Young's modulus E, Poisson rate ν and the in-plane contravariant metric coefficients $a^{\alpha\beta}$'s, $a^{\alpha\beta} = \boldsymbol{a}^\alpha \cdot \boldsymbol{a}^\beta$, as follow:

$$C^{\alpha\beta\gamma\eta} = \frac{E}{2(1+\nu)} (a^{\alpha\gamma} a^{\beta\eta} + a^{\alpha\eta} a^{\beta\gamma} + \frac{2\nu}{1-\nu} a^{\alpha\beta} a^{\gamma\eta}) \tag{12}$$

and $N^{\alpha\beta}$, $M^{\alpha\beta}$ are the components of membrane and bending stresses given by:

$$N^{\alpha\beta} = h C^{\alpha\beta\gamma\eta} \varepsilon_{\gamma\eta}, \quad M^{\alpha\beta} = \frac{h^3}{12} C^{\alpha\beta\gamma\eta} \kappa_{\gamma\eta} \tag{13}$$

Essential boundaries on the edges and corners denoted by Γ_v, Γ_θ and C_v are naturally existed in complementary energy functional, $\bar{\boldsymbol{v}}$, $\bar{\theta}_{\boldsymbol{n}}$ are the corresponding prescribed displacement and normal rotation. $\boldsymbol{T}$, M_{nn} and P can be determined by Euler-Lagrange equations of shell problem [**?**] as follows:

$$\boldsymbol{T} = \boldsymbol{T}_N + \boldsymbol{T}_M \ \rightarrow \begin{cases} \boldsymbol{T}_N = \boldsymbol{a}_\alpha N^{\alpha\beta} n_\beta \\ \boldsymbol{T}_M = (\boldsymbol{a}_3 M^{\alpha\beta} s_\alpha n_\beta)_{,\gamma} s^\gamma + (\boldsymbol{a}_3 M^{\alpha\beta})|_\beta n_\alpha \end{cases} \tag{14}$$

$$M_{\boldsymbol{nn}} = M^{\alpha\beta} n_\alpha n_\beta \tag{15}$$

$$P = -[[M^{\alpha\beta} s_\alpha n_\beta]] \tag{16}$$

where $\boldsymbol{n} = n^\alpha \boldsymbol{a}_\alpha = n_\alpha \boldsymbol{a}^\alpha$ and $\boldsymbol{s} = s^\alpha \boldsymbol{a}_\alpha = s_\alpha \boldsymbol{a}^\alpha$ are the outward normal and tangent directions on boundaries. $[[f]]$ is the jump operator defined by:

$$[[f]]_{\boldsymbol{x}=\boldsymbol{x}_c} = \lim_{\boldsymbol{\epsilon}\rightarrow\boldsymbol{0}+} (f(\boldsymbol{x}_c + \boldsymbol{\epsilon}) - f(\boldsymbol{x}_c - \boldsymbol{\epsilon})), \boldsymbol{x}_c \in \Gamma \tag{17}$$

where f is an arbitrary function on Γ.

Moreover, the natural boundary conditions should be applied by Lagrangian multiplier method with displacement $\boldsymbol{v}$ regarded as multiplier. Thus then the new complementary energy functional namely Π is given by:

$$\begin{aligned} &\Pi(\boldsymbol{v}, \varepsilon_{\alpha\beta}, \kappa_{\alpha\beta}, N^{\alpha\beta}, M^{\alpha\beta}) \\ =&\Pi_C(\varepsilon_{\alpha\beta}, \kappa_{\alpha\beta}, N^{\alpha\beta}, M^{\alpha\beta}) + \int_{\Gamma_M} \theta_{\boldsymbol{n}} (M_{\boldsymbol{nn}} - \bar{M}_{\boldsymbol{nn}}) d\Gamma \\ -&\int_{\Gamma_T} \boldsymbol{v} \cdot (\boldsymbol{T} - \bar{\boldsymbol{T}}) d\Gamma - \boldsymbol{v} \cdot \boldsymbol{a}_3 (P - \bar{P})_{\boldsymbol{x}\in C_P} - \int_\Omega \boldsymbol{v} \cdot (\boldsymbol{b} - \bar{\boldsymbol{b}}) d\Omega \end{aligned} \tag{18}$$

where $\bar{\boldsymbol{T}}$, $\bar{M}_{\boldsymbol{nn}}$ and $\bar{P}$ are the corresponding prescribed traction, bending moment and concentrated force on edges Γ_T, Γ_M and corner C_P respectively. All the boundaries meet the following geometric relationships:

$$\begin{cases} \Gamma = \Gamma_v \cup \Gamma_T \cup \Gamma_\theta \cup \Gamma_M, \quad C = C_v \cup C_P, \\ \Gamma_v \cap \Gamma_T = \Gamma_\theta \cap \Gamma_M = C_v \cap C_P = \varnothing \end{cases} \tag{19}$$

and $\bar{\boldsymbol{b}}$ stands for the prescribed body force in Ω, $\boldsymbol{b}$ also can be given based upon Euler-Lagrange equations [**?**] as:

$$\boldsymbol{b} = \boldsymbol{b}_N + \boldsymbol{b}_M \rightarrow \begin{cases} \boldsymbol{b}_N = (\boldsymbol{a}_\alpha N^{\alpha\beta})|_\beta \\ \boldsymbol{b}_M = (\boldsymbol{a}_3 M^{\alpha\beta})|_{\alpha\beta} \end{cases} \tag{20}$$

Introducing a standard variational argument to Eq. (18), $\delta\Pi = 0$, and considering the arbitrariness of virtual variables, $\delta\boldsymbol{v}$, $\delta\varepsilon_{\alpha\beta}$, $\delta\kappa_{\alpha\beta}$, $N^{\alpha\beta}$, $M^{\alpha\beta}$ lead to the following weak form:

$$-\int_\Omega h\delta\varepsilon_{\alpha\beta} C^{\alpha\beta\gamma\eta} \varepsilon_{\gamma\eta} d\Omega + \int_\Omega \delta\varepsilon_{\alpha\beta} N^{\alpha\beta} d\Omega = 0 \tag{21a}$$

$$-\int_\Omega \frac{h^3}{12}\delta\kappa_{\alpha\beta}C^{\alpha\beta\gamma\eta}\kappa_{\gamma\eta}d\Omega + \int_\Omega \delta\kappa_{\alpha\beta}M^{\alpha\beta}d\Omega = 0 \tag{21b}$$

$$\begin{aligned}\int_\Omega \delta N^{\alpha\beta}\varepsilon_{\alpha\beta}d\Omega - \int_\Gamma \delta\boldsymbol{T}_N\cdot\boldsymbol{v}d\Gamma + \int_\Omega \delta\boldsymbol{b}_N\cdot\boldsymbol{v}d\Omega \\ + \int_{\Gamma_v}\delta\boldsymbol{T}_N\cdot\boldsymbol{v}d\Gamma = \int_{\Gamma_v}\delta\boldsymbol{T}_N\cdot\bar{\boldsymbol{v}}d\Gamma\end{aligned} \tag{21c}$$

$$\begin{aligned}\int_\Omega \delta M^{\alpha\beta}\kappa_{\alpha\beta}d\Omega - \int_\Gamma \delta M_{\boldsymbol{nn}}\theta_{\boldsymbol{n}}d\Gamma + \int_\Gamma \delta\boldsymbol{T}_M\cdot\boldsymbol{v}d\Gamma + (\delta P\boldsymbol{a}_3\cdot\boldsymbol{v})_{\boldsymbol{x}\in C} + \int_\Omega \delta\boldsymbol{b}_M\cdot\boldsymbol{v}d\Omega \\ + \int_{\Gamma_\theta}\delta M_{\boldsymbol{nn}}\theta_{\boldsymbol{n}}d\Gamma - \int_{\Gamma_v}\delta\boldsymbol{T}_M\cdot\boldsymbol{v}d\Gamma - (\delta P\boldsymbol{a}_3\cdot\boldsymbol{v})_{\boldsymbol{x}\in C_v} \\ = \int_{\Gamma_\theta}\delta M_{\boldsymbol{nn}}\bar{\theta}_{\boldsymbol{n}}d\Gamma - \int_{\Gamma_v}\delta\boldsymbol{T}_M\cdot\bar{\boldsymbol{v}}d\Gamma - (\delta P\boldsymbol{a}_3\cdot\bar{\boldsymbol{v}})_{\boldsymbol{x}\in C_v}\end{aligned} \tag{21d}$$

$$\begin{aligned}\int_\Gamma \delta\theta_{\boldsymbol{n}}M_{\boldsymbol{nn}}d\Gamma - \int_\Gamma \delta\boldsymbol{v}\cdot\boldsymbol{T}d\Gamma - (\delta\boldsymbol{v}\cdot\boldsymbol{a}_3P)_{\boldsymbol{x}\in C} + \int_\Omega \delta\boldsymbol{v}\cdot\boldsymbol{b}d\Omega \\ -\int_{\Gamma_\theta}\delta\theta_{\boldsymbol{n}}M_{\boldsymbol{nn}}d\Gamma + \int_{\Gamma_v}\delta\boldsymbol{v}\cdot\boldsymbol{T}d\Gamma + (\delta\boldsymbol{v}\cdot\boldsymbol{a}_3P)_{\boldsymbol{x}\in C_v} = -\int_{\Gamma_T}\delta\boldsymbol{v}\cdot\bar{\boldsymbol{t}}d\Gamma - \int_\Omega \delta\boldsymbol{v}\cdot\bar{\boldsymbol{b}}d\Omega\end{aligned} \tag{21e}$$

where the geometric relationships of Eq. (19) is used herein.

3. Mixed meshfree formulation for modified Hellinger-Reissner weak form

3.1. Reproducing kernel approximation for displacement

In this study, the displacement is approximated by traditional reproducing kernel approximation. As shown in Fig, the mid-surface of the shell Ω is discretized by a set of meshfree nodes $\{\boldsymbol{\xi}_I\}_{I=1}^{n_p}$ in parametric configuration, where n_p is the total number of meshfree nodes. The approximated displacement namely $\boldsymbol{v}^h$ can be expressed by:

$$\boldsymbol{v}(\boldsymbol{\xi}) = \sum_{I=1}^{n_p} \Psi_I(\boldsymbol{\xi}) \boldsymbol{d}_I \tag{22}$$

in which Ψ_I and $\boldsymbol{d}_I$ is the shape function and nodal coefficient tensor related by node $\boldsymbol{\xi}_I$. According to reproducing kernel approximation [**?**], the shape function takes the following form:

$$\Psi_I(\boldsymbol{\xi}) = \boldsymbol{p}^T(\boldsymbol{\xi})\boldsymbol{c}(\boldsymbol{\xi})\phi(\boldsymbol{\xi}_I - \boldsymbol{\xi}) \tag{23}$$

where $\boldsymbol{p}$ is the basis function vector, and in this study, the following quadratic basis function is considered:

$$\boldsymbol{p} = \{1,\ \xi^1,\ \xi^2,\ (\xi^1)^2, \xi^1\xi^2, (\xi^2)^2\}^T \tag{24}$$

The kernel function denoted byϕ controls the support and smoothness of meshfree shape functions. The quantic B-spline function with square support is used herein as the kernel function:

$$\phi(\boldsymbol{\xi}_I - \boldsymbol{\xi}) = \phi(\hat{s}_1)\phi(\hat{s}_2), \quad \hat{s}_\alpha = \frac{|\xi_I^\alpha - \xi^\alpha|}{s_{\alpha I}} \tag{25}$$

with

$$\phi(\hat{s}_\alpha) = \frac{1}{5!}\begin{cases} (3-3\hat{s}_\alpha)^5 - 6(2-3\hat{s}_\alpha)^5 + 15(1-3\hat{s}_\alpha)^5 & \hat{s}_\alpha \leq \frac{1}{3} \\ (3-3\hat{s}_\alpha)^5 - 6(2-3\hat{s}_\alpha)^5 & \frac{1}{3} < \hat{s}_\alpha \leq \frac{2}{3} \\ (3-3\hat{s}_\alpha)^5 & \frac{2}{3} < \hat{s}_\alpha \leq 1 \\ 0 & \hat{s}_\alpha > 1 \end{cases} \tag{26}$$

and $\hat{s}_{\alpha I}$ means the characterized size of support for meshfree shape function Ψ_I.

The unknown vector $\boldsymbol{c}$ in shape function are determined by the fulfillment of the so-call consistency condition:

$$\sum_{I=1}^{n_p} \Psi_I(\boldsymbol{\xi})\boldsymbol{p}(\boldsymbol{\xi}_I) = \boldsymbol{p}(\boldsymbol{\xi}) \tag{27}$$

or equivalently

$$\sum_{I=1}^{n_p} \Psi_I(\boldsymbol{\xi})\boldsymbol{p}(\boldsymbol{\xi}_I - \boldsymbol{\xi}) = \boldsymbol{p}(\boldsymbol{0}) \tag{28}$$

Substituting Eq. (22) into (28), yields:

$$\boldsymbol{A}(\boldsymbol{\xi})\boldsymbol{c}(\boldsymbol{\xi}) = \boldsymbol{p}(\boldsymbol{0}) \quad \Rightarrow \quad \boldsymbol{c}(\boldsymbol{\xi}) = \boldsymbol{A}^{-1}(\boldsymbol{\xi})\boldsymbol{p}(\boldsymbol{0}) \tag{29}$$

where $\boldsymbol{A}$ is the moment matrix:

$$\boldsymbol{A}(\boldsymbol{\xi}) = \sum_{I=1}^{n_p} \phi(\boldsymbol{\xi}_I - \boldsymbol{\xi})\boldsymbol{p}(\boldsymbol{\xi}_I - \boldsymbol{\xi})\boldsymbol{p}^T(\boldsymbol{\xi}_I - \boldsymbol{\xi}) \tag{30}$$

Taking Eq. (29) back into Eq. (22), the expression of meshfree shape function can be given by:

$$\Psi_I(\boldsymbol{\xi}) = \boldsymbol{p}^T(\boldsymbol{\xi}_I - \boldsymbol{\xi})\boldsymbol{A}^{-1}(\boldsymbol{\xi})\boldsymbol{p}(\boldsymbol{0})\phi(\boldsymbol{\xi}_I - \boldsymbol{\xi}) \tag{31}$$

3.2. Reproducing kernel gradient smoothing approximation for effective stress and strain

In Galerkin meshfree formulation, the mid-plane of thin shell Ω is split by a set of integration cells Ω_C's, $\cup_{C=1}^{n_e}\Omega_C \approx \Omega$. With the inspiration of reproducing kernel smoothing framework, the Cartesian and covariant derivatives of displacement, $\boldsymbol{v}_{,\alpha}$ and $-\boldsymbol{v}_{,\alpha}|_\beta$, in strains $\varepsilon_{\alpha\beta}$, $\kappa_{\alpha\beta}$ are approximated by $(p-1)$-th order polynomials in each integration cells. In integration cell Ω_C, the approximated derivatives and strains denoted by $\boldsymbol{v}^h_{,\alpha}$, $\varepsilon^h_{\alpha\beta}$ and $-\boldsymbol{v}^h_{,\alpha}|_\beta$, $\kappa^h_{\alpha\beta}$ can be expressed by:

$$\boldsymbol{v}^h_{,\alpha}(\boldsymbol{\xi}) = \boldsymbol{q}^T(\boldsymbol{\xi})\boldsymbol{d}^\varepsilon_\alpha, \quad \varepsilon^h_{\alpha\beta}(\boldsymbol{\xi}) = \boldsymbol{q}^T(\boldsymbol{\xi})\frac{1}{2}(\boldsymbol{a}_\alpha \cdot \boldsymbol{d}^\varepsilon_\beta + \boldsymbol{a}_\beta \cdot \boldsymbol{d}^\varepsilon_\alpha) \tag{32}$$

$$-\boldsymbol{v}^h_{,\alpha}|_\beta(\boldsymbol{\xi}) = \boldsymbol{q}^T(\boldsymbol{\xi})\boldsymbol{d}^\kappa_{\alpha\beta}, \quad \kappa^h_{\alpha\beta}(\boldsymbol{\xi}) = \boldsymbol{q}^T(\boldsymbol{\xi})\boldsymbol{a}_3 \cdot \boldsymbol{d}^\kappa_{\alpha\beta} \tag{33}$$

where $\boldsymbol{q}$ is the $(p-1)$th order polynomial vector and has the following form:

$$\boldsymbol{q} = \{1,\ \xi^1,\ \xi^2,\ \ldots, (\xi^2)^{p-1}\}^T \tag{34}$$

and the $\boldsymbol{d}^\varepsilon_\alpha$, $\boldsymbol{d}^\kappa_{\alpha\beta}$ are the corresponding coefficient vector tensors. For the conciseness, the mixed usage of tensor and vector is introduced in this study, for example, the component of coefficient tensor vector $\boldsymbol{d}^\varepsilon_{\alpha I}$, $\boldsymbol{d}^\varepsilon_\alpha = \{\boldsymbol{d}^\varepsilon_{\alpha I}\}$, is a three dimensional tensor, $\dim \boldsymbol{d}^\varepsilon_{\alpha I} = \dim \boldsymbol{v}$.

In order to meet the integration constraint of thin shell problem, the approximated stresses $N^{\alpha\beta h}$, $M^{\alpha\beta h}$ are assumed to be a similar form with strains, yields:

$$N^{\alpha\beta h}(\boldsymbol{\xi}) = \boldsymbol{q}^T(\boldsymbol{\xi})\boldsymbol{a}^\alpha \cdot \boldsymbol{d}^N_\beta, \quad \boldsymbol{a}_\alpha N^{\alpha\beta h}(\boldsymbol{\xi}) = \boldsymbol{q}^T(\boldsymbol{\xi})\boldsymbol{d}^N_\beta \tag{35}$$

$$M^{\alpha\beta h}(\boldsymbol{\xi}) = \boldsymbol{q}^T(\boldsymbol{\xi})\boldsymbol{a}_3 \cdot \boldsymbol{d}^M_{\alpha\beta}, \quad \boldsymbol{a}_3 M^{\alpha\beta h}(\boldsymbol{\xi}) = \boldsymbol{q}^T(\boldsymbol{\xi})\boldsymbol{d}^M_{\alpha\beta} \tag{36}$$

substituting the approximations of Eqs. (22), (32), (33), (35), (36) into Eqs. (21c), (21d) can express $\boldsymbol{d}^\varepsilon_\beta$ and $\boldsymbol{d}^\kappa_{\alpha\beta}$ by $\boldsymbol{d}$ as:

$$\boldsymbol{d}^\varepsilon_\beta = \boldsymbol{G}^{-1}\left(\sum_{I=1}^{n_p}(\tilde{\boldsymbol{g}}_{\beta I} - \bar{\boldsymbol{g}}_{\beta I})\boldsymbol{d}_I + \hat{\boldsymbol{g}}_\beta\right) \tag{37}$$

$$\boldsymbol{d}^{\kappa}_{\alpha\beta} = \boldsymbol{G}^{-1}\left(\sum_{I=1}^{n_p}(\tilde{\boldsymbol{g}}_{\alpha\beta I} - \bar{\boldsymbol{g}}_{\alpha\beta I})\boldsymbol{d}_I + \hat{\boldsymbol{g}}_{\alpha\beta}\right) \tag{38}$$

with

$$\boldsymbol{G} = \int_{\Omega_C} \boldsymbol{q}^T\boldsymbol{q}\, d\Omega \tag{39}$$

$$\tilde{\boldsymbol{g}}_{\beta I} = \int_{\Gamma_C} \Psi_I \boldsymbol{q} n_\beta d\Gamma - \int_{\Omega_C} \Psi_I \boldsymbol{q}^*|_\beta d\Omega \tag{40a}$$

$$\bar{\boldsymbol{g}}_{\beta I} = \int_{\Gamma_C\cap\Gamma_v} \Psi_I \boldsymbol{q} n_\beta d\Gamma \tag{40b}$$

$$\hat{\boldsymbol{g}}_{\beta} = \int_{\Gamma_C\cap\Gamma_v} \boldsymbol{q} n_\beta \bar{\boldsymbol{v}} d\Gamma \tag{40c}$$

$$\begin{aligned}\tilde{\boldsymbol{g}}_{\alpha\beta I} = &\int_{\Gamma_C} \Psi_{I,\gamma} n^\gamma \boldsymbol{q} n_\alpha n_\beta d\Gamma - \int_{\Gamma_C} \Psi_I (\boldsymbol{q}^{**}|_\beta n_\alpha + (\boldsymbol{q} s_\alpha n_\beta)_{,\gamma} s^\gamma) d\Gamma \\ &+ [[\Psi_I \boldsymbol{q} s_\alpha n_\beta]]_{\boldsymbol{x}\in C_C} - \int_{\Omega_C} \Psi \boldsymbol{q}^{**}_{,\alpha}|_\beta d\Omega\end{aligned} \tag{41a}$$

$$\begin{aligned}\bar{\boldsymbol{g}}_{\alpha\beta I} = &\int_{\Gamma_C\cap\Gamma_\theta} \Psi_{I,\gamma} n^\gamma \boldsymbol{q} n_\alpha n_\beta d\Gamma - \int_{\Gamma_C\cap\Gamma_v} \Psi_I (\boldsymbol{q}^{**}|_\beta n_\alpha + (\boldsymbol{q} s_\alpha n_\beta)_{,\gamma} s^\gamma) d\Gamma \\ &+ [[\Psi_I \boldsymbol{q} s_\alpha n_\beta]]_{\boldsymbol{x}\in C_C\cap C_v}\end{aligned} \tag{41b}$$

$$\begin{aligned}\hat{\boldsymbol{g}}_{\alpha\beta} = &\int_{\Gamma_C\cap\Gamma_\theta} \boldsymbol{q} n_\alpha n_\beta \boldsymbol{a}_3 \bar{\theta}_{\boldsymbol{n}} d\Gamma - \int_{\Gamma_C\cap\Gamma_v} (\boldsymbol{q}^{**}|_\beta n_\alpha + (\boldsymbol{q} s_\alpha n_\beta)_{,\gamma} s^\gamma) \bar{\boldsymbol{v}} d\Gamma \\ &+ [[\boldsymbol{q} s_\alpha n_\beta \bar{\boldsymbol{v}}]]_{\boldsymbol{x}\in C_C\cap C_v}\end{aligned} \tag{41c}$$

plugging Eqs. (37) and (38) back into Eqs. (32) and (33) respectively gives the final expression of $\boldsymbol{v}^h_{,\alpha}$, $\varepsilon^h_{\alpha\beta}$ and $-\boldsymbol{v}^h_{,\alpha\beta}$, $\boldsymbol{\kappa}^h_{\alpha\beta}$ as:

$$\boldsymbol{v}^h_{,\alpha} = \sum_{I=1}^{n_p}(\tilde{\Psi}_{I,\alpha} - \bar{\Psi}_{I,\alpha})\boldsymbol{d}_I + \boldsymbol{q}^T\boldsymbol{G}^{-1}\hat{\boldsymbol{g}}_\alpha \tag{42a}$$

$$\begin{aligned}\varepsilon^h_{\alpha\beta} = &\sum_{I=1}^{n_p}\frac{1}{2}(\boldsymbol{a}_\alpha\tilde{\Psi}_{I,\beta} + \boldsymbol{a}_\beta\tilde{\Psi}_{I,\alpha})\cdot\boldsymbol{d}_I - \sum_{I=1}^{n_p}\frac{1}{2}(\boldsymbol{a}_\alpha\bar{\Psi}_{I,\beta} + \boldsymbol{a}_\beta\bar{\Psi}_{I,\alpha})\cdot\boldsymbol{d}_I \\ &+ \boldsymbol{q}^T\boldsymbol{G}^{-1}\frac{1}{2}(\boldsymbol{a}_\alpha\cdot\hat{\boldsymbol{g}}_\beta + \boldsymbol{a}_\beta\cdot\hat{\boldsymbol{g}}_\alpha) \\ &= \tilde{\varepsilon}^h_{\alpha\beta} - \bar{\varepsilon}^h_{\alpha\beta} + \hat{\varepsilon}^h_{\alpha\beta}\end{aligned} \tag{42b}$$

$$-\boldsymbol{v}^h_{,\alpha}|_\beta = \sum_{I=1}^{n_p}(\tilde{\Psi}_{I,\alpha\beta} - \bar{\Psi}_{I,\alpha\beta})\boldsymbol{d}_I + \boldsymbol{q}^T\boldsymbol{G}^{-1}\hat{\boldsymbol{g}}_{\alpha\beta} \tag{43a}$$

$$\begin{aligned}\kappa^h_{\alpha\beta} = &\sum_{I=1}^{n_p}\tilde{\Psi}_{I,\alpha\beta}\boldsymbol{a}_3\cdot\boldsymbol{d}_I - \sum_{I=1}^{n_p}\bar{\Psi}_{I,\alpha\beta}\boldsymbol{a}_3\cdot\boldsymbol{d}_I + \boldsymbol{q}^T\boldsymbol{G}^{-1}\boldsymbol{a}_3\cdot\hat{\boldsymbol{g}}_{\alpha\beta} \\ &= \tilde{\kappa}^h_{\alpha\beta} - \bar{\kappa}^h_{\alpha\beta} + \hat{\kappa}^h_{\alpha\beta}\end{aligned} \tag{43b}$$

with

$$\tilde{\Psi}_{I,\alpha}(\boldsymbol{\xi}) = \boldsymbol{q}^T(\boldsymbol{\xi})\boldsymbol{G}^{-1}\tilde{\boldsymbol{g}}_{\alpha I}, \quad \bar{\Psi}_{I,\alpha}(\boldsymbol{\xi}) = \boldsymbol{q}^T(\boldsymbol{\xi})\boldsymbol{G}^{-1}\tilde{\boldsymbol{g}}_{\alpha I} \tag{44}$$

$$\begin{cases} \tilde{\varepsilon}^h_{\alpha\beta} = \sum_{I=1}^{n_p} \frac{1}{2}(\boldsymbol{a}_\alpha \tilde{\Psi}_{I,\beta} + \boldsymbol{a}_\beta \tilde{\Psi}_{I,\alpha}) \cdot \boldsymbol{d}_I \\ \bar{\varepsilon}^h_{\alpha\beta} = \sum_{I=1}^{n_p} \frac{1}{2}(\boldsymbol{a}_\alpha \bar{\Psi}_{I,\beta} + \boldsymbol{a}_\beta \bar{\Psi}_{I,\alpha}) \cdot \boldsymbol{d}_I \\ \hat{\varepsilon}^h_{\alpha\beta} = \boldsymbol{q}^T\boldsymbol{G}^{-1}\frac{1}{2}(\boldsymbol{a}_\alpha \cdot \hat{\boldsymbol{g}}_\beta + \boldsymbol{a}_\beta \cdot \hat{\boldsymbol{g}}_\alpha) \end{cases} \tag{45}$$

$$\tilde{\Psi}_{I,\alpha\beta}(\boldsymbol{\xi}) = \boldsymbol{q}^T(\boldsymbol{\xi})\boldsymbol{G}^{-1}\tilde{\boldsymbol{g}}_{\alpha\beta I}, \quad \bar{\Psi}_{I,\alpha\beta}(\boldsymbol{\xi}) = \boldsymbol{q}^T(\boldsymbol{\xi})\boldsymbol{G}^{-1}\tilde{\boldsymbol{g}}_{\alpha\beta I} \tag{46}$$

$$\begin{cases} \tilde{\kappa}^h_{\alpha\beta} = \sum_{I=1}^{n_p} \tilde{\Psi}_{I,\alpha\beta}\boldsymbol{a}_3 \cdot \boldsymbol{d}_I \\ \bar{\kappa}^h_{\alpha\beta} = \sum_{I=1}^{n_p} \bar{\Psi}_{I,\alpha\beta}\boldsymbol{a}_3 \cdot \boldsymbol{d}_I \\ \hat{\kappa}^h_{\alpha\beta} = \boldsymbol{q}^T\boldsymbol{G}^{-1}\boldsymbol{a}_3 \cdot \hat{\boldsymbol{g}}_{\alpha\beta} \end{cases} \tag{47}$$

Furthermore, taking Eqs. (32) and (33) into Eqs.(21a) and (21b) can obtain the approximated effective stresses $N^{\alpha\beta h}$, $M^{\alpha\beta h}$ and their coefficients $\boldsymbol{d}^N_\beta$, $\boldsymbol{d}^M_{\alpha\beta}$ as:

$$\begin{aligned} &\frac{1}{2}(\delta\boldsymbol{d}^\varepsilon_\alpha \cdot \boldsymbol{a}_\beta + \delta\boldsymbol{d}^\varepsilon_\beta \ \cdot \boldsymbol{a}_\alpha)hC^{\alpha\beta\gamma\eta}\frac{1}{2}(\boldsymbol{a}_\gamma \cdot \boldsymbol{d}^\varepsilon_\eta + \boldsymbol{a}_\gamma \cdot \boldsymbol{d}^\varepsilon_\eta)\boldsymbol{G} \\ =&\frac{1}{2}(\delta\boldsymbol{d}^\varepsilon_\alpha \cdot \boldsymbol{d}^N_\beta + \delta\boldsymbol{d}^\varepsilon_\beta \cdot \boldsymbol{d}^N_\alpha)\boldsymbol{G} \\ \Rightarrow \ \boldsymbol{d}^\beta_N =&\boldsymbol{a}_\beta hC^{\alpha\beta\gamma\eta}\frac{1}{2}(\boldsymbol{a}_\gamma \cdot \boldsymbol{d}^\varepsilon_\eta + \boldsymbol{a}_\eta \cdot \boldsymbol{d}^\varepsilon_\gamma) \end{aligned} \tag{48}$$

$$\begin{aligned} \delta\boldsymbol{d}^\kappa_{\alpha\beta} \cdot \boldsymbol{a}_3\frac{h^3}{12}C^{\alpha\beta\gamma\eta}\boldsymbol{a}_3 \cdot \boldsymbol{d}^\kappa_{\gamma\eta}\boldsymbol{G} = \delta\boldsymbol{d}^\kappa_{\alpha\beta} \cdot \boldsymbol{d}^M_{\alpha\beta}\boldsymbol{G} \\ \Rightarrow \boldsymbol{d}^{\alpha\beta}_M = \boldsymbol{a}_3\frac{h^3}{12}C^{\alpha\beta\gamma\eta}\boldsymbol{a}_3 \cdot \boldsymbol{d}^\kappa_{\gamma\eta} \end{aligned} \tag{49}$$

$$N^{\alpha\beta h} = hC^{\alpha\beta\gamma\eta}(\tilde{\varepsilon}^h_{\gamma\eta} - \bar{\varepsilon}^h_{\gamma\eta} + \hat{\varepsilon}^h_{\gamma\eta}) = \tilde{N}^{\alpha\beta h} - \bar{N}^{\alpha\beta h} + \hat{N}^{\alpha\beta h} \tag{50}$$

$$M^{\alpha\beta h} = \frac{h^3}{12}C^{\alpha\beta\gamma\eta}(\tilde{\kappa}^h_{\gamma\eta} - \bar{\kappa}^h_{\gamma\eta} + \hat{\kappa}^h_{\gamma\eta}) = \tilde{M}^{\alpha\beta h} - \bar{M}^{\alpha\beta h} + \hat{M}^{\alpha\beta h} \tag{51}$$

with

$$\tilde{N}^{\alpha\beta h} = hC^{\alpha\beta\gamma\eta}\tilde{\varepsilon}^h_{\gamma\eta}, \quad \bar{N}^{\alpha\beta h} = hC^{\alpha\beta\gamma\eta}\bar{\varepsilon}^h_{\gamma\eta}, \quad \hat{N}^{\alpha\beta h} = hC^{\alpha\beta\gamma\eta}\hat{\varepsilon}^h_{\gamma\eta} \tag{52}$$

$$\tilde{M}^{\alpha\beta h} = \frac{h^3}{12}C^{\alpha\beta\gamma\eta}\tilde{\kappa}^h_{\gamma\eta}, \quad \bar{M}^{\alpha\beta h} = \frac{h^3}{12}C^{\alpha\beta\gamma\eta}\bar{\kappa}^h_{\gamma\eta}, \quad \hat{M}^{\alpha\beta h} = \frac{h^3}{12}C^{\alpha\beta\gamma\eta}\hat{\kappa}^h_{\gamma\eta} \tag{53}$$

It is noted that, referring to reproducing kernel gradient smoothing framework [**?**], $\tilde{\Psi}_{I,\alpha}$, $\tilde{\Psi}_{I,\alpha\beta}$ are actually the first and second order smoothed gradients in curvilinear coordinates. $\tilde{\boldsymbol{g}}_{\alpha I}$ and $\tilde{\boldsymbol{g}}_{\alpha\beta I}$ are the right hand side integration constraints for first and second order gradients, then this formulation can meet the variational consistency for the p-th order polynomials. It should be known that, in curved model, the variational consistency for non-polynomial functions, like trigonometric functions, should be required for the polynomial solution. Even with p-th order variational consistency, the proposed formulation can not exactly reproduce the solution spanned by basis functions, however the accuracy of reproducing kernel smoothed gradients is still better that traditonal meshfree formulation, this will be evidenced by numerical examples in further section.

4. Naturally variational enforcement for essential boundary conditions

4.1. Discrete equilibrium equations

With the approximated effective stresses and strains, the last equation of weak form becomes:

$$-\sum_{C=1}^{n_e}(\tilde{\boldsymbol{g}}_{\alpha I}^T-\bar{\boldsymbol{g}}_{\alpha I}^T)\boldsymbol{d}_N^{\alpha}-\sum_{C=1}^{n_e}(\tilde{\boldsymbol{g}}_{\alpha\beta I}^T-\bar{\boldsymbol{g}}_{\alpha\beta I}^T)\boldsymbol{d}_M^{\alpha\beta}=\boldsymbol{f}_I \tag{54}$$

where $\boldsymbol{f}_I$'s are the components of the traditional force vector:

$$\boldsymbol{f}_I=\int_{\Gamma_t}\Psi_I\bar{\boldsymbol{t}}d\Gamma-\int_{\Gamma_M}\Psi_{I,\gamma}n^{\gamma}\bar{M}_{\boldsymbol{nn}}d\Gamma+[[\Psi_I\boldsymbol{a}_3\bar{P}]]_{\boldsymbol{x}\in C_P}+\int_{\Omega}\Psi_I\bar{\boldsymbol{b}}d\Omega \tag{55}$$

and further substituting coefficients $\boldsymbol{d}_N^{\alpha}$, $\boldsymbol{d}_M^{\alpha\beta}$ into Eq. (54) gives the final discrete equilibrium equations:

$$\begin{aligned}
&-\sum_{C=1}^{n_c}(\tilde{\boldsymbol{g}}_{\alpha I}^T-\bar{\boldsymbol{g}}_{\alpha I}^T)\boldsymbol{d}_N^{\alpha}-\sum_{C=1}^{n_c}(\tilde{\boldsymbol{g}}_{\alpha\beta I}^T-\bar{\boldsymbol{g}}_{\alpha\beta I}^T)\boldsymbol{d}_M^{\alpha\beta}\\
&=\sum_{C=1}^{n_e}\sum_{J=1}^{n_p}\begin{pmatrix}
\boldsymbol{a}_\alpha\tilde{\boldsymbol{g}}_{\beta I}^T hC^{\alpha\beta\gamma\eta}\boldsymbol{a}_\gamma\tilde{\boldsymbol{g}}_{\eta J}+\tilde{\boldsymbol{g}}_{\alpha\beta I}^T\boldsymbol{a}_3\dfrac{h^3}{12}C^{\alpha\beta\gamma\eta}\boldsymbol{a}_3\tilde{\boldsymbol{g}}_{\gamma\eta}\\
-\boldsymbol{a}_\alpha\bar{\boldsymbol{g}}_{\beta I}^T hC^{\alpha\beta\gamma\eta}\boldsymbol{a}_\gamma\tilde{\boldsymbol{g}}_{\eta J}-\boldsymbol{a}_\alpha\tilde{\boldsymbol{g}}_{\beta I}^T hC^{\alpha\beta\gamma\eta}\boldsymbol{a}_\gamma\bar{\boldsymbol{g}}_{\eta J}\\
-\bar{\boldsymbol{g}}_{\alpha\beta I}^T\boldsymbol{a}_3\dfrac{h^3}{12}C^{\alpha\beta\gamma\eta}\boldsymbol{a}_3\tilde{\boldsymbol{g}}_{\gamma\eta J}-\tilde{\boldsymbol{g}}_{\alpha\beta I}^T\boldsymbol{a}_3\dfrac{h^3}{12}C^{\alpha\beta\gamma\eta}\boldsymbol{a}_3\bar{\boldsymbol{g}}_{\gamma\eta J}\\
+\boldsymbol{a}_\alpha\tilde{\boldsymbol{g}}_{\beta I}^T hC^{\alpha\beta\gamma\eta}\boldsymbol{a}_\gamma\hat{\boldsymbol{g}}_{\eta J}-\boldsymbol{a}_\alpha\bar{\boldsymbol{g}}_{\beta I}^T hC^{\alpha\beta\gamma\eta}\boldsymbol{a}_\gamma\hat{\boldsymbol{g}}_{\eta J}\\
+\tilde{\boldsymbol{g}}_{\alpha\beta I}^T\boldsymbol{a}_3\dfrac{h^3}{12}C^{\alpha\beta\gamma\eta}\boldsymbol{a}_3\hat{\boldsymbol{g}}_{\gamma\eta J}-\bar{\boldsymbol{g}}_{\alpha\beta I}^T\boldsymbol{a}_3\dfrac{h^3}{12}C^{\alpha\beta\gamma\eta}\boldsymbol{a}_3\hat{\boldsymbol{g}}_{\gamma\eta J}\\
+\boldsymbol{a}_\alpha\bar{\boldsymbol{g}}_{\beta I}^T hC^{\alpha\beta\gamma\eta}\boldsymbol{a}_\gamma\bar{\boldsymbol{g}}_{\eta J}+\bar{\boldsymbol{g}}_{\alpha\beta I}^T\boldsymbol{a}_3\dfrac{h^3}{12}C^{\alpha\beta\gamma\eta}\boldsymbol{a}_3\bar{\boldsymbol{g}}_{\gamma\eta J}
\end{pmatrix}\\
&=\sum_{J=1}^{n_p}(\boldsymbol{K}_{IJ}+\tilde{\boldsymbol{K}}_{IJ}+\bar{\boldsymbol{K}}_{IJ})\cdot\boldsymbol{d}_J-\tilde{\boldsymbol{f}}_I-\bar{\boldsymbol{f}}_I
\end{aligned} \tag{56}$$

where

$$\boldsymbol{K}_{IJ}=\int_{\Omega}\tilde{\boldsymbol{\varepsilon}}_{\alpha\beta I}\tilde{\boldsymbol{N}}_J^{\alpha\beta}d\Omega+\int_{\Omega}\tilde{\boldsymbol{\kappa}}_{\alpha\beta I}\tilde{\boldsymbol{M}}_J^{\alpha\beta}d\Omega \tag{57}$$

$$\begin{aligned}
\tilde{\boldsymbol{K}}_{IJ}=&-\int_{\Gamma_v}(\Psi_I\tilde{\boldsymbol{t}}_J+\tilde{\boldsymbol{t}}_I\Psi_J)d\Gamma\\
&+\int_{\Gamma_\theta}(\Psi_{I,\gamma}n^{\gamma}\boldsymbol{a}_3\tilde{M}_{\boldsymbol{nn}J}+\boldsymbol{a}_3\tilde{M}_{\boldsymbol{nn}I}\Psi_{I,\gamma}n^{\gamma})d\Gamma\\
&+([[\Psi_I\boldsymbol{a}_3P_J]]+[[P_I\boldsymbol{a}_3\Psi_J]])_{\boldsymbol{x}\in C_v}
\end{aligned} \tag{58a}$$

$$\tilde{\boldsymbol{f}}_I=-\int_{\Gamma_v}\tilde{\boldsymbol{t}}_I\cdot\bar{\boldsymbol{v}}d\Gamma+\int_{\Gamma_\theta}\tilde{M}_{\boldsymbol{nn}}\bar{\theta}_{\boldsymbol{n}}d\Gamma+[[\tilde{P}_I\boldsymbol{a}_3\cdot\bar{\boldsymbol{v}}]]_{\boldsymbol{x}\in C_v} \tag{58b}$$

$$\bar{\boldsymbol{K}}_{IJ} = -\int_{\Gamma_v} \bar{\boldsymbol{t}}_I \Psi_J d\Gamma + \int_{\Gamma_\theta} \boldsymbol{a}_3 \bar{M}_{\boldsymbol{nn}I} \Psi_{J,\gamma} n^\gamma d\Gamma + [[\bar{P}_I \boldsymbol{a}_3 \Psi_J]]_{\boldsymbol{x}\in C_v} \tag{59a}$$

$$\bar{\boldsymbol{f}}_I = -\int_{\Gamma_v} \bar{\boldsymbol{t}}_I \cdot \bar{\boldsymbol{v}} d\Gamma + \int_{\Gamma_\theta} \bar{M}_{\boldsymbol{nn}} \bar{\theta}_{\boldsymbol{n}} d\Gamma + [[\bar{P}_I \boldsymbol{a}_3 \cdot \bar{\boldsymbol{v}}]]_{\boldsymbol{x}\in C_v} \tag{59b}$$

The detailed derivations of Eqs (57)-(59) are listed in the Appendix. As shown in these equations, the Eq. (57) is the conventional stiffness matrix evaluated by smoothed gradients $\tilde{\Psi}_{I,\alpha}$, $\tilde{\Psi}_{I,\alpha}|_\beta$, and the Eqs. (58) and (59) contribute for the enforcement of essential boundary.

4.2. Comparison with Nitsche's method

The Nitsche's method for enforcing essential boundary can be regarded as a combination of Lagrangian multiplier method and penalty method, in which the Lagrangian multiplier is represented by the approximated displacement. The corresponding total potential energy functional Π_P is given by:

$$\begin{aligned}
\Pi_P(\boldsymbol{v}) = & \int_\Omega \frac{1}{2}\varepsilon_{\alpha\beta} N^{\alpha\beta} d + \int_\Omega \frac{1}{2}\kappa_{\alpha\beta} M^{\alpha\beta} d\Omega \\
& - \int_{\Gamma_t} \boldsymbol{v} \cdot \bar{\boldsymbol{t}} d\Gamma + \int_{\Gamma_M} \boldsymbol{v}_{,\gamma} n^\gamma \boldsymbol{a}_3 M_{\boldsymbol{nn}} d\Gamma + (\boldsymbol{v} \cdot \boldsymbol{a}_3 P)_{\boldsymbol{x}\in C_P} - \int_\Omega \boldsymbol{v} \cdot \bar{\boldsymbol{b}} d\Omega \\
& \underbrace{- \int_{\Gamma_v} \boldsymbol{t} \cdot (\boldsymbol{v} - \bar{\boldsymbol{v}}) d\Gamma + \int_{\Gamma_\theta} M_{\boldsymbol{nn}} (\theta_{\boldsymbol{n}} - \bar{\theta}_{\boldsymbol{n}}) d\Gamma + (P\boldsymbol{a}_3 \cdot (\boldsymbol{v} - \bar{\boldsymbol{v}}))_{\boldsymbol{x}\in C_v}}_{\text{consistent term}} \\
& \underbrace{+ \frac{\alpha_v}{2} \int_{\Gamma_v} \boldsymbol{v} \cdot \boldsymbol{v} d\Gamma + \frac{\alpha_\theta}{2} \int_{\Gamma_\theta} \theta_{\boldsymbol{n}}^2 d\Gamma + \frac{\alpha_C}{2} (\boldsymbol{v} \cdot \boldsymbol{v})_{\boldsymbol{x}\in C_v}}_{\text{stabilized term}}
\end{aligned} \tag{60}$$

where the consistent term rephrased from Lagrangian multiplier method contributes to enforce the essential boundary and meet the variational consistency condition. However the consistent term can not always ensure the coercivity of stiffness, so the penalty method is introduced to be regarded as a stabilized term. With a standard variational argument, the corresponding weak form can be stated as:

$$
\begin{aligned}
\delta\Pi_P(\boldsymbol{v}) = &\int_\Omega \delta\varepsilon_{\alpha\beta}N^{\alpha\beta}d + \int_\Omega \delta\kappa_{\alpha\beta}M^{\alpha\beta}d\Omega \\
&- \int_{\Gamma_t} \delta\boldsymbol{v}\cdot\bar{\boldsymbol{t}}d\Gamma + \int_{\Gamma_M} \delta\boldsymbol{v}_{,\gamma}n^\gamma\boldsymbol{a}_3M_{\boldsymbol{nn}}d\Gamma + (\delta\boldsymbol{v}\cdot\boldsymbol{a}_3P)_{\boldsymbol{x}\in C_P} - \int_\Omega \delta\boldsymbol{v}\cdot\bar{\boldsymbol{b}}d\Omega \\
&- \int_{\Gamma_v} \delta\boldsymbol{v}\cdot\boldsymbol{t}d\Gamma + \int_{\Gamma_\theta} \delta\theta_{\boldsymbol{n}}M_{\boldsymbol{nn}}d\Gamma + (\boldsymbol{v}\cdot\boldsymbol{a}_3P)_{\boldsymbol{x}\in C_v} \\
&- \int_{\Gamma_v} \delta\boldsymbol{t}\cdot(\boldsymbol{v}-\bar{\boldsymbol{v}})d\Gamma + \int_{\Gamma_\theta} \delta M_{\boldsymbol{nn}}(\theta_{\boldsymbol{n}}-\bar{\theta}_{\boldsymbol{n}})d\Gamma + (\delta P\boldsymbol{a}_3\cdot(\boldsymbol{v}-\bar{\boldsymbol{v}}))_{\boldsymbol{x}\in C_v} \\
&+ \alpha_v\int_{\Gamma_v} \delta\boldsymbol{v}\cdot\boldsymbol{v}d\Gamma + \alpha_\theta\int_{\Gamma_\theta} \delta\theta_{\boldsymbol{n}}\theta_{\boldsymbol{n}}d\Gamma + \alpha_C(\delta\boldsymbol{v}\cdot\boldsymbol{v})_{\boldsymbol{x}\in C_v} \\
&= 0
\end{aligned}
\tag{61}
$$

in which α_v, α_θ and α_C are experimental artificial parameters. Further invoking the conventional reproducing kernel approximation of Eq. (22) leads to the following discrete equilibrium equations:

$$
\sum_{J=1}^{n_p}(\boldsymbol{K}_{IJ}+\boldsymbol{K}^c_{IJ}+\boldsymbol{K}^s_{IJ})\boldsymbol{d}_J = \boldsymbol{f}_I + \boldsymbol{f}^c + \boldsymbol{f}^s \tag{62}
$$

where the stiffness $\boldsymbol{K}_{IJ}$ is identical with Eq. (57). $\boldsymbol{K}^c_{IJ}$ and $\boldsymbol{K}^s_{IJ}$ are the stiffness matrix for consistent and stabilized terms respectively, and have the following forms:

$$
\begin{aligned}
\boldsymbol{K}^c_{IJ} = &-\int_{\Gamma_v}\Big((\boldsymbol{\mathcal{T}}^\alpha\Psi_{I,\alpha}+\boldsymbol{\mathcal{V}}^{\alpha\beta}\Psi_{I,\alpha}|_\beta)\Psi_J + \Psi_I(\boldsymbol{\mathcal{T}}^\alpha\Psi_{J,\alpha}+\boldsymbol{\mathcal{V}}^{\alpha\beta}\Psi_{J,\alpha}|_\beta)\Big)\,d\Gamma \\
&+\int_{\Gamma_M}(\mathcal{M}^{\alpha\beta}\Psi_{I,\alpha}|_\beta\boldsymbol{a}_3\Psi_{J,\gamma}n^\gamma + \Psi_{I,\gamma}n^\gamma\boldsymbol{a}_3\mathcal{M}^{\alpha\beta}\Psi_{I,\alpha}|_\beta)d\Gamma
\end{aligned}
\tag{63a}
$$

5. Numerical examples

In this section, several examples are carried out to verify proposed method, which employs the consistent reproducing kernel gradient smoothing integration scheme (RKGSI) and the non-consistent Gauss integration scheme (GI) with penalty method, Nitsche's method and the proposed Hu-Washizu formulation (HW) to enforce the essential boundary conditions. A normalized support size of 2.5 is used for all methods to ensure the requirement of quadratic base meshfree approximation. To eliminate the influence of integration, the Gauss integration scheme use 6 Gauss points for domain integration and 3 points for boundary integration, and such that the number of integration points are identical between Gauss scheme and RKGSI scheme. The error estimates of displacement namely L_2-Error and energy namely H_e-Error is used here:

$$
\begin{aligned}
L_2\text{-Error} &= \frac{\sqrt{\int_\Omega (\boldsymbol{v}-\boldsymbol{v}^h)\cdot(\boldsymbol{v}-\boldsymbol{v}^h) d\Omega}}{\sqrt{\boldsymbol{v}\cdot\boldsymbol{v}}} \\
H_e\text{-Error} &= \frac{\sqrt{\int_\Omega \left((\varepsilon_{\alpha\beta}-\varepsilon^h_{\alpha\beta})(N^{\alpha\beta}-N^{\alpha\beta h}) + \int_\Omega (\kappa_{\alpha\beta}-\kappa^h_{\alpha\beta})(M^{\alpha\beta}-M^{\alpha\beta h}) \right) d\Omega}}{\sqrt{\int_\Omega (\varepsilon_{\alpha\beta}N^{\alpha\beta}+\kappa_{\alpha\beta}M^{\alpha\beta}) d\Omega}}
\end{aligned}
\tag{64}
$$

5.1. Patch tests

The linear and quadratic patch tests for flat and curved thin shell are firstly study to verify the variational consistency of the proposed method. As shown in Fig. 1, the flat and curved model is depicted by an identical parametric domain $\Omega = (0,1)\otimes(0,1)$, where the cylindrical coordinate system with radius $R = 1$ is employed to describe the curved model, and the whole domain Ω is discretized by 165 meshfree nodes. All the boundaries are enforced as essential boundary conditions with the following manufactured exact solution:

$$
\boldsymbol{v} = \begin{Bmatrix} (\xi^1+2\xi^2)^n \\ (3\xi^1+4\xi^2)^n \\ (5\xi^1+6\xi^2)^n \end{Bmatrix}, \quad n = \begin{cases} 1 & \text{Linear patch test} \\ 2 & \text{Quadratic patch test} \end{cases}
\tag{65}
$$

Figure 1: Meshfree discretization for patch test

Table 1 lists the L_2- and H_e-Error results of patch test with flat model, where the RKGSI with variational consistent essential boundary enforcement, i.e. RKGSI-Nitsche and RKGSI-HW, can pass the linear and quadratic path test. Due to the loss of variational consistency condition, even with Nitsche's method, Gauss meshfree formulations show noticeable errors. Table 2 shows the results for curved model, which indicated that all the mehtods cannot pass the patch test, which mainly because the proposed smoothed gradient of Eqs. (35),

(36) is unable to exactly reproduce the non-polynomial membrane and bending stress. However, the RKGSI-HW and RKGSI-Nitsche also performance better accuracy than other methods due to the fulfillment of first two order variational consistency. Meanwhile, the bending moment contours of M^{12} are listed in Fig. 3, which further verify that the proposed method obtain a satisfactory result comparing with exact solution, the conventional Gauss meshree formulations show observable errors.

Table 1: Results of patch test for flat model

	Linear patch test		Quadratic patch test	
	L_2-Error	H_e-Error	L_2-Error	H_e-Error
GI-Penalty	$4.45E-4$	$1.35E-2$	$2.01E-3$	$1.63E-2$
GI-Nitsche	$4.51E-4$	$1.42E-2$	$1.22E-3$	$1.68E-2$
RKGSI-Penalty	$3.64E-9$	$6.77E-8$	$4.54E-9$	$6.57E-8$
RKGSI-Nitsche	$3.31E-12$	$1.34E-11$	$5.98E-12$	$1.21E-11$
RKGSI-HR	$6.67E-13$	$1.50E-11$	$1.07E-12$	$1.26E-11$

Table 2: Results of patch test for curved model.

	Linear patch test		Quadratic patch test	
	L_2-Error	H_e-Error	L_2-Error	H_e-Error
GI-Penalty	$3.79E-4$	$1.30E-2$	$1.74E-3$	$1.37E-2$
GI-Nitsche	$4.04E-4$	$1.42E-2$	$1.15E-3$	$1.49E-2$
RKGSI-Penalty	$1.47E-4$	$5.39E-3$	$2.26E-4$	$2.09E-3$
RKGSI-Nitsche	$2.41E-6$	$7.37E-5$	$2.47E-6$	$2.89E-5$
RKGSI-HR	$4.28E-6$	$1.30E-4$	$9.69E-6$	$2.41E-4$

Figure 2: Contour plots of M^{12} for curved shell patch test.

5.2. Scordelis-Lo roof

This example consider the classical Scordelis-Lo roof problem, as shown in Fig., the cylindrical roof has the radius $R = 25$, length $L = 50$, thickness $h = 0.25$, Young's modulus $E = 4.32 \times 10^8$ and Poisson rate $\nu = 0.0$. An uniform body force of $b_z = -90$ are enforced in whole roof and the curved edges are enforced by $v_x = v_z = 0$, and the straight edges are free.

Due to the symmetry, only a quadrant of the model is considered for meshfree analysis, which is discretized by the 5×8, 11×16, 17×24 and 23×32 meshfree nodes.

Figure 3: Description of Scordelis-Lo roof problem.

6. Conclusion

Appendix A. Green's theorems for in-plane vector

This Appendix discuss two kinds of Green's theorems used for the development of the method. For an arbitrary vector v^α and a scalar function f, with the Green's theorem for in-plane vector, the first Green's theorem is list as follow [**?**]:

$$\begin{aligned}\int_\Omega f_{,\alpha} v^\alpha d\Omega &= \int_\Gamma f v^\alpha n_\alpha d\Gamma - \int_\Omega f(v^\alpha_{,\alpha} + \Gamma^\beta_{\beta\alpha} v^\alpha) d\Omega \\ &= \int_\Gamma f v^\alpha n_\alpha d\Gamma - \int_\Omega f v^\alpha|_\alpha d\Omega\end{aligned} \tag{A.1}$$

where $\Gamma^\gamma_{\alpha\beta} = \boldsymbol{a}_{\alpha,\beta} \cdot \boldsymbol{a}^\gamma$ denotes the Christoffel symbol of the second kind. $v^\alpha|_\alpha$ can be regarded as the in-plane covariant derivative of the vector v^α:

$$v^\alpha|_\alpha = v^\alpha_{,\alpha} + \Gamma^\beta_{\beta\alpha} v^\alpha \tag{A.2}$$

The second Green's theorem is established with a mixed form of second order derivative, let $A^{\alpha\beta}$ be an arbitrary symmetric second order tensor, the Green's theorem yields [**?**]:

$$\begin{aligned}\int_\Omega f_{,\alpha}|_\beta A^{\alpha\beta} d\Omega = &\int_\Gamma f_{,\gamma} n^\gamma A^{\alpha\beta} n_\alpha n_\beta d\Gamma - \int_\Gamma f(A^{\alpha\beta} s_\alpha n_\beta)_{,\gamma} s^\gamma d\Gamma + [[f A^{\alpha\beta} s_\alpha n_\beta]]_{\boldsymbol{x}\in C} \\ &- \int_\Gamma f(A^{\alpha\beta}_{,\beta} n_\alpha + \Gamma^\gamma_{\alpha\beta} A^{\alpha\beta} n_\gamma + \Gamma^\gamma_{\gamma\beta} A^{\alpha\beta} n_\alpha) d\Gamma \\ &+ \int_\Omega f \begin{pmatrix} \Gamma^\gamma_{\alpha\beta,\gamma} A^{\alpha\beta} + \Gamma^\gamma_{\alpha\beta} A^{\alpha\beta}_{,\gamma} + \Gamma^\eta_{\eta\gamma} \Gamma^\gamma_{\alpha\beta} A^{\alpha\beta} \\ + A^{\alpha\beta}_{,\alpha\beta} + \Gamma^\gamma_{\gamma\beta,\alpha} A^{\alpha\beta} + 2\Gamma^\gamma_{\gamma\alpha} A^{\alpha\beta}_{,\beta} + \Gamma^\gamma_{\gamma\alpha} \Gamma^\eta_{\eta\beta} A^{\alpha\beta} \end{pmatrix} d\Omega \\ = &\int_\Gamma f_{,\gamma} n^\gamma A^{\alpha\beta} n_\alpha n_\beta d\Gamma - \int_\Gamma f(A^{\alpha\beta} s_\alpha n_\beta)_{,\gamma} s^\gamma d\Gamma + [[f A^{\alpha\beta} s_\alpha n_\beta]]_{\boldsymbol{x}\in C} \\ &- \int_\Gamma f A^{\alpha\beta}|_\beta n_\alpha d\Gamma + \int_\Omega f A^{\alpha\beta}|_{\alpha\beta} d\Omega\end{aligned} \tag{A.3}$$

with

$$A^{\alpha\beta}|_\beta = A^{\alpha\beta}_{,\beta} + \Gamma^\alpha_{\beta\gamma} A^{\beta\gamma} + \Gamma^\gamma_{\gamma\beta} A^{\alpha\beta} \tag{A.4}$$

$$\begin{aligned}A^{\alpha\beta}|_{\alpha\beta} = &\Gamma^\gamma_{\alpha\beta,\gamma} A^{\alpha\beta} + \Gamma^\gamma_{\alpha\beta} A^{\alpha\beta}_{,\gamma} + \Gamma^\eta_{\eta\gamma} \Gamma^\gamma_{\alpha\beta} A^{\alpha\beta} \\ &+ A^{\alpha\beta}_{,\alpha\beta} + \Gamma^\gamma_{\gamma\beta,\alpha} A^{\alpha\beta} + 2\Gamma^\gamma_{\gamma\alpha} A^{\alpha\beta}_{,\beta} + \Gamma^\gamma_{\gamma\alpha} \Gamma^\eta_{\eta\beta} A^{\alpha\beta}\end{aligned} \tag{A.5}$$

For the sake of brevity, the notion of covariant derivative is extended to scalar function as:

$$f_{|\alpha} = f_{,\alpha} + \Gamma^\beta_{\beta\alpha} f \tag{A.6}$$

$$f_{|\beta} n_\alpha = f_{,\beta} n_\alpha + \Gamma^\gamma_{\alpha\beta} f n_\gamma + \Gamma^\gamma_{\gamma\beta} f n_\alpha \tag{A.7}$$

$$\begin{aligned}f_{|\alpha\beta} = &\Gamma^\gamma_{\alpha\beta,\gamma} f + \Gamma^\gamma_{\alpha\beta} f_{,\gamma} + \Gamma^\eta_{\eta\gamma} \Gamma^\gamma_{\alpha\beta} f \\ &+ f_{,\alpha\beta} + \Gamma^\gamma_{\gamma\beta,\alpha} f + 2\Gamma^\gamma_{\gamma\alpha} f_{,\beta} + \Gamma^\gamma_{\gamma\alpha} \Gamma^\eta_{\eta\beta} f\end{aligned} \tag{A.8}$$

Appendix B. Derivations for stiffness metrics and force vectors

This Appendix details the derivations of stiffness

References

[1] H. Dah-wei, A method for establishing generalized variational principle 6 (6) 501–509. `doi:10.1007/BF01876390`.
URL `http://link.springer.com/10.1007/BF01876390`