1. Introduction

2. Hu-Washizu's formulation of complementary energy for thin shell

2.1. Kinematics for thin shell

$$\varepsilon_{\alpha\beta} = \frac{1}{2}(\boldsymbol{a}_\alpha \cdot \boldsymbol{v}_{,\beta} + \boldsymbol{a}_\beta \cdot \boldsymbol{v}_{,\alpha}) \tag{1}$$

$$\theta_{\boldsymbol{n}} = \boldsymbol{a}_3 \cdot \boldsymbol{v}_{,\alpha} n^\alpha \tag{2}$$

$$\kappa_{\alpha\beta} = (\Gamma^\gamma_{\alpha\beta}\boldsymbol{v}_{,\gamma} - \boldsymbol{v}_{,\alpha\beta}) \cdot \boldsymbol{a}_3 = -\boldsymbol{v}_{,\alpha}|_\beta \cdot \boldsymbol{a}_3 \tag{3}$$

$$\boldsymbol{t} = \boldsymbol{t}_N + \boldsymbol{t}_M \tag{4}$$

$$\boldsymbol{t}_N = \boldsymbol{a}_\alpha N^{\alpha\beta} n_\beta \tag{5}$$

$$\boldsymbol{t}_M = (\boldsymbol{a}_3 M^{\alpha\beta})|_\beta n_\alpha + (\boldsymbol{a}_3 M^{\alpha\beta} s_\alpha n_\beta)_{,\gamma} s^\gamma \tag{6}$$

$$M_{\boldsymbol{nn}} = M^{\alpha\beta} n_\alpha n_\beta \tag{7}$$

$$\boldsymbol{b} = \boldsymbol{b}_N + \boldsymbol{b}_M \tag{8}$$

$$\boldsymbol{b}_N = (\boldsymbol{a}_\alpha N^{\alpha\beta})|_\beta \tag{9}$$

$$\boldsymbol{b}_M = (\boldsymbol{a}_3 M^{\alpha\beta})_{,\alpha}|_\beta \tag{10}$$

$$P = -[[M^{\alpha\beta} s_\alpha n_\beta]] \tag{11}$$

2.2. Galerkin weak form for Hu-Washizu principle of complementary energy

In accordance with the Hu-Washizu variational principle of complementary energy [1], the corresponding complementary functional, denoted by Π, is listed as follow:

$$\begin{aligned}
&\Pi(\boldsymbol{v}, \varepsilon_{\alpha\beta}, \kappa_{\alpha\beta}, N^{\alpha\beta}, M^{\alpha\beta}) \\
&= \int_\Omega \frac{h}{2}\varepsilon_{\alpha\beta} C^{\alpha\beta\gamma\eta}\varepsilon_{\gamma\eta} d\Omega + \int_\Omega \frac{h^3}{24}\kappa_{\alpha\beta} C^{\alpha\beta\gamma\eta}\kappa_{\gamma\eta} d\Omega \\
&+ \int_\Omega \varepsilon_{\alpha\beta}(N^{\alpha\beta} - hC^{\alpha\beta\gamma\eta}\varepsilon_{\gamma\eta}) d\Omega + \int_\Omega \kappa_{\alpha\beta}(M^{\alpha\beta} - \frac{h^3}{12}C^{\alpha\beta\gamma\eta}\kappa_{\gamma\eta}) d\Omega \\
&- \int_{\Gamma_v} \boldsymbol{t} \cdot \bar{\boldsymbol{v}} d\Gamma + \int_{\Gamma_\theta} M_{\boldsymbol{nn}} \bar{\theta}_{\boldsymbol{n}} d\Gamma - (P\boldsymbol{a}_3 \cdot \bar{\boldsymbol{v}})_{\boldsymbol{x}\in C_w} \\
&+ \int_{\Gamma_M} \theta_{\boldsymbol{n}}(M_{\boldsymbol{nn}} - \bar{M}_{\boldsymbol{nn}}) d\Gamma - \int_{\Gamma_t} \boldsymbol{v} \cdot (\boldsymbol{t} - \bar{\boldsymbol{t}}) d\Gamma - \boldsymbol{v} \cdot \boldsymbol{a}_3 (P - \bar{P})_{\boldsymbol{x}\in C_P} \\
&- \int_\Omega \boldsymbol{v} \cdot (\boldsymbol{b} - \bar{\boldsymbol{b}}) d\Omega
\end{aligned} \tag{12}$$

Introducing a standard variational argument to Eq. (12), $\delta\Pi = 0$, and considering the arbitrariness of virtual variables, $\delta\boldsymbol{v}$, $\delta\varepsilon_{\alpha\beta}$, $\delta\kappa_{\alpha\beta}$, $N^{\alpha\beta}$, $M^{\alpha\beta}$ lead to the following weak form:

$$-\int_\Omega h\delta\varepsilon_{\alpha\beta}C^{\alpha\beta\gamma\eta}\varepsilon_{\gamma\eta}d\Omega + \int_\Omega \delta\varepsilon_{\alpha\beta}N^{\alpha\beta}d\Omega = 0 \tag{13a}$$

$$-\int_\Omega \frac{h^3}{12}\delta\kappa_{\alpha\beta}C^{\alpha\beta\gamma\eta}\kappa_{\gamma\eta}d\Omega + \int_\Omega \delta\kappa_{\alpha\beta}M^{\alpha\beta}d\Omega = 0 \tag{13b}$$

$$\begin{aligned}\int_\Omega \delta N^{\alpha\beta}\varepsilon_{\alpha\beta}d\Omega - \int_\Gamma \delta\boldsymbol{t}_N\cdot\boldsymbol{v}d\Gamma + \int_\Omega \delta\boldsymbol{b}_N\cdot\boldsymbol{v}d\Omega \\ + \int_{\Gamma_v}\delta\boldsymbol{t}_N\cdot\boldsymbol{v}d\Gamma = \int_{\Gamma_v}\delta\boldsymbol{t}_N\cdot\bar{\boldsymbol{v}}d\Gamma\end{aligned} \tag{13c}$$

$$\begin{aligned}\int_\Omega \delta M^{\alpha\beta}\kappa_{\alpha\beta}d\Omega - \int_\Gamma \delta M_{\boldsymbol{nn}}\theta_{\boldsymbol{n}}d\Gamma + \int_\Gamma \delta\boldsymbol{t}_M\cdot\boldsymbol{v}d\Gamma + (\delta P\boldsymbol{a}_3\cdot\boldsymbol{v})_{\boldsymbol{x}\in C} + \int_\Omega \delta\boldsymbol{b}_M\cdot\boldsymbol{v}d\Omega \\ + \int_{\Gamma_\theta}\delta M_{\boldsymbol{nn}}\theta_{\boldsymbol{n}}d\Gamma - \int_{\Gamma_v}\delta\boldsymbol{t}_M\cdot\boldsymbol{v}d\Gamma - (\delta P\boldsymbol{a}_3\cdot\boldsymbol{v})_{\boldsymbol{x}\in C_v} \\ = \int_{\Gamma_\theta}\delta M_{\boldsymbol{nn}}\bar{\theta}_{\boldsymbol{n}}d\Gamma - \int_{\Gamma_v}\delta\boldsymbol{t}_M\cdot\bar{\boldsymbol{v}}d\Gamma - (\delta P\boldsymbol{a}_3\cdot\bar{\boldsymbol{v}})_{\boldsymbol{x}\in C_v}\end{aligned} \tag{13d}$$

$$\begin{aligned}\int_\Gamma \delta\theta_{\boldsymbol{n}}M_{\boldsymbol{nn}}d\Gamma - \int_\Gamma \delta\boldsymbol{v}\cdot\boldsymbol{t}d\Gamma - (\delta\boldsymbol{v}\cdot\boldsymbol{a}_3P)_{\boldsymbol{x}\in C} + \int_\Omega \delta\boldsymbol{v}\cdot\boldsymbol{b}d\Omega \\ -\int_{\Gamma_\theta}\delta\theta_{\boldsymbol{n}}M_{\boldsymbol{nn}}d\Gamma + \int_{\Gamma_v}\delta\boldsymbol{v}\cdot\boldsymbol{t}d\Gamma + (\delta\boldsymbol{v}\cdot\boldsymbol{a}_3P)_{\boldsymbol{x}\in C_v} = -\int_{\Gamma_t}\delta\boldsymbol{v}\cdot\bar{\boldsymbol{t}}d\Gamma - \int_\Omega \delta\boldsymbol{v}\cdot\bar{\boldsymbol{b}}d\Omega\end{aligned} \tag{13e}$$

where the geometric relationships of Eq. () is used herein.

3. Mixed meshfree formulation for modified Hellinger-Reissner weak form

3.1. Reproducing kernel approximation for displacement

In this study, the displacement is approximated by traditional reproducing kernel approximation. As shown in Fig,

$$\boldsymbol{v}(\boldsymbol{\xi}) = \sum_{I=1}^{n_p} \Psi_I(\boldsymbol{\xi}) \boldsymbol{d}_I \tag{14}$$

3.2. Reproducing kernel gradient smoothing approximation for effective stress and strain

In Galerkin meshfree formulation, the mid-plane of thin shell Ω is split by a set of integration cells Ω_C's, $\cup_{C=1}^{n_e} \Omega_C \approx \Omega$. With the inspiration of reproducing kernel smoothing framework, the Cartesian and covariant derivatives of displacement, $\boldsymbol{v}_{,\alpha}$ and $-\boldsymbol{v}_{,\alpha}|_\beta$, in strains $\varepsilon_{\alpha\beta}$, $\kappa_{\alpha\beta}$ are approximated by $(p-1)$-th order polynomials in each integration cells. In integration cell Ω_C, the approximated derivatives and strains denoted by $\boldsymbol{v}^h_{,\alpha}$, $\varepsilon^h_{\alpha\beta}$ and $-\boldsymbol{v}^h_{,\alpha}|_\beta$, $\kappa^h_{\alpha\beta}$ can be expressed by:

$$\boldsymbol{v}^h_{,\alpha}(\boldsymbol{\xi}) = \boldsymbol{q}^T(\boldsymbol{\xi}) \boldsymbol{d}^\varepsilon_\alpha, \quad \varepsilon^h_{\alpha\beta}(\boldsymbol{\xi}) = \boldsymbol{q}^T(\boldsymbol{\xi}) \frac{1}{2} (\boldsymbol{a}_\alpha \cdot \boldsymbol{d}^\varepsilon_\beta + \boldsymbol{a}_\beta \cdot \boldsymbol{d}^\varepsilon_\alpha) \tag{15}$$

$$-\boldsymbol{v}^h_{,\alpha}|_\beta(\boldsymbol{\xi}) = \boldsymbol{q}^T(\boldsymbol{\xi}) \boldsymbol{d}^\kappa_{\alpha\beta}, \quad \kappa^h_{\alpha\beta}(\boldsymbol{\xi}) = \boldsymbol{q}^T(\boldsymbol{\xi}) \boldsymbol{a}_3 \cdot \boldsymbol{d}^\kappa_{\alpha\beta} \tag{16}$$

where $\boldsymbol{q}$ is the $(p-1)$th order polynomial vector and has the following form:

$$\boldsymbol{q} = \{1,\ \xi^1,\ \xi^2,\ \dots, (\xi^2)^{p-1}\}^T \tag{17}$$

and the $\boldsymbol{d}^\varepsilon_\alpha$, $\boldsymbol{d}^\kappa_{\alpha\beta}$ are the corresponding coefficient vector tensors. For the conciseness, the mixed usage of tensor and vector is introduced in this study, for example, the component of coefficient tensor vector $\boldsymbol{d}^\varepsilon_{\alpha I}$, $\boldsymbol{d}^\varepsilon_\alpha = \{\boldsymbol{d}^\varepsilon_{\alpha I}\}$, is a three dimensional tensor, $\dim \boldsymbol{d}^\varepsilon_{\alpha I} = \dim \boldsymbol{v}$.

In order to meet the integration constraint of thin shell problem, the approximated stresses $N^{\alpha\beta h}$, $M^{\alpha\beta h}$ are assumed to be a similar form with strains, yields:

$$N^{\alpha\beta h}(\boldsymbol{\xi}) = \boldsymbol{q}^T(\boldsymbol{\xi}) \boldsymbol{a}^\alpha \cdot \boldsymbol{d}^N_\beta, \quad \boldsymbol{a}_\alpha N^{\alpha\beta h}(\boldsymbol{\xi}) = \boldsymbol{q}^T(\boldsymbol{\xi}) \boldsymbol{d}^N_\beta \tag{18}$$

$$M^{\alpha\beta h}(\boldsymbol{\xi}) = \boldsymbol{q}^T(\boldsymbol{\xi}) \boldsymbol{a}_3 \cdot \boldsymbol{d}^M_{\alpha\beta}, \quad \boldsymbol{a}_3 M^{\alpha\beta h}(\boldsymbol{\xi}) = \boldsymbol{q}^T(\boldsymbol{\xi}) \boldsymbol{d}^M_{\alpha\beta} \tag{19}$$

substituting the approximations of Eqs. (14), (15), (16), (18), (19) into Eqs. (13c), (13d) can express $\boldsymbol{d}^\varepsilon_\beta$ and $\boldsymbol{d}^\kappa_{\alpha\beta}$ by $\boldsymbol{d}$ as:

$$\boldsymbol{d}^\varepsilon_\beta = \boldsymbol{G}^{-1} \left(\sum_{I=1}^{n_p} (\tilde{\boldsymbol{g}}_{\beta I} - \bar{\boldsymbol{g}}_{\beta I}) \boldsymbol{d}_I + \hat{\boldsymbol{g}}_\beta \right) \tag{20}$$

$$\boldsymbol{d}^\kappa_{\alpha\beta} = \boldsymbol{G}^{-1} \left(\sum_{I=1}^{n_p} (\tilde{\boldsymbol{g}}_{\alpha\beta I} - \bar{\boldsymbol{g}}_{\alpha\beta I}) \boldsymbol{d}_I + \hat{\boldsymbol{g}}_{\alpha\beta} \right) \tag{21}$$

with

$$\boldsymbol{G} = \int_{\Omega_C} \boldsymbol{q}^T \boldsymbol{q} d\Omega \tag{22}$$

$$\tilde{\boldsymbol{g}}_{\beta I} = \int_{\Gamma_C} \Psi_I \boldsymbol{q} n_\beta d\Gamma - \int_{\Omega_C} \Psi_I \boldsymbol{q}^*|_\beta d\Omega \tag{23a}$$

$$\bar{\boldsymbol{g}}_{\beta I} = \int_{\Gamma_C \cap \Gamma_v} \Psi_I \boldsymbol{q} n_\beta d\Gamma \tag{23b}$$

$$\hat{\boldsymbol{g}}_\beta = \int_{\Gamma_C \cap \Gamma_v} \boldsymbol{q} n_\beta \bar{\boldsymbol{v}} d\Gamma \tag{23c}$$

$$\begin{aligned}\tilde{\boldsymbol{g}}_{\alpha\beta I} = & \int_{\Gamma_C} \Psi_{I,\gamma} n^\gamma \boldsymbol{q} n_\alpha n_\beta d\Gamma - \int_{\Gamma_C} \Psi_I (\boldsymbol{q}^{**}|_\beta n_\alpha + (\boldsymbol{q} s_\alpha n_\beta)_{,\gamma} s^\gamma) d\Gamma \\ & + [[\Psi_I \boldsymbol{q} s_\alpha n_\beta]]_{\boldsymbol{x} \in C_C} - \int_{\Omega_C} \Psi \boldsymbol{q}^{**}_{,\alpha}|_\beta d\Omega\end{aligned} \tag{24a}$$

$$\begin{aligned}\bar{\boldsymbol{g}}_{\alpha\beta I} = & \int_{\Gamma_C \cap \Gamma_\theta} \Psi_{I,\gamma} n^\gamma \boldsymbol{q} n_\alpha n_\beta d\Gamma - \int_{\Gamma_C \cap \Gamma_v} \Psi_I (\boldsymbol{q}^{**}|_\beta n_\alpha + (\boldsymbol{q} s_\alpha n_\beta)_{,\gamma} s^\gamma) d\Gamma \\ & + [[\Psi_I \boldsymbol{q} s_\alpha n_\beta]]_{\boldsymbol{x} \in C_C \cap C_v}\end{aligned} \tag{24b}$$

$$\begin{aligned}\hat{\boldsymbol{g}}_{\alpha\beta} = & \int_{\Gamma_C \cap \Gamma_\theta} \boldsymbol{q} n_\alpha n_\beta \boldsymbol{a}_3 \bar{\theta}_{\boldsymbol{n}} d\Gamma - \int_{\Gamma_C \cap \Gamma_v} (\boldsymbol{q}^{**}|_\beta n_\alpha + (\boldsymbol{q} s_\alpha n_\beta)_{,\gamma} s^\gamma) \bar{\boldsymbol{v}} d\Gamma \\ & + [[\boldsymbol{q} s_\alpha n_\beta \bar{\boldsymbol{v}}]]_{\boldsymbol{x} \in C_C \cap C_v}\end{aligned} \tag{24c}$$

plugging Eqs. (20) and (21) back into Eqs. (15) and (16) respectively gives the final expression of $\boldsymbol{v}^h_{,\alpha}$, $\varepsilon^h_{\alpha\beta}$ and $-\boldsymbol{v}^h_{,\alpha\beta}$, $\boldsymbol{\kappa}^h_{\alpha\beta}$ as:

$$\boldsymbol{v}^h_{,\alpha} = \sum_{I=1}^{n_p} (\tilde{\Psi}_{I,\alpha} - \bar{\Psi}_{I,\alpha} \boldsymbol{d}_I + \boldsymbol{q}^T(\boldsymbol{\xi}) \boldsymbol{G}^{-1} \hat{\boldsymbol{g}}_\alpha \tag{25a}$$

$$\begin{aligned}\varepsilon^h_{\alpha\beta} = & \sum_{I=1}^{n_p} \frac{1}{2} \left(\boldsymbol{a}_\alpha (\tilde{\Psi}_{I,\beta} - \bar{\Psi}_{I,\beta}) + \boldsymbol{a}_\beta (\tilde{\Psi}_{I,\alpha} - \bar{\Psi}_{I,\alpha}) \right) \cdot \boldsymbol{d}_I \\ & + \boldsymbol{q}^T \boldsymbol{G}^{-1} \frac{1}{2} (\boldsymbol{a}_\alpha \cdot \hat{\boldsymbol{g}}_\beta + \boldsymbol{a}_\beta \cdot \hat{\boldsymbol{g}}_\alpha)\end{aligned} \tag{25b}$$

$$-\boldsymbol{v}^h_{,\alpha}|_\beta = \sum_{I=1}^{n_p} (\tilde{\Psi}_{I,\alpha\beta} - \bar{\Psi}_{I,\alpha\beta}) \boldsymbol{d}_I + \boldsymbol{q}^T(\boldsymbol{\xi}) \boldsymbol{G}^{-1} \hat{\boldsymbol{g}}_{\alpha\beta} \tag{26a}$$

$$\kappa^h_{\alpha\beta} = \sum_{I=1}^{n_p} (\tilde{\Psi}_{I,\alpha\beta} - \bar{\Psi}_{I,\alpha\beta}) \boldsymbol{a}_3 \cdot \boldsymbol{d}_I + \boldsymbol{q}^T(\boldsymbol{\xi}) \boldsymbol{G}^{-1} \boldsymbol{a}_3 \cdot \hat{\boldsymbol{g}}_{\alpha\beta} \tag{26b}$$

with

$$\tilde{\Psi}_{I,\alpha}(\boldsymbol{\xi}) = \boldsymbol{q}^T(\boldsymbol{\xi}) \boldsymbol{G}^{-1} \tilde{\boldsymbol{g}}_{\alpha I}, \quad \bar{\Psi}_{I,\alpha}(\boldsymbol{\xi}) = \boldsymbol{q}^T(\boldsymbol{\xi}) \boldsymbol{G}^{-1} \tilde{\boldsymbol{g}}_{\alpha I} \tag{27}$$

$$\tilde{\Psi}_{I,\alpha\beta}(\boldsymbol{\xi}) = \boldsymbol{q}^T(\boldsymbol{\xi})\boldsymbol{G}^{-1}\tilde{\boldsymbol{g}}_{\alpha\beta I}, \quad \bar{\Psi}_{I,\alpha\beta}(\boldsymbol{\xi}) = \boldsymbol{q}^T(\boldsymbol{\xi})\boldsymbol{G}^{-1}\tilde{\boldsymbol{g}}_{\alpha\beta I} \tag{28}$$

It is noted that, referring to reproducing kernel gradient smoothing framework [**?**], $\tilde{\Psi}_{I,\alpha}$, $\tilde{\Psi}_{I,\alpha\beta}$ are actually the first and second order smoothed gradients in curvilinear coordinates. $\tilde{\boldsymbol{g}}_{\alpha I}$ and $\tilde{\boldsymbol{g}}_{\alpha\beta I}$ are the right hand side integration constraints for first and second order gradients, then this formulation can meet the variational consistency for the p-th order polynomials. It should be known that, in curved model, the variational consistency for non-polynomial functions, like trigonometric functions, should be required for the polynomial solution. Even with p-th order variational consistency, the proposed formulation can not exactly reproduce the solution spanned by basis functions, however the accuracy of reproducing kernel smoothed gradients is still better that traditonal meshfree formulation, this will be evidenced by numerical examples in further section.

Furthermore, taking Eqs. (15) and (16) into Eqs.(13a) and (13b) can obtain the approximated effective stresses $N^{\alpha\beta h}$, $M^{\alpha\beta h}$ as:

$$\begin{aligned}\frac{1}{2}(\delta\boldsymbol{d}^{\varepsilon}_{\alpha}\cdot\boldsymbol{a}_{\beta}+\delta\boldsymbol{d}^{\varepsilon}_{\beta}\ \cdot\boldsymbol{a}_{\alpha})hC^{\alpha\beta\gamma\eta}\frac{1}{2}(\boldsymbol{a}_{\gamma}\cdot\boldsymbol{d}^{\varepsilon}_{\eta}+\boldsymbol{a}_{\gamma}\cdot\boldsymbol{d}^{\varepsilon}_{\eta})\boldsymbol{G}&\\ =\frac{1}{2}(\delta\boldsymbol{d}^{\varepsilon}_{\alpha}\cdot\boldsymbol{d}^{N}_{\beta}+\delta\boldsymbol{d}^{\varepsilon}_{\beta}\cdot\boldsymbol{d}^{N}_{\alpha})\boldsymbol{G}&\end{aligned} \tag{29}$$

4. Naturally variational enforcement for essential boundary conditions

5. Numerical examples

	Linear patch test		Quadratic patch test	
	L_2-Error	H_e-Error	L_2-Error	H_e-Error
GI-Penalty				
GI-Nitsche				
RKGSI-Penalty				
RKGSI-Nitsche				
RKGSI-HR				

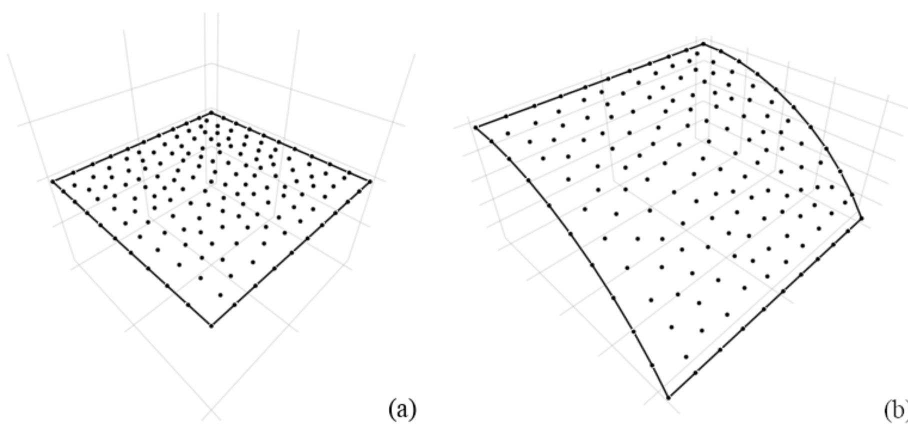

Figure 1: Meshfree discretization for patch test

References

[1] H. Dah-wei, A method for establishing generalized variational principle 6 (6) 501–509. doi:10.1007/BF01876390.
URL http://link.springer.com/10.1007/BF01876390