

1 Quasi-consistent efficient meshfree thin shell
2 formulation with naturally stabilized enforced essential
3 boundary conditions

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5 **Abstract**

This research proposed an efficient and quasi-consistent meshfree thin shell formulation with naturally stabilized enforcement of essential boundary conditions. Within the framework of the Hu-Washizu variational principle, a mixed formulation of displacements, strains and stresses is employed in this approach, where the displacements are discretized using meshfree shape functions, and the strains and stresses are expressed using smoothed gradients and covariant bases. The smoothed gradients satisfy the first second-order integration constraint and observed variational consistency for polynomial strains and stresses. Owing to Hu-Washizu variational principle, the essential boundary conditions automatically arise in its weak form. As a result, the suggested technique's enforcement of essential boundary conditions resembles that of the traditional Nitsche's method. Contrary to Nitsche's method, the costly higher order derivatives of conventional meshfree shape functions are replaced by the smoothed gradients with fast computation, which improve the efficiency. Meanwhile, the proposed formulation features a naturally stabilized term without adding any artificial stabilization factors, which eliminates the application of penalty method as a stabilization. Further, the efficacy of the proposed Hu-Washizu meshfree thin shell formulation is illustrated by a set of classical standard thin shell problems.

6 *Keywords:* Meshfree, Thin shell, Hu-Washizu variational principle,
7 Reproducing kernel gradient smoothing, Essential boundary condition

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8 1. Introduction

9 Thin shell structures generally adhere to the Kirchhoff hypothesis [1], that
10 neglects the shear deformation can be described using Galerkin formulation
11 which requires to have at least C^1 continuity. Traditional finite element meth-
12 ods typically employ C^0 continuous shape functions, and it prefers hybrid and
13 mixed models, like linear and nonlinear Mindlin model [2, 3] and the one in-
14 extensible director model [4], in simulation of shell structure. Over the past
15 thirty years, various novel formulations with high order smoothed shape func-
16 tions have been applied to thin shell formulations. These include element-free
17 Galerkin method [5], maximum-entropy method [6], Hermite reproducing kernel
18 particle method [7], peridynamics [8], isogeometric [9], and others. For a more
19 comprehensive review of advances and applications of meshfree techniques in
20 various scientific and engineering fields, refer to [10, 11, 12, 13, 14, 15, 16, 16].
21 Among these approaches, Galerkin meshfree methods with moving least square
22 approximation (MLS) [17] or reproducing kernel approximation (RK) [18] es-
23 tablished the shape functions based on a collection of dispersed nodes, and high
24 order continuity of shape functions can be easily achieved even with low-order
25 basis functions. For thin shell analysis, high order meshfree approximation can
26 also further alleviate the membrane locking caused by the mismatched approxi-
27 mation order of membrane strain and bending strain [5]. Moreover, node-based
28 MLS/RK approximations generally offer the flexibility of local refinement and
29 can relieve the burden of mesh distortion. However, the high order smoothed
30 meshfree shape functions accompany the enlarged and overlapping supports,
31 which may potentially cause many problems for shape functions. One of the
32 issues is the loss of the Kronecker delta property, which means that, unlike the
33 finite element methods, the necessary boundary conditions cannot be directly
34 enforced [19]. Another issue is that the variational consistency or said integra-
35 tion constraint, which is a condition that requires the formulation to exactly
36 reproduce the solution spanned by the basis functions, cannot be satisfied. This
37 issue is mainly caused by the misalignment between the numerical integration
38 domains and supports of shape functions. Thus, the shape functions exhibit a
39 piecewise nature in each integration domain. Besides, it has to be noted that
40 the traditional integration rules like Gauss scheme cannot ensure the integration
41 accuracy in Galerkin weak form [20, 21]. Therefore, variational consistency is
42 vital to the solution accuracy in the Galerkin meshfree formulations.

43 Various ways have been presented to enforce the necessary boundary for
44 Galerkin meshfree methods directly, including the boundary singular kernel
45 method [22], mixed transformation method [22], and interpolation element-free
46 method [23] for recovering shape functions' Kronecker property. However, these
47 methods were not based on variational setting and cannot guarantee variational
48 consistency. The accuracy maybe poor at locations away from the sample points.
49 In contrast, enforcing the essential boundary conditions using a variational
50 approach is preferred for Galerkin meshfree methods. The variational consis-
51 tent Lagrange multiplier approach was initially used to the Galerkin meshfree
52 method by Belytschko et al. [17]. In this method, the extra degrees of freedom

are used to determine the discretization of Lagrange multiplier. Ivannikov et al. [24] extended this approach to geometrically nonlinear thin shells. Lu et al. [25] suggested the modified variational essential boundary enforcement approach and expressed the Lagrange multiplier by equivalent tractions to eliminate the excess degrees of freedom. However, the coercivity of this approach is not always ensured and potentially reduces the accuracy. Zhu and Atluri [26] pioneered the penalty method for meshfree method, making it a straightforward approach to enforce essential boundary conditions via Galerkin weak form. However, the penalty method lacks variational consistency and requires experimental artificial parameters whose optimal value is hard to determine. Fernández-Méndez and Huerta [19] imposed necessary boundary conditions using Nitsche's approach in the meshfree formulation. This approach can be seen as a hybrid combination of the modified variational method and the penalty method because the modified variational method generates variational consistency through the use of a consistent term, and the penalty method is used as a stabilized term to recover the coercivity. Skatulla and Sansour [27] extended Nitsche's thin shell analysis method and proposed an iteration algorithm to determine artificial parameters at each integration point. In addition, the Nitsche's method has been successfully used in multiple patches problems [28] and composited material problems [29] to ensure the variational consistency between different geometrical or material domains.

In order to address the issue of numerical integration, a series of consistent integration schemes have been developed for Galerkin meshfree methods. Among these include stabilized conforming nodal integration [30], variational consistent integration [31], quadratic consistent integration [32], reproducing kernel gradient smoothing integration [33], and consistent projection integration [34]. The assumed strain approach establishes the most consistent integration scheme, while the smoothed gradient replaces the costly higher order derivatives of traditional meshfree shape functions and shows a high efficiency. Moreover, to achieve global variational consistency, a consistent essential boundary condition enforcement must be combined with the consistent integration scheme. The combination of consistent integration scheme and Nitsche's method for treating essential boundary conditions may demonstrate better performance since both the methods can satisfy the coercivity without requiring additional degrees of freedom. Nevertheless, Nitsche's approach still retains the artificial parameters in the stabilized terms, and it is essential to remain cautious of the costly higher order derivatives, particularly for thin plate and thin shell problems. Recently, Wu et al. [35, 36] proposed an efficient and stabilized essential boundary condition enforcement method based upon the Hellinger-Reissner variational principle, where a mixed formulation in Hellinger-Reissner weak form recasts the reproducing kernel gradient smoothing integration. The terms required for enforcing essential boundary conditions are identical to the Nitsche's method, and both have consistent and stabilized terms. However, the stabilized term of this method naturally exists in the Hellinger-Reissner weak form and no longer needs the artificial parameters, even for essential boundary enforcement. Instead all of the higher order derivatives are represented by the smoothed gradients and

99 their derivatives.

100 In this study, an efficient and stabilized variational consistent meshfree
101 method that naturally enforces the essential boundary conditions is developed
102 for thin shell structures. Following the concept of the Hellinger-Reissner prin-
103 ciple base consistent meshfree method, the Hu-Washizu variational principle of
104 complementary energy with variables of displacement, strains, and stresses were
105 employed. The displacement is approximated by conventional meshfree shape
106 functions, and the strains and stresses were expressed by smoothed gradients
107 with covariant bases. It is important to note that although the first second-order
108 integration requirements were naturally embedded in the smoothed gradients,
109 their fulfillment resulted in a quasi-satisfaction of variational consistency. This
110 is mainly because of the non-polynomial nature of the stresses. Hu-Washizu's
111 weak form was used to evaluate all the essential boundary conditions regard-
112 ing displacements and rotations. This type of formulation is similar to the
113 Nitsche's method but does not require any artificial parameters. Compared
114 with Nitsche's method, conventional reproducing smoothed gradients and its
115 direct derivatives replace the costly higher order derivatives. By utilizing the
116 advantages of a replicating kernel gradient smoothing framework, the smoothed
117 gradients showed better performance compared to conventional derivatives of
118 shape functions, hence increasing the meshfree formulation's computational ef-
119 ficiency.

120 The remainder of this research article is structured as follows: The kinemat-
121 ics of the thin shell structure and the weak form of the associated Hu-Washizu
122 principle are briefly described in Section 2. The mixed formulation regarding
123 the displacements, strains and stresses in accordance with Hu-Washizu weak
124 form are presented in Section 3. The discrete equilibrium equations are derived
125 in Section 4 using the naturally occurring accommodation of essential. Subse-
126 quently, they are compared to the equations obtained using Nitsche's method.
127 The numerical results in Section 5 validate the efficacy of the proposed Hu-
128 Washizu meshfree thin shell formulation. Lastly, the concluding remarks are
129 presented in Section 6.

2. Hu-Washizu's formulation of complementary energy for thin shell

2.1. Kinematics for thin shell

Consider the configuration of a shell $\bar{\Omega}$, as shown in Fig. 1, which can be easily described by a parametric curvilinear coordinate system $\boldsymbol{\xi} = \{\xi^i\}_{i=1,2,3}$. The mid-surface of the shell denoted by Ω is specified by the in-plane coordinates $\boldsymbol{\xi} = \{\xi^\alpha\}_{\alpha=1,2}$, as the thickness direction of shell is by ξ^3 , $-\frac{h}{2} \leq \xi^3 \leq \frac{h}{2}$, h is the thickness of shell. In this work, Latin indices take the values from 1 to 3, and Greek indices are evaluated by 1 or 2. For the Kirchhoff hypothesis [5], the position $\mathbf{x} \in \bar{\Omega}$ is defined by linear functions with respect to ξ^3 :

$$\mathbf{x}(\xi^1, \xi^2, \xi^3) = \mathbf{r}(\xi^1, \xi^2) + \xi^3 \mathbf{a}_3(\xi^1, \xi^2) \quad (1)$$

in which \mathbf{r} means the position on the mid-surface of shell, and \mathbf{a}_3 is correspond-

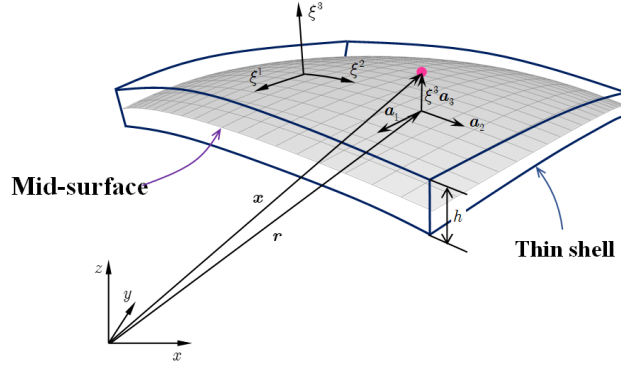


Figure 1: Kinematics for thin shell.

ing normal direction. For the mid-surface of shell, the in-plane covariant base vector with respect to ξ^α can be derived by a trivial partial differentiation to \mathbf{r} :

$$\mathbf{a}_\alpha = \frac{\partial \mathbf{r}}{\partial \xi^\alpha} = \mathbf{r}_{,\alpha}, \alpha = 1, 2 \quad (2)$$

to provide for a clear expression, the subscript comma denotes the partial differentiation operation with respect to in-plane coordinates ξ^α , and the normal vector \mathbf{a}_3 can be obtained by the normalized cross product of \mathbf{a}_α 's as follows:

$$\mathbf{a}_3 = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{\|\mathbf{a}_1 \times \mathbf{a}_2\|} \quad (3)$$

where $\|\bullet\|$ is the Euclidean norm operator.

With the assumption of infinitesimal deformation, the strain components with respect to the global contravariant base can be stated as:

$$\epsilon_{ij} = \frac{1}{2}(\mathbf{x}_{,i} \cdot \mathbf{u}_{,j} + \mathbf{u}_{,i} \cdot \mathbf{x}_{,j}) \quad (4)$$

where \mathbf{u} represents the displacement for the shell deformation. To satisfy the Kirchhoff hypothesis, the displacement is assumed to be of the following form:

$$\mathbf{u}(\xi^1, \xi^2, \xi^3) = \mathbf{v}(\xi^1, \xi^2) + \boldsymbol{\theta}(\xi^1, \xi^2)\xi^3 \quad (5)$$

in which the quadratic and higher order terms are neglected. \mathbf{v} , $\boldsymbol{\theta}$ represent the displacement and rotation in mid-surface, respectively.

Subsequently, plugging Eqs. (1) and (5) into Eq. (4) and neglecting the quadratic terms, the strain components can be rephrased as follows:

$$\begin{aligned} \epsilon_{\alpha\beta} &= \frac{1}{2}(\mathbf{a}_\alpha \cdot \mathbf{v}_{,\beta} + \mathbf{v}_{,\alpha} \cdot \mathbf{a}_\beta) \\ &+ \frac{1}{2}(\mathbf{a}_{3,\alpha} \cdot \mathbf{v}_{,\beta} + \mathbf{v}_{,\alpha} \cdot \mathbf{a}_{3,\beta} + \mathbf{a}_\alpha \cdot \boldsymbol{\theta}_{,\beta} + \boldsymbol{\theta}_{,\alpha} \cdot \mathbf{a}_\beta)\xi^3 \\ &= \varepsilon_{\alpha\beta} + \kappa_{\alpha\beta}\xi^3 \end{aligned} \quad (6a)$$

$$\epsilon_{\alpha 3} = \frac{1}{2}(\mathbf{a}_\alpha \cdot \boldsymbol{\theta} + \mathbf{v}_{,\alpha} \cdot \mathbf{a}_3) + \frac{1}{2}(\mathbf{a}_3 \cdot \boldsymbol{\theta})_{,\alpha}\xi^3 \quad (6b)$$

$$\epsilon_{33} = \mathbf{a}_3 \cdot \boldsymbol{\theta} \quad (6c)$$

where $\varepsilon_{\alpha\beta}$, $\kappa_{\alpha\beta}$ represent membrane and bending strains, respectively, and are given as follows:

$$\varepsilon_{\alpha\beta} = \frac{1}{2}(\mathbf{a}_\alpha \cdot \mathbf{v}_{,\beta} + \mathbf{v}_{,\alpha} \cdot \mathbf{a}_\beta) \quad (7)$$

$$\kappa_{\alpha\beta} = \frac{1}{2}(\mathbf{a}_{3,\alpha} \cdot \mathbf{v}_{,\beta} + \mathbf{v}_{,\alpha} \cdot \mathbf{a}_{3,\beta} + \mathbf{a}_\alpha \cdot \boldsymbol{\theta}_{,\beta} + \boldsymbol{\theta}_{,\alpha} \cdot \mathbf{a}_\beta) \quad (8)$$

In accordance with the Kirchhoff hypothesis, the thickness of shell will not change, and the deformation related with direction of ξ^3 will vanish, i.e. $\epsilon_{3i} = 0$. Thus, the rotation $\boldsymbol{\theta}$ can be rewritten as:

$$\epsilon_{3i} = 0 \Rightarrow \begin{cases} \boldsymbol{\theta} \cdot \mathbf{a}_\alpha = -\mathbf{v}_{,\alpha} \cdot \mathbf{a}_3 \\ \boldsymbol{\theta} \cdot \mathbf{a}_3 = 0 \end{cases} \Rightarrow \boldsymbol{\theta} = -\mathbf{v}_{,\alpha} \cdot \mathbf{a}_3 \mathbf{a}^\alpha \quad (9)$$

where \mathbf{a}^α 's is the in-plane contravariant base vector, $\mathbf{a}^\alpha \cdot \mathbf{a}_\beta = \delta^\alpha_\beta$, δ is the Kronecker delta function. The detailed derivation of Eq. 9 can be found in [37].

Furthermore, on substituting Eq. (9) into Eq. (8) leads to:

$$\kappa_{\alpha\beta} = (\Gamma_{\alpha\beta}^\gamma \mathbf{v}_{,\gamma} - \mathbf{v}_{,\alpha\beta}) \cdot \mathbf{a}_3 = -\mathbf{v}_{,\alpha}|_\beta \cdot \mathbf{a}_3 \quad (10)$$

in which $\Gamma_{\alpha\beta}^\gamma = \mathbf{a}_{\alpha,\beta} \cdot \mathbf{a}^\gamma$ is namely the Christoffel symbol of the second kind, and $\mathbf{v}_{,\alpha}|_\beta$ is the in-plane covariant derivative of $\mathbf{v}_{,\alpha}$, i.e. $\mathbf{v}_{,\alpha}|_\beta = \Gamma_{\alpha\beta}^\gamma \mathbf{v}_{,\gamma} - \mathbf{v}_{,\alpha\beta}$.

2.2. Galerkin weak form for Hu-Washizu principle of complementary energy

In this study, the Hu-Washizu variational principle of complementary energy [38] was adopted for the development of the proposed analytical approach, the

168 corresponding complementary functional, denoted by Π_C , is listed as follows:

$$\begin{aligned}
& \Pi_C(\varepsilon_{\alpha\beta}, \kappa_{\alpha\beta}, N^{\alpha\beta}, M^{\alpha\beta}) \\
&= \int_{\Omega} \frac{h}{2} \varepsilon_{\alpha\beta} C^{\alpha\beta\gamma\eta} \varepsilon_{\gamma\eta} d\Omega + \int_{\Omega} \frac{h^3}{24} \kappa_{\alpha\beta} C^{\alpha\beta\gamma\eta} \kappa_{\gamma\eta} d\Omega \\
&+ \int_{\Omega} \varepsilon_{\alpha\beta} (N^{\alpha\beta} - h C^{\alpha\beta\gamma\eta} \varepsilon_{\gamma\eta}) d\Omega + \int_{\Omega} \kappa_{\alpha\beta} (M^{\alpha\beta} - \frac{h^3}{12} C^{\alpha\beta\gamma\eta} \kappa_{\gamma\eta}) d\Omega \\
&- \int_{\Gamma_v} \mathbf{T} \cdot \bar{\mathbf{v}} d\Gamma + \int_{\Gamma_{\theta}} M_{\mathbf{n}\mathbf{n}} \bar{\theta}_{\mathbf{n}} d\Gamma - (P \mathbf{a}_3 \cdot \bar{\mathbf{v}})_{\mathbf{x} \in C_w}
\end{aligned} \tag{11}$$

169 where $C^{\alpha\beta\gamma\eta}$'s represent the components of fourth order elasticity tensor with
170 respect to the covariant base and plane stress assumption, and it can be ex-
171 pressed by Young's modulus E , Poisson's ratio ν and the in-plane contravariant
172 metric coefficients $a^{\alpha\beta}$'s, $a^{\alpha\beta} = \mathbf{a}^{\alpha} \cdot \mathbf{a}^{\beta}$, as follows:

$$C^{\alpha\beta\gamma\eta} = \frac{E}{2(1+\nu)} (a^{\alpha\gamma} a^{\beta\eta} + a^{\alpha\eta} a^{\beta\gamma} + \frac{2\nu}{1-\nu} a^{\alpha\beta} a^{\gamma\eta}) \tag{12}$$

173 and $N^{\alpha\beta}$, $M^{\alpha\beta}$ represent the components of membrane- and bending- stresses
174 which are given by:

$$N^{\alpha\beta} = h C^{\alpha\beta\gamma\eta} \varepsilon_{\gamma\eta}, \quad M^{\alpha\beta} = \frac{h^3}{12} C^{\alpha\beta\gamma\eta} \kappa_{\gamma\eta} \tag{13}$$

175 Essential boundaries on the edges and corners denoted by Γ_v , Γ_{θ} and C_w
176 are naturally existed in complementary energy functional, and $\bar{\mathbf{v}}$, $\bar{\theta}_{\mathbf{n}}$ are the
177 corresponding prescribed displacement and normal rotation, respectively. \mathbf{T} ,
178 $M_{\mathbf{n}\mathbf{n}}$ and P can be determined by Euler-Lagrange equations of shell problem
179 [37] as follows:

$$\mathbf{T} = \mathbf{T}_N + \mathbf{T}_M \rightarrow \begin{cases} \mathbf{T}_N = \mathbf{a}_{\alpha} N^{\alpha\beta} n_{\beta} \\ \mathbf{T}_M = (\mathbf{a}_3 M^{\alpha\beta} s_{\alpha} n_{\beta})_{,\gamma} s^{\gamma} + (\mathbf{a}_3 M^{\alpha\beta})|_{\beta} n_{\alpha} \end{cases} \tag{14}$$

$$M_{\mathbf{n}\mathbf{n}} = M^{\alpha\beta} n_{\alpha} n_{\beta} \tag{15}$$

$$P = -[[M^{\alpha\beta} s_{\alpha} n_{\beta}]] \tag{16}$$

182 where $\mathbf{n} = n^{\alpha} \mathbf{a}_{\alpha} = n_{\alpha} \mathbf{a}^{\alpha}$ and $\mathbf{s} = s^{\alpha} \mathbf{a}_{\alpha} = s_{\alpha} \mathbf{a}^{\alpha}$ are the outward normal and
183 tangent directions on boundaries. $[[f]]$ is the jump operator defined by:

$$[[f]]_{\mathbf{x}=\mathbf{x}_c} = \lim_{\epsilon \rightarrow \mathbf{0}^+} (f(\mathbf{x}_c + \epsilon) - f(\mathbf{x}_c - \epsilon)), \mathbf{x}_c \in \Gamma \tag{17}$$

184 where f is an arbitrary function on Γ .

185 Moreover, the natural boundary conditions should be applied by Lagrangian
186 multiplier method with displacement \mathbf{v} regarded as multiplier. Thus, then the

new complementary energy functional namely Π is given by:

$$\begin{aligned} & \Pi(\mathbf{v}, \varepsilon_{\alpha\beta}, \kappa_{\alpha\beta}, N^{\alpha\beta}, M^{\alpha\beta}) \\ &= \Pi_C(\varepsilon_{\alpha\beta}, \kappa_{\alpha\beta}, N^{\alpha\beta}, M^{\alpha\beta}) + \int_{\Gamma_M} \theta_{\mathbf{n}}(M_{\mathbf{n}\mathbf{n}} - \bar{M}_{\mathbf{n}\mathbf{n}}) d\Gamma \\ & - \int_{\Gamma_T} \mathbf{v} \cdot (\mathbf{T} - \bar{\mathbf{T}}) d\Gamma - \mathbf{v} \cdot \mathbf{a}_3(P - \bar{P})_{\mathbf{x} \in C_P} - \int_{\Omega} \mathbf{v} \cdot (\mathbf{b} - \bar{\mathbf{b}}) d\Omega \end{aligned} \quad (18)$$

where $\bar{\mathbf{T}}$, $\bar{M}_{\mathbf{n}\mathbf{n}}$ and \bar{P} are the prescribed traction, bending moment and concentrated force on edges Γ_T , Γ_M and corner C_P respectively. All the specified boundaries meet the following geometric relationships:

$$\begin{cases} \Gamma = \Gamma_v \cup \Gamma_T \cup \Gamma_\theta \cup \Gamma_M, & C = C_v \cup C_P, \\ \Gamma_v \cap \Gamma_T = \Gamma_\theta \cap \Gamma_M = C_v \cap C_P = \emptyset \end{cases} \quad (19)$$

and $\bar{\mathbf{b}}$ stands for the prescribed body force in Ω , \mathbf{b} can be written based on Euler-Lagrange equations [37] as:

$$\mathbf{b} = \mathbf{b}_N + \mathbf{b}_M \rightarrow \begin{cases} \mathbf{b}_N = (\mathbf{a}_\alpha N^{\alpha\beta})|_\beta \\ \mathbf{b}_M = (\mathbf{a}_3 M^{\alpha\beta})|_{\alpha\beta} \end{cases} \quad (20)$$

Introducing a standard variational argument to Eq. (18), $\delta\Pi = 0$, and considering the arbitrariness of virtual variables, $\delta\mathbf{v}$, $\delta\varepsilon_{\alpha\beta}$, $\delta\kappa_{\alpha\beta}$, $N^{\alpha\beta}$, $M^{\alpha\beta}$ lead to the following weak form:

$$- \int_{\Omega} h \delta\varepsilon_{\alpha\beta} C^{\alpha\beta\gamma\eta} \varepsilon_{\gamma\eta} d\Omega + \int_{\Omega} \delta\varepsilon_{\alpha\beta} N^{\alpha\beta} d\Omega = 0 \quad (21a)$$

$$- \int_{\Omega} \frac{h^3}{12} \delta\kappa_{\alpha\beta} C^{\alpha\beta\gamma\eta} \kappa_{\gamma\eta} d\Omega + \int_{\Omega} \delta\kappa_{\alpha\beta} M^{\alpha\beta} d\Omega = 0 \quad (21b)$$

$$\begin{aligned} & \int_{\Omega} \delta N^{\alpha\beta} \varepsilon_{\alpha\beta} d\Omega - \int_{\Gamma} \delta \mathbf{T}_N \cdot \mathbf{v} d\Gamma + \int_{\Omega} \delta \mathbf{b}_N \cdot \mathbf{v} d\Omega \\ & + \int_{\Gamma_v} \delta \mathbf{T}_N \cdot \mathbf{v} d\Gamma = \int_{\Gamma_v} \delta \mathbf{T}_N \cdot \bar{\mathbf{v}} d\Gamma \end{aligned} \quad (21c)$$

$$\begin{aligned} & \int_{\Omega} \delta M^{\alpha\beta} \kappa_{\alpha\beta} d\Omega - \int_{\Gamma} \delta M_{\mathbf{n}\mathbf{n}} \theta_{\mathbf{n}} d\Gamma + \int_{\Gamma} \delta \mathbf{T}_M \cdot \mathbf{v} d\Gamma + (\delta P \mathbf{a}_3 \cdot \mathbf{v})_{\mathbf{x} \in C} + \int_{\Omega} \delta \mathbf{b}_M \cdot \mathbf{v} d\Omega \\ & + \int_{\Gamma_\theta} \delta M_{\mathbf{n}\mathbf{n}} \theta_{\mathbf{n}} d\Gamma - \int_{\Gamma_v} \delta \mathbf{T}_M \cdot \mathbf{v} d\Gamma - (\delta P \mathbf{a}_3 \cdot \mathbf{v})_{\mathbf{x} \in C_v} \\ & = \int_{\Gamma_\theta} \delta M_{\mathbf{n}\mathbf{n}} \bar{\theta}_{\mathbf{n}} d\Gamma - \int_{\Gamma_v} \delta \mathbf{T}_M \cdot \bar{\mathbf{v}} d\Gamma - (\delta P \mathbf{a}_3 \cdot \bar{\mathbf{v}})_{\mathbf{x} \in C_v} \end{aligned} \quad (21d)$$

$$\begin{aligned}
& \int_{\Gamma} \delta \theta_{\mathbf{n}} M_{\mathbf{nn}} d\Gamma - \int_{\Gamma} \delta \mathbf{v} \cdot \mathbf{T} d\Gamma - (\delta \mathbf{v} \cdot \mathbf{a}_3 P)_{\mathbf{x} \in C} + \int_{\Omega} \delta \mathbf{v} \cdot \mathbf{b} d\Omega \\
& - \int_{\Gamma_{\theta}} \delta \theta_{\mathbf{n}} M_{\mathbf{nn}} d\Gamma + \int_{\Gamma_v} \delta \mathbf{v} \cdot \mathbf{T} d\Gamma + (\delta \mathbf{v} \cdot \mathbf{a}_3 P)_{\mathbf{x} \in C_v} = - \int_{\Gamma_T} \delta \mathbf{v} \cdot \bar{\mathbf{t}} d\Gamma - \int_{\Omega} \delta \mathbf{v} \cdot \bar{\mathbf{b}} d\Omega
\end{aligned} \tag{21e}$$

200 where the geometric relationships of Eq. (19) is used herein.

201 3. Mixed meshfree formulation for modified Hu-Washizu's weak form

202 3.1. Reproducing kernel approximation for displacement

203 This study approximates the displacement by adopting reproducing kernel
 204 approximation. As shown in Fig. 2, the mid-surface of the shell Ω is discretized
 205 by a set of meshfree nodes $\{\boldsymbol{\xi}_I\}_{I=1}^{n_p}$ in parametric configuration, where n_p is the
 206 total number of meshfree nodes. The approximated displacement namely \mathbf{v}^h
 207 can be expressed as:

$$\mathbf{v}(\boldsymbol{\xi}) = \sum_{I=1}^{n_p} \Psi_I(\boldsymbol{\xi}) \mathbf{d}_I \quad (22)$$

208 where Ψ_I and \mathbf{d}_I represent the shape function and nodal coefficient tensor re-
 209 lated by node $\boldsymbol{\xi}_I$. According to reproducing kernel approximation [18], the shape
 210 function takes the following form:

$$\Psi_I(\boldsymbol{\xi}) = \mathbf{p}^T(\boldsymbol{\xi}) \mathbf{c}(\boldsymbol{\xi}) \phi(\boldsymbol{\xi}_I - \boldsymbol{\xi}) \quad (23)$$

211 where \mathbf{p} is the basis function vector represented using the following quadratic
 212 function as:

$$\mathbf{p} = \{1, \xi^1, \xi^2, (\xi^1)^2, \xi^1 \xi^2, (\xi^2)^2\}^T \quad (24)$$

213 The kernel function denoted by ϕ controls the support and smoothness of
 214 meshfree shape functions. The quintic B-spline function with square support is
 215 used herein as the kernel function:

$$\phi(\boldsymbol{\xi}_I - \boldsymbol{\xi}) = \phi(\hat{s}_1) \phi(\hat{s}_2), \quad \hat{s}_\alpha = \frac{|\xi_I^\alpha - \xi^\alpha|}{s_{\alpha I}} \quad (25)$$

216 with

$$\phi(\hat{s}_\alpha) = \frac{1}{5!} \begin{cases} (3 - 3\hat{s}_\alpha)^5 - 6(2 - 3\hat{s}_\alpha)^5 + 15(1 - 3\hat{s}_\alpha)^5 & \hat{s}_\alpha \leq \frac{1}{3} \\ (3 - 3\hat{s}_\alpha)^5 - 6(2 - 3\hat{s}_\alpha)^5 & \frac{1}{3} < \hat{s}_\alpha \leq \frac{2}{3} \\ (3 - 3\hat{s}_\alpha)^5 & \frac{2}{3} < \hat{s}_\alpha \leq 1 \\ 0 & \hat{s}_\alpha > 1 \end{cases} \quad (26)$$

217 and $s_{\alpha I}$ means the support size of meshfree shape function Ψ_I .

218 The unknown vector \mathbf{c} in shape function are determined by the fulfillment
 219 of the so-called consistency condition:

$$\sum_{I=1}^{n_p} \Psi_I(\boldsymbol{\xi}) \mathbf{p}(\boldsymbol{\xi}_I) = \mathbf{p}(\boldsymbol{\xi}) \quad (27)$$

220 or equivalently

$$\sum_{I=1}^{n_p} \Psi_I(\boldsymbol{\xi}) \mathbf{p}(\boldsymbol{\xi}_I - \boldsymbol{\xi}) = \mathbf{p}(\mathbf{0}) \quad (28)$$

221 Substituting Eq. (22) into (28), yields:

$$\mathbf{A}(\boldsymbol{\xi}) \mathbf{c}(\boldsymbol{\xi}) = \mathbf{p}(\mathbf{0}) \quad \Rightarrow \quad \mathbf{c}(\boldsymbol{\xi}) = \mathbf{A}^{-1}(\boldsymbol{\xi}) \mathbf{p}(\mathbf{0}) \quad (29)$$

where \mathbf{A} is the moment matrix:

$$\mathbf{A}(\boldsymbol{\xi}) = \sum_{I=1}^{n_p} \phi(\boldsymbol{\xi}_I - \boldsymbol{\xi}) \mathbf{p}(\boldsymbol{\xi}_I - \boldsymbol{\xi}) \mathbf{p}^T(\boldsymbol{\xi}_I - \boldsymbol{\xi}) \quad (30)$$

Substituting Eq. (29) back into Eq. (22), the expression of meshfree shape function can be written as:

$$\Psi_I(\boldsymbol{\xi}) = \mathbf{p}^T(\boldsymbol{\xi}_I - \boldsymbol{\xi}) \mathbf{A}^{-1}(\boldsymbol{\xi}) \mathbf{p}(\mathbf{0}) \phi(\boldsymbol{\xi}_I - \boldsymbol{\xi}) \quad (31)$$

3.2. Reproducing kernel gradient smoothing approximation for effective stress and strain

In Galerkin meshfree formulation, the mid-plane of thin shell Ω is split by a set of integration cells Ω_C 's, $\cup_{C=1}^{n_c} \Omega_C \approx \Omega$, as shown in Fig. 2. With the inspiration of reproducing kernel smoothing framework, the Cartesian and covariant derivatives of displacement, $\mathbf{v}_{,\alpha}$ and $-\mathbf{v}_{,\alpha}|_{\beta}$, in strains $\varepsilon_{\alpha\beta}$, $\kappa_{\alpha\beta}$ are approximated by $(p-1)$ -th order polynomials in each integration cells. In integration cell Ω_C , the approximated derivatives and strains denoted by $\mathbf{v}_{,\alpha}^h$, $\varepsilon_{\alpha\beta}^h$ and $-\mathbf{v}_{,\alpha}^h|_{\beta}$, $\kappa_{\alpha\beta}^h$ can be expressed by:

$$\mathbf{v}_{,\alpha}^h(\boldsymbol{\xi}) = \mathbf{q}^T(\boldsymbol{\xi}) \mathbf{d}_{\alpha}^{\varepsilon}, \quad \varepsilon_{\alpha\beta}^h(\boldsymbol{\xi}) = \mathbf{q}^T(\boldsymbol{\xi}) \frac{1}{2} (\mathbf{a}_{\alpha} \cdot \mathbf{d}_{\beta}^{\varepsilon} + \mathbf{a}_{\beta} \cdot \mathbf{d}_{\alpha}^{\varepsilon}) \quad (32)$$

$$-\mathbf{v}_{,\alpha}^h|_{\beta}(\boldsymbol{\xi}) = \mathbf{q}^T(\boldsymbol{\xi}) \mathbf{d}_{\alpha\beta}^{\kappa}, \quad \kappa_{\alpha\beta}^h(\boldsymbol{\xi}) = \mathbf{q}^T(\boldsymbol{\xi}) \mathbf{a}_3 \cdot \mathbf{d}_{\alpha\beta}^{\kappa} \quad (33)$$

where \mathbf{q} is the linear polynomial vector and has the following form:

$$\mathbf{q} = \{1, \xi^1, \xi^2\}^T \quad (34)$$

and the $\mathbf{d}_{\alpha}^{\varepsilon}$, $\mathbf{d}_{\alpha\beta}^{\kappa}$ are the corresponding coefficient vector tensors. For the conciseness, the mixed usage of tensor and vector is introduced in this study. For instance, the component of coefficient tensor vector $\mathbf{d}_{\alpha I}^{\varepsilon}$, $\mathbf{d}_{\alpha}^{\varepsilon} = \{\mathbf{d}_{\alpha I}^{\varepsilon}\}$, is a three dimensional tensor, $\dim \mathbf{d}_{\alpha I}^{\varepsilon} = \dim \mathbf{v}$.

To satisfy the integration constraint of thin shell problem, the approximated stresses $N^{\alpha\beta h}$, $M^{\alpha\beta h}$ were assumed to have a comparable form to strains, and yields:

$$N^{\alpha\beta h}(\boldsymbol{\xi}) = \mathbf{q}^T(\boldsymbol{\xi}) \mathbf{a}^{\alpha} \cdot \mathbf{d}_N^{\beta}, \quad \mathbf{a}_{\alpha} N^{\alpha\beta h}(\boldsymbol{\xi}) = \mathbf{q}^T(\boldsymbol{\xi}) \mathbf{d}_N^{\beta} \quad (35)$$

$$M^{\alpha\beta h}(\boldsymbol{\xi}) = \mathbf{q}^T(\boldsymbol{\xi}) \mathbf{a}_3 \cdot \mathbf{d}_M^{\alpha\beta}, \quad \mathbf{a}_3 M^{\alpha\beta h}(\boldsymbol{\xi}) = \mathbf{q}^T(\boldsymbol{\xi}) \mathbf{d}_M^{\alpha\beta} \quad (36)$$

substituting the approximations of Eqs. (22), (32), (33), (35), (36) into Eqs. (21c), (21d) can express $\mathbf{d}_{\beta}^{\varepsilon}$ and $\mathbf{d}_{\alpha\beta}^{\kappa}$ by \mathbf{d} as:

$$\mathbf{d}_{\beta}^{\varepsilon} = \mathbf{G}^{-1} \left(\sum_{I=1}^{n_p} (\tilde{\mathbf{g}}_{\beta I} - \bar{\mathbf{g}}_{\beta I}) \mathbf{d}_I + \hat{\mathbf{g}}_{\beta} \right) \quad (37)$$

$$\mathbf{d}_{\alpha\beta}^{\kappa} = \mathbf{G}^{-1} \left(\sum_{I=1}^{n_p} (\tilde{\mathbf{g}}_{\alpha\beta I} - \bar{\mathbf{g}}_{\alpha\beta I}) \mathbf{d}_I + \hat{\mathbf{g}}_{\alpha\beta} \right) \quad (38)$$

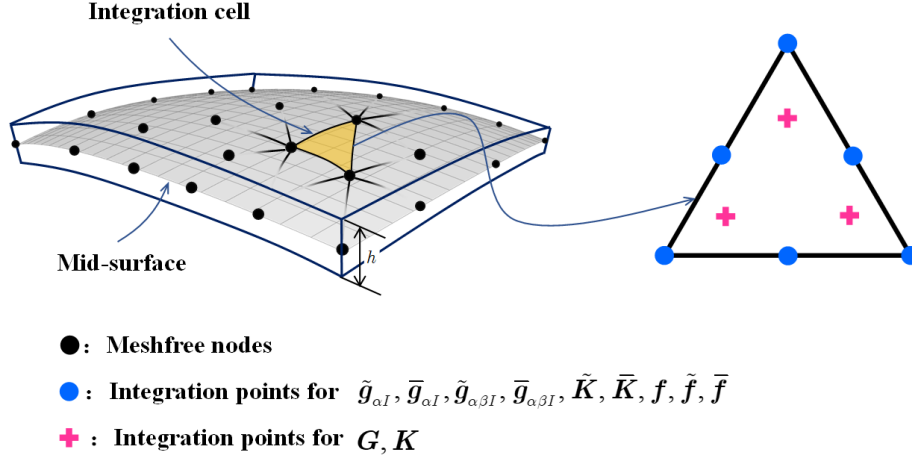


Figure 2: Integration scheme for Hu-Washizu weak form.

247 with

$$G = \int_{\Omega_C} \mathbf{q}^T \mathbf{q} d\Omega \quad (39)$$

248

$$\tilde{g}_{\beta I} = \int_{\Gamma_C} \Psi_I \mathbf{q} n_\beta d\Gamma - \int_{\Omega_C} \Psi_I \mathbf{q}_{|\beta} d\Omega \quad (40a)$$

$$\bar{g}_{\beta I} = \int_{\Gamma_C \cap \Gamma_v} \Psi_I \mathbf{q} n_\beta d\Gamma \quad (40b)$$

$$\hat{g}_\beta = \int_{\Gamma_C \cap \Gamma_v} \mathbf{q} n_\beta \bar{v} d\Gamma \quad (40c)$$

249

$$\begin{aligned} \tilde{g}_{\alpha\beta I} &= \int_{\Gamma_C} \Psi_{I,\gamma} n^\gamma \mathbf{q} n_\alpha n_\beta d\Gamma - \int_{\Gamma_C} \Psi_I (\mathbf{q}_{|\beta} n_\alpha + (\mathbf{q} s_\alpha n_\beta)_{,\gamma} s^\gamma) d\Gamma \\ &\quad + [[\Psi_I \mathbf{q} s_\alpha n_\beta]]_{\mathbf{x} \in C_C} - \int_{\Omega_C} \Psi \mathbf{q}_{,\alpha|\beta} d\Omega \end{aligned} \quad (41a)$$

$$\begin{aligned} \bar{g}_{\alpha\beta I} &= \int_{\Gamma_C \cap \Gamma_\theta} \Psi_{I,\gamma} n^\gamma \mathbf{q} n_\alpha n_\beta d\Gamma - \int_{\Gamma_C \cap \Gamma_v} \Psi_I (\mathbf{q}_{|\beta} n_\alpha + (\mathbf{q} s_\alpha n_\beta)_{,\gamma} s^\gamma) d\Gamma \\ &\quad + [[\Psi_I \mathbf{q} s_\alpha n_\beta]]_{\mathbf{x} \in C_C \cap C_v} \end{aligned} \quad (41b)$$

$$\begin{aligned} \hat{g}_{\alpha\beta} &= \int_{\Gamma_C \cap \Gamma_\theta} \mathbf{q} n_\alpha n_\beta \mathbf{a}_3 \bar{\theta}_n d\Gamma - \int_{\Gamma_C \cap \Gamma_v} (\mathbf{q}_{|\beta} n_\alpha + (\mathbf{q} s_\alpha n_\beta)_{,\gamma} s^\gamma) \bar{v} d\Gamma \\ &\quad + [[\mathbf{q} s_\alpha n_\beta \bar{v}]]_{\mathbf{x} \in C_C \cap C_v} \end{aligned} \quad (41c)$$

250 where evaluations of $\mathbf{q}_{|\beta}$, $\mathbf{q}_{,\alpha|\beta}$ are discussed in Appendix A. Further plugging
 251 Eqs. (37) and (38) back into Eqs. (32) and (33) respectively gives the final

expression of $\mathbf{v}_{,\alpha}^h$, $\varepsilon_{\alpha\beta}^h$ and $-\mathbf{v}_{,\alpha}^h|_{\beta}$, $\kappa_{\alpha\beta}^h$ as:

$$\mathbf{v}_{,\alpha}^h = \sum_{I=1}^{n_p} (\tilde{\Psi}_{I,\alpha} - \bar{\Psi}_{I,\alpha}) \mathbf{d}_I + \mathbf{q}^T \mathbf{G}^{-1} \hat{\mathbf{g}}_{\alpha} \quad (42a)$$

$$\begin{aligned} \varepsilon_{\alpha\beta}^h &= \sum_{I=1}^{n_p} \frac{1}{2} (\mathbf{a}_{\alpha} \tilde{\Psi}_{I,\beta} + \mathbf{a}_{\beta} \tilde{\Psi}_{I,\alpha}) \cdot \mathbf{d}_I - \sum_{I=1}^{n_p} \frac{1}{2} (\mathbf{a}_{\alpha} \bar{\Psi}_{I,\beta} + \mathbf{a}_{\beta} \bar{\Psi}_{I,\alpha}) \cdot \mathbf{d}_I \\ &\quad + \mathbf{q}^T \mathbf{G}^{-1} \frac{1}{2} (\mathbf{a}_{\alpha} \cdot \hat{\mathbf{g}}_{\beta} + \mathbf{a}_{\beta} \cdot \hat{\mathbf{g}}_{\alpha}) \\ &= \tilde{\varepsilon}_{\alpha\beta}^h - \bar{\varepsilon}_{\alpha\beta}^h + \hat{\varepsilon}_{\alpha\beta}^h \end{aligned} \quad (42b)$$

$$-\mathbf{v}_{,\alpha}^h|_{\beta} = \sum_{I=1}^{n_p} (\tilde{\Psi}_{I,\alpha\beta} - \bar{\Psi}_{I,\alpha\beta}) \mathbf{d}_I + \mathbf{q}^T \mathbf{G}^{-1} \hat{\mathbf{g}}_{\alpha\beta} \quad (43a)$$

$$\begin{aligned} \kappa_{\alpha\beta}^h &= \sum_{I=1}^{n_p} \tilde{\Psi}_{I,\alpha\beta} \mathbf{a}_3 \cdot \mathbf{d}_I - \sum_{I=1}^{n_p} \bar{\Psi}_{I,\alpha\beta} \mathbf{a}_3 \cdot \mathbf{d}_I + \mathbf{q}^T \mathbf{G}^{-1} \mathbf{a}_3 \cdot \hat{\mathbf{g}}_{\alpha\beta} \\ &= \tilde{\kappa}_{\alpha\beta}^h - \bar{\kappa}_{\alpha\beta}^h + \hat{\kappa}_{\alpha\beta}^h \end{aligned} \quad (43b)$$

with

$$\begin{cases} \tilde{\varepsilon}_{\alpha\beta}^h = \sum_{I=1}^{n_p} \frac{1}{2} (\mathbf{a}_{\alpha} \tilde{\Psi}_{I,\beta} + \mathbf{a}_{\beta} \tilde{\Psi}_{I,\alpha}) \cdot \mathbf{d}_I = \sum_{I=1}^{n_p} \tilde{\varepsilon}_{\alpha\beta I} \cdot \mathbf{d}_I \\ \bar{\varepsilon}_{\alpha\beta}^h = \sum_{I=1}^{n_p} \frac{1}{2} (\mathbf{a}_{\alpha} \bar{\Psi}_{I,\beta} + \mathbf{a}_{\beta} \bar{\Psi}_{I,\alpha}) \cdot \mathbf{d}_I = \sum_{I=1}^{n_p} \bar{\varepsilon}_{\alpha\beta I} \cdot \mathbf{d}_I \\ \hat{\varepsilon}_{\alpha\beta}^h = \mathbf{q}^T \mathbf{G}^{-1} \frac{1}{2} (\mathbf{a}_{\alpha} \cdot \hat{\mathbf{g}}_{\beta} + \mathbf{a}_{\beta} \cdot \hat{\mathbf{g}}_{\alpha}) \end{cases} \quad (44)$$

$$\begin{cases} \tilde{\Psi}_{I,\alpha}(\boldsymbol{\xi}) = \mathbf{q}^T(\boldsymbol{\xi}) \mathbf{G}^{-1} \tilde{\mathbf{g}}_{\alpha I} \\ \bar{\Psi}_{I,\alpha}(\boldsymbol{\xi}) = \mathbf{q}^T(\boldsymbol{\xi}) \mathbf{G}^{-1} \bar{\mathbf{g}}_{\alpha I} \\ \tilde{\varepsilon}_{\alpha\beta I} = \frac{1}{2} (\mathbf{a}_{\alpha} \tilde{\Psi}_{I,\beta} + \mathbf{a}_{\beta} \tilde{\Psi}_{I,\alpha}) \\ \bar{\varepsilon}_{\alpha\beta I} = \frac{1}{2} (\mathbf{a}_{\alpha} \bar{\Psi}_{I,\beta} + \mathbf{a}_{\beta} \bar{\Psi}_{I,\alpha}) \end{cases} \quad (45)$$

$$\begin{cases} \tilde{\kappa}_{\alpha\beta}^h = \sum_{I=1}^{n_p} \tilde{\Psi}_{I,\alpha\beta} \mathbf{a}_3 \cdot \mathbf{d}_I = \sum_{I=1}^{n_p} \tilde{\kappa}_{\alpha\beta I} \cdot \mathbf{d}_I \\ \bar{\kappa}_{\alpha\beta}^h = \sum_{I=1}^{n_p} \bar{\Psi}_{I,\alpha\beta} \mathbf{a}_3 \cdot \mathbf{d}_I = \sum_{I=1}^{n_p} \bar{\kappa}_{\alpha\beta I} \cdot \mathbf{d}_I \\ \hat{\kappa}_{\alpha\beta}^h = \mathbf{q}^T \mathbf{G}^{-1} \mathbf{a}_3 \cdot \hat{\mathbf{g}}_{\alpha\beta} \end{cases} \quad (46)$$

$$\begin{cases} \tilde{\Psi}_{I,\alpha\beta}(\boldsymbol{\xi}) = \mathbf{q}^T(\boldsymbol{\xi}) \mathbf{G}^{-1} \tilde{\mathbf{g}}_{\alpha\beta I} \\ \bar{\Psi}_{I,\alpha\beta}(\boldsymbol{\xi}) = \mathbf{q}^T(\boldsymbol{\xi}) \mathbf{G}^{-1} \bar{\mathbf{g}}_{\alpha\beta I} \\ \tilde{\kappa}_{\alpha\beta I} = \tilde{\Psi}_{I,\alpha\beta} \mathbf{a}_3 \\ \bar{\kappa}_{\alpha\beta I} = \bar{\Psi}_{I,\alpha\beta} \mathbf{a}_3 \end{cases} \quad (47)$$

260 It has to be noted that, referring to reproducing kernel gradient smoothing
 261 framework [33], $\tilde{\Psi}_{I,\alpha}$, $\tilde{\Psi}_{I,\alpha\beta}$ are actually the first and second order smoothed
 262 gradients in curvilinear coordinates. If the right hand side integration constraints
 263 for first and second order gradients are $\tilde{\mathbf{g}}_{\alpha I}$ and $\tilde{\mathbf{g}}_{\alpha\beta I}$, respectively, then this for-
 264 mulation can satisfy the variational consistency for the second order polynomi-
 265 als. It should be mentioned that in curved model, the variational consistency for
 266 non-polynomial functions, such as trigonometric functions, should be required
 267 for the polynomial solution. Even with high order polynomial variational consis-
 268 tency, the proposed formulation cannot exactly reproduce the solution spanned
 269 by the basis functions. However, the accuracy of reproducing kernel smoothed
 270 gradients is still superior than the traditional meshfree formulation. The nu-
 271 merical examples in the following section will better demonstrate the precision
 272 of the reproducing kernel smoothed gradients.

273 **4. Naturally variational enforcement for essential boundary condi-**
 274 **tions**

275 *4.1. Discrete equilibrium equations*

276 With the approximated effective stresses and strains, the last equation of
 277 weak form Eq. (21e) becomes:

$$-\sum_{C=1}^{n_e} \sum_{I=1}^{n_p} \delta \mathbf{d}_I \cdot \left((\tilde{\mathbf{g}}_{\alpha I}^T - \bar{\mathbf{g}}_{\alpha I}^T) \mathbf{d}_N^\alpha + (\tilde{\mathbf{g}}_{\alpha \beta I}^T - \bar{\mathbf{g}}_{\alpha \beta I}^T) \mathbf{d}_M^{\alpha \beta} \right) = -\sum_{I=1}^{n_p} \delta \mathbf{d}_I \cdot \mathbf{f}_I \quad (48)$$

278 where \mathbf{f}_I 's denote the components of the traditional force vector:

$$\mathbf{f}_I = \int_{\Gamma_t} \Psi_I \bar{\mathbf{t}} d\Gamma - \int_{\Gamma_M} \Psi_{I,\gamma} n^\gamma \bar{M}_{nn} d\Gamma + [[\Psi_I \mathbf{a}_3 \bar{P}]]_{\mathbf{x} \in C_P} + \int_{\Omega} \Psi_I \bar{\mathbf{b}} d\Omega \quad (49)$$

279 The left side of Eq. (48) can be simplified using the following steps. For clarity,
 280 the derivation of first term in Eq. (48) taken as an example is given by:

$$\begin{aligned} \sum_{I=1}^{n_p} \delta \mathbf{d}_I \cdot \tilde{\mathbf{g}}_{\alpha I}^T \mathbf{d}_N^\alpha &= \sum_{I=1}^{n_p} \delta \mathbf{d}_I \cdot (\mathbf{G}^{-1} \tilde{\mathbf{g}}_{\alpha I})^T \mathbf{G} \mathbf{d}_N^\alpha \\ &= \int_{\Omega_C} \sum_{I=1}^{n_p} \delta \mathbf{d}_I \cdot (\mathbf{q}^T \mathbf{G}^{-1} \tilde{\mathbf{g}}_{\alpha I}) \mathbf{q}^T \mathbf{d}_N^\alpha d\Omega \\ &= \int_{\Omega_C} \sum_{I=1}^{n_p} \delta \mathbf{d}_I \cdot \mathbf{a}_\beta (\mathbf{q}^T \mathbf{G}^{-1} \tilde{\mathbf{g}}_{\alpha I}) N^{\alpha \beta h} d\Omega \\ &= \int_{\Omega_C} \delta \tilde{\varepsilon}_{\alpha \beta}^h N^{\alpha \beta h} d\Omega \end{aligned} \quad (50)$$

281 following the above procedure and including the weak form of Eqs. (21a), (21b),
 282 the left side of Eq. (48) in Ω_C becomes:

$$\begin{aligned}
 & \sum_{I=1}^{n_p} \delta \mathbf{d}_I \cdot \left((\tilde{\mathbf{g}}_{\alpha I}^T - \bar{\mathbf{g}}_{\alpha I}^T) \mathbf{d}_N^\alpha + (\tilde{\mathbf{g}}_{\alpha \beta I}^T - \bar{\mathbf{g}}_{\alpha \beta I}^T) \mathbf{d}_M^{\alpha \beta} \right) \\
 &= \int_{\Omega_C} ((\delta \tilde{\varepsilon}_{\alpha \beta}^h - \delta \bar{\varepsilon}_{\alpha \beta}^h) N^{\alpha \beta h} + (\delta \tilde{\kappa}_{\alpha \beta}^h - \delta \bar{\kappa}_{\alpha \beta}^h) M^{\alpha \beta h}) d\Omega \\
 &= \int_{\Omega_C} (\delta \tilde{\varepsilon}_{\alpha \beta}^h - \delta \bar{\varepsilon}_{\alpha \beta}^h) h C^{\alpha \beta \gamma \eta} \varepsilon_{\gamma \eta}^h + (\delta \tilde{\kappa}_{\alpha \beta}^h - \delta \bar{\kappa}_{\alpha \beta}^h) \frac{h^3}{12} C^{\alpha \beta \gamma \eta} \kappa_{\gamma \eta}^h \\
 &= \int_{\Omega_C} \delta \tilde{\varepsilon}_{\alpha \beta}^h h C^{\alpha \beta \gamma \eta} \varepsilon_{\gamma \eta}^h d\Omega + \int_{\Omega_C} \delta \tilde{\kappa}_{\alpha \beta}^h \frac{h^3}{12} C^{\alpha \beta \gamma \eta} \kappa_{\gamma \eta}^h d\Omega \\
 &\quad - \int_{\Omega_C} \delta \bar{\varepsilon}_{\alpha \beta}^h h C^{\alpha \beta \gamma \eta} \varepsilon_{\gamma \eta}^h d\Omega - \int_{\Omega_C} \delta \bar{\kappa}_{\alpha \beta}^h h C^{\alpha \beta \gamma \eta} \varepsilon_{\gamma \eta}^h d\Omega \\
 &\quad - \int_{\Omega_C} \delta \tilde{\kappa}_{\alpha \beta}^h \frac{h^3}{12} C^{\alpha \beta \gamma \eta} \kappa_{\gamma \eta}^h d\Omega - \int_{\Omega_C} \delta \bar{\kappa}_{\alpha \beta}^h \frac{h^3}{12} C^{\alpha \beta \gamma \eta} \kappa_{\gamma \eta}^h d\Omega \\
 &\quad + \int_{\Omega_C} \delta \bar{\varepsilon}_{\alpha \beta}^h h C^{\alpha \beta \gamma \eta} \varepsilon_{\gamma \eta}^h d\Omega + \int_{\Omega_C} \delta \bar{\kappa}_{\alpha \beta}^h \frac{h^3}{12} C^{\alpha \beta \gamma \eta} \kappa_{\gamma \eta}^h d\Omega \\
 &\quad + \int_{\Omega_C} (\delta \tilde{\varepsilon}_{\alpha \beta}^h - \delta \bar{\varepsilon}_{\alpha \beta}^h) h C^{\alpha \beta \gamma \eta} \varepsilon_{\gamma \eta}^h d\Omega + \int_{\Omega_C} (\delta \tilde{\kappa}_{\alpha \beta}^h - \delta \bar{\kappa}_{\alpha \beta}^h) \frac{h^3}{12} C^{\alpha \beta \gamma \eta} \kappa_{\gamma \eta}^h d\Omega
 \end{aligned} \tag{51}$$

283 The complete discrete equilibrium equations can be obtained by further substituting
 284 Eqs. (44) and (46) into above equation, respectively:

$$(\mathbf{K} + \tilde{\mathbf{K}} + \bar{\mathbf{K}}) \mathbf{d} = \mathbf{f} + \tilde{\mathbf{f}} + \bar{\mathbf{f}} \tag{52}$$

285 where the components of stiffness matrices and force vectors in discrete equilibrium
 286 equations can be evaluated as follows:

$$\mathbf{K}_{IJ} = \int_{\Omega} \tilde{\varepsilon}_{\alpha \beta I} h C^{\alpha \beta \gamma \eta} \tilde{\varepsilon}_{\gamma \eta J} d\Omega + \int_{\Omega} \tilde{\kappa}_{\alpha \beta I} \frac{h^3}{12} C^{\alpha \beta \gamma \eta} \tilde{\kappa}_{\gamma \eta J} d\Omega \tag{53}$$

287

$$\begin{aligned}
 \tilde{\mathbf{K}}_{IJ} &= - \int_{\Gamma_v} (\Psi_I \tilde{\mathbf{T}}_{NJ} + \tilde{\mathbf{T}}_{NI} \Psi_J) d\Gamma \\
 &\quad + \int_{\Gamma_\theta} (\Psi_{I,\gamma} n^\gamma \mathbf{a}_3 \tilde{\mathbf{M}}_{nnJ} + \mathbf{a}_3 \tilde{\mathbf{M}}_{nnI} \Psi_{J,\gamma} n^\gamma) d\Gamma \\
 &\quad + ([[\Psi_I \mathbf{a}_3 \tilde{\mathbf{P}}_J]] + [[\tilde{\mathbf{P}}_I \mathbf{a}_3 \Psi_J]])_{\mathbf{x} \in C_v}
 \end{aligned} \tag{54a}$$

$$\tilde{\mathbf{f}}_I = - \int_{\Gamma_v} \tilde{\mathbf{T}}_{NI} \cdot \bar{\mathbf{v}} d\Gamma + \int_{\Gamma_\theta} \tilde{\mathbf{M}}_{nnI} \bar{\theta}_n d\Gamma + [[\tilde{\mathbf{P}}_I \mathbf{a}_3 \cdot \bar{\mathbf{v}}]]_{\mathbf{x} \in C_v} \tag{54b}$$

288

$$\bar{\mathbf{K}}_{IJ} = - \int_{\Gamma_v} \bar{\mathbf{T}}_{MI} \Psi_J d\Gamma + \int_{\Gamma_\theta} \mathbf{a}_3 \bar{\mathbf{M}}_{nnI} \Psi_{J,\gamma} n^\gamma d\Gamma + [[\bar{\mathbf{P}}_I \mathbf{a}_3 \Psi_J]]_{\mathbf{x} \in C_v} \tag{55a}$$

$$\bar{\mathbf{f}}_I = - \int_{\Gamma_v} \bar{\mathbf{T}}_{MI} \cdot \bar{\mathbf{v}} d\Gamma + \int_{\Gamma_\theta} \bar{\mathbf{M}}_{nnI} \bar{\theta}_n d\Gamma + [[\bar{\mathbf{P}}_I \mathbf{a}_3 \cdot \bar{\mathbf{v}}]]_{\mathbf{x} \in C_v} \tag{55b}$$

289 The detailed derivations of Eqs (53)-(55) are listed in the Appendix B. As
 290 shown in these equations, Eq. (53) is the conventional stiffness matrix evalu-
 291 ated by smoothed gradients $\tilde{\Psi}_{I,\alpha}$, $\tilde{\Psi}_{I,\alpha|\beta}$, and the Eqs. (54) and (55) contribute
 292 for the enforcement of essential boundary. It should be noticed that, in accor-
 293 dance with reproducing kernel smoothed gradient framework, the integration
 294 scheme of Eqs. (53-55) should be aligned with those used in the construction of
 295 smoothed gradients. The integration scheme used for the proposed method is
 296 shown in Fig. 2, in which the total number of the blue circular integration points
 297 has been optimized from a global point of view, aiming to reduce the compu-
 298 tation of traditional meshfree shape functions and its first order derivatives. In
 299 contrast, for assembly stiffness matrix \mathbf{K} , the low order Gauss integration rule
 300 is suitable to ensure the accuracy due to the inherently variational consistency
 301 in the smoothed gradients. The detailed positions and weight of the integration
 302 points and the efficiency demonstration of this optimized integration scheme
 303 can be found in [33, 39]. Examining Eqs. (54) and (55), closely reveal that the
 304 structure of the suggested approach to enforce essential boundary conditions is
 305 identical to that of the conventional Nitsche's method, with both having the
 306 consistent and stabilized terms. Thus, a review of Nitsche's method and a com-
 307 parison with the proposed approach will be provided in the next subsection.

308 4.2. Comparison with Nitsche's method

309 The Nitsche's method for enforcing essential boundaries can be regarded as a
 310 combination of Lagrangian multiplier method and penalty method, in which the
 311 Lagrangian multiplier is represented by the approximated displacement. The
 312 corresponding total potential energy functional Π_P is given by:

$$\begin{aligned}
 \Pi_P(\mathbf{v}) = & \int_{\Omega} \frac{1}{2} \varepsilon_{\alpha\beta} N^{\alpha\beta} d\Omega + \int_{\Omega} \frac{1}{2} \kappa_{\alpha\beta} M^{\alpha\beta} d\Omega \\
 & - \int_{\Gamma_t} \mathbf{v} \cdot \bar{\mathbf{t}} d\Gamma + \int_{\Gamma_M} \mathbf{v}_{,\gamma} n^{\gamma} \mathbf{a}_3 M_{nn} d\Gamma + (\mathbf{v} \cdot \mathbf{a}_3 P)_{\mathbf{x} \in C_P} - \int_{\Omega} \mathbf{v} \cdot \bar{\mathbf{b}} d\Omega \\
 & - \underbrace{\int_{\Gamma_v} \mathbf{t} \cdot (\mathbf{v} - \bar{\mathbf{v}}) d\Gamma + \int_{\Gamma_{\theta}} M_{nn} (\theta_n - \bar{\theta}_n) d\Gamma + (P \mathbf{a}_3 \cdot (\mathbf{v} - \bar{\mathbf{v}}))_{\mathbf{x} \in C_v}}_{\text{consistent term}} \quad (56) \\
 & + \underbrace{\sum_{i=1}^3 \frac{\alpha_{vi}}{2} \int_{\Gamma_v} (\mathbf{v} \cdot \mathbf{a}_i)^2 d\Gamma + \frac{\alpha_{\theta}}{2} \int_{\Gamma_{\theta}} \theta_n^2 d\Gamma + \frac{\alpha_C}{2} (\mathbf{v} \cdot \mathbf{a}_3)_{\mathbf{x} \in C_v}^2}_{\text{stabilized term}}
 \end{aligned}$$

313 where the consistent term generated from the Lagrangian multiplier method
 314 contributes to enforce the essential boundary, and meet the variational con-
 315 sistency condition. However, the consistent term can not always ensure the
 316 coercivity of stiffness, so the penalty method is introduced to serve as a sta-
 317 bilized term, in which α_{vi} is the experimental artificial parameter to enforce
 318 the displacement towards the \mathbf{a}_i direction, α_{θ} and α_C are parameters to en-
 319 force rotation and corner deflection, respectively. With a standard variational

argument, the corresponding weak form can be stated as:

$$\begin{aligned}
\delta \Pi_P(\mathbf{v}) = & \int_{\Omega} \delta \varepsilon_{\alpha\beta} N^{\alpha\beta} d\Omega + \int_{\Omega} \delta \kappa_{\alpha\beta} M^{\alpha\beta} d\Omega \\
& - \int_{\Gamma_t} \delta \mathbf{v} \cdot \bar{\mathbf{t}} d\Gamma + \int_{\Gamma_M} \delta \mathbf{v}_{,\gamma} n^{\gamma} \mathbf{a}_3 M_{nn} d\Gamma + (\delta \mathbf{v} \cdot \mathbf{a}_3 P)_{\mathbf{x} \in C_P} - \int_{\Omega} \delta \mathbf{v} \cdot \bar{\mathbf{b}} d\Omega \\
& - \int_{\Gamma_v} \delta \mathbf{v} \cdot \mathbf{t} d\Gamma + \int_{\Gamma_{\theta}} \delta \theta_{\mathbf{n}} M_{nn} d\Gamma + (\mathbf{v} \cdot \mathbf{a}_3 P)_{\mathbf{x} \in C_v} \\
& - \int_{\Gamma_v} \delta \mathbf{t} \cdot (\mathbf{v} - \bar{\mathbf{v}}) d\Gamma + \int_{\Gamma_{\theta}} \delta M_{nn} (\theta_{\mathbf{n}} - \bar{\theta}_{\mathbf{n}}) d\Gamma + (\delta P \mathbf{a}_3 \cdot (\mathbf{v} - \bar{\mathbf{v}}))_{\mathbf{x} \in C_v} \\
& + \sum_{i=1}^3 \alpha_{vi} \int_{\Gamma_v} (\delta \mathbf{v} \cdot \mathbf{a}_i) (\mathbf{a}_i \cdot \mathbf{v}) d\Gamma + \alpha_{\theta} \int_{\Gamma_{\theta}} \delta \theta_{\mathbf{n}} \theta_{\mathbf{n}} d\Gamma + \alpha_C (\delta \mathbf{v} \cdot \mathbf{a}_3 \mathbf{a}_3 \cdot \mathbf{v})_{\mathbf{x} \in C_v} \\
& = 0
\end{aligned} \tag{57}$$

Upon further invoking the conventional reproducing kernel approximation of Eq. (22), the subsequent discrete equilibrium equations can be obtained:

$$(\mathbf{K} + \mathbf{K}^c + \mathbf{K}^s) \mathbf{d} = \mathbf{f} + \mathbf{f}^c + \mathbf{f}^s \tag{58}$$

where the stiffness \mathbf{K} is identical with Eq. (53). \mathbf{K}^c and \mathbf{K}^s are the stiffness matrices for consistent and stabilized terms, respectively, and their components have the following form:

$$\begin{aligned}
\mathbf{K}_{IJ}^c = & - \int_{\Gamma_v} (\Psi_I \mathbf{T}_{NJ} + \mathbf{T}_{NI} \Psi_J) d\Gamma \\
& + \int_{\Gamma_{\theta}} (\Psi_{I,\gamma} n^{\gamma} \mathbf{a}_3 M_{nnJ} + \mathbf{a}_3 M_{nnI} \Psi_{J,\gamma} n^{\gamma}) d\Gamma \\
& + ([\Psi_I \mathbf{a}_3 P_J] + [P_I \mathbf{a}_3 \Psi_J])_{\mathbf{x} \in C_v}
\end{aligned} \tag{59a}$$

$$\mathbf{f}_I^c = - \int_{\Gamma_v} \mathbf{T}_I \cdot \bar{\mathbf{v}} d\Gamma + \int_{\Gamma_{\theta}} M_{nnI} \bar{\theta}_{\mathbf{n}} d\Gamma + [P_I \mathbf{a}_3 \cdot \bar{\mathbf{v}}]_{\mathbf{x} \in C_v} \tag{59b}$$

$$\mathbf{K}_{IJ}^s = \alpha_v \int_{\Gamma_v} \Psi_I \Psi_J d\Gamma + \alpha_{\theta} \int_{\Gamma_{\theta}} \Psi_{I,\eta} n^{\eta} \mathbf{a}_3 \mathbf{a}_3 n^{\gamma} \Psi_{J,\gamma} d\Gamma + \alpha_C [\Psi_I \mathbf{a}_3 \mathbf{a}_3 \Psi_J]_{\mathbf{x} \in C_v} \tag{60a}$$

$$\mathbf{f}_I^s = \alpha_v \int_{\Gamma_v} \Psi_I \bar{\mathbf{v}} d\Gamma + \alpha_{\theta} \int_{\Gamma_{\theta}} \Psi_{I,\eta} n^{\eta} \mathbf{a}_3 \bar{\theta}_{\mathbf{n}} d\Gamma + \alpha_C [\Psi_I \mathbf{a}_3 \mathbf{a}_3 \cdot \bar{\mathbf{v}}]_{\mathbf{x} \in C_v} \tag{60b}$$

with

$$\boldsymbol{\alpha}_v = \begin{bmatrix} \alpha_{v1} & 0 & 0 \\ 0 & \alpha_{v2} & 0 \\ 0 & 0 & \alpha_{v3} \end{bmatrix} \tag{61}$$

On comparing with the consistent terms of Eqs. (54) and (59), the expressions were almost identical, the major difference is that the higher order

330 derivatives of shape functions have been replaced by the smoothed gradients.
 331 Owing to the reproducing kernel framework, the construction of the smoothed
 332 gradients only concerned about the computation of traditional meshfree shape
 333 functions and their first order derivatives, which avoid the costly computation
 334 of higher order derivatives. Moreover, the stabilized terms in Eq. (60) em-
 335 ploys the penalty method with big enough artificial parameters to ensure the
 336 coercivity of stiffness. Besides, the optimal values of these artificial parame-
 337 ters are proportional to the grid size of discrete model that can be represented
 338 by the support size in meshfree approximation, where $\alpha_{v\alpha} \propto s^{-1}$, $\alpha_{v3} \propto s^{-3}$,
 339 $\alpha_\theta \propto s^{-1}$, $\alpha_C \propto s^{-2}$ [37], and $s = \min\{s_{\alpha I}\}$. In contrast, the stabilized term of
 340 Eq. (55) naturally exists in its weak form, and can stabilize the result without
 341 considering any artificial parameters.

342 5. Numerical examples

343 In this section, the suggested method is validated through several exam-
 344 ples using the Nitsche's method, the consistent reproducing kernel gradient
 345 smoothing integration scheme (RKGSI), and the non-consistent Gauss inte-
 346 gration scheme (GI) with penalty method, as well as the proposed Hu-Washizu
 347 formulation (HW) to enforce the necessary boundary conditions. A normalized
 348 support size of 2.5 is used for all the considered methods to ensure the require-
 349 ment of quadratic base meshfree approximation. To eliminate the influence of
 350 integration error, the Gauss integration scheme uses 6 Gauss points for domain
 351 integration and 3 points for boundary integration, so as to maintain the same
 352 integration accuracy between domain and boundaries. Moreover, the number
 353 of integration points are identical between the Gauss and RKGSI schemes. The
 354 error estimates of displacement (L_2 -Error) and energy (H_e -Error) is used here:

$$\begin{aligned}
 L_2\text{-Error} &= \frac{\sqrt{\int_{\Omega} (\mathbf{v} - \mathbf{v}^h) \cdot (\mathbf{v} - \mathbf{v}^h) d\Omega}}{\sqrt{\mathbf{v} \cdot \mathbf{v}}} \\
 H_e\text{-Error} &= \frac{\sqrt{\int_{\Omega} \left((\varepsilon_{\alpha\beta} - \varepsilon_{\alpha\beta}^h)(N^{\alpha\beta} - N^{\alpha\beta h}) + \int_{\Omega} (\kappa_{\alpha\beta} - \kappa_{\alpha\beta}^h)(M^{\alpha\beta} - M^{\alpha\beta h}) \right) d\Omega}}{\sqrt{\int_{\Omega} (\varepsilon_{\alpha\beta} N^{\alpha\beta} + \kappa_{\alpha\beta} M^{\alpha\beta}) d\Omega}}
 \end{aligned}
 \tag{62}$$

355 5.1. Patch tests

356 The linear and quadratic patch tests for flat and curved thin shells are firstly
 357 studied to verify the variational consistency of the proposed method. As shown
 358 in Fig. 3, the flat and curved models are depicted by an identical parametric
 359 domain $\Omega = (0, 1) \otimes (0, 1)$, where the cylindrical coordinate system with radius
 360 $R = 1$, thickness $h = 0.05$ is employed to describe the curved model, and
 361 the whole domain Ω is discretized by the 165 meshfree nodes. The Young's
 362 modulus and Poisson's ratio of thin shell are set to $E = 1$, $\nu = 0$. The artificial
 363 parameters of $\alpha_v = 10^5 \times E$, $\alpha_{\theta} = 10^3 \times E$, $\alpha_C = 10^5 \times E$ and $\alpha_v = 10^9 \times$
 364 E , $\alpha_{\theta} = 10^9 \times E$, $\alpha_C = 10^9 \times E$ were adopted in Nitsche's- and penalty- method,
 365 respectively. All the boundaries are enforced as essential boundary conditions
 366 with the following manufactured exact solution:

$$\mathbf{v} = \begin{Bmatrix} (\xi^1 + 2\xi^2)^n \\ (3\xi^1 + 4\xi^2)^n \\ (5\xi^1 + 6\xi^2)^n \end{Bmatrix}, \quad n = \begin{cases} 1 & \text{Linear patch test} \\ 2 & \text{Quadratic patch test} \end{cases}
 \tag{63}$$

367 Table 1 lists the L_2 - and H_e -Error results of patch test with flat model, where
 368 the RKGSI scheme with variational consistent essential boundary enforcement,
 369 i.e. RKGSI-Nitsche and RKGSI-HW, can pass the linear and quadratic patch
 370 test. In contrast, the RKGSI-Penalty cannot pass the patch test since the
 371 Penalty method is unable to ensure the variational consistency. Due to the
 372 loss of variational consistency condition, even with the Nitsche's method, Gauss

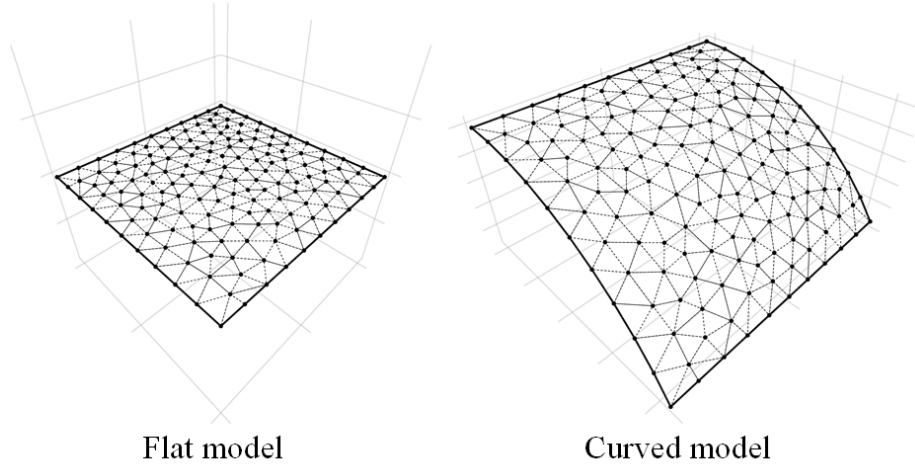


Figure 3: Meshfree discretization for patch test

373 meshfree formulations show noticeable errors. Table 2 shows the results for
 374 curved model, which indicated that all the considered methods cannot pass
 375 the patch test. This is mainly because the proposed smoothed gradient of
 376 Eqs. (35) and (36) could not exactly reproduce the non-polynomial membrane
 377 and bending stresses. On the other hand, the RKGSI-HW and RKGSI-Nitsche
 378 methods provide better accuracy compared to the other approaches due to the
 379 fulfillment of first second-order variational consistency. Even only with local
 380 variational consistency, the RKGSI-Penalty obtained a better result than the
 381 traditional Gauss scheme. Meanwhile, the bending moment contours of M^{12}
 382 are listed in Fig. 4, which further verify that the proposed method provided a
 383 satisfactory result compared to the exact solution. Contrarily, both the RKGSI-
 384 Penalty and the conventional Gauss meshree formulations observed errors.

Table 1: Results of patch test for flat model.

	Linear patch test		Quadratic patch test	
	L_2 -Error	H_e -Error	L_2 -Error	H_e -Error
GI-Penalty	1.41E-04	4.62E-03	1.97E-03	7.17E-03
GI-Nitsche	1.73E-04	5.61E-03	1.85E-03	7.76E-03
RKGSI-Penalty	5.04E-09	1.02E-07	3.01E-09	3.41E-08
RKGSI-Nitsche	9.75E-12	8.98E-11	1.29E-12	1.06E-12
RKGSI-HR	6.15E-13	6.91E-12	7.51E-13	8.36E-12

385 5.2. Scordelis-Lo roof

386 This example considers the classical Scordelis-Lo roof problem, as depicted
 387 in Fig. 5. The cylindrical roof has dimensions $R = 25$, $L = 50$, $h = 0.25$,

Table 2: Results of patch test for cylindrical model.

	Linear patch test		Quadratic patch test	
	L_2 -Error	H_e -Error	L_2 -Error	H_e -Error
GI-Penalty	1.75E-04	4.50E-03	1.08E-03	5.83E-03
GI-Nitsche	1.77E-04	5.36E-03	1.07E-03	6.33E-03
RKGSi-Penalty	8.59E-05	9.11E-04	4.28E-04	2.08E-03
RKGSi-Nitsche	1.27E-05	5.32E-05	1.88E-05	5.6E-04
RKGSi-HR	1.43E-05	1.60E-04	2.93E-05	2.85E-04

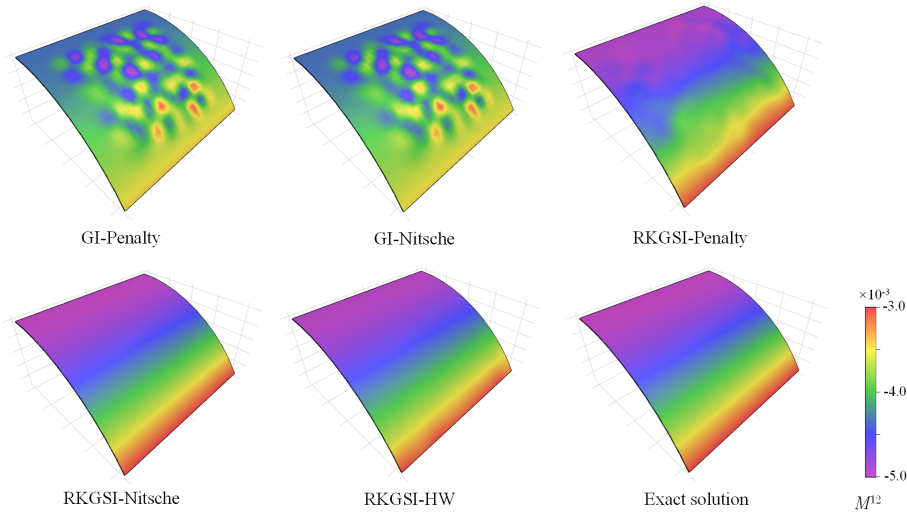


Figure 4: Contour plots of M^{12} for curved shell patch test.

388 Young's modulus $E = 4.32 \times 10^8$ and Poisson's ratio $\nu = 0.0$. The entire roof
389 is subjected to an uniform body force of $b_z = -90$, with the straight edges
390 remaining free and the the curved edges are enforced by $v_x = v_z = 0$.

391 Due to the symmetry, only a quadrant of the model is considered for meshfree
392 analysis, which is discretized by the 11×16 , 13×20 , 17×24 and 19×28 meshfree
393 nodes, as listed in Fig. 6. The comparison of the displacement in z -direction
394 at node A , v_{A3} , is used as the investigated quantity, with the reference value
395 0.3006 given by [9]. Firstly, Fig. 7 presents a sensitivity study for the artificial
396 parameters of α_{vi} 's and α_θ 's in the RKGSi meshfree formulations with the
397 Nitsche's- and penalty- method, where all of the parameters are scaled by the
398 support size as, $\alpha_{v\alpha} = s^{-1}\bar{\alpha}_v$, $\alpha_{v3} = s^{-3}\bar{\alpha}_v$ and $\alpha_\theta = s^{-1}\bar{\alpha}_\theta$. For a better
399 comparison, the result of the proposed RKGSi-HW is also listed in this figure.
400 The results of Fig. 7 revealed, that Nitsche's method observed less artificial
401 sensitivity. However, both the methods cannot trivially determine the optimal
402 values of the artificial parameters. The optimal artificial parameters from Fig.

7 are adopted for the convergence study in Fig. 8. The convergence result showed that the RKGSi method get satisfactory results while the traditional Gauss methods demonstrated noticeable errors.

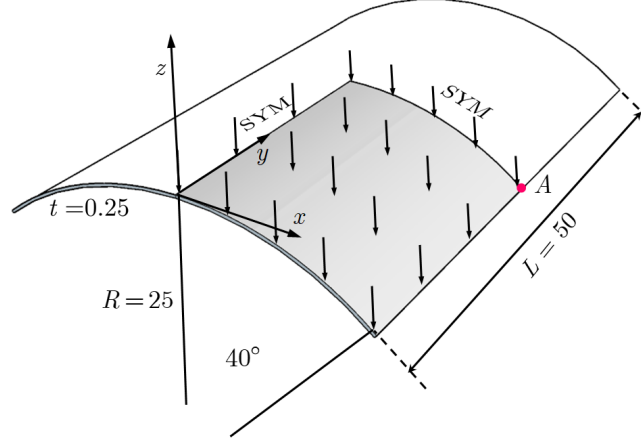


Figure 5: Description of Scordelis-Lo roof problem.

5.3. Pinched Hemispherical shell

Consider the hemispherical shell shown in Fig. 9, which is loaded at four points $P = \pm 2$ at 90° interval at its bottom. The hemispherical shell has an radius $R = 10$, thickness $h = 0.04$, Young's modulus $E = 6.825 \times 10^7$ and Poisson's ratio $\nu = 0.3$.

Due to symmetry, only quadrant model, where the 16×16 , 24×24 , 32×32 and 40×40 meshfree nodes have been discretized as shown in Fig. (10), were considered. The quantity under investigation for convergence is the displacement at x -direction on point A, $v_{A1} = 0.094$ [40]. Fig. 11 displays the corresponding convergence results, indicating the RKGSi scheme performed significantly better compared to the GI meshfree formulation. Meanwhile, the efficiency comparison for this problem is also shown in Fig. 12, in which the CPU time for assembly and calculation of shape functions are considered. Fig. 12(a) indicates that the RKGSi scheme observed high efficiency in assembly. This is due to the variational inconsistent Gauss meshfree formulation which require more Gaussian points to get satisfactory results. Fig. 12(b) lists the CPU time spent on enforcing essential boundary conditions for the penalty method, Nitsche's method and proposed HW method. The results highlighted that the proposed HW method consumed comparable CPU time in assembly compared to Nitsche's method. However, less time was spent to calculate the shape functions. Since both the HW method and penalty method were developed considering the shape functions first order derivatives. For this reason, both the methods shared an almost identical time in computing the shape functions.

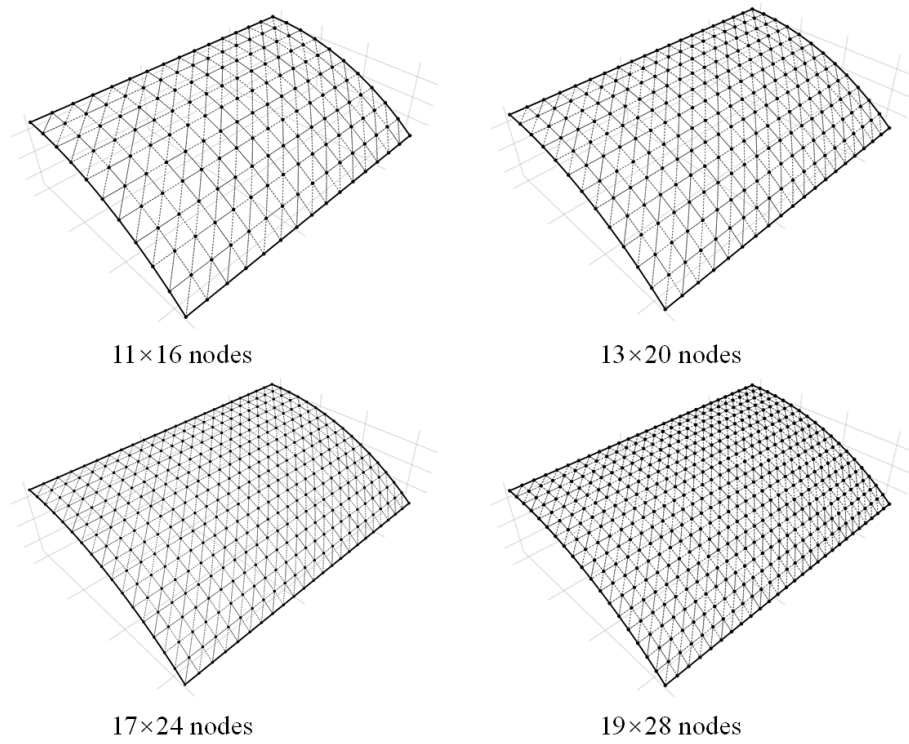


Figure 6: Meshfree discretizations for Scordelis-Lo roof problem.

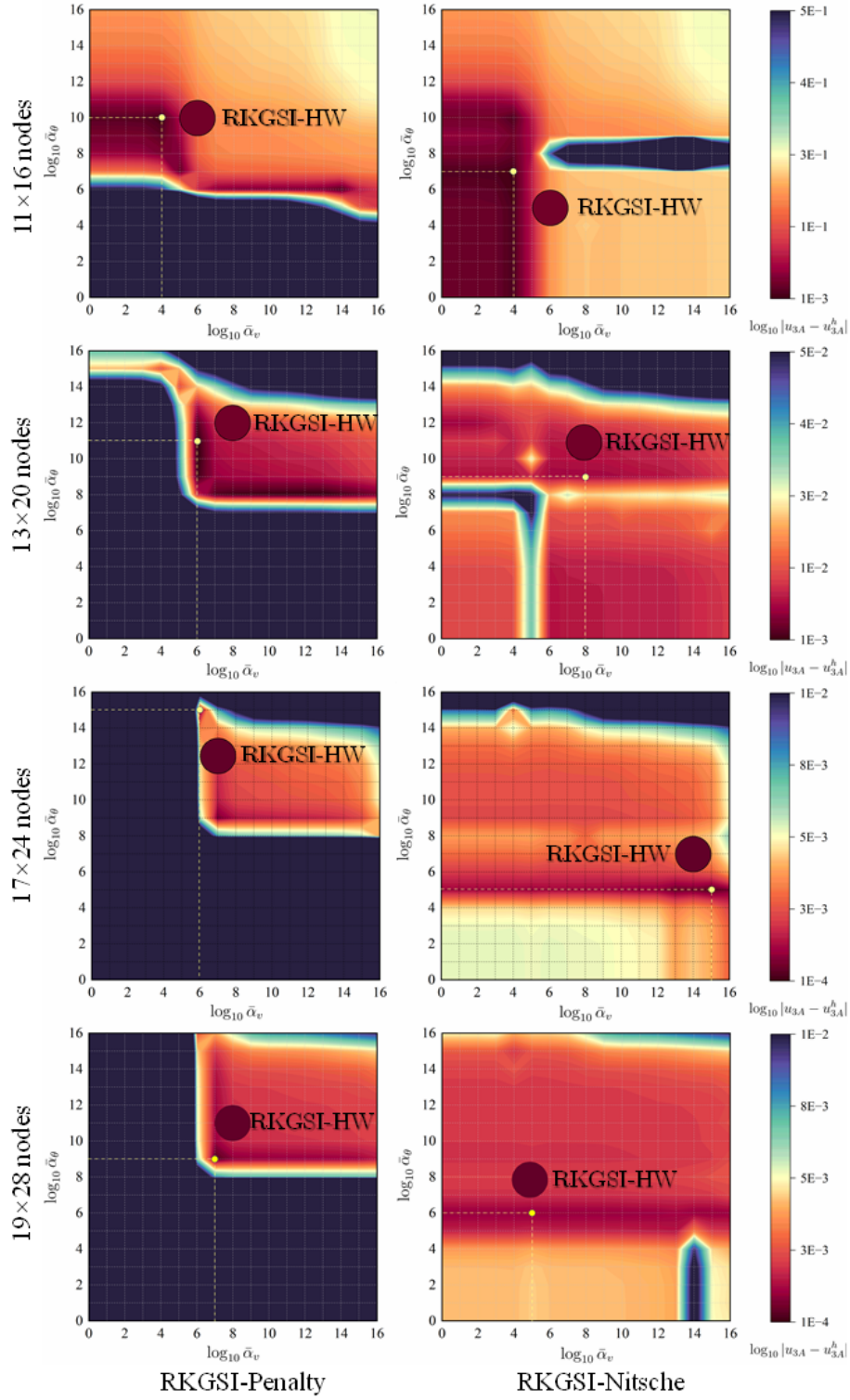


Figure 7: Sensitivity comparison of α_v and α_θ for Scordelis-Lo problem.

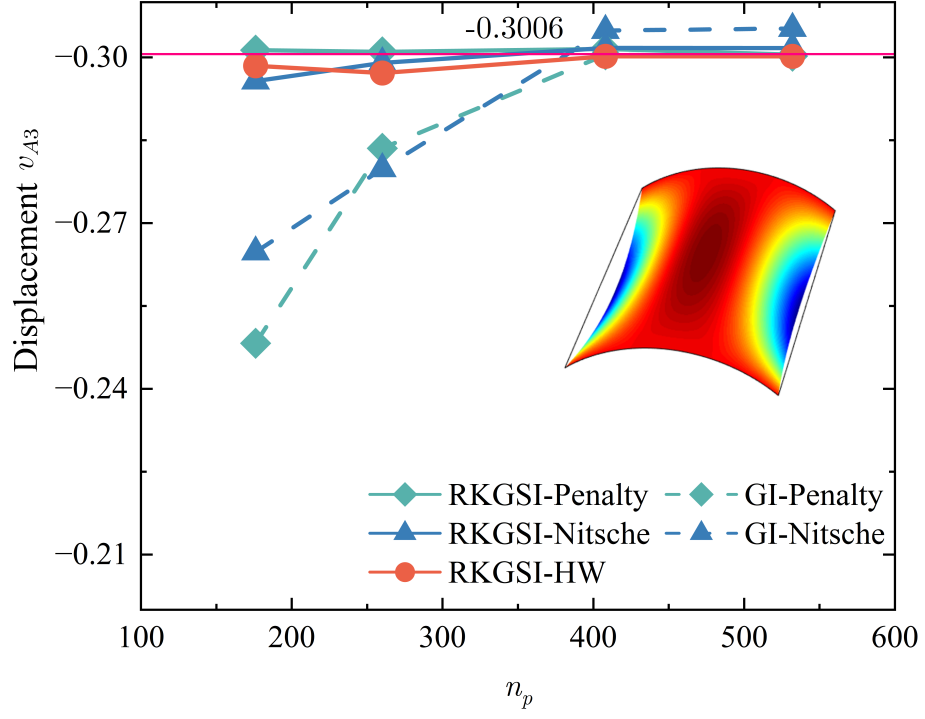


Figure 8: Displacement convergence for Scordelis-Lo roof problem.

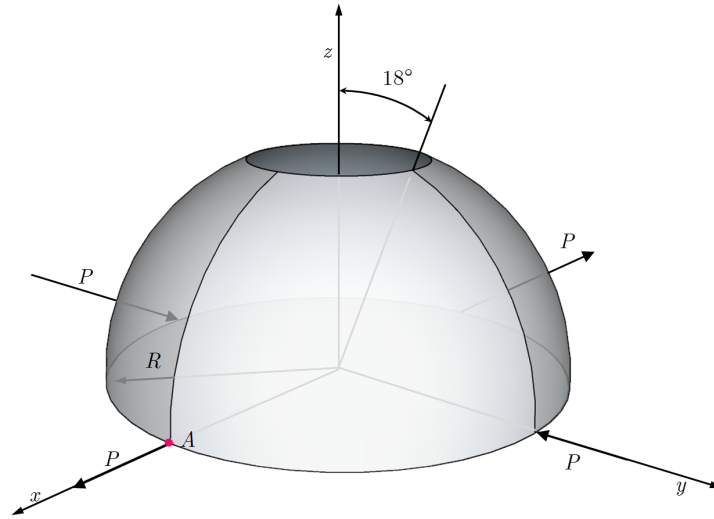
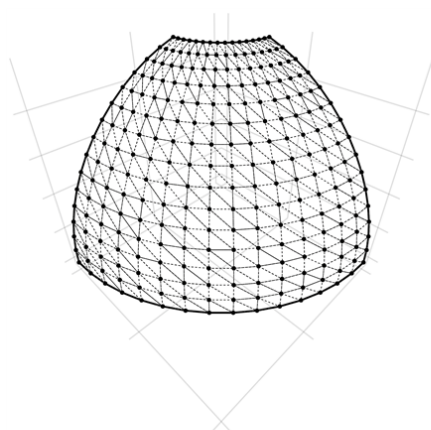
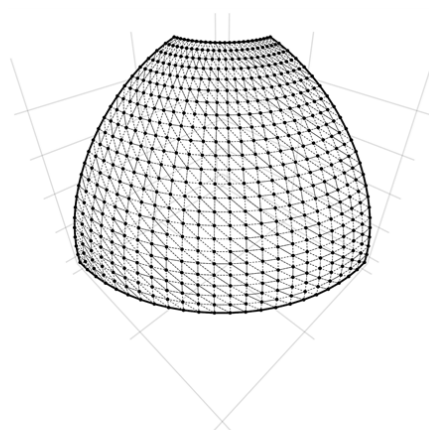


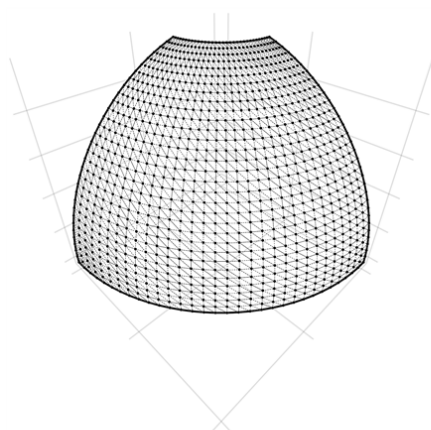
Figure 9: Description of pinched hemispherical shell problem.



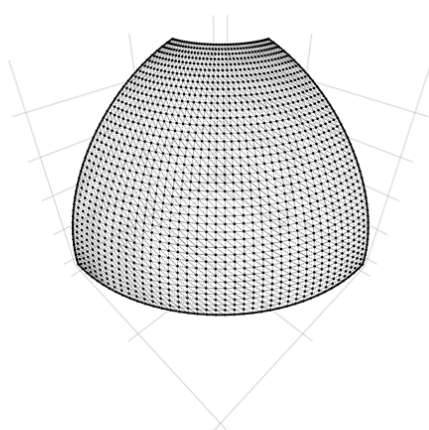
16×16 nodes



24×24 nodes



32×32 nodes



40×40 nodes

Figure 10: Meshfree discretizations for pinched hemispherical shell problem.

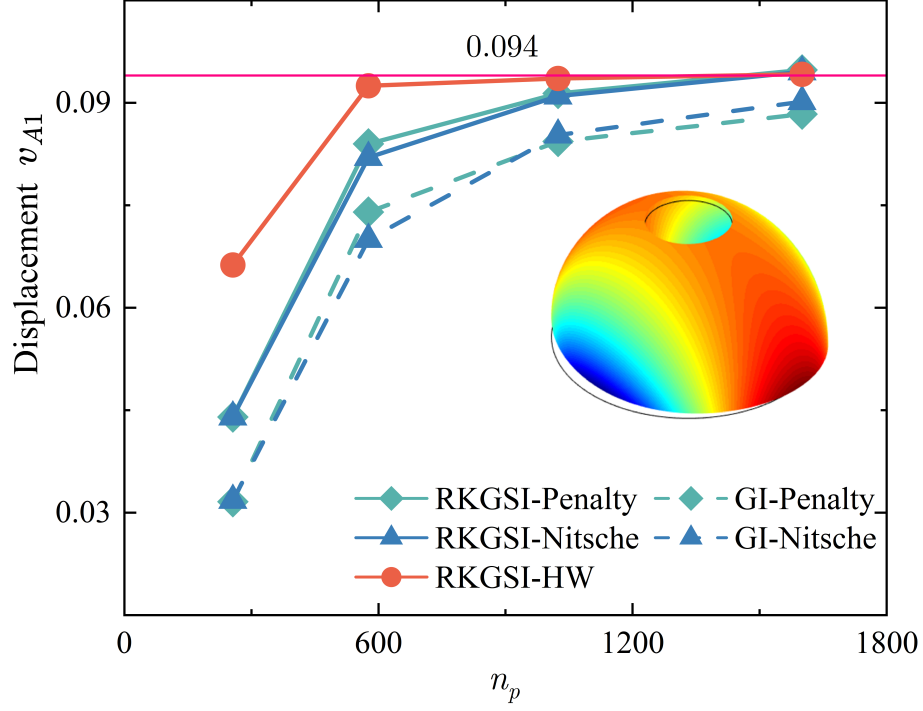


Figure 11: Displacement convergence for pinched hemispherical shell problem.

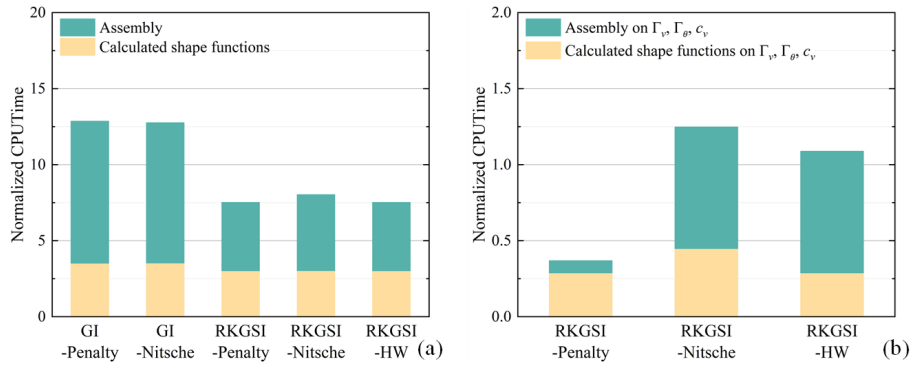


Figure 12: Efficiency comparison for pinched hemispherical shell problem: (a) Whole domain; (b) Essential boundaries

429 6. Conclusion

430 In this study, an efficient and quasi-consistent meshfree thin shell formu-
431 lation was presented to naturally enforce the essential boundary conditions.
432 Mixed formulation with the Hu-Washizu principle weak form is adopted, where
433 the traditional meshfree shape functions discretized the displacement, and the
434 strains and stresses were expressed by the reproducing kernel smoothed gradi-
435 ents and the covariant bases, respectively. The smoothed gradient naturally
436 embedded the first second-order integration constraints and has a quasi varia-
437 tional consistency for the curved models in each integration cell. Owing to the
438 Hu-Washizu variational principle, the essential boundary condition enforcement
439 has a similar form with the conventional Nitsche’s method; both have consistent
440 and stabilized terms. The costly high order derivatives in the Nitsche’s consis-
441 tent term have been replaced by the smoothed gradients, which improved the
442 computational speed due to the reproducing kernel gradient smoothing frame-
443 work. Furthermore, the stabilized term naturally existed in the Hu-Washizu
444 weak form, and the artificial parameter needed in Nitsche’s stabilized term has
445 vanished, which can automatically maintain the coercivity for the stiffness ma-
446 trix. Based on general reproducing kernel gradient smoothing framework, the
447 proposed methodology can be trivially extended to high order basis meshfree for-
448 mulation. The numerical results demonstrated that the proposed Hu-Washizu
449 quasi-consistent meshfree thin shell formulation showed excellent accuracy, ef-
450 ficiency, and stability.

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455 Appendix A. Green's theorems for in-plane vector

456 This Appendix discusses two kinds of Green's theorems used for the devel-
 457 opment of the proposed meshfree method. For an arbitrary vectors v^α and a
 458 scalar function f , with Green's theorem for in-plane vector, the first Green's
 459 theorem is listed as follows [37]:

$$\begin{aligned} \int_{\Omega} f_{,\alpha} v^\alpha d\Omega &= \int_{\Gamma} f v^\alpha n_\alpha d\Gamma - \int_{\Omega} f (v_{,\alpha}^\alpha + \Gamma_{\beta\alpha}^\beta v^\alpha) d\Omega \\ &= \int_{\Gamma} f v^\alpha n_\alpha d\Gamma - \int_{\Omega} f v^\alpha|_\alpha d\Omega \end{aligned} \quad (\text{A.1})$$

460 where $\Gamma_{\alpha\beta}^\gamma = \mathbf{a}_{\alpha,\beta} \cdot \mathbf{a}^\gamma$ denotes the Christoffel symbol of the second kind. $v^\alpha|_\alpha$
 461 can be represented as the in-plane covariant derivative of the vector v^α :

$$v^\alpha|_\alpha = v_{,\alpha}^\alpha + \Gamma_{\beta\alpha}^\beta v^\alpha \quad (\text{A.2})$$

462 The second Green's theorem is established with a mixed form of second
 463 order derivative. Let $A^{\alpha\beta}$ can be an arbitrary symmetric second order tensor,
 464 the Green's theorem yields [37]:

$$\begin{aligned} \int_{\Omega} f_{,\alpha}|_\beta A^{\alpha\beta} d\Omega &= \int_{\Gamma} f_{,\gamma} n^\gamma A^{\alpha\beta} n_\alpha n_\beta d\Gamma - \int_{\Gamma} f (A^{\alpha\beta} s_\alpha n_\beta)_{,\gamma} s^\gamma d\Gamma + [[f A^{\alpha\beta} s_\alpha n_\beta]]_{\mathbf{x} \in C} \\ &\quad - \int_{\Gamma} f (A_{,\beta}^{\alpha\beta} n_\alpha + \Gamma_{\alpha\beta}^\gamma A^{\alpha\beta} n_\gamma + \Gamma_{\gamma\beta}^\gamma A^{\alpha\beta} n_\alpha) d\Gamma \\ &\quad + \int_{\Omega} f \left(\Gamma_{\alpha\beta,\gamma}^\gamma A^{\alpha\beta} + \Gamma_{\alpha\beta}^\gamma A_{,\gamma}^{\alpha\beta} + \Gamma_{\eta\gamma}^\eta \Gamma_{\alpha\beta}^\gamma A^{\alpha\beta} \right. \\ &\quad \left. + A_{,\alpha\beta}^{\alpha\beta} + \Gamma_{\gamma\beta,\alpha}^\gamma A^{\alpha\beta} + 2\Gamma_{\gamma\alpha}^\gamma A_{,\beta}^{\alpha\beta} + \Gamma_{\gamma\alpha}^\gamma \Gamma_{\eta\beta}^\eta A^{\alpha\beta} \right) d\Omega \\ &= \int_{\Gamma} f_{,\gamma} n^\gamma A^{\alpha\beta} n_\alpha n_\beta d\Gamma - \int_{\Gamma} f (A^{\alpha\beta} s_\alpha n_\beta)_{,\gamma} s^\gamma d\Gamma + [[f A^{\alpha\beta} s_\alpha n_\beta]]_{\mathbf{x} \in C} \\ &\quad - \int_{\Gamma} f A^{\alpha\beta}|_\beta n_\alpha d\Gamma + \int_{\Omega} f A^{\alpha\beta}|_{\alpha\beta} d\Omega \end{aligned} \quad (\text{A.3})$$

465 with

$$A^{\alpha\beta}|_\beta = A_{,\beta}^{\alpha\beta} + \Gamma_{\beta\gamma}^\alpha A^{\beta\gamma} + \Gamma_{\gamma\beta}^\gamma A^{\alpha\beta} \quad (\text{A.4})$$

$$\begin{aligned} A^{\alpha\beta}|_{\alpha\beta} &= \Gamma_{\alpha\beta,\gamma}^\gamma A^{\alpha\beta} + \Gamma_{\alpha\beta}^\gamma A_{,\gamma}^{\alpha\beta} + \Gamma_{\eta\gamma}^\eta \Gamma_{\alpha\beta}^\gamma A^{\alpha\beta} \\ &\quad + A_{,\alpha\beta}^{\alpha\beta} + \Gamma_{\gamma\beta,\alpha}^\gamma A^{\alpha\beta} + 2\Gamma_{\gamma\alpha}^\gamma A_{,\beta}^{\alpha\beta} + \Gamma_{\gamma\alpha}^\gamma \Gamma_{\eta\beta}^\eta A^{\alpha\beta} \end{aligned} \quad (\text{A.5})$$

467 For the sake of brevity, the notion of covariant derivative is extended to a
 468 scalar function as:

$$f|_\alpha = f_{,\alpha} + \Gamma_{\beta\alpha}^\beta f \quad (\text{A.6})$$

$$f|_\beta n_\alpha = f_{,\beta} n_\alpha + \Gamma_{\alpha\beta}^\gamma f n_\gamma + \Gamma_{\gamma\beta}^\gamma f n_\alpha \quad (\text{A.7})$$

$$\begin{aligned} f|_{\alpha\beta} &= \Gamma_{\alpha\beta,\gamma}^\gamma f + \Gamma_{\alpha\beta}^\gamma f_{,\gamma} + \Gamma_{\eta\gamma}^\eta \Gamma_{\alpha\beta}^\gamma f \\ &\quad + f_{,\alpha\beta} + \Gamma_{\gamma\beta,\alpha}^\gamma f + 2\Gamma_{\gamma\alpha}^\gamma f_{,\beta} + \Gamma_{\gamma\alpha}^\gamma \Gamma_{\eta\beta}^\eta f \end{aligned} \quad (\text{A.8})$$

471 Appendix B. Derivations for stiffness metrics and force vectors

472 This Appendix details the derivations of stiffness matrices and force vectors
 473 in Eqs. (53)-(55), where the relationships of Eqs. (40), (41), (44) and (46) are
 474 used herein. Firstly, the membrane strain terms are considered as follows:

$$\begin{aligned}
 & \sum_{C=1}^{n_e} \int_{\Omega_C} \delta \tilde{\varepsilon}_{\alpha\beta}^h h C^{\alpha\beta\gamma\eta} \tilde{\varepsilon}_{\gamma\eta}^h d\Omega \\
 &= \sum_{C=1}^{n_e} \sum_{I,J=1}^{n_p} \delta \mathbf{d}_I \cdot \underbrace{\int_{\Omega_C} \tilde{\varepsilon}_{\alpha\beta I} h C^{\alpha\beta\gamma\eta} \mathbf{a}_\gamma \mathbf{q}^T d\Omega \mathbf{G}^{-1} \bar{\mathbf{g}}_{\eta J}}_{\tilde{\mathbf{g}}_I^{\eta T}} \cdot \mathbf{d}_J \\
 &= \sum_{C=1}^{n_e} \sum_{I,J=1}^{n_p} \delta \mathbf{d}_I \cdot \int_{\Gamma_C \cap \Gamma_v} \Psi_J \mathbf{q}^T \underbrace{\mathbf{G}^{-1} \tilde{\mathbf{g}}_I^\alpha n_\alpha}_{\tilde{\mathbf{T}}_{NI}} d\Gamma \cdot \mathbf{d}_J \\
 &= \sum_{I,J=1}^{n_p} \delta \mathbf{d}_I \cdot \int_{\Gamma_v} \tilde{\mathbf{T}}_{NI} \Psi_J d\Gamma \cdot \mathbf{d}_J
 \end{aligned} \tag{B.1}$$

475 with

$$476 \quad \tilde{\mathbf{g}}_I^\alpha = \mathbf{q} \mathbf{a}_\beta h C^{\alpha\beta\gamma\eta} \tilde{\varepsilon}_{\gamma\eta I} \tag{B.2}$$

$$477 \quad \tilde{\mathbf{T}}_{NI} = \mathbf{q}^T \mathbf{G}^{-1} \tilde{\mathbf{g}}_I^\alpha n_\alpha \tag{B.3}$$

Following this path, the bending strain terms can be reorganized by:

$$\begin{aligned}
 & \sum_{C=1}^{n_e} \int_{\Omega_C} \delta \tilde{\kappa}_{\alpha\beta}^h \frac{h^3}{12} C^{\alpha\beta\gamma\eta} \tilde{\kappa}_{\gamma\eta}^h d\Omega \\
 &= \sum_{C=1}^{n_e} \sum_{I,J=1}^{n_p} \delta \mathbf{d}_I \cdot \underbrace{\int_{\Omega_C} \tilde{\kappa}_{\alpha\beta I} \frac{h^3}{12} C^{\alpha\beta\gamma\eta} \mathbf{a}_3 \mathbf{q}^T d\Omega \mathbf{G}^{-1} \bar{\mathbf{g}}_{\gamma\eta J}}_{\tilde{\mathbf{g}}_I^{\gamma\eta T}} \cdot \mathbf{d}_J \\
 &= \sum_{C=1}^{n_e} \sum_{I,J=1}^{n_p} \delta \mathbf{d}_I \cdot \left(\begin{aligned} & \int_{\Gamma_C \cap \Gamma_\theta} \underbrace{\mathbf{q}^T \mathbf{G}^{-1} \tilde{\mathbf{g}}_I^{\alpha\beta} n_\alpha n_\beta}_{\tilde{\mathbf{M}}_{nnI}} n^\gamma \Psi_{J,\gamma} d\Gamma \\ & - \int_{\Gamma_C \cap \Gamma_v} \underbrace{(\mathbf{q}_{|\beta}^T \mathbf{G}^{-1} \tilde{\mathbf{g}}_I^{\alpha\beta} n_\alpha + (\mathbf{q}^T \mathbf{G}^{-1} \tilde{\mathbf{g}}_I^{\alpha\beta} s_\alpha n_\beta)_{,\gamma} s^\gamma)}_{\tilde{\mathbf{T}}_{MI}} \Psi_J d\Gamma \\ & + [[\underbrace{\mathbf{q}^T \mathbf{G}^{-1} \tilde{\mathbf{g}}_I^{\alpha\beta} s_\alpha n_\beta}_{\tilde{\mathbf{P}}_I \mathbf{a}_3} \Psi_J]]_{\mathbf{x} \in C_C \cap C_v} \end{aligned} \right) \cdot \mathbf{d}_J \\
 &= \sum_{I,J=1}^{n_p} \delta \mathbf{d}_I \cdot \left(\int_{\Gamma_\theta} \tilde{\mathbf{M}}_{nnI} n^\gamma \Psi_{J,\gamma} d\Gamma - \int_{\Gamma_v} \tilde{\mathbf{T}}_{MI} \Psi_J d\Gamma + [[\tilde{\mathbf{P}}_I \Psi_J]]_{\mathbf{x} \in C_v} \right)
 \end{aligned} \tag{B.4}$$

478 with

$$\tilde{\mathbf{g}}_I^{\alpha\beta} = \int_{\Omega_C} \mathbf{q} \frac{h^3}{12} C^{\alpha\beta\gamma\eta} \mathbf{a}_3 \tilde{\kappa}_{\gamma\eta I} d\Omega \quad (\text{B.5})$$

479

$$\begin{cases} \tilde{M}_{nnI} = \mathbf{q}^T \mathbf{G}^{-1} \tilde{\mathbf{g}}_I^{\alpha\beta} n_\alpha n_\beta \\ \tilde{\mathbf{T}}_{MI} = \mathbf{q}_{|\beta}^T \mathbf{G}^{-1} \tilde{\mathbf{g}}_I^{\alpha\beta} n_\alpha + (\mathbf{q}^T \mathbf{G}^{-1} \tilde{\mathbf{g}}_I^{\alpha\beta} s_\alpha n_\beta)_{,\gamma} s^\gamma \\ \tilde{\mathbf{P}}_I = \mathbf{q}^T \mathbf{G}^{-1} \tilde{\mathbf{g}}_I^{\alpha\beta} s_\alpha n_\beta \cdot \mathbf{a}_3 \end{cases} \quad (\text{B.6})$$

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