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Author1[1,], Author2[1]

[a] *Key Laboratory for Intelligent Infrastructure and Monitoring of Fujian Province, College of Civil Engineering, Huaqiao University , Xiamen, Fujian, 361021, China*

Abstract

This is the abstract.

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*Corresponding author

1. Introduction

2. Mixed and penalty formulations for nearly-incompressible elasticity problems

2.1. Penalty formulation

Consider a body $\Omega \in \mathbb{R}^{n_d}$ with boundary Γ in n_d-dimension, where the Γ_t and Γ_g denotes its natural boundary and essential boundary such that $\Gamma_t \cup \Gamma_g = \Gamma$, $\Gamma_t \cap \Gamma_g = \varnothing$. The corresponding governing equations are given by:

$$\begin{cases} \nabla \cdot \boldsymbol{\sigma} + \boldsymbol{b} = \boldsymbol{0} & \text{in } \Omega \\ \boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{t} & \text{on } \Gamma_t \\ \boldsymbol{u} = \boldsymbol{g} & \text{on } \Gamma_g \end{cases} \tag{1}$$

in which $\boldsymbol{\sigma}$ denotes to stress tensor and, for isotropic linear elastic material, can be expressed by:

$$\boldsymbol{\sigma}(\boldsymbol{u}) = 3\kappa\boldsymbol{\varepsilon}^v(\boldsymbol{u}) + 2\mu\boldsymbol{\varepsilon}^d(\boldsymbol{u}) \tag{2}$$

where $\boldsymbol{\varepsilon}^v$ and $\boldsymbol{\varepsilon}^d$ are the volumetric(dilatation) and deviatoric parts of strain tensor $\boldsymbol{\varepsilon}$, and these are evaluated by:

$$\boldsymbol{\varepsilon}^v(\boldsymbol{u}) = \frac{1}{3}\nabla \cdot \boldsymbol{u}\ \boldsymbol{1}, \quad \boldsymbol{\varepsilon}^d(\boldsymbol{u}) = \frac{1}{2}(\boldsymbol{u}\nabla + \nabla\boldsymbol{u}) - \boldsymbol{\varepsilon}^v, \quad \boldsymbol{\varepsilon}^v : \boldsymbol{\varepsilon}^d = 0 \tag{3}$$

where $\boldsymbol{1} = \delta_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j$ is second order identity tensor. κ, μ are the bulk modulus and shear modulus, and they can be represented by Young's modulus E and Poisson's ratio ν:

$$\kappa = \frac{E}{2(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)} \tag{4}$$

And $\boldsymbol{b}$ denotes to prescribed body force in Ω. $\boldsymbol{t}$, $\boldsymbol{g}$ are prescribed traction and displacement on natural and essential boundaries respectively.

In accordance with Galerkin formulation, the displacement denoted by $\boldsymbol{u}$ can be got by the following weak problem: Find $\boldsymbol{u} \in V$

$$\int_\Omega 2\mu\delta\boldsymbol{\varepsilon}^d : \boldsymbol{\varepsilon}^d d\Omega + \int_\Omega 3\kappa\delta\boldsymbol{\varepsilon}^v : \boldsymbol{\varepsilon}^v d\Omega = \int_{\Gamma_t} \delta\boldsymbol{u}\cdot\boldsymbol{t}d\Gamma + \int_\Omega \delta\boldsymbol{u}\cdot\boldsymbol{b}d\Omega, \quad \forall\delta\boldsymbol{u} \in V \tag{5}$$

where V is the spaces defined by $V = \{\boldsymbol{v} \in H^1(\Omega)^2 \mid \boldsymbol{v} = \boldsymbol{g},\ \text{on } \Gamma_g\}$. $\delta\boldsymbol{u}$ is the virtual counterpart of $\boldsymbol{u}$, and $\delta\boldsymbol{\varepsilon}^v$ and $\delta\boldsymbol{\varepsilon}^d$ are the corresponding volumetric and deviatoric strain evaluated by $\delta\boldsymbol{u}$.

In traditional finite element formulation, the entire domain Ω is discretized by a set of construct mesh with vertices $\{\boldsymbol{x}_I\}_{I=1}^{n_u}$ [?], where n_u is the total number of vertices. Then, the displacement and its virtual counterpart can be approximated by the nodal coefficient and shape functions at $\boldsymbol{x}_I$'s, the approximated displacement and its virtual counterpart, namely $\boldsymbol{u}_h, \delta\boldsymbol{u}_h$ have the following form:

$$\boldsymbol{u}_h(\boldsymbol{x}) = \sum_{I=1}^{n_u} N_I(\boldsymbol{x})\boldsymbol{u}_I, \quad \delta\boldsymbol{u}_h(\boldsymbol{x}) = \sum_{I=1}^{n_u} N_I(\boldsymbol{x})\delta\boldsymbol{u}_I \tag{6}$$

where N_I and $\boldsymbol{u}_I$ are the shape function and nodal coefficient tensor at node $\boldsymbol{x}_I$. Introducing Eq. (**??**) to weak form of Eq. (**??**) leads to the following Ritz-Galerkin problem: Find $\boldsymbol{u}_h \in V_h$,

$$\int_\Omega 2\mu\delta\boldsymbol{\varepsilon}_h^d : \boldsymbol{\varepsilon}_h^d d\Omega + \int_\Omega 3\kappa\delta\boldsymbol{\varepsilon}_h^v : \boldsymbol{\varepsilon}_h^v d\Omega = \int_{\Gamma_t} \delta\boldsymbol{u}_h \cdot \boldsymbol{t} d\Gamma + \int_\Omega \delta\boldsymbol{u}_h \cdot \boldsymbol{b} d\Omega, \quad \forall \delta\boldsymbol{u}_h \in V_h \tag{7}$$

where the approximate spaces $V_h \subseteq V$,

$$V_h = \{\boldsymbol{v}_h \in (\text{span}\{N_I\}_{I=1}^{n_u})^2 | \boldsymbol{v}^h = \boldsymbol{g}, \text{ on } \Gamma_g\} \tag{8}$$

For the arbitrariness of $\delta\boldsymbol{u}_h$, the above equation can be reduced by elimination of $\delta\boldsymbol{u}_I$'s as the following discrete equilibrium equation:

$$(2\mu\boldsymbol{K}^d + 3\kappa\boldsymbol{K}^v)\boldsymbol{d}^u = \boldsymbol{f} \tag{9}$$

where $\boldsymbol{K}^v$ and $\boldsymbol{K}^d$ are the volumetric and deviatoric stiffness matrices, and their components has the following forms:

$$\boldsymbol{K}_{IJ}^v = \int_\Omega \boldsymbol{B}_I^{vT} \boldsymbol{B}_J^v d\Omega \tag{10}$$

$$\boldsymbol{K}_{IJ}^d = \int_\Omega \boldsymbol{B}_I^{dT} \boldsymbol{B}_J^d d\Omega \tag{11}$$

with and $\boldsymbol{f}$ is the force vector and its components can be expressed by:

$$\boldsymbol{f}_I = \int_{\Gamma_t} N_I \boldsymbol{t} d\Gamma + \int_\Omega N_I \boldsymbol{b} d\Omega \tag{12}$$

$\boldsymbol{d}^u$ is the coefficient vector containning $\boldsymbol{u}_I$'s.

It can be observed from Eq. (**??**) that, for a nearly-incompressible material, i.e. $\nu \to 0.5$, $\kappa \to \infty$. As a result, the volumetric stiffness matrix $\boldsymbol{K}^v$ of (**??**) services as an enforcement like penalty method to enforce the volumetric deformation to be zero, $\nabla \cdot \boldsymbol{u} = 0$, while the bulking modulus κ can be regarded as a penalty parameter. Traditional finite element formulations suffer severe volumetric locking due to this enforcement, and this is so-called volumetric locking. To reduce the burden of volumetric locking, the reduced the integration points in volumetric stiffness matrix. For clarity, substituting numerical integration to the volumetric part of weak form in Eq. (**??**) leads to:

$$\int_\Omega 3\kappa\delta\boldsymbol{\varepsilon}_h^v : \boldsymbol{\varepsilon}_h^v d\Omega \approx \sum_{C=1}^{n_e} \sum_{G=1}^{n_g} 3\kappa \nabla \cdot \delta\boldsymbol{u}_h(\boldsymbol{x}_G) \nabla \cdot \boldsymbol{u}_h(\boldsymbol{x}_G) w_G \tag{13}$$

The corresponding components of volumetric stiffness $\boldsymbol{K}^v$ in Eq. (**??**) yields:

$$\boldsymbol{K}_{IJ}^v \approx \bar{\boldsymbol{K}}_{IJ}^v = \sum_{C=1}^{n_e} \sum_{G=1}^{n_g} \boldsymbol{B}_I^{vT}(\boldsymbol{x}_G) \boldsymbol{B}_J^v(\boldsymbol{x}_G) w_G \tag{14}$$

where $\boldsymbol{x}_G$'s and w_G's are the positions and weights of integration points. n_g is the total number of integration points in each element, thus the total integration point is $n_c \times n_g$. The reduced integration formulations use less number of integration points compared with traditional full integration scheme. For instance, the conventional quadrilateral element use 2×2 Gauss integration points as full integration, the full integration means that the stiffness matrix is exactly evaluated by this integration scheme. And for reduced integration formulation, the number of integration points is reduced from 4 to 1.

2.2. Mixed formulation

Another approach to alleviate the volumetric locking is using the mixed-formulation. In this approach, the pressure is approximated by another way as follows:

$$\boldsymbol{\sigma}(\boldsymbol{u}, p) = p\mathbf{1} + 2\mu\boldsymbol{\varepsilon}^d(\boldsymbol{u}) \tag{15}$$

The strong form for mixed-formulation can be rephrased as:

$$\begin{cases} \nabla \cdot \boldsymbol{\sigma} + \boldsymbol{b} = \mathbf{0} & \text{in } \Omega \\ \frac{p}{\kappa} + \nabla \cdot \boldsymbol{u} = 0 & \text{in } \Omega \\ \boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{t} & \text{on } \Gamma_t \\ \boldsymbol{u} = \boldsymbol{g} & \text{on } \Gamma_g \end{cases} \tag{16}$$

where $p \in Q$, $Q = \{q \in L^2(\Omega) | \int_\Omega q d\Omega = 0\}$.

In traditional mixed formulations, the pressure p are discretized by different sets of controlled nodes, namely displacement nodes $\{\boldsymbol{x}_I\}_{I=1}^{n_d}$ and pressure nodes $\{\boldsymbol{x}_K\}_{K=1}^{n_p}$, where n_d and n_p are the total number of displacement nodes and pressure nodes. And then the approximate displacement denoted by $\boldsymbol{u}_h$ and approximate pressure denoted by p_h can be expressed by

$$p_h(\boldsymbol{x}) = \sum_{K=1}^{n_p} \Psi_K(\boldsymbol{x}) p_K, \quad \delta p_h(\boldsymbol{x}) = \sum_{K=1}^{n_p} \Psi_K(\boldsymbol{x}) \delta p_K \tag{17}$$

where p_K's are the coefficients. and N_I^d, N_K^p are the corresponding shape functions. the corresponding Ritz-Galerkin problem is that: Find $\boldsymbol{u}_h \in V_h$, $p_h \in Q_h$

$$\int_\Omega 2\mu\delta\boldsymbol{\varepsilon}_h^d : \boldsymbol{\varepsilon}_h^d d\Omega + \int_\Omega \nabla \cdot \delta\boldsymbol{u}_h p_h d\Omega = \int_{\Gamma_t} \delta\boldsymbol{u}_h \cdot \boldsymbol{t} d\Gamma + \int_\Omega \delta\boldsymbol{u}_h \cdot \boldsymbol{b} d\Omega, \quad \forall \delta\boldsymbol{u}_h \in V_h \tag{18a}$$

$$\int_\Omega \delta p_h \nabla \cdot \boldsymbol{u}_h d\Omega - \int_\Omega \frac{1}{3\kappa} \delta p_h p_h d\Omega = 0, \qquad \forall \delta p_h \in Q_h \tag{18b}$$

where $Q_h \subseteq Q$ are defined by:

$$Q_h = \{q_h \in \text{span}\{\Psi_K\}_{K=1}^{n_p} | \int_\Omega q_h d\Omega = 0\} \tag{19}$$

With the arbitrariness of $\boldsymbol{v}_h$ and q_h, the Eq.(**??**) leads to the following discrete governing equations:

$$\begin{bmatrix} 2\mu\boldsymbol{K}^{uu} & \boldsymbol{K}^{up} \\ (\boldsymbol{K}^{up})^T & -\frac{1}{3\kappa}\boldsymbol{K}^{pp} \end{bmatrix} \begin{Bmatrix} \boldsymbol{d}^u \\ \boldsymbol{d}^p \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{f} \\ \boldsymbol{0} \end{Bmatrix} \tag{20}$$

where $\boldsymbol{K}^{uu} = \boldsymbol{K}^d$.

From the second equation of governing equilibrium equations in Eq. (**??**), the coefficient vector $\boldsymbol{d}^p$ can be expressed by $\boldsymbol{d}^u$ as follows:

$$\boldsymbol{d}^p = 3\kappa(\boldsymbol{K}^{pp})^{-1}(\boldsymbol{K}^{up})^T\boldsymbol{d}^u \tag{21}$$

Further substituting the above equation into first equation of Eq. (**??**) leads to:

$$\begin{aligned} &(2\mu\underbrace{\boldsymbol{K}^{uu}}_{\boldsymbol{K}^d} + 3\kappa\underbrace{\boldsymbol{K}^{up}(\boldsymbol{K}^{pp})^{-1}(\boldsymbol{K}^{up})^T}_{\tilde{\boldsymbol{K}}^v})\boldsymbol{d}^u = \boldsymbol{f} \\ \Rightarrow\ &(2\mu\boldsymbol{K}^d + 3\kappa\tilde{\boldsymbol{K}}^v) = \boldsymbol{f} \end{aligned} \tag{22}$$

2.3. Equivalence between penalty– and mixed–formulation

It can be observed from the weak form for mixed-formulation in Eq. (**??**) or the discrete equation shown in Eq. (**??**) that, the solution of pressure p_h is an orthogonal projection of $3\kappa\nabla\cdot\boldsymbol{u}_h$. Let $P_h : V_h \to P_h(V_h)$ such that $P_h(V_h) \subseteq Q_h$, where $P_h(V_h) = \text{Im}\, P_h$ is the image of linear operator P_h [**?**]. Under this circumstance, $p_h = P_h(3\kappa\nabla\cdot\boldsymbol{u}_h) = 3\kappa\tilde{\nabla}\cdot\boldsymbol{u}_h$, and the Eq. (**??**) can be rephrased as:

$$\int_\Omega q_h(\nabla\cdot\boldsymbol{u}_h - \tilde{\nabla}\cdot\boldsymbol{u}_h)d\Omega = 0, \quad \forall q_h \in Q_h \tag{23}$$

Accordingly, the corresponding volumetric part of weak form turns to:

$$\begin{aligned} \int_\Omega \nabla\cdot\delta\boldsymbol{u}_h p_h d\Omega &= \underbrace{\int_\Omega (\nabla\cdot\boldsymbol{u}_h - \tilde{\nabla}\cdot\delta\boldsymbol{u}_h)p_h d\Omega}_{0} + \int_\Omega \tilde{\nabla}\cdot\delta\boldsymbol{u}_h \underbrace{p_h}_{\tilde{\nabla}\cdot\boldsymbol{u}_h} d\Omega \\ &= \int_\Omega 3\kappa\tilde{\nabla}\cdot\delta\boldsymbol{u}_h\tilde{\nabla}\cdot\boldsymbol{u}_h d\Omega \end{aligned} \tag{24}$$

and the Ritz-Galerkin formulation becomes to: Find $\boldsymbol{u}_h \in V_h$

$$\int_\Omega 2\mu\delta\boldsymbol{\varepsilon}_h^d : \boldsymbol{\varepsilon}_h^d d\Omega + \int_\Omega 3\kappa\tilde{\nabla}\cdot\delta\boldsymbol{u}_h\tilde{\nabla}\cdot\boldsymbol{u}_h d\Omega = \int_{\Gamma_t} \delta\boldsymbol{u}_h\cdot\boldsymbol{t}d\Gamma + \int_\Omega \delta\boldsymbol{u}_h\cdot\boldsymbol{b}d\Omega, \quad \forall\boldsymbol{u}_h \in V_h \tag{25}$$

In contrast, for penalty formulation, the reduced numerical integration also can be regarded as a projection. Let ϱ_i be the orthogonal polynomials,

$$\int_{\Omega_C} \varrho_i\varrho_j d\Omega = \begin{cases} J_C w_i & i = j \\ 0 & i \neq j \end{cases} \tag{26}$$

The orthogonal interpolation $T^k : V \to W^k$, where W^k is the interpolation space spanned by k orthogonal polynomials:

$$W^k := \operatorname{span}\{\varrho_i\}_{i=1}^k \tag{27}$$

For traditional Gauss-type integration scheme, $\varrho_i(\boldsymbol{x}_j) = \delta_{ij}$, $\boldsymbol{x}_j$'s are the positions of integration points. The volumetric strain can be depicted by orthogonal interpolation as:

$$\nabla\cdot\boldsymbol{u}_h(\boldsymbol{x}) \approx \bar{\nabla}\cdot\boldsymbol{u}_h(\boldsymbol{x}) = \sum_{G=1}^{n_g} \varrho_G(\boldsymbol{x})\nabla\cdot\boldsymbol{u}_h(\boldsymbol{x}_G), \quad \nabla\cdot\boldsymbol{u}_h(\boldsymbol{x}_G) = \bar{\nabla}\cdot\boldsymbol{u}_h(\boldsymbol{x}_G) \tag{28}$$

while the integration points are regarded as interpolation coefficients. While the total number of integration points n_g is lower than full integration, these means $\nabla\cdot\boldsymbol{u}_h$ projects to a subspace.

$$\begin{aligned}\int_\Omega 3\kappa\bar{\nabla}\cdot\delta\boldsymbol{u}_h\bar{\nabla}\cdot\boldsymbol{u}_h d\Omega &= \sum_{C=1}^{n_e}\sum_{G,L=1}^{n_g} 3\kappa\nabla\cdot\delta\boldsymbol{u}_h(\boldsymbol{x}_G)\nabla\cdot\boldsymbol{u}_h(\boldsymbol{x}_L)\int_\Omega \varrho_G\varrho_L d\Omega \\ &= \sum_{C=1}^{n_e}\sum_{G=1}^{n_g} 3\kappa\nabla\cdot\delta\boldsymbol{u}_h(\boldsymbol{x}_G)\nabla\cdot\boldsymbol{u}_h(\boldsymbol{x}_G)J_C w_G\end{aligned} \tag{29}$$

With comparison of Eqs. (**??**) and (**??**), the penalty formulation is actually equivalence with mixed formulation that, all approaches can be described by projection format.

3. Optimal polynomial–wise constraint count

3.1. Inf-sup value estimator

The problem of Eqs.(??) the approximations of Eq.(??) should satisfy the so-call Ladyzhenskaya–Babuška–Brezzi(LBB) condition or inf-sup condition [?] to ensure the formulation's accuracy:

$$\beta_h = \inf_{q_h \in Q_h} \sup_{\boldsymbol{v}_h \in V_h} \frac{\int_\Omega q_h \nabla \cdot \boldsymbol{v}_h d\Omega}{\|q_h\|_{L^2} \|\boldsymbol{v}_h\|_{H^1}} \geq \beta > 0 \tag{30}$$

in which β_h is namely inf-sup value, β stands for a constant independent of characterized element size h.

Theorem 1. *Let L the polynomial space of degree at most k, $\mathcal{L}_{(k)}$ is defined as follows:*

$$\mathcal{L}_{(k)} := ?? \tag{31}$$

such that, the inf-sup value β_h Inf-sup value has the following relationship:

$$\beta_h \leq \sqrt{\lambda_{n_{dof} - n_p + 1}} + Ch\|\boldsymbol{v}\| \tag{32}$$

with

$$\lambda_k = \inf_{U \subset (W^{n_u})_k^2} \sup_{\boldsymbol{v} \in W} \frac{\int_\Omega (\nabla \cdot \boldsymbol{v})^2 d\Omega}{\|\boldsymbol{v}\|_{H^1}^2} \tag{33}$$

Proof.

assuming that, $\boldsymbol{v}$ is a polynomial in polynomial space $\mathcal{L}$ of degree at most $2n_u$, i.e. $\boldsymbol{v} \in \mathcal{L}_{(2n_u)}$. As $\mathcal{L}_{2n_u} \subset V$, $\boldsymbol{v}$ can satisfied with problem of Eq.(??). In accordance with $Q_h \subseteq Q$, the following relationship hold true:

$$b(\boldsymbol{v}, q_h) = \int_\Omega q_h \nabla \cdot \boldsymbol{v} d\Omega = 0, \quad \forall q_h \in Q_h \tag{34}$$

A direct subtraction between the second equation of Eq.(??) and Eq.(??) yields:

$$b(\boldsymbol{v} - \boldsymbol{v}_h, q_h) = \int_\Omega q_h (\nabla \cdot \boldsymbol{v} - \nabla \cdot \boldsymbol{v}_h) d\Omega = 0, \quad \forall q_h \in Q_h \tag{35}$$

The above relation means that $\nabla \cdot \boldsymbol{v}_h$ is the elliptic projection of $\nabla \cdot \boldsymbol{v}$ onto Q_h, $\nabla \cdot \boldsymbol{v} \in Q_h(V_h)$ To further development of this methodology, the inf-sup value β_h can be firstly derived as follows:

$$\begin{aligned}
\lambda_h^{(k)} &= \inf_{W \subset Q_h^{(k)}(V_h)} \sup_{\boldsymbol{v}_h \in W} \frac{\int_\Omega (\nabla \cdot \boldsymbol{v}_h)^2 d\Omega}{\|\boldsymbol{v}_h\|^2} \\
&\leq \inf_{W \subset L_{(n_u)}^{(k)}(V)} \sup_{\boldsymbol{v} \in W} \frac{\int_\Omega (\nabla \cdot (\Pi_h \boldsymbol{v}))^2 d\Omega}{\|\Pi_h \boldsymbol{v}\|^2} \\
&= \inf_{W \subset L_{(n_u)}^{(k)}(V)} \sup_{\boldsymbol{v} \in W} \frac{\int_\Omega (\nabla \cdot \boldsymbol{v})^2 d\Omega}{\|\boldsymbol{v}\|^2} \frac{\|\boldsymbol{v}\|^2}{\|\Pi \boldsymbol{v}\|^2}
\end{aligned} \tag{36}$$

$$\|\Pi_h \boldsymbol{v} - \boldsymbol{v}\| \leq \|\Pi_h \boldsymbol{v}\| - \|\boldsymbol{v}\| \leq Ch^{p+1}\|\boldsymbol{v}\| \tag{37}$$

$$\|\Pi_h \boldsymbol{v}\| \leq (1 + Ch^{p+1})\|\boldsymbol{v}\| \tag{38}$$

$$\|\boldsymbol{v} - \Pi_h \boldsymbol{v}\|_{H(div)} \leq \|\boldsymbol{v}\| - \|\Pi_h \boldsymbol{v}\| \leq Ch^{p}\|\boldsymbol{v}\| \tag{39}$$

$$\frac{\|\boldsymbol{v}\|_1^2}{\|\Pi_h \boldsymbol{v}\|_1^2} \leq \frac{1}{(1 - Ch^{p+1})^2} \approx 1 + 2Ch^{p+1} \tag{40}$$

$$\mathcal{L}^{(n_u)} \tag{41}$$

$$\|\sum_{k=1}^{n_u} \mathcal{L}^{(k)} \boldsymbol{v}\| \leq Ch \tag{42}$$

$$\begin{aligned}
\beta_h &= \inf_{q_h \in Q_h} \sup_{\boldsymbol{v}_h \in V_h} \frac{\int_\Omega q_h \nabla \cdot \boldsymbol{v}_h d\Omega}{\|q_h\|_{L^2}\|\boldsymbol{v}_h\|_{H^1}} \\
&\leq \inf_{q_h \in \mathrm{Im} P_h} \sup_{\boldsymbol{v}_h \in V_h} \frac{\int_\Omega q_h \nabla \cdot \boldsymbol{v}_h d\Omega}{\|q_h\|_{L^2}\|\boldsymbol{v}_h\|_{H^1}} \\
&= \inf_{q_h \in \mathrm{Im} P_h} \sup_{\boldsymbol{v}_h \in V_h \setminus \ker P_h} \frac{\underbrace{\int_\Omega q_h (\nabla \cdot \boldsymbol{v}_h - \tilde{\nabla} \cdot \boldsymbol{v}_h) d\Omega}_{0} + \int_\Omega q_h \tilde{\nabla} \cdot \boldsymbol{v}_h d\Omega}{\|q_h\|_{L^2}\|\boldsymbol{v}_h\|_{H^1}} \\
&= \inf_{\boldsymbol{u}_h \in V_h \setminus \ker P_h} \sup_{\boldsymbol{v}_h \in V_h \setminus \ker P_h} \frac{\int_\Omega \tilde{\nabla} \cdot \boldsymbol{u}_h \tilde{\nabla} \cdot \boldsymbol{v}_h d\Omega}{\|\tilde{\nabla} \cdot \boldsymbol{u}_h\|_{L^2}\|\boldsymbol{v}_h\|_{H^1}} \\
&\leq
\end{aligned} \tag{43}$$

As $\mathrm{Im}P_h \subseteq Q_h$, the discrete inf-sup value β_h can be rephrased as:

$$\beta_h = \inf_{q_h \in Q_h} \sup_{\boldsymbol{v}_h \in V_h} \frac{\int_\Omega q_h \nabla \cdot \boldsymbol{v}_h d\Omega}{\|q_h\|_{L^2}\|\boldsymbol{v}_h\|_{H^1}} \leq \inf_{q_h \in \mathrm{Im} P_h} \sup_{\boldsymbol{v}_h \in V_h} \frac{\int_\Omega q_h \nabla \cdot \boldsymbol{v}_h d\Omega}{\|q_h\|_{L^2}\|\boldsymbol{v}_h\|_{H^1}} \tag{44}$$

In accordance with Hilbert projection theorem [**?**], there exists an orthgonal projection of $\nabla \cdot \boldsymbol{v}_h$, namely $\tilde{\nabla} \boldsymbol{v}_h = P_h \nabla \boldsymbol{v}_h$, such that:

$$\int_\Omega q_h (\nabla \boldsymbol{v} - \tilde{\nabla} \boldsymbol{v}) d\Omega = 0 \tag{45}$$

for the above relationship, the Eq. (**??**) is restated by:

$$
\begin{aligned}
&\inf_{q_h \in \mathrm{Im} P_h} \sup_{\boldsymbol{v}_h \in V_h} \frac{\int_\Omega q_h \nabla \cdot \boldsymbol{v}_h d\Omega}{\|q_h\|_{L^2} \|\boldsymbol{v}_h\|_{H^1}} \\
= &\inf_{q_h \in \mathrm{Im} P_h} \sup_{\boldsymbol{v}_h \in V_h} \frac{\underbrace{\int_\Omega q_h (\nabla \cdot \boldsymbol{v}_h - \tilde{\nabla} \cdot \boldsymbol{v}_h) d\Omega}_{0} + \int_\Omega q_h \tilde{\nabla} \cdot \boldsymbol{v}_h d\Omega}{\|q_h\|_{L^2} \|\boldsymbol{v}_h\|_{H^1}} \\
= &\inf_{q_h \in \mathrm{Im} P_h} \sup_{\boldsymbol{v}_h \in V_h \setminus \ker P_h} \frac{\int_\Omega q_h \tilde{\nabla} \cdot \boldsymbol{v}_h d\Omega}{\|q_h\|_{L^2} \|\boldsymbol{v}_h\|_{H^1}}
\end{aligned} \tag{46}
$$

Assuming that a $\boldsymbol{u}_h \in V_h \setminus \ker P_h$ such that

$$
\inf_{\boldsymbol{u}_h \in V_h \setminus \ker P_h} \sup_{\boldsymbol{v}_h \in V_h \setminus \ker P_h} \frac{\int_\Omega \tilde{\nabla} \cdot \boldsymbol{u}_h \tilde{\nabla} \cdot \boldsymbol{v}_h d\Omega}{\|\tilde{\nabla} \cdot \boldsymbol{u}_h\|_{L^2} \|\boldsymbol{v}_h\|_{H^1}} \tag{47}
$$

Since V_h is a finite dimensional Hilbert space, there always an orthogonal basis $\{\boldsymbol{v}_h^{(k)}\}_{k=1}^{2n_p}$ for V_h (Bryan2008 P61). Let $V_h^{(n)} = \mathrm{span}\{\boldsymbol{v}_h^{(k)}\}_{k=1}^{n}$. Let $\boldsymbol{u}^h = \sum_{i=1}^n c_i \boldsymbol{v}_h^{(i)}$, such that:

$$
\begin{aligned}
\inf_{\boldsymbol{u}_h \in \bar{V}_h} \sup_{\boldsymbol{v}_h \in \bar{V}_h} \frac{\int_\Omega \tilde{\nabla} \cdot \boldsymbol{u}_h \tilde{\nabla} \cdot \boldsymbol{v}_h d\Omega}{\|\tilde{\nabla} \cdot \boldsymbol{u}_h\|_{L^2} \|\boldsymbol{v}_h\|_{H^1}} &= \inf_{\boldsymbol{u}_h \in \bar{V}_h} \sup_{\boldsymbol{v}_h \in \bar{V}_h} \frac{\sum_{k=1}^n \int_\Omega \tilde{\nabla} \cdot \boldsymbol{u}_h^{(k)} \tilde{\nabla} \cdot \boldsymbol{v}_h d\Omega}{\|\tilde{\nabla} \cdot \boldsymbol{u}_h\|_{L^2} \|\boldsymbol{v}_h\|_{H^1}} \\
&= \inf_{\boldsymbol{u}_h \in \bar{V}_h} \sup_{\boldsymbol{v}_h \in \bar{V}_h} \frac{\sum_{k=1}^n \int_\Omega \tilde{\nabla} \cdot \boldsymbol{u}_h^{(k)} \tilde{\nabla} \cdot \boldsymbol{v}_h^{(k)} d\Omega}{\|\tilde{\nabla} \cdot \boldsymbol{u}_h\|_{L^2} \|\boldsymbol{v}_h\|_{H^1}} \\
&\leq \inf_{W_h \subset V_h^{(n_p+1)}} \sup_{\boldsymbol{v}_h \in W_h} \frac{\|\tilde{\nabla} \cdot \boldsymbol{v}_h\|_{L^2}}{\|\boldsymbol{v}_h\|_{H^1}}
\end{aligned} \tag{48}
$$

$$
\|\nabla \cdot \boldsymbol{v} - \tilde{\nabla} \cdot \boldsymbol{v}_h\|_{L^2} \leq Ch \tag{49}
$$

Interpolation error for polynomials,

Triangular inequality

$$
\begin{aligned}
&\|\tilde{\nabla} \cdot \boldsymbol{v}_h\|_{L^2} - \|\nabla \cdot \boldsymbol{v}\|_{L^2} \\
\leq &\|\nabla \cdot \boldsymbol{v} - \tilde{\nabla} \cdot \boldsymbol{v}_h\|_{L^2} \leq Ch^p \|\nabla \cdot \boldsymbol{v}\|_{H^{p+1}} \leq C' h^p \|\nabla \cdot \boldsymbol{v}\|_{L^2} \\
\Rightarrow &\|\tilde{\nabla} \cdot \boldsymbol{v}_h\|_{L^2} \leq (1 + C') h^p \|\nabla \cdot \boldsymbol{v}\|_{L^2} = C'' h^p \|\nabla \cdot \boldsymbol{v}\|_{L^2}
\end{aligned} \tag{50}
$$

$$
\begin{aligned}
&\|\nabla \cdot \boldsymbol{v}\|_{L^2} - \|\tilde{\nabla} \cdot \boldsymbol{v}_h\|_{L^2} \\
\leq &\|\nabla \cdot \boldsymbol{v} - \tilde{\nabla} \cdot \boldsymbol{v}_h\|_{L^2} \leq Ch^p \|\nabla \cdot \boldsymbol{v}\|_{H^{p+1}} \leq C' h^p \|\nabla \cdot \boldsymbol{v}\|_{L^2} \\
\Rightarrow &\|\tilde{\nabla} \cdot \boldsymbol{v}_h\|_{L^2} \geq (1 - C') h^p \|\nabla \cdot \boldsymbol{v}\|_{L^2} = C'' h^p \|\nabla \cdot \boldsymbol{v}\|_{L^2}
\end{aligned} \tag{51}
$$

$$
\begin{aligned}
\frac{\|\tilde{\nabla} \cdot \boldsymbol{v}_h\|_{L^2}}{\|\boldsymbol{v}_h\|_{H^1}} &= \frac{\|\nabla \cdot \boldsymbol{v}\|_{L^2}}{\|\boldsymbol{v}\|_{H^1}} \frac{\|\tilde{\nabla} \cdot \boldsymbol{v}\|_{L^2}}{\|\nabla \cdot \boldsymbol{v}\|_{L^2}} \frac{\|\boldsymbol{v}\|_{H^1}}{\|\boldsymbol{v}_h\|_{H^1}} \\
&\leq \frac{\|\nabla \cdot (\mathcal{L}\boldsymbol{v})\|_{L^2}}{\|\mathcal{L}\boldsymbol{v}\|_{H^1}} (C_1 + C_2 h^p)
\end{aligned}
\tag{52}
$$

3.2. Optimal constraint count

Table 1: Degrees of freedom and volumetric constraint

p	$2n_u$	c
1	6	5
2	12	9
3	20	14
4	30	20
p	$\sum_{i=1}^{p+1} 2i$	$\sum_{i=1}^{p+1} i+1$

3.3. Equivalence with inf-sup test

4. Numerical examples

4.1. Inf–sup– and patch–tests for nearly–incompressible elastic material

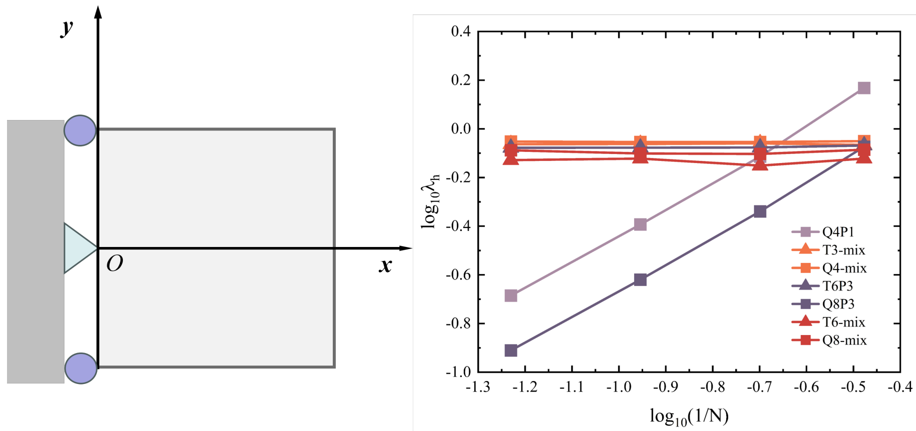

Figure 1: Illustration of inf–sup– and patch–tests

4.1.1. Inf–sup test

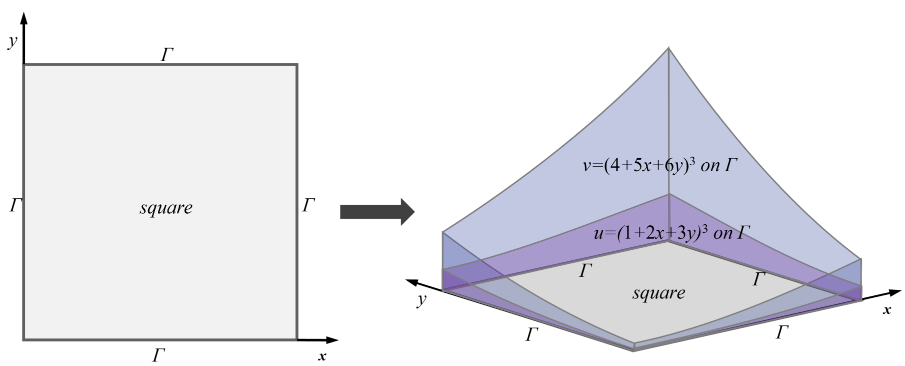

Figure 2: Inf–sup test for various finite element formulations

beginfigure[!ht]

4.2. Cantilever beam problem ?? (convergence study)

4.3. Cook membrane problem

4.4. Block under compression problem

Problem reference [**?**], Neo-Hooke model, T4 and Q8 elements.

Table 2: Results of Nearly–incompressible elasticity patch test

	Linear patch test		Quadratic patch test	
	L_2-Error	H_e-Error	L_2-Error	H_e-Error
T3-stripe				
T3-cross				
T3-mix				
Q4				
Q4R1				
Q4-mix				
T6				
T6P3				
T6-mix				
Q8				
Q8P3				
Q8-mix				

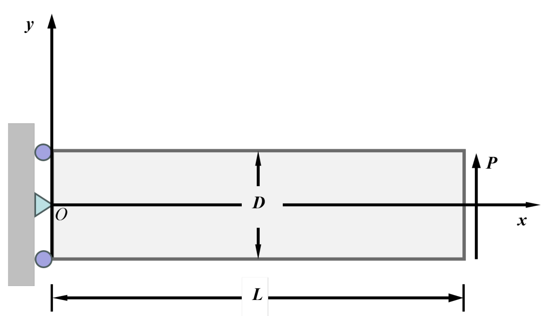

Figure 3: Illustration of cantilever beam problem

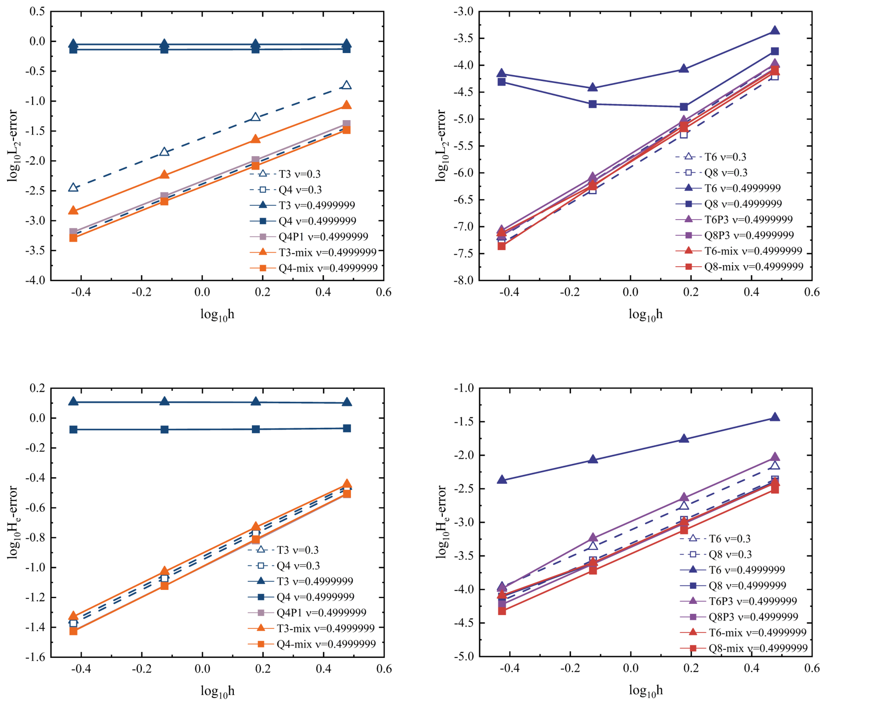

Figure 4: Convergence comparison of cantilever beam problem: **??** L^2-Error; **??** H^e-Error

Figure 5: Contour plots of cantilever beam problem

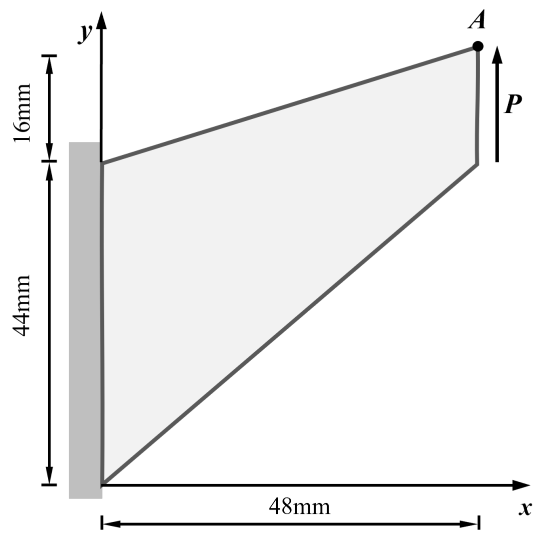

Figure 6: Illustration of cook membrane problem

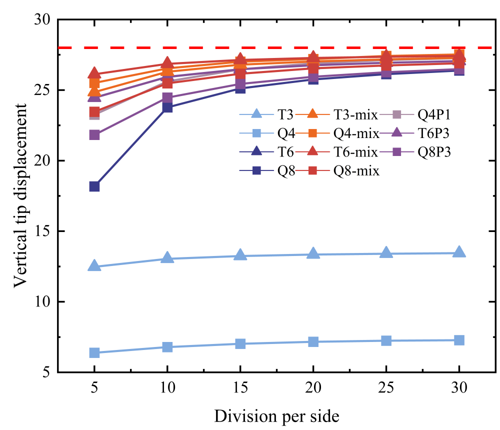

Figure 7: Convergence comparison of cook membrane problem

Figure 8: Contour plots of cook membrane problem

Figure 9: Illustration of block under compression problem

Figure 10: Convergence comparison of block under compression problem

Figure 11: Contour plots of block under compression problem

Appendix A. Hilbert space

In this appendix, some functional analysis tools are introduced here. For a variable v, its norm regarding to L^2-norm is defined by:

$$\|v\|_{L^2} := \left(\int_\Omega |v|^2 d\Omega \right)^{1/2} \tag{A.1}$$

$$H(\text{div}) := \{\boldsymbol{v} \in \mathcal{V} \mid \nabla \cdot \boldsymbol{v} \in L^2\} \tag{A.2}$$

$$\|\boldsymbol{v}\|_{H(div)} := \left(\|\boldsymbol{v}\|_{L^2} + \|\nabla \cdot \boldsymbol{v}\|_{L^2}\right)^{1/2} \tag{A.3}$$