

**Problem 1:** Suppose  $P$  is a probability measure defined on  $S = \{1, 2, 3, 4, 5\}$  with mass function

$s$	$p(s)$
1	0.1
2	0.3
3	0.2
4	0.3
5	0.1

Compute the information content  $I(s)$  of each sample point and the entropy  $H(P)$ .

We compute

$s$	$p(s)$	$I(s)$
1	0.1	3.322
2	0.3	1.737
3	0.2	2.322
4	0.3	1.737
5	0.1	3.322

where the surprisals are rounded to three places after the decimal point. The entropy is  $H(P) \approx 2.171$ .

**Problem 2:** Let  $X \sim \mathcal{Ber}(\theta)$  for  $\theta \in [0, 1]$ . Compute a formula for  $H(X)$  in terms of  $\theta$ .

We compute:

$$H(X) = - \sum_{x=0}^1 p(x) \log_2(p(x)) = -(1 - \theta) \log_2(1 - \theta) - \theta \log_2(\theta).$$

**Problem 3:** Suppose  $P$  and  $Q$  are probability measures defined on  $S = \{1, 2, 3, 4, 5\}$  with mass functions

$s$	$p(s)$	$q(s)$
1	0.1	0.05
2	0.3	0.15
3	0.2	0.7
4	0.3	0.03
5	0.1	0.07

Compute the cross entropies  $H_P(Q)$  and  $H_Q(P)$ .

$$H_P(Q) = - \sum_{s=1}^5 p(s) \log_2(q(s)) \approx 3.258, \quad H_Q(P) = - \sum_{s=1}^5 q(s) \log_2(p(s)) \approx 2.337.$$

**Problem 4:** Suppose  $P$  and  $Q$  are probability measures defined on  $S = \{1, 2, 3, 4, 5\}$  with mass functions

$s$	$p(s)$	$q(s)$
1	0.1	0.05
2	0.3	0.15
3	0.2	0.7
4	0.3	0.03
5	0.1	0.07

Compute  $D(P \parallel Q)$  and  $D(Q \parallel P)$ .

$$D(P \parallel Q) = \sum_{s=1}^5 p(s) \log_2 \left( \frac{p(s)}{q(s)} \right) \approx 1.087 \quad \text{and} \quad D(Q \parallel P) = \sum_{s=1}^5 q(s) \log_2 \left( \frac{q(s)}{p(s)} \right) \approx 0.929.$$

**Problem 5:** While it is **not** true that  $D(P \parallel Q) = D(Q \parallel P)$  for **all**  $P$  and  $Q$ , there are examples of special distributions  $P$  and  $Q$  for which equality does hold. Find examples.

For  $\theta \in [0, 1]$ , define  $P_\theta$  and  $Q_\theta$  on the sample space  $S = \{0, 1\}$  by

$s$	$p_\theta(s)$	$q_\theta(s)$
0	$\theta$	$1 - \theta$
1	$1 - \theta$	$\theta$

One then easily proves  $D(P_\theta \parallel Q_\theta) = D(Q_\theta \parallel P_\theta)$ .

**Problem 6:** For each  $\phi \in [0, 1]$ , consider the communication channel with transition matrix

$$\mathbf{K} = \begin{bmatrix} 1 - \phi & \phi \\ \phi & 1 - \phi \end{bmatrix}$$

These are called *binary symmetric channels*. Suppose that  $X \sim \mathcal{Ber}(\alpha)$  for some  $\alpha \in [0, 1]$ , with the range of  $X$  enumerated as  $x_0 = 0$  and  $x_1 = 1$ . Show that  $X$  and the communication channel determine a random variable  $Y$  with range  $y_0 = 0$  and  $y_1 = 1$ . Determine its distribution.

Since the range of  $Y$  is  $\{0, 1\}$ , it must be Bernoulli, with  $Y \sim \mathcal{Ber}(\beta)$  for some  $\beta \in [0, 1]$ . We need to determine the parameter  $\beta$ . But notice that the probability vectors encoding the mass functions of  $X$  and  $Y$  have the form

$$\boldsymbol{\pi}(X)^\top = [1 - \alpha \quad \alpha] \quad \text{and} \quad \boldsymbol{\pi}(Y)^\top = [1 - \beta \quad \beta].$$

So, if we conceptualize the entries in the transition matrix as the conditional probabilities, then by the Law of Total Probability we must have

$$\boldsymbol{\pi}(Y)^\top = \boldsymbol{\pi}(X)^\top \mathbf{K} = [1 - \alpha \quad \alpha] \begin{bmatrix} 1 - \phi & \phi \\ \phi & 1 - \phi \end{bmatrix} = [(1 - \alpha)(1 - \phi) + \alpha\phi \quad (1 - \alpha)\phi + \alpha(1 - \phi)].$$

Thus,  $\beta = \phi + \alpha - 2\phi\alpha$ .

**Problem 7:** Suppose  $X$  and  $Y$  are Bernoulli random variables with joint mass function given by

$p(x, y)$	$y = 0$	$y = 1$
$x = 0$	0.3	0.1
$x = 1$	0.36	0.24

- (a) Compute the transition matrix of the communication channel induced from  $X$  and  $Y$ .

This just means that we need to compute the conditional mass function  $p(y|x)$ . We begin by computing the marginal mass  $p(x)$ :

$x$	$p(x)$
0	0.4
1	0.6

Then the transition matrix is given by

$$\mathbf{K} = \left\{ \begin{array}{c|cc} p(y|x) & y = 0 & y = 1 \\ \hline x = 0 & 0.75 & 0.25 \\ x = 1 & 0.6 & 0.4 \end{array} \right\}.$$

- (b) Compute the mutual information  $I(X, Y)$ .

$$I(X, Y) \approx 0.018$$