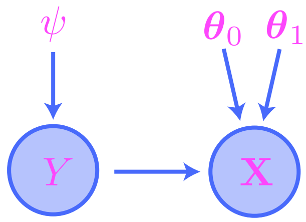
Problem 1: Consider a *Naive Bayes model* as described in the programming assignment for chapter 12. The underlying graph is of the form



where $\mathbf{X} \in \mathbb{R}^n$. The parameters are given by a number $\psi \in [0,1]$ which parametrizes the distribution of $Y \sim \mathcal{B}er(\psi)$, as well as two vectors $\boldsymbol{\theta}_0, \boldsymbol{\theta}_1 \in [0,1]^n$. The link function at $\mathbf{X}$ is given by

$$p(\mathbf{x} \mid y;\ \boldsymbol{\theta}_0, \boldsymbol{\theta}_1) = \prod_{j=1}^{n} \phi_j^{x_j}(1-\phi_j)^{1-x_j}$$

where

$$\boldsymbol{\phi} = (1-y)\boldsymbol{\theta}_0 + y\boldsymbol{\theta}_1$$

and $\boldsymbol{\phi}^\intercal = (\phi_1, \ldots, \phi_n)$.

(a) Assuming that Naive Bayes models are trained as **generative** models, write down a formula for the model likelihood function $\mathcal{L}_{\text{model}}(\psi, \boldsymbol{\theta}_0, \boldsymbol{\theta}_1)$. For simplicity, your formula should contain the ϕ_j's rather than the parameters $\boldsymbol{\theta}_0$ and $\boldsymbol{\theta}_0$ themselves.

We have

$$\begin{aligned}
\mathcal{L}_{\text{model}}(\psi, \boldsymbol{\theta}_0, \boldsymbol{\theta}_1;\ \mathbf{x}, y) &= p(\mathbf{x}, y;\ \psi, \boldsymbol{\theta}_0, \boldsymbol{\theta}_1) \\
&= p(y; \psi)p(\mathbf{x} \mid y;\ \boldsymbol{\theta}_0, \boldsymbol{\theta}_1) \\
&= \psi^y(1-\psi)^{1-y} \prod_{j=1}^{n} \phi_j^{x_j}(1-\phi_j)^{1-x_j}.
\end{aligned}$$

(b) Using your answer from part (a), write down a formula for the model surprisal function $\mathcal{I}_{\text{model}}(\psi, \boldsymbol{\theta}_0, \boldsymbol{\theta}_1)$. For simplicity, your formula should contain the ϕ_j's rather than the parameters $\boldsymbol{\theta}_0$ and $\boldsymbol{\theta}_0$ themselves.

We have

$$\begin{aligned}
\mathcal{I}_{\text{model}}(\psi, \boldsymbol{\theta}_0, \boldsymbol{\theta}_1;\ \mathbf{x}, y) &= -\log \mathcal{L}_{\text{model}}(\psi, \boldsymbol{\theta}_0, \boldsymbol{\theta}_1;\ \mathbf{x}, y) \\
&= -y\log\psi - (1-y)\log(1-\psi) - \sum_{j=1}^{n}\left[x_j \log \phi_j + (1-x_j)\log(1-\phi_j)\right].
\end{aligned}$$

(c) Using your answer from part (b), write down an explicit formula for the cross entropy stochastic objective function $J(\psi, \boldsymbol{\theta}_0, \boldsymbol{\theta}_1)$ for a dataset of size m.

Letting

$$(\mathbf{x}_1, y_1), (\mathbf{x}_2, y_2), \ldots, (\mathbf{x}_m, y_m) \in \{0,1\}^m \times \{0,1\}$$

be the dataset, by definition we have

$$\begin{aligned} J(\psi, \boldsymbol{\theta}_0, \boldsymbol{\theta}_1) &= \frac{1}{m}\sum_{i=1}^{m} \mathcal{I}_{\text{model}}(\psi, \boldsymbol{\theta}_0, \boldsymbol{\theta}_1;\ \mathbf{x}_i, y_i) \\ &= \frac{1}{m}\sum_{i=1}^{m} \left\{ -y_i \log\psi - (1-y_i)\log(1-\psi) - \sum_{j=1}^{n} \left[x_{ij}\log\phi_j + (1-x_{ij})\log(1-\phi_j)\right] \right\}. \end{aligned}$$

Problem 2: Consider the observed dataset

$$(0,0), (1,1), (2,3) \in \mathbb{R}^2.$$

Using this dataset, compute the exact MLEs for the parameters β_0 and β_1 of a simple linear regression model (with known variance).

We have $\bar{x} = 1$ and $\bar{y} = 4/3$, so

$$(\beta_1)^\star_{\text{MLE}} = \frac{(0-1)(0-4/3) + (1-1)(1-4/3) + (2-1)(3-4/3)}{(0-1)^2 + (1-1)^2 + (2-1)^2} = \frac{3}{2}$$

and

$$(\beta_0)^\star_{\text{MLE}} = \frac{4}{3} - \frac{3}{2}\cdot 1 = -\frac{1}{6}.$$

Problem 3: For the neural network trained in Section 13.5, compute the following:

(a) The number of gradient steps per epoch.

The dataset has size $m = 3{,}072$ and the mini-batch size is $k = 128$. Thus, the algorithm takes $3{,}072/128 = 24$ gradient steps per epoch.

(b) The *exact* number of gradient steps over all epochs.

The algorithm takes a total of $80 \times 24 = 1{,}920$ gradient steps.

(c) The number of trainable parameters in the network.

The network has three hidden layers of widths 8, 8, and 4, and an input layer of width 2. This means we have *four* parameter groups:

$$(\mathbf{W}_1, \mathbf{b}_1), (\mathbf{W}_2, \mathbf{b}_2), (\mathbf{W}_3, \mathbf{b}_3), (\mathbf{w}_4, b_4),$$

with

$$\mathbf{W}_1 \in \mathbb{R}^{2\times 8},\ \mathbf{b}_1 \in \mathbb{R}^8,\ \mathbf{W}_2 \in \mathbb{R}^{8\times 8},\ \mathbf{b}_2 \in \mathbb{R}^8,\ \mathbf{W}_3 \in \mathbb{R}^{8\times 4},\ \mathbf{b}_3 \in \mathbb{R}^4,\ \mathbf{w}_4 \in \mathbb{R}^4,\ b_4 \in \mathbb{R}.$$

So, there are

$$(2\cdot 8) + 8 + (8\cdot 8) + 8 + (8\cdot 4) + 4 + 4 + 1 = 137$$

trainable parameters.