8. More probability theory

8.1. Expectations and joint distributions

Theorem 8.1 (Bivariate Law of the Unconscious Statistician (LotUS))

Let X and Y be two random variables and $g : \mathbb{R}^2 \to \mathbb{R}$ a function.

1. If X and Y are jointly discrete with mass function $p(x, y)$, then

$$E\left(g(X, Y)\right) = \sum_{(x,y)\in\mathbb{R}^2} g(x, y)p(x, y).$$

2. If X and Y are jointly continuous with density function $f(x, y)$, then

$$E\left(g(X, Y)\right) = \iint_{\mathbb{R}^2} g(x, y)f(x, y)\,\mathrm{d}x\mathrm{d}y.$$

Theorem 8.2 (Independence and expectations)

If X and Y are independent random variables, then $E(XY) = E(X)E(Y)$.

Theorem 8.3 (Linearity of Expectations)

Let X and Y be two random variables and let $c \in \mathbb{R}$ be a constant. Then:

$$E(X + Y) = E(X) + E(Y), \tag{8.1}$$

and

$$E(cX) = cE(X). \tag{8.2}$$

Problem Prompt

Do problem 1 on the worksheet.

8.2. Expectations and conditional distributions

Definition 8.1

Let X and Y be two random variables.

1. If Y and X are jointly discrete with conditional mass function $p(y|x)$, then the *conditional expected value* of Y given $X = x$ is the sum

$$E(Y \mid X = x) \stackrel{\text{def}}{=} \sum_{y \in \mathbb{R}} y p(y|x).$$

2. If Y and X are jointly continuous with conditional density function $f(y|x)$, then the *conditional expected value* of Y given $X = x$ is the integral

$$E(Y \mid X = x) \stackrel{\text{def}}{=} \int_{\mathbb{R}} y f(y|x) \, \mathrm{d}y.$$

Problem Prompt

Do problem 2 on the worksheet.

Problem Prompt

Do problem 3 on the worksheet.

Theorem 8.4 (The Law of Total Expectation)

Let X and Y be two random variables that are either jointly discrete or jointly continuous. Then

$$E\big[E(Y \mid X)\big] = E(Y).$$

Problem Prompt

Do problem 4 on the worksheet.