Problem 1: Suppose that X and Y are jointly continuous random variables with density

$$f(x,y) = \begin{cases} 24xy & : 0 \le x \le 1,\ 0 \le y \le 1,\ x+y \le 1, \\ 0 & : \text{otherwise.} \end{cases}$$

Compute the expectation $E(XY)$.

Using the bivariate LotUS, we get:

$$E(XY) = \iint_{\mathbb{R}^2} xyf(x,y)\ \mathrm{d}y\mathrm{d}x = 24\int_0^1 \int_0^{1-x} x^2y^2\ \mathrm{d}y\mathrm{d}x = \frac{2}{15}.$$

Problem 2: Suppose X and Y are jointly continuous random variables with the same density from Problem 1. Compute a formula for the conditional expectation $E(Y \mid X = x)$. Take care to precisely state the domain of this function.

We first get the marginal density for X:

$$f(x) = \int_{\mathbb{R}} f(x,y)\ \mathrm{d}y = 24\int_0^{1-x} xy\ \mathrm{d}y = 12x(1-x)^2,$$

where the second inequality holds for all $0 \le x \le 1$; otherwise, we have $f(x) = 0$. Note that the marginal $f(x)$ is nonzero for $0 < x < 1$. For these particular x-values, the conditional density is defined and is given by

$$f(y|x) = \frac{f(x,y)}{f(x)} = \begin{cases} \frac{2y}{(1-x)^2} & : 0 \le x \le 1,\ 0 \le y \le 1,\ x+y \le 1, \\ 0 & : \text{otherwise.} \end{cases}$$

Therefore, for all x with $0 < x < 1$, we have

$$E(Y \mid X = x) = \int_{\mathbb{R}} yf(y|x)\ \mathrm{d}y = \frac{2}{(1-x)^2}\int_0^{1-x} y^2\ \mathrm{d}y = \frac{2}{3}(1-x).$$

Problem 3: Let X and Y be two random variables on the probability space $S = \{a,b,c\}$. Suppose that the probability distribution P on S has mass function $p(s)$ and that X and Y are defined according to the following table:

s	$p(s)$	$X(s)$	$Y(s)$
a	0.2	1	2
b	0.5	2	1
c	0.3	1	1

Compute the random variable $E(Y \mid X)$.

Our first goal is to get the joint mass function $p(x,y)$. We compute:

$p(x,y)$	$y=1$	$y=2$
$x=1$	0.3	0.2
$x=2$	0.5	0

By adding across the rows, we get the marginal:

x	$p(x)$
1	0.5
2	0.5

Then, from the formula $p(y|x) = p(x,y)/p(x)$, we get

$p(y\|x)$	$y=1$	$y=2$
$x=1$	0.6	0.4
$x=2$	1	0

If $x \neq 1$ or 2, then the conditional mass $p(y|x)$ is undefined. For $x = 1$ or 2, we have

$$E(Y \mid X = x) = \sum_{y=1}^{2} y p(y|x) = \begin{cases} 1 \cdot 0.6 + 2 \cdot 0.4 = 1.4 & : x = 1, \\ 1 \cdot 1 + 2 \cdot 0 = 1 & : x = 2. \end{cases}$$

Finally, recall that the random variable $E(Y \mid X)$ is defined as the composite $h(X) = h \circ X$, where $h(x) = E(Y \mid X = x)$ for $x = 1, 2$. Hence, we conclude the problem with:

s	$E(Y \mid X)(s)$
a	1.4
b	1
c	1.4

Problem 4: Suppose that a point $X = x$ is chosen uniformly in the interval $(0, 1)$. After x has been chosen, suppose that a second point $Y = y$ is chosen uniformly in the interval $[x, 1]$. Compute the expectation $E(Y)$.

We will compute the expectation via the Law of Total Expectation:

$$E(Y) = E\Big[E(Y \mid X)\Big].$$

So, we begin by noting that $E(Y \mid X = x) = (x+1)/2 = x/2 + 1/2$, since the mean of a uniform distribution over an interval is the midpoint between its boundary points. Therefore $E(Y \mid X) = X/2 + 1/2$, and so

$$E(Y) = E\Big(X/2 + 1/2\Big) = E(X)/2 + 1/2 = 1/4 + 1/2 = 3/4.$$

Problem 5: The waiting time X in minutes between calls to a 911 center is exponentially distributed with mean $\mu = 2$ minutes. Compute the distribution of the transformed random variable $Y = 60X$ that measures the waiting time in seconds.

Since the mean of $X \sim \mathcal{E}xp(\lambda)$ is $1/\lambda$, we see that $\lambda = 1/2$ and so

$$f_X(x) = \begin{cases} \frac{1}{2} e^{-x/2} & : x > 0, \\ 0 & : \text{otherwise}. \end{cases}$$

In the notation of the Density Transformation Theorem, the support of the density is $T = (0, \infty)$. If we define $r : T \to \mathbb{R}$ by $r(x) = 60x$, then the range U of r is the open interval $(0, \infty)$. Note that the inverse function $s : U \to \mathbb{R}$ is given by $s(y) = y/60$, which is continuously differentiable. Therefore, we have everything that we need to apply the Density Transformation Theorem. We get:

$$f_Y(y) = \begin{cases} \frac{1}{120} e^{-y/120} & : y > 0, \\ 0 & : \text{otherwise}, \end{cases}$$

and thus $Y \sim \mathcal{E}xp(1/120)$.

Problem 6: Suppose that X and Y are two random variables such that $Y = e^X$ and $X \sim \mathcal{N}(\mu, \sigma^2)$. Compute the density of Y.

The support of the density of X is all of $\mathbb{R}$; so, in the notation of the Density Transformation Theorem we have $T = \mathbb{R}$. Define $r : T \to \mathbb{R}$ by setting $r(x) = e^x$, and note that the image U of r is the open interval $(0, \infty)$. The inverse function $s : U \to \mathbb{R}$ is given by $s(y) = \log y$, which is continuously differentiable. Therefore, we have everything that we need in order to use the Density Transformation Theorem. We compute:

$$f_Y(y) = \begin{cases} \frac{1}{y\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2\sigma^2}\left(\log y - \mu\right)^2\right] & : y > 0, \\ 0 & : \text{otherwise.} \end{cases}$$

Hence, Y is a *lognormal* random variable.

Problem 7: Suppose that $\mathbf{X} = (X_1, X_2)$ is a two-dimensional continuous random vector with density

$$f(x_1, x_2) = \begin{cases} 4x_1x_2 & : 0 < x_1 < 1,\ 0 < x_2 < 1, \\ 0 & : \text{otherwise.} \end{cases}$$

Letting T be the support of the density, define $r : T \to \mathbb{R}^2$ by setting

$$r(x_1, x_2) = \left(\frac{x_1}{x_2}, x_1x_2\right)$$

for $(x_1, x_2) \in \mathbb{R}^2$. Compute the density of the random vector $\mathbf{Y} = r(\mathbf{X})$.

As I will explain during class, the image U of r is the open region in the (y_1, y_2)-plane bounded by the curves

$$y_1 = y_2, \quad y_2 = 0, \quad y_2 = \frac{1}{y_2}.$$

To compute an inverse $s : U \to \mathbb{R}^2$ of r, notice that the equations

$$y_1 = \frac{x_1}{x_2} \quad \text{and} \quad y_2 = x_1x_2$$

can be solved uniquely for x_1 and x_2 yielding the solutions

$$x_1 = \sqrt{y_1y_2} \quad \text{and} \quad x_2 = \sqrt{\frac{y_2}{y_1}}.$$

Thus we may define s by setting

$$s(y_1, y_2) = \left(\sqrt{y_1y_2}, \sqrt{\frac{y_2}{y_1}}\right)$$

for all $(y_1, y_2) \in U$. To verify that s is continuously differentiable, we compute its Jacobian matrix:

$$\frac{\partial(s_1, s_2)}{\partial(y_1, y_2)}(y_1, y_2) = \begin{bmatrix} \frac{1}{2}\sqrt{\frac{y_2}{y_1}} & \frac{1}{2}\sqrt{\frac{y_1}{y_2}} \\ -\frac{1}{2}\sqrt{\frac{y_2}{y_1^3}} & \frac{1}{2}\sqrt{\frac{1}{y_1y_2}} \end{bmatrix}.$$

The four partial derivatives are all continuous on U, and thus s is continuously differentiable. We therefore have everything that we need to apply the Density Transformation Theorem. First, we compute the determinant of the Jacobian matrix:

$$\det \frac{\partial(s_1, s_2)}{\partial(y_1, y_2)}(y_1, y_2) = \frac{1}{2y_1}.$$

Then, we have

$$f\left(\sqrt{y_1 y_2}, \sqrt{\frac{y_2}{y_1}}\right) \cdot \frac{1}{2y_1} = 4\sqrt{y_1 y_2} \cdot \sqrt{\frac{y_2}{y_1}} \cdot \frac{1}{2y_1} = \frac{2y_2}{y_1}$$

for $(y_1, y_2) \in U$, and thus

$$f(y_1, y_2) = \begin{cases} \frac{2y_2}{y_1} & : (y_1, y_2) \in U, \\ 0 & : \text{otherwise.} \end{cases}$$

Problem 8: Suppose that X is a continuous random variable with uniform distribution on $[a, b]$. Compute its moment generating function $\psi(t)$, and then find all moments $E(X^k)$, for $k \geq 1$.

We have

$$f(x) = \frac{1}{b-a}$$

for all $x \in [a, b]$ and $f(x) = 0$ otherwise. Thus, by the LotUS, we have

$$\psi(t) = E(e^{tX}) = \frac{1}{b-a} \int_a^b e^{tx} \, \mathrm{d}x = \frac{e^{tb} - e^{ta}}{t(b-a)}.$$

If we expand the exponentials, we get

$$\begin{aligned} \psi(t) &= \frac{1}{t(b-a)} \left[\sum_{k=0}^{\infty} \frac{t^k b^k}{k!} - \sum_{k=0}^{\infty} \frac{t^k a^k}{k!} \right] \\ &= \frac{1}{b-a} \sum_{k=0}^{\infty} \frac{b^{k+1} - a^{k+1}}{(k+1)!} t^k \end{aligned}$$

for all $t \in \mathbb{R}$. Thus,

$$E(X^k) = \psi^{(k)}(0) = \frac{b^{k+1} - a^{k+1}}{(k+1)(b-a)} = \frac{a^k + a^{k-1}b + \cdots + ab^{k-1} + b^k}{k+1}.$$

Problem 9: Use moment generating functions to confirm that the mean and variance of a random variable $X \sim \mathcal{N}(\mu, \sigma^2)$ are indeed μ and σ^2.

Letting $\psi(t)$ be the moment generating function of X, we have

$$\psi'(t) = \psi(t)(\mu + \sigma^2 t),$$

and so

$$E(X) = \psi'(0) = \mu.$$

To compute the variance, we first compute

$$\psi''(t) = \psi(t)(\mu + \sigma^2 t)^2 + \psi(t)\sigma^2.$$

Then

$$E(X^2) = \psi''(0) = \mu^2 + \sigma^2,$$

and so

$$V(X) = E(X^2) - E(X)^2 = \mu^2 + \sigma^2 - \mu^2 = \sigma^2.$$

Problem 10: Suppose that X and Y are random variables with the joint density function

$$f(x, y) = \begin{cases} 2xy + 0.5 & : 0 \leq x \leq 1, \ 0 \leq y \leq 1, \\ 0 & : \text{otherwise.} \end{cases}$$

Compute the covariance of X and Y.

Using the Shortcut Formula for Covariance, we compute:

$$\sigma_{XY} = E(XY) - E(X)E(Y).$$

But first, let's grab the expectations of X and Y. To do this, we integrate out y to get the density of x:

$$f(x) = \int_{\mathbb{R}} f(x,y)\ \mathrm{d}y = \int_0^1 (2xy + 0.5)\ \mathrm{d}y = x + 0.5$$

for $0 \leq x \leq 1$, and $f(x) = 0$ otherwise. Then, we compute:

$$E(X) = \int_{\mathbb{R}} xf(x)\ \mathrm{d}x = \int_0^1 (x^2 + 0.5x)\ \mathrm{d}x = \frac{7}{12}.$$

Now, if you look at the joint density function, you'll notice that it is symmetric in x and y. This means that $E(Y) = 7/12$, as well. Finally, we compute the covariance from the shortcut formula:

$$\begin{aligned}
\sigma_{XY} &= E(XY) - \frac{7^2}{12^2} \\
&= \iint_{\mathbb{R}^2} xyf(x,y)\ \mathrm{d}y\mathrm{d}x - \frac{7^2}{12^2} \\
&= \int_0^1 \int_0^1 (2x^2y^2 + 0.5xy)\ \mathrm{d}y\mathrm{d}x - \frac{7^2}{12^2} \\
&= \frac{25}{72} - \frac{7^2}{12^2} \\
&= \frac{1}{144} \\
&\approx 0.007.
\end{aligned}$$

Problem 11: Compute the correlation ρ_{XY} of the random variables in the previous problem.

First, we compute the variances:

$$\sigma_X^2 = \int_{\mathbb{R}} x^2 f(x)\ \mathrm{d}x = \int_0^1 x^2(x + 0.5)\ \mathrm{d}x = \frac{5}{12}.$$

By symmetry, we also have $\sigma_Y^2 = 5/12$. Thus,

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{1/144}{5/12} = \frac{1}{60} \approx 0.017.$$

Problem 12: Many students applying for college take the SAT, which consists of math and verbal components (the latter is currently called evidence-based reading and writing). Let X and Y denote the math and verbal scores, respectively, for a randomly selected student. According to the College Board, the population of students taking the exam in 2017 had the following results:

$$\mu_X = 527, \quad \sigma_X = 107, \quad \mu_Y = 533, \quad \sigma_Y = 100, \quad \rho_{XY} = 0.77.$$

Supposing that $(X, Y) \sim \mathcal{N}_2(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, determine the probability that a student's total score $X + Y$ exceeds 1250, the minimum admission score for a particular university.

From class, we learned that the random variable $Z = X+Y$ is normal. Using linearity of expectation, we compute its mean

$$E(Z) = \mu_X + \mu_Y = 527 + 533 = 1060$$

and using bilinearity and symmetry of covariance we compute its variance

$$\begin{aligned}
\sigma_Z^2 &= \operatorname{Cov}(X+Y, X+Y) \\
&= \operatorname{Cov}(X,X) + \operatorname{Cov}(Y,Y) + 2\operatorname{Cov}(X,Y) \\
&= \sigma_X^2 + \sigma_Y^2 + 2\rho_{XY}\sigma_X\sigma_Y \\
&= 107^2 + 100^2 + 2(0.77)(107)(100) \\
&= 37{,}927.
\end{aligned}$$

Thus, $Z \sim \mathcal{N}(1060, 37{,}927)$. Using technology, we then compute

$$P(X+Y > 1250) = P(Z > 1250) = 1 - F_Z(1250) \approx 0.165.$$