Problem 1: Let the sample space S represent the current population of the United States. Suppose also that to every individual $s \in S$ we assign the state $X(s)$ in which they currently reside. So, for example, I would write

$$X(\text{John}) = \text{New York}.$$

Does this function

$$X : S \to \{\text{Alabama}, \text{Alaska}, \ldots, \text{Wyoming}\}$$

qualify as a random variable on S? If not, how could you make it fit the template of a random variable?

No, this is not a valid random variable since random variables are supposed to output numbers. We could change X into a random variable by assigning a number to each state. While this would technically work, it might be undesirable since it introduces an artificial order on the states.

Problem 2: Suppose that we flip a fair coin twice. As usual, we model the situation using a uniform probability space with sample space

$$S = \{HH, HT, TH, TT\}.$$

Describe some random variables on S.

We could let X be a random variable that represents the number of heads that we obtain. Or, X could be the number of tails.

Problem 3: Suppose that we toss a pair of fair six-sided dice. As usual, we model the situation using a uniform probability space with sample space

$$S = \{(1,1), (1,2), \ldots, (5,6), (6,6)\}.$$

Describe some random variables on S.

We could let X be a random variable that represents the first number that we obtain. Or, it could represent the second number. Or, it could be the *sum* of the two numbers.

Problem 4: Let

$$S = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$$

be the solid unit disk in $\mathbb{R}^2$. Consider drawing a point uniformly at random from S.

(a) Define an appropriate probability space that models this scenario.

The sample space is the disk S. The probability measure is continuous and uniform, with density function given by

$$f(x,y) = \frac{1}{\pi}$$

for all $(x,y) \in S$.

(b) Describe some random variables in this scenario.

This is similar to the previous problem. We could let X be a random variable that represents the x-value that we obtain. Or, it could represent the y-value. Or, it could be the *sum* of the x- and y-values.

Problem 5: You and a friend play a game where you each toss a fair coin. If both coins land tails, you win \$1; if they both land heads, you win \$2; if the coins do not match (one lands a head, the other a tail), you lose \$1 (win $-\$1$).

(a) Describe an appropriate probability space that models the situation.

As usual, the sample space is

$$S = \{HH, HT, TH, TT\}$$

The probability measure is uniform, with the probability of any simple event occurring equal to $1/4$.

(b) Let X be the random variable which gives your winnings on a single play of the game. Describe the probability distribution of X.

We have

$$\begin{aligned} P(X = 1) &= 1/4, \\ P(X = 2) &= 1/4, \\ P(X = -1) &= 1/2. \end{aligned}$$

Problem 6: Five balls, numbered 1, 2, 3, 4, and 5, are placed in an urn. Two balls are randomly selected from the five, and their numbers noted. (The order of the selected balls does not matter.)

(a) Describe an appropriate probability space that models the situation.

The sample space S is all two-element combinations chosen from $\{1, 2, 3, 4, 5\}$. The probability measure is uniform, and since the cardinality of S is $\binom{5}{2} = 10$, the probability of any simple event is $1/10$.

(b) Let X = the largest of the two numbers. Describe the probability distribution of X.

We have

$$\begin{aligned} P(X = 2) &= 1/10, \\ P(X = 3) &= 2/10, \\ P(X = 4) &= 3/10, \\ P(X = 5) &= 4/10. \end{aligned}$$

(c) Let Y = the sum of the two numbers. Describe the probability distribution of Y.

We have

$$
\begin{aligned}
P(Y = 3) &= 1/10, \\
P(Y = 4) &= 1/10, \\
P(Y = 5) &= 2/10, \\
P(Y = 6) &= 2/10, \\
P(Y = 7) &= 2/10, \\
P(Y = 8) &= 1/10, \\
P(Y = 9) &= 1/10.
\end{aligned}
$$

Problem 7: Let S be the discrete probability space $S = \{1, 2, 3, 4\}$ with probability function

s	$p(s)$
1	1/3
2	1/3
3	1/6
4	1/6

Define the random variable $X : S \to \mathbb{R}$ by

$$
X(s) = \begin{cases} 0 & : s = 1, 4, \\ 1 & : s = 2, \\ 2 & : s = 3. \end{cases}
$$

Describe the probability distribution of X.

We have

$$
\begin{aligned}
P(X = 0) &= 1/3 + 1/6 = 1/2, \\
P(X = 1) &= 1/3, \\
P(X = 2) &= 1/6.
\end{aligned}
$$

Problem 8: Which of the random variables in problems 5-7 are discrete? Which are continuous?

They are *all* discrete, since they all have ranges that contain only finitely many values!

Problem 9: A gas station operates two pumps, each of which can pump up to 10,000 gallons of gas in a month. The total amount of gas pumped at the station in a month is a continuous random variable X (measured in 10,000 gallons) with a probability density function given by

$$
f(x) = \begin{cases} x & : 0 < x < 1, \\ 2 - x & : 1 \le x < 2, \\ 0 & : \text{otherwise}. \end{cases}
$$

(a) Find the probability that the station will pump between 8000 and 12,000 gallons in a particular month.

We have
$$P(0.8 \leq X \leq 1.2) = \int_{0.8}^{1.2} f(x)\ \mathrm{d}x = \int_{0.8}^{1} x\ \mathrm{d}x + \int_{1}^{1.2} (2-x)\ \mathrm{d}x = 0.36.$$

(b) Given that the station pumped more than 10,000 gallons in a particular month, find the probability that the station pumped more than 15,000 gallons during a month.

We are asked to compute the conditional probability
$$P(X \geq 1.5 \mid X \geq 1) = \frac{P(X \geq 1.5 \text{ and } X \geq 1)}{P(X \geq 1)} = \frac{P(X \geq 1.5)}{P(X \geq 1)}.$$

But we have
$$P(X \geq 1) = \int_{1}^{\infty} f(x)\ \mathrm{d}x = \int_{1}^{2} (2-x)\ \mathrm{d}x = 1/2$$

and
$$P(X \geq 1.5) = \int_{1.5}^{\infty} f(x)\ \mathrm{d}x = \int_{1.5}^{2} (2-x)\ \mathrm{d}x = 1/8.$$

Hence
$$P(X \geq 1.5 \mid x \geq 1) = 1/4.$$

Problem 10: The length of time to failure (in hundreds of hours) for a transistor is a random variable X with distribution function given by

$$F(x) = \begin{cases} 1 - e^{-x^2} & : x \geq 0, \\ 0 & : \text{otherwise.} \end{cases}$$

(a) Compute the quantile $Q(0.3)$.

We compute the value $x = Q(0.3)$ by solving $F(x) = 0.3$ for x:
$$F(x) = 0.3 \quad \Rightarrow \quad 1 - e^{-x^2} = 0.3 \quad \Rightarrow \quad x = \sqrt{-\ln 0.7} \approx 0.60.$$

(b) Compute the density $f(x)$ of X.

We use the Fundamental Theorem of Calculus to compute:
$$f(x) = F'(x) = \begin{cases} 2xe^{-x^2} & : x > 0, \\ 0 & : x < 0. \end{cases}$$

(c) Find the probability that the transistor operates for at least 200 hours.

We compute:
$$P(X \geq 2) = 1 - P(X < 2) = 1 - F(2) \approx 0.02.$$

(d) Compute the conditional probability $P(X > 1 \mid X \leq 2)$.

We compute:

$$P(X > 1 \mid X \leq 2) = \frac{P(X > 1 \text{ and } X \leq 2)}{P(X \leq 2)} = \frac{F(2) - F(1)}{F(2)} \approx 0.36.$$

Problem 11: Compute the expected value $E(X)$ of the random variable X in problem 5.

We have:

$$E(Y) = 1\left(\frac{1}{4}\right) + 2\left(\frac{1}{4}\right) - 1\left(\frac{1}{2}\right) = 0.25.$$

Problem 12: Suppose X is a discrete random variable distributed uniformly on its range

$$\{1, 2, \ldots, n\},$$

for some integer $n \geq 1$. Compute the mean value of X.

Since X is distributed uniformly, its probability function is $p(x) = 1/n$ for all x in the range. Thus:

$$\mu_X = \sum_{x=1}^{n} x \cdot \frac{1}{n} = \frac{n(n+1)}{2} \cdot \frac{1}{n} = \frac{n+1}{2}.$$

On the other hand, if we think of μ_X as the "center of mass" of the probability distribution, then μ_X should be halfway between 1 and n. But the midpoint between these two numbers is precisely their average:

$$\mu_X = \frac{n+1}{2}.$$

Problem 13: Let X be the number of interviews that a student has prior to getting a job. Suppose that the probability function of X is given by

$$p(x) = \begin{cases} k/x^2 & : x = 1, 2, \ldots, \\ 0 & : \text{otherwise}, \end{cases}$$

where k is a constant such that $\sum_{x=1}^{\infty} k/x^2 = 1$. Compute the mean μ_X (if it exists).

By definition, we have:

$$\mu_X = \sum_{x=1}^{\infty} x \cdot \frac{k}{x^2} = k \cdot \sum_{x=1}^{\infty} \frac{1}{x}.$$

However, the series on the right is the infamous harmonic series, which—as you well know from your training in calculus—diverges. Thus, μ_X does not exist.

Problem 14: Let X be a continuous random variable with density function

$$f(x) = \begin{cases} (3/8)x^2 & : 0 \leq x \leq 2, \\ 0 & : \text{otherwise}. \end{cases}$$

Compute the mean value of X.

We have

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x) \, \mathrm{d}x = \int_0^2 (3/8)x^3 \, \mathrm{d}x = 3/2.$$

Problem 15: The proportion of time per day that all checkout counters in a supermarket are busy is a random variable X with density function

$$f(x) = \begin{cases} cx^2(1-x)^4 & : 0 \leq x \leq 1, \\ 0 & : \text{otherwise.} \end{cases}$$

(a) Find the value of c that makes $f(x)$ a valid density function.

We must solve the equation

$$\int_{-\infty}^{\infty} f(x) \, \mathrm{d}x = 1$$

for c. But

$$\int_{-\infty}^{\infty} f(x) \, \mathrm{d}x = \int_0^1 cx^2(1-x)^4 \, \mathrm{d}x = \frac{c}{105},$$

and hence we must have $c = 105$.

(b) Compute the expected value $E(X)$.

We compute:

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x) \, \mathrm{d}x = 105 \int_0^1 x^3(1-x)^4 \, \mathrm{d}x = 3/8.$$

Problem 16: Consider the random variables X and Y defined in lecture via the table:

s	$X(s)$	$Y(s)$	$(X+Y)(s)$
1	-1	0	-1
2	1	2	3
3	3	-1	2
4	0	3	3

Suppose that the probability distribution on S is uniform.

(a) Compute the probability distributions of X, Y, and $X+Y$.

For X, we have:

$$\begin{aligned} P(X=-1) &= 1/4, \\ P(X=0) &= 1/4, \\ P(X=1) &= 1/4, \\ P(X=3) &= 1/4. \end{aligned}$$

For Y, we have:

$$P(Y = -1) = 1/4,$$
$$P(Y = 0) = 1/4,$$
$$P(Y = 2) = 1/4,$$
$$P(Y = 3) = 1/4.$$

For $X + Y$, we have:

$$P(X + Y = -1) = 1/4,$$
$$P(X + Y = 2) = 1/4,$$
$$P(X + Y = 3) = 1/2.$$

In particular, notice that both X and Y are distributed uniformly on their ranges, but $X+Y$ is not!

(b) Without me even telling you, I bet you can guess the definition of the pointwise product XY. List the outputs of XY, and compute its probability distribution.

The pointwise product is just the row-wise product:

s	$X(s)$	$Y(s)$	$(XY)(s)$
1	-1	0	0
2	1	2	2
3	3	-1	-3
4	0	3	0

Then:

$$P(XY = -3) = 1/4,$$
$$P(XY = 0) = 1/2,$$
$$P(XY = 2) = 1/4.$$

Problem 17: Suppose that we have a random variable X on the finite sample space $S = \{1, 2, 3, 4, 5\}$ with

s	$X(s)$
1	0
2	$\pi/2$
3	π
4	$3\pi/2$
5	2π

(a) Compute the random variable $\sin(X)$.

We have:

s	$X(s)$	$\sin(X)(s)$
1	0	0
2	$\pi/2$	1
3	π	0
4	$3\pi/2$	-1
5	2π	0

(b) Assume that the (discrete) probability distribution on S has probability function $p(s)$ with

s	$p(s)$
1	0
2	1/8
3	1/2
4	1/4
5	1/8

Compute the probability distribution of $\sin(X)$.

We have:

$$\begin{aligned} P(\sin(X) = -1) &= 1/4, \\ P(\sin(X) = 0) &= 5/8, \\ P(\sin(X) = 1) &= 1/8. \end{aligned}$$

Problem 18: Compute the expectation of the random variable $\sin(X)$ in the previous problem.

Let's do this two ways. First, by using the definition of the expectation,

$$E(\sin X) = \sum_{y \in \mathbb{R}} y \cdot p_{\sin X}(y),$$

where $p_{\sin X}(y)$ is the mass function of the random variable $Y = \sin X$. Using the results of the previous problem, we get:

$$E(\sin(X)) = 1 \cdot (1/8) - 1 \cdot (1/4) = -1/8.$$

On the other hand, we can also use the LotUS:

$$E(\sin X) = \sum_{x \in \mathbb{R}} \sin(x) \cdot p_X(x).$$

Now, we compute:

x	$p_X(x)$
0	0
$\pi/2$	1/8
π	1/2
$3\pi/2$	1/4
2π	1/8

Then, using the previously displayed formula, we get:

$$\begin{aligned} E(\sin X) &= \sin(0) \cdot 0 + \sin(\pi/2) \cdot (1/8) + \sin(\pi) \cdot (1/2) + \sin(3\pi/2) \cdot (1/4) + \sin(2\pi) \cdot (1/8) \\ &= 1/8 - 1/4 \\ &= -1/8. \end{aligned}$$

Problem 19: Let X be a continuous random variable with density function

$$f(x) = \begin{cases} (3/8)x^2 & : 0 \leq x \leq 2, \\ 0 & : \text{otherwise.} \end{cases}$$

Compute the mean value of $2X^2 + 1$.

By the LotUS with $g(x) = 2x^2 + 1$, we have

$$E(2X^2 + 1) = \int_{-\infty}^{\infty} (2x^2 + 1) f(x) \, dx = \frac{3}{8} \int_0^2 (2x^2 + 1)x^2 \, dx = 5.8.$$

Problem 20: Let X be a random variable on the same sample space $S = \{1, 2, 3, 4\}$ with the following values:

s	$X(s)$
1	0
2	1
3	0
4	3

Supposing that the probability distribution on S has probability function given by

s	$p(s)$
1	1/3
2	1/9
3	1/3
4	2/9

compute the expectation of the random variable $X^2 + X + 2$.

We will do this in two ways: First, I want to use "weak" linearity of expectations to compute the desired expectation like this:

$$E(X^2 + X + 2) = E(X^2) + E(X) + E(2).$$

Of course, we know that $E(2) = 2$ since 2 is a *constant* random variable. As for the other two expectations $E(X)$ and $E(X^2)$, we first compute:

s	$X(s)$	$X^2(s)$
1	0	0
2	1	1
3	0	0
4	3	9

Then we have

$$E(X) = 1 \cdot (1/9) + 3 \cdot (2/9) \quad \text{and} \quad E(X^2) = 1 \cdot (1/9) + 9 \cdot (2/9),$$

so

$$E(X^2 + X + 2) = \frac{44}{9} \approx 4.89.$$

However, we could have also computed the expectation by using the LotUS with the function $g(x) = x^2 + x + 2$. In this case, we would compute:

s	$X(s)$	$(X^2 + X + 2)(s)$
1	0	2
2	1	4
3	0	2
4	3	14

Then, the LotUS gives us

$$E(X^2 + X + 2) = 2 \cdot (1/3 + 1/3) + 4 \cdot (1/9) + 14 \cdot (2/9) = \frac{44}{9} \approx 4.89,$$

which matches our first answer.

Problem 21: Suppose that X is a discrete random variable with probability distribution

x	$p(x)$
0	1/8
1	1/4
2	3/8
3	1/4

Compute the expectation, variance, and standard deviation of X.

We compute:

$$\mu = 0 \cdot (1/8) + 1 \cdot (1/4) + 2 \cdot (3/8) + 3 \cdot (1/4) = 7/4 = 1.75,$$

and

$$\sigma^2 = (0 - 1.75)^2(1/8) + (1 - 1.75)^2(1/4) + (2 - 1.75)^2(3/8) + (3 - 1.75)^2(1/4) = 15/16 = 0.9375,$$

and

$$\sigma = \sqrt{15/16} \approx 0.97.$$

Problem 22: A single fair six-sided die is tossed once and let X be the number facing up. Find the expected value, variance, and standard deviation of X.

The random variable X is distributed uniformly on its range with $p(x) = 1/6$ for all $x = 1, 2, 3, 4, 5, 6$. Then

$$E(X) = \frac{1}{6} \cdot (1 + 2 + 3 + 4 + 5 + 6) = 3.5,$$

and

$$V(X) = \frac{1}{6} \cdot \left[(1 - 3.5)^2 + (2 - 3.5)^2 + (3 - 3.5)^2 + (4 - 3.5)^2 + (5 - 3.5)^2 + (6 - 3.5)^2\right] \approx 2.92,$$

and

$$\sigma \approx \sqrt{2.92} \approx 1.71.$$

Problem 23: Suppose that X is a continuous random variable with density function

$$f(x) = \begin{cases} (3/2)x^2 + x & : 0 \le x \le 1, \\ 0 & : \text{otherwise.} \end{cases}$$

Compute the expectation, variance, and standard deviation of X.

We compute:

$$\mu = \int_{-\infty}^{\infty} x \cdot f(x)\ \mathrm{d}x = \int_0^1 x\Big((3/2)x^2 + x\Big)\ \mathrm{d}x = 17/24 \approx 0.71,$$

and

$$V(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\ \mathrm{d}x = \int_0^1 (x - 17/24)^2\Big((3/2)x^2 + x\Big)\ \mathrm{d}x \approx 0.05,$$

and

$$\sigma \approx \sqrt{0.05} \approx 0.22.$$

Problem 24:

(a) For a certain random variable X it is known that $E(X) = 2$ and $V(X) = 3$. What is $E(X^2)$?

From the shortcut formula for variance, we get:

$$E(X^2) = V(X) + E(X)^2 = 3 + 2^2 = 7.$$

(b) Let X be a random variable with $E(X) = 2$ and $V(X) = 4$. Compute the expectation and variance of $3 - 2X$.

Using "weak" linearity of expectations, we get

$$E(3-2X) = E(3) - 2E(X) = 3 - 2 \cdot 2 = -1.$$

For the variance, notice that $3 - 2X$ is an affine transformation of X. Then:

$$V(3-2X) = (-2)^2 V(X) = 4 \cdot 4 = 16.$$

Problem 25: Approximately 10% of the glass bottles coming off a production line have serious flaws in the glass. If two bottles are randomly selected, find the mean, variance, and standard deviation of the number of bottles that have serious flaws.

Let X be the number of bottles that have serious flaws. The probability mass function of X is

x	$p(x)$
0	$(0.9)^2 = 0.81$
1	$2(0.1)(0.9) = 0.18$
2	$(0.1)^2 = 0.01$

Given this, we compute

$$\mu_X = 0 \cdot (0.81) + 1 \cdot (0.18) + 2 \cdot (0.01) = 0.2,$$

and

$$\sigma_X^2 = E(X)^2 - \mu_X^2 = 0^2 \cdot (0.81) + 1^2 \cdot (0.18) + 2^2 \cdot (0.01) - (0.2)^2 = 0.18,$$

and

$$\sigma_X = \sqrt{0.18} \approx 0.42.$$