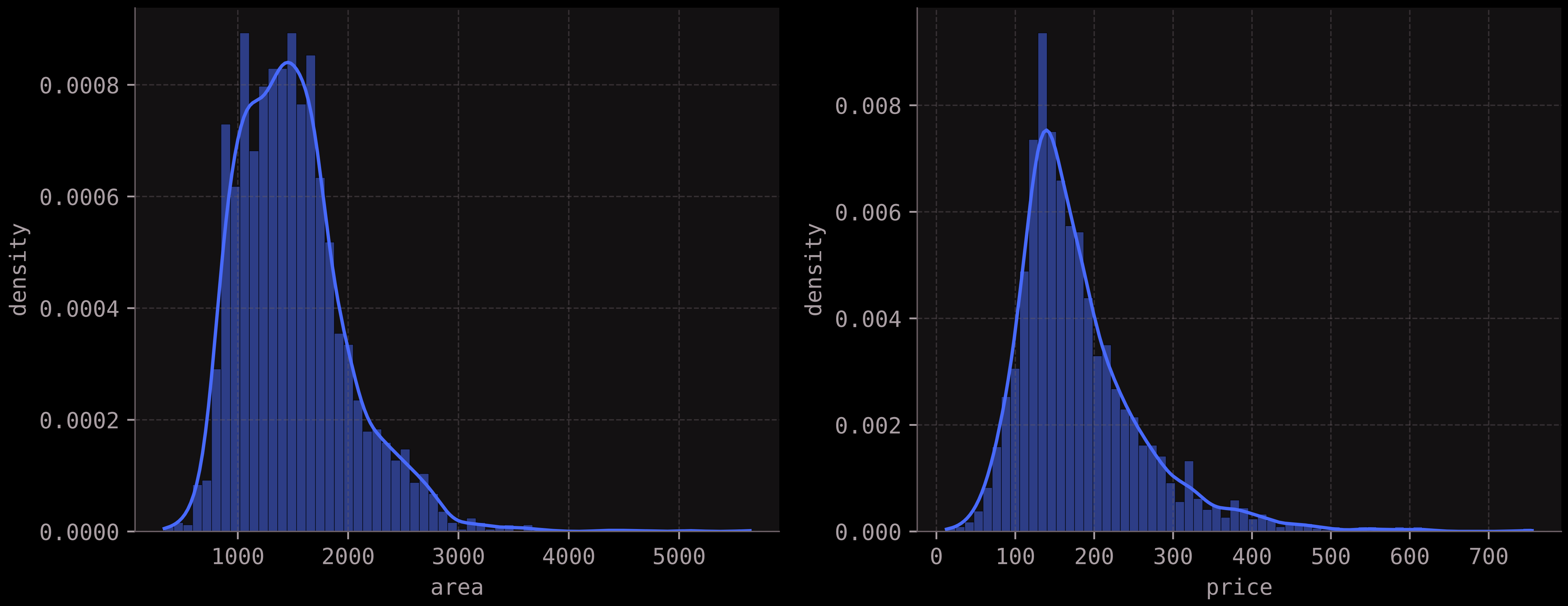
7. Random vectors

7.1. Motivation


0.0008
0.0006
0.0004
0.0002
0.0000
density
1000
2000
3000
4000
5000
area
0.008
0.006
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density
0
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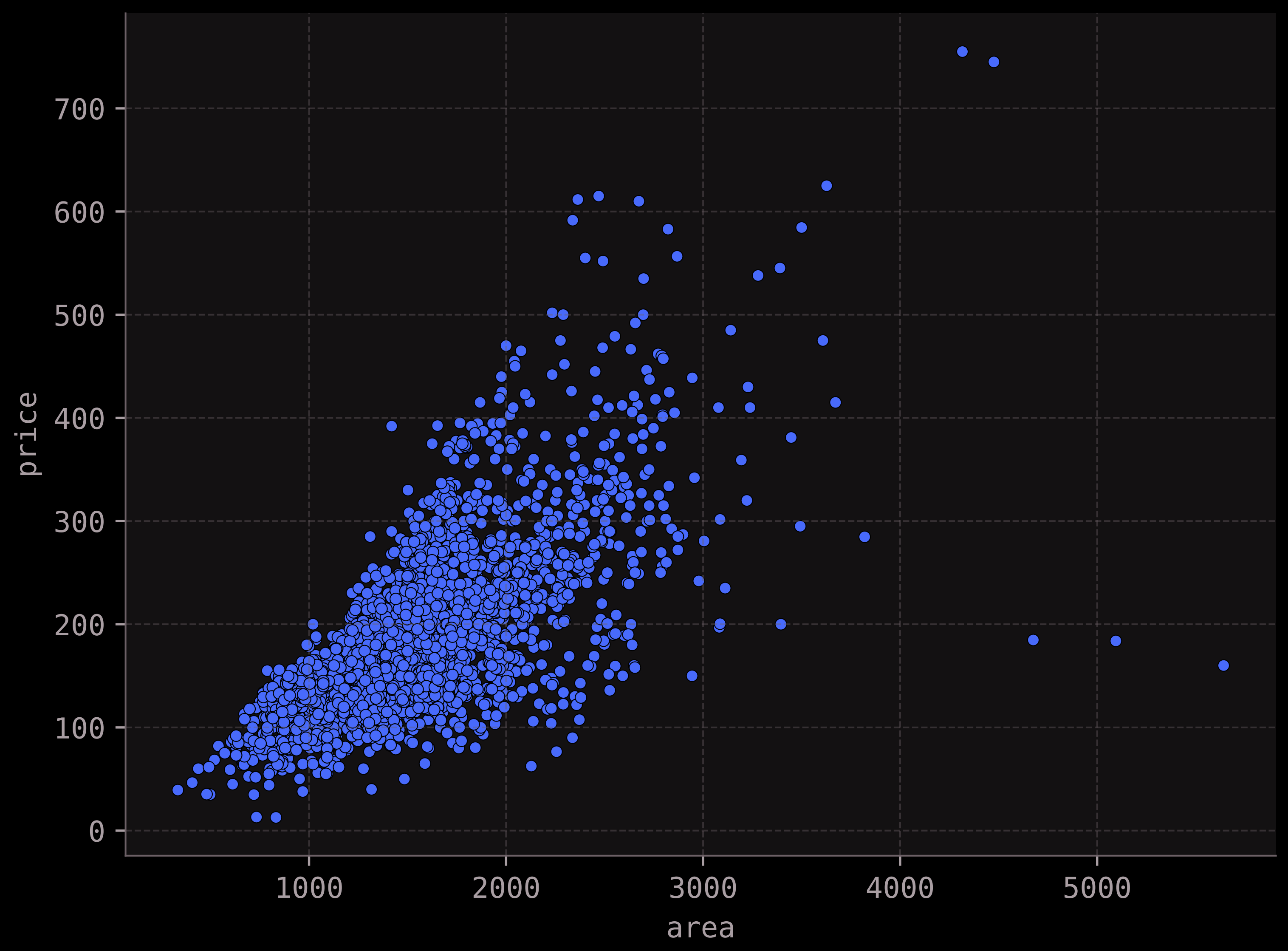
400
500
600
700
price



0
100
200
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price
1000
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5000
area

7.2. 2-dimensional random vectors

Definition 7.1

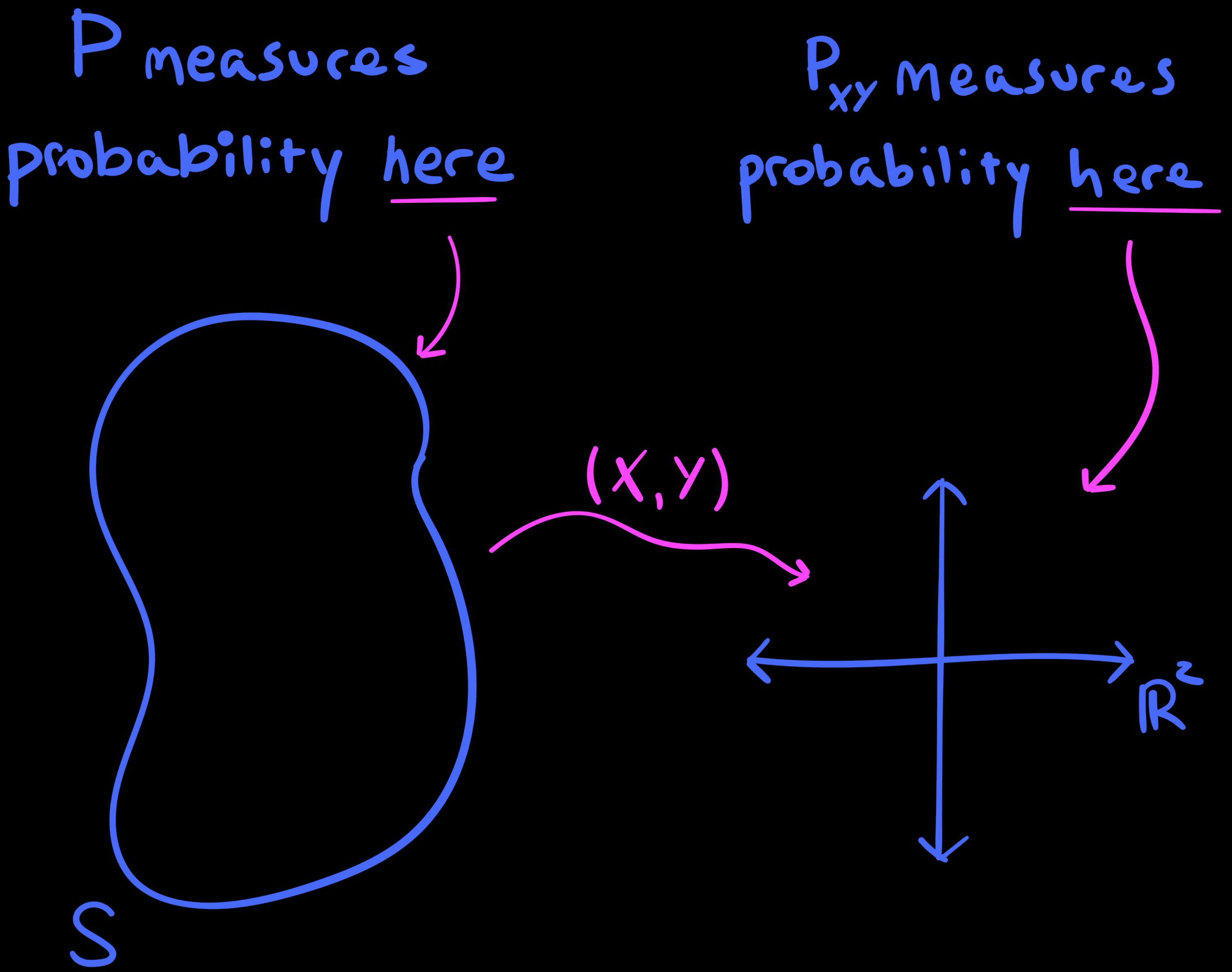
Let S be a probability space. A 2-*dimensional random vector* is a function

$$\mathbf{X} : S \to \mathbb{R}^2.$$

Thus, we may write $\mathbf{X}(s) = (X_1(s), X_2(s))$ for each sample point $s \in S$, where

$$X_1 : S \to \mathbb{R} \quad \text{and} \quad X_2 : S \to \mathbb{R}$$

are random variables. When we do so, the random variables X_1 and X_2 are called the *components* of the random vector $\mathbf{X}$.

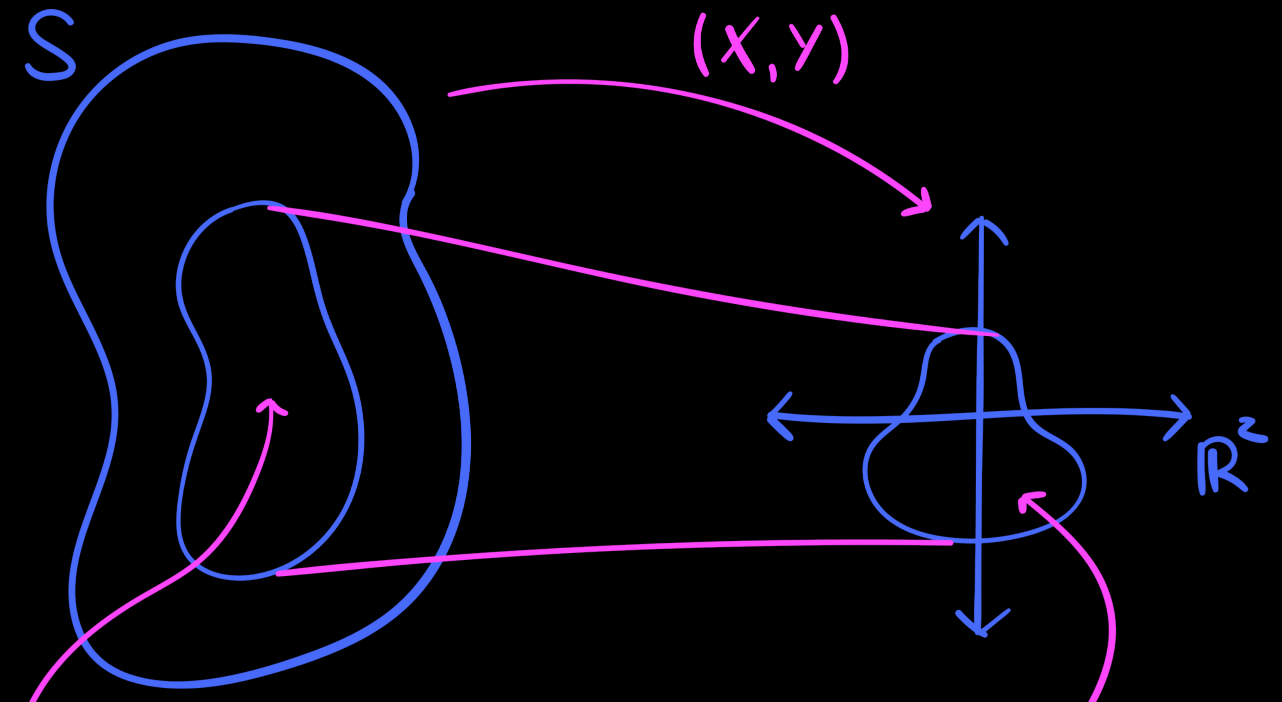

P measures probability here
P_{XY} measures probability here
(X, Y)
S
$\mathbb{R}^2$

Definition 7.2

Let $(X, Y) : S \to \mathbb{R}^2$ be a 2-dimensional random vector on a probability space S with probability measure P. We define the *probability measure* of (X, Y), denoted P_{XY}, via the formula

$$P_{XY}(C) = P(\{s \in S : (X(s), Y(s)) \in C\}), \tag{7.1}$$

for all events $C \subset \mathbb{R}^2$. The probability measure P_{XY} is also called the *joint distribution* or the *bivariate distribution* of X and Y.



$\{s \in S : (X(s), Y(s)) \in C\}$ $\qquad\qquad$ C

these have equal probabilities by definition!

Problem Prompt

Do problem 1 on the worksheet.

Definition 7.3

Let (X, Y) be a 2-dimensional random vector.

- We shall say (X, Y) is *discrete*, or that X and Y are *jointly discrete*, if the joint probability distribution P_{XY} is discrete. In other words, we require that there exists a *joint probability mass function* $p(x, y)$ such that

$$P((X, Y) \in C) = \sum_{(x,y)\in C} p(x, y)$$

  for all events $C \subset \mathbb{R}^2$.

- We shall say (X, Y) is *continuous*, or that X and Y are *jointly continuous*, if the joint probability distribution P_{XY} is continuous. In other words, we require that there exists a *joint probability density function* $f(x, y)$ such that

$$P\left((X, Y) \in C\right) = \iint_C f(x, y)\, \mathrm{d}x\mathrm{d}y$$

  for all events $C \subset \mathbb{R}^2$.

Theorem 7.1

Let (X, Y) be a 2-dimensional random vector.

1. The random vector (X, Y) is discrete if and only if both X and Y are discrete.
2. If (X, Y) is continuous, then X and Y are both continuous. However, it does *not* nececessarily follow that if both X and Y are continuous, then so too is (X, Y).

Problem Prompt

Do problems 2-4 on the worksheet.

7.3. Bivariate distribution functions

Definition 7.4

Let (X, Y) be a 2-dimensional random vector. The *distribution function* of (X, Y) is the function $F : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$F(x, y) = P(X \leq x, Y \leq y).$$

In particular:

1. If (X, Y) is discrete with probability mass function $p(x, y)$, then

$$F(x, y) = \sum_{x^\star \leq x,\, y^\star \leq y} p(x^\star, y^\star).$$

2. If (X, Y) is continuous with probability density function $f(x, y)$, then

$$F(x, y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(x^\star, y^\star)\, \mathrm{d}x^\star \mathrm{d}y^\star.$$

Problem Prompt

Do problem 5 on the worksheet.