

Problem 1: Suppose that we flip a loaded coin twice, with probability p of landing heads. Let $X = 1$ if the first flip lands heads, and $X = 0$ if it lands tails. Similarly, let $Y = 1$ if the second flip lands heads, and $Y = 0$ if it lands tails. Compute the joint distribution of (X, Y) . Verify that your answer is correct by checking that all probabilities sum to 1.

We may specify the joint distribution by computing the four probabilities

$$P(X = 0, Y = 0), P(X = 1, Y = 0), P(X = 0, Y = 1), P(X = 1, Y = 1).$$

It will be convenient to display these in a table:

$X \backslash Y$	0	1
0		
1		

However, since the two events are independent (the first flip and the second), we may compute the probabilities by multiplying the probabilities of X and Y . But since $X, Y \sim \text{Ber}(p)$, we have

$X \backslash Y$	0	1
0	$(1-p)^2$	$p(1-p)$
1	$p(1-p)$	p^2

To verify that our probabilities are correct, we sum them to obtain:

$$(1-p)^2 + 2p(1-p) + p^2 = 1 - 2p + p^2 + 2p - 2p^2 + p^2 = 1.$$

Problem 2: Let (X, Y) be discrete with probability mass function $p(x, y)$ given in the following table:

$x \backslash y$	0	1	2	3	4
0	0.08	0.07	0.06	0.01	0.01
1	0.06	0.10	0.12	0.05	0.02
2	0.05	0.06	0.09	0.04	0.03
3	0.02	0.03	0.03	0.03	0.04

(a) Compute $P(X \leq 2, Y \leq 2)$.

We have

$$\begin{aligned} P(X \leq 2, Y \leq 2) &= \sum_{x \leq 2, y \leq 2} p(x, y) \\ &= 0.08 + 0.07 + 0.06 + 0.06 + 0.10 + 0.12 + 0.05 + 0.06 + 0.09 \\ &= 0.69. \end{aligned}$$

(b) Compute $P(X = Y)$.

We have

$$P(X = Y) = \sum_{x=y} p(x, y) = 0.08 + 0.10 + 0.09 + 0.03 = 0.3.$$

(c) Compute $P(X > Y)$.

We have

$$P(X > Y) = \sum_{x > y} p(x, y) = 0.06 + 0.05 + 0.06 + 0.02 + 0.03 + 0.03 = 0.25.$$

Problem 3: Suppose that (X, Y) is continuous with probability density function

$$f(x, y) = \begin{cases} cx^2y & : x^2 \leq y \leq 1, \\ 0 & : \text{otherwise.} \end{cases}$$

(a) Determine the value of c that makes f a valid density.

We have

$$\iint_{\mathbb{R}^2} f(x, y) \, dydx = \int_{-1}^1 \int_{x^2}^1 cx^2y \, dydx = \frac{4}{21}c,$$

so that we must have $c = 21/4$.

(b) Compute $P(X \geq Y)$.

We have

$$P(X \geq Y) = \iint_{\{x \geq y\}} f(x, y) \, dydx = \frac{21}{4} \int_0^1 \int_{x^2}^x x^2y \, dydx = \frac{3}{20}.$$

Problem 4: Suppose that the continuous random vector (X, Y) is uniformly distributed over the triangle in \mathbb{R}^2 with vertices $(-1, 0)$, $(1, 0)$, and $(0, 1)$.

(a) Compute the density function of (X, Y) .

The area of the triangle is $\frac{1}{2} \times 2 \times 1 = 1$. Since the volume under the density surface and above this triangle must be 1, we must have

$$f(x, y) = \begin{cases} 1 & : (x, y) \text{ is in the triangle,} \\ 0 & : \text{otherwise.} \end{cases}$$

(b) Compute $P(X \leq 3/4, Y \leq 3/4)$.

This problem is best solved by drawing a picture. See class notes. We may compute the probability by removing two triangles from the triangle of area 1, of areas $1/16$ and $1/32$. The answer is therefore $29/32$.

Problem 5: Suppose that (X, Y) is continuous with probability density function

$$f(x, y) = \begin{cases} 30xy^2 & : x - 1 \leq y \leq 1 - x, \ 0 \leq x \leq 1, \\ 0 & : \text{otherwise.} \end{cases}$$

Compute $F(1/2, 1/2)$.

We have

$$F(1/2, 1/2) = \int_{-\infty}^{1/2} \int_{-\infty}^{1/2} f(x, y) \, dy dx = 30 \int_0^{1/2} \int_{x-1}^{1/2} xy^2 \, dy dx = \frac{9}{16}.$$

Problem 6: Compute the marginal probability mass function distribution $p_X(x)$ of the discrete random vector (X, Y) with probability mass function

$x \backslash y$	0	1	2	3	4
0	0.08	0.07	0.06	0.01	0.01
1	0.06	0.10	0.12	0.05	0.02
2	0.05	0.06	0.09	0.04	0.03
3	0.02	0.03	0.03	0.03	0.04

Verify that your computations are correct by making sure all marginal probabilities sum to 1. How would you compute the other marginal mass function $p_Y(y)$?

To compute $p_X(x)$, we simply sum across all y -values (i.e., rows):

$$p_X(0) = \sum_{y=0}^4 p(0, y) = 0.08 + 0.07 + 0.06 + 0.01 + 0.01 = 0.23.$$

Similarly:

$$p_X(1) = 0.35,$$

$$p_X(2) = 0.27,$$

$$p_X(3) = 0.15.$$

To check our work, we verify that these probabilities sum to 1:

$$p_X(0) + p_X(1) + p_X(2) + p_X(3) = 0.23 + 0.35 + 0.27 + 0.15 = 1.$$

To compute $p_Y(y)$, we would sum instead over all x -values (i.e., columns).

Problem 7: Suppose that (X, Y) is continuous with probability density function

$$f(x, y) = \begin{cases} \frac{21}{4}x^2y & : x^2 \leq y \leq 1, \\ 0 & : \text{otherwise.} \end{cases}$$

Compute the marginal density functions $f_X(x)$ and $f_Y(y)$.

By “integrating out” the dependence on y , we compute

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) \, dy = \frac{21}{4} \int_{x^2}^1 x^2 y \, dy = \frac{21}{8} x^2 (1 - x^4).$$

Note that this is valid as long as $-1 \leq x \leq 1$; otherwise, we have $f_X(x) = 0$. Then, by “integrating out” the dependence on x , we compute

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y) \, dx = \frac{21}{4} \int_{-\sqrt{y}}^{\sqrt{y}} x^2 y \, dx = \frac{7}{2} y^{5/2}.$$

Note that this is valid as long as $0 \leq y \leq 1$; otherwise, we have $f_Y(y) = 0$.

Problem 8: Suppose the joint PMF of two discrete random variables X and Y is given by

$x \backslash y$	1	2	3	4
1	0.1	0	0.1	0
2	0.3	0	0.1	0.2
3	0	0.2	0	0

Determine the conditional mass function $p_{X|Y}(x|1)$.

We have

$$\begin{aligned} p_{X|Y}(1|1) &= \frac{p_{XY}(1,1)}{p_Y(1)} = \frac{0.1}{0.1 + 0.3 + 0} = 0.25 \\ p_{X|Y}(2|1) &= \frac{p_{XY}(2,1)}{p_Y(1)} = \frac{0.3}{0.1 + 0.3 + 0} = 0.75 \\ p_{X|Y}(3|1) &= \frac{p_{XY}(3,1)}{p_Y(1)} = \frac{0}{0.1 + 0.3 + 0} = 0. \end{aligned}$$

Problem 9: Suppose that (X, Y) is continuous with probability density function

$$f(x, y) = \begin{cases} \frac{21}{4}x^2y & : x^2 \leq y \leq 1, \\ 0 & : \text{otherwise.} \end{cases}$$

Compute the conditional density function $f_{Y|X}(y|x)$.

We have that

$$f_{Y|X}(y|x) = \frac{f_{XY}(x, y)}{f_X(x)}.$$

But from a previous problem above we computed

$$f_X(x) = \begin{cases} \frac{21}{8}x^2(1 - x^4) & : -1 \leq x \leq 1, \\ 0 & : \text{otherwise.} \end{cases}$$

Thus, we have $f_X(x) \neq 0$ only when $-1 < x < 1$ and $x \neq 0$. For these x -values, we have

$$f_{Y|X}(y|x) = \begin{cases} \frac{2y}{1-x^4} & : x^2 \leq y \leq 1, \\ 0 & : \text{otherwise.} \end{cases}$$

Problem 10: A soft-drink machine has a random amount Y in supply at the beginning of a given day and dispenses a random amount X during the day (with measurements in gallons). It is not resupplied during the day, hence $X \leq Y$. It has been observed that X and Y have joint density given by

$$f(x, y) = \begin{cases} 1/2, & : 0 \leq x \leq y \leq 2, \\ 0 & : \text{otherwise.} \end{cases}$$

Find the conditional density $f(x|y)$ and evaluate the probability that less than 1/2 gallon will be sold, given that the machine contains 1.5 gallons at the start of the day.

We first need to compute the marginal density of Y :

$$f(y) = \int_{-\infty}^{\infty} f(x, y) \, dx = \int_0^y \frac{1}{2} \, dx = \frac{1}{2}y.$$

Notice that this is only valid if $0 \leq y \leq 2$; otherwise, we have $f(y) = 0$. In particular, we have $f(y) \neq 0$ only when $0 < y \leq 2$. For these y -values, we thus have

$$f(x|y) = \frac{f(x, y)}{f(y)} = \begin{cases} \frac{1}{y} & : 0 \leq x \leq y \leq 2, \\ 0 & : \text{otherwise.} \end{cases}$$

Then:

$$P(X \leq 1/2 | Y = 1.5) = \int_{-\infty}^{1/2} f(x|1.5) \, dx = \frac{2}{3} \int_0^{1/2} dx = \frac{1}{3}.$$

Problem 11: Suppose that a person's score X on a mathematics aptitude test is a number between 0 and 1, and that their score Y on a music aptitude test is also a number between 0 and 1. Suppose further that in the population of all college students in the United States, the scores X and Y are distributed according to the following joint PDF

$$f(x, y) = \begin{cases} \frac{2}{5}(2x + 3y) & : 0 \leq x, y \leq 1, \\ 0 & : \text{otherwise.} \end{cases}$$

- (a) What proportion of college students obtain a score greater than 0.8 on the mathematics test?

It will be convenient to first get the marginal densities for both X and Y :

$$f_X(x) = \int_{\mathbb{R}} f(x, y) \, dy = \frac{2}{5} \int_0^1 (2x + 3y) \, dy = \frac{1}{5}(4x + 3), \quad 0 \leq x \leq 1,$$

and

$$f_Y(y) = \int_{\mathbb{R}} f(x, y) \, dx = \frac{2}{5} \int_0^1 (2x + 3y) \, dx = \frac{1}{5}(6y + 2), \quad 0 \leq y \leq 1.$$

Then, to answer this part of the problem, we compute:

$$P(X \geq 0.8) = \int_{0.8}^{\infty} f_X(x) \, dx = \frac{1}{5} \int_{0.8}^1 (4x + 3) \, dx = 0.264.$$

- (b) If a student's score on the music test is 0.3, what is the probability that their score on the mathematics test will be greater than 0.8?

We are asked to compute $P(X \geq 0.8 | Y = 0.3)$. So, we compute:

$$\begin{aligned} P(X \geq 0.8 | Y = 0.3) &= \int_{0.8}^{\infty} f_{X|Y}(x|0.3) \, dx \\ &= \int_{0.8}^{\infty} \frac{f(x, 0.3)}{f_Y(0.3)} \, dx \\ &= \int_{0.8}^1 \frac{\frac{2}{5}(2x + 3 \cdot 0.3)}{\frac{1}{5}(6 \cdot 0.3 + 2)} \, dx \\ &\approx 0.114. \end{aligned}$$

- (c) If a student's score on the mathematics test is 0.3, what is the probability that their score on the music test will be greater than 0.8?

We are asked to compute $P(Y \geq 0.8|X = 0.3)$. So, we compute:

$$\begin{aligned} P(Y \geq 0.8|X = 0.3) &= \int_{0.8}^{\infty} f_{Y|X}(y|0.3) \, dy \\ &= \int_{0.8}^{\infty} \frac{f(0.3, y)}{f_X(0.3)} \, dy \\ &= \int_{0.8}^1 \frac{\frac{2}{5}(2 \cdot 0.3 + 3y)}{\frac{1}{5}(4 \cdot 0.3 + 3)} \, dy \\ &\approx 0.314. \end{aligned}$$

Problem 12: Let X be the number of heads obtained from a single flip of a coin, so that $X \sim \text{Ber}(\theta)$ for some unknown probability θ . Suppose further that θ is an observed value of a $\text{Beta}(2, 2)$ random variable. If we flip the coin and obtain $x = 1$, how should we “update” the distribution of θ ?

This is a problem in Bayesian statistics. The idea is that we have a *prior probability distribution* of the probability of heads θ given by $\text{Beta}(2, 2)$. In particular, the expected value of θ at the outset is 0.5. But when we see that we obtain a head on a single flip, this might suggest to us that θ is likely to be greater than 0.5. We want to “update” our prior probability distribution $\text{Beta}(2, 2)$ to reflect the arrival of the new data point $x = 1$. In the Bayesian lingo, this is the process of “updating the prior distribution to obtain the posterior distribution,” where the *posterior distribution* is the updated one.

Let's begin. We are told that

$$p(x|\theta) = \theta^x(1 - \theta)^{1-x}, \quad x = 0, 1,$$

and

$$f(\theta) \propto \theta(1 - \theta), \quad 0 < \theta < 1.$$

The density function of the posterior distribution is (by definition) the conditional density $f(\theta|x)$. According to Bayes' Theorem, it is obtained via the formula

$$f(\theta|x) = \frac{p(x|\theta)f(\theta)}{p(x)}.$$

But we observed $x = 1$, so in fact we have

$$f(\theta|1) = \frac{p(1|\theta)f(\theta)}{p(1)} \propto \theta^2(1 - \theta), \quad 0 < \theta < 1.$$

Now, technically we don't know the posterior density $f(\theta|1)$ *exactly*, we only know it up to some proportionality constant. But here's a trick that is often used in Bayesian statistics: The variable part of the density $\theta^2(1 - \theta)$ is recognizable as the variable part of a $\text{Beta}(3, 2)$ density. So, whatever the proportionality constant of $f(\theta|1)$ is, it must be the same one as a $\text{Beta}(3, 2)$ density since both densities must integrate to 1 over $[0, 1]$. Thus, the posterior distribution must be a $\text{Beta}(3, 2)$ distribution! In particular, the posterior expected value of θ increases to $3/(3 + 2) = 0.6$.

Problem 13: Suppose that three random vectors X , Y , and Z are jointly continuous with density function

$$f(x, y, z) = \begin{cases} c(x + 2y + 3z) & : 0 \leq x, y, z \leq 1, \\ 0 & : \text{otherwise.} \end{cases}$$

- (a) Determine the value of c that makes $f(x, y, z)$ a valid density function.

As usual, we solve

$$\iiint_{\mathbb{R}^3} f(x, y, z) \, dx dy dz = 1$$

for c . But the integral on the left-hand side is

$$\int_0^1 \int_0^1 \int_0^1 c(x + 2y + 3z) \, dx dy dz = 3c,$$

so we must have $c = 1/3$.

- (b) Compute the marginal density $f_{XY}(x, y)$.

The marginal density is obtained from the joint one by “integrating out” the variable z . So:

$$f_{XY}(x, y) = \int_{\mathbb{R}} f(x, y, z) \, dz = \frac{1}{3} \int_0^1 (x + 2y + 3z) \, dz = \frac{1}{3} (x + 2y + 3/2),$$

for $0 \leq x, y \leq 1$, and $f_{XY}(x, y) = 0$ otherwise.

- (c) Compute the probability $P(Z < 1/2 \mid X = 1/4, Y = 3/4)$.

We have:

$$\begin{aligned} P(Z < 1/2 \mid X = 1/4, Y = 3/4) &= \int_{-\infty}^{1/2} f_{Z|XY}(z \mid 1/4, 3/4) \, dz \\ &= \int_{-\infty}^{1/2} \frac{f(1/4, 3/4, z)}{f_{XY}(1/4, 3/4)} \, dz \\ &= \int_0^{1/2} \frac{\frac{1}{3}(1/4 + 2(3/4) + 3z)}{\frac{1}{3}(1/4 + 2(3/4) + 3/2)} \, dz \\ &= \frac{5}{13} \\ &\approx 0.385. \end{aligned}$$

Problem 14: Suppose that X , Y , and Z have joint “mixed density” function

$$f(x, y, z) = \begin{cases} cx^{1+y+z}(1-x)^{3-y-z} & : 0 < x < 1, \, y, z \in \{0, 1\}, \\ 0 & : \text{otherwise.} \end{cases}$$

- (a) Determine the value of c .

We solve

$$\sum_{y, z \in \{0, 1\}} \int_{\mathbb{R}} f(x, y, z) \, dx = 1$$

for c . But the expression on the left-hand side is

$$\sum_{y,z \in \{0,1\}} \int_0^1 cx^{1+y+z}(1-x)^{3-y-z} dx = \frac{1}{6}c,$$

so we must have $c = 6$.

(b) Compute the marginal “density” $f_{XY}(x, y)$.

We have

$$f_{XY}(x, y) = \sum_{z \in \{0,1\}} f(x, y, z) = 6x^{1+y}(1-x)^{3-y} + 6x^{2+y}(1-x)^{2-y}$$

for $y \in \{0, 1\}$ and $0 < x < 1$, and $f_{XY}(x, y) = 0$ otherwise.

(c) Compute the conditional “density” $f_{Z|XY}(z|1/4, 1)$.

We have

$$f_{Z|XY}(z|1/4, 1) = \frac{f(1/4, 1, z)}{f_{XY}(1/4, 1)} = \frac{3^{3-z}/128}{9/32} = \frac{3^{1-z}}{4},$$

and so

$$f_{Z|XY}(z|1/4, 1) = \begin{cases} 3/4 & : z = 0, \\ 1/4 & : z = 1, \\ 0 & : \text{otherwise.} \end{cases}$$

Problem 15: Suppose that two measurements X and Y are made of the rainfall at a certain location on May 1 of two consecutive years. Supposing that X and Y are independent and that their marginal density functions are each given by

$$f_X(x) = \begin{cases} 2x & : 0 \leq x \leq 1, \\ 0 & : \text{otherwise,} \end{cases} \quad \text{and} \quad f_Y(y) = \begin{cases} 2y & : 0 \leq y \leq 1, \\ 0 & : \text{otherwise,} \end{cases}$$

determine their joint density and compute the probability $P(X + Y \leq 1)$.

Since the variables are independent, their joint density is

$$f(x, y) = f_X(x)f_Y(y) = \begin{cases} 4xy & : 0 \leq x, y \leq 1, \\ 0 & : \text{otherwise.} \end{cases}$$

Then

$$P(X + Y \leq 1) = \iint_{\{x+y \leq 1\}} f(x, y) dx dy = 4 \int_0^1 \int_0^{-x+1} xy dy dx = \frac{1}{6}.$$

Problem 16: Suppose that the joint density function of two continuous random variables X and Y is given by

$$f(x, y) = \begin{cases} kx^2y^2 & : x^2 + y^2 \leq 1, \\ 0 & : \text{otherwise,} \end{cases}$$

for some constant k . Prove that X and Y are dependent.

We will show that the joint density *cannot* be factored into the marginal densities. For this, we first compute

$$f_X(x) = \int_{\mathbb{R}} f(x, y) \, dy = \frac{2}{3} k x^2 (1 - x^2)^{3/2}$$

for $-1 \leq x \leq 1$, and $f_X(x) = 0$ otherwise. Likewise, we have

$$f_Y(y) = \int_{\mathbb{R}} f(x, y) \, dx = \frac{2}{3} k y^2 (1 - y^2)^{3/2}$$

for $-1 \leq y \leq 1$ and $f_Y(y) = 0$ otherwise. But notice that

$$f(0.9, 0.9) = 0$$

since the point $(0.9, 0.9)$ lies outside the unit circle, while

$$f_X(0.9) \neq 0 \quad \text{and} \quad f_Y(0.9) \neq 0.$$

Thus, we have $f(0.9, 0.9) \neq f_X(0.9)f_Y(0.9)$, which proves the variables are dependent.

Problem 17: Suppose that a point (X, Y) is chosen at random from the rectangle R defined as follows:

$$R = \{(x, y) : 0 \leq x \leq 2, 1 \leq y \leq 4\}.$$

- (a) Determine the joint density of X and Y , the marginal density of X , and the marginal density of Y .

The joint distribution must be uniform over the rectangle, thus the density surface is a horizontal plane over R . Since the volume underneath it must be 1 and the rectangle R has area 6, we must have $f(x, y) = 1/6$ for all $(x, y) \in R$, and $f(x, y) = 0$ otherwise. Then, we have

$$f_X(x) = \int_{\mathbb{R}} f(x, y) \, dy = \frac{1}{6} \int_1^4 dy = \frac{1}{2}$$

for $0 \leq x \leq 2$ and $f_X(x) = 0$ otherwise, as well as

$$f_Y(y) = \int_{\mathbb{R}} f(x, y) \, dx = \frac{1}{6} \int_0^2 dx = \frac{1}{3}$$

for $1 \leq y \leq 4$ and $f_Y(y) = 0$ otherwise.

- (b) Are X and Y independent?

Yes. For proof, notice that from part (a) we have

$$f_X(x)f_Y(y) = \begin{cases} \frac{1}{6} & : (x, y) \in R, \\ 0 & : (x, y) \notin R, \end{cases}$$

which is exactly the formula for the joint density $f(x, y)$.

Problem 18: Let $n \geq 1$ be an integer and suppose $X \sim \mathcal{Gam}(n+1, 1)$. Suppose that Y_1, Y_2, \dots, Y_n is an identically distributed random sample that is independent given X with density

$$f(y|x) = \begin{cases} \frac{1}{x} & : 0 < y < x, \\ 0 & : \text{otherwise.} \end{cases}$$

- (a) Determine the joint density of the random sample.

Note that

$$f(x) = \begin{cases} \frac{1}{n!}x^n e^{-x} & : x > 0 \\ 0 & : \text{otherwise.} \end{cases}$$

We are asked to compute $f(y_1, \dots, y_n)$. To do this, we “integrate out” the dependence on x of the joint density $f(x, y_1, \dots, y_n)$. But, by conditional independence of the random sample, we have

$$f(x, y_1, \dots, y_n) = f(y_1, \dots, y_n|x)f(x) = \prod_{i=1}^n f(y_i|x)f(x) = \frac{1}{n!}e^{-x}$$

for $0 < y_1, \dots, y_n < x$ and $x > 0$, and $f(x, y_1, \dots, y_n) = 0$ otherwise. Thus, for $y_1, \dots, y_n > 0$ we must have

$$f(y_1, \dots, y_n) = \int_{\mathbb{R}} f(x, y_1, \dots, y_n) \, dx = \frac{1}{n!} \int_M^{\infty} e^{-x} \, dx = \frac{1}{n!}e^{-M}$$

where M is the maximum of the y_i 's, and $f(y_1, \dots, y_n) = 0$ otherwise.

- (b) Determine the conditional density of X for any given observed values of the random sample.

We have

$$f(x|y_1, \dots, y_n) = \frac{f(x, y_1, \dots, y_n)}{f(y_1, \dots, y_n)}$$

for those y_i 's such that the denominator is not zero. But from (a), we have $f(y_1, \dots, y_n) \neq 0$ if $y_1, \dots, y_n > 0$, and for these y_i 's we have

$$f(x|y_1, \dots, y_n) = \frac{\frac{1}{n!}e^{-x}}{\frac{1}{n!}e^{-M}} = e^{-x+M}$$

for $x > M$ and $f(x|y_1, \dots, y_n) = 0$ otherwise.