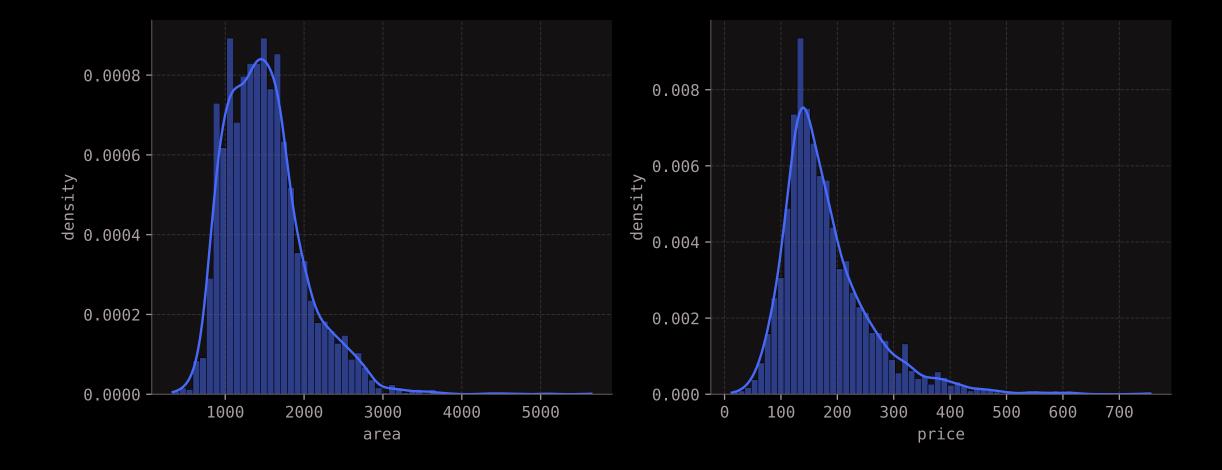
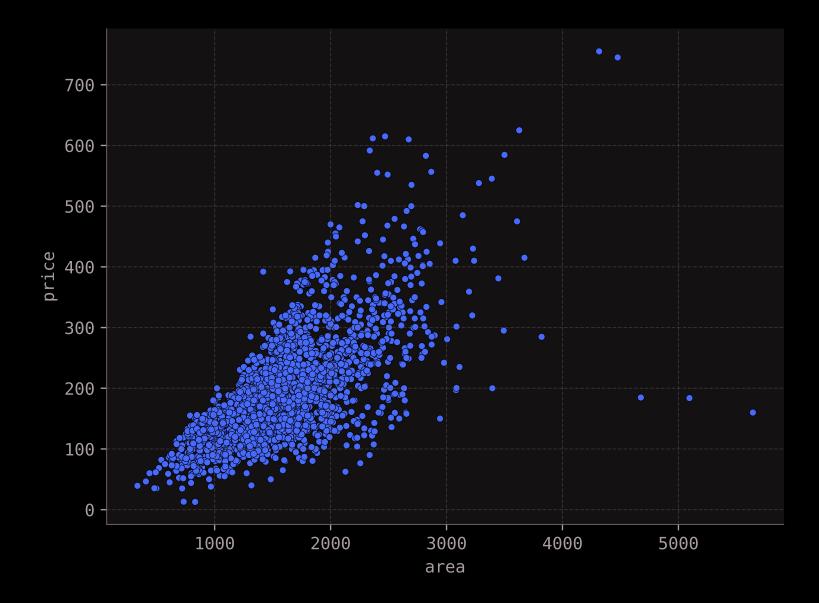
7. Random vectors

7.1. Motivation





7.2. 2-dimensional random vectors

Definition 7.1

Let S be a probability space. A 2-dimensional random vector is a function

$$\mathbf{X}:S o\mathbb{R}^2.$$

Thus, we may write $\mathbf{X}(s) = (X_1(s), X_2(s))$ for each sample point $s \in S$, where

$$X_1:S o\mathbb{R}\quad ext{and}\quad X_2:S o\mathbb{R}$$

are random variables. When we do so, the random variables X_1 and X_2 are called the components of the random vector \mathbf{X} .

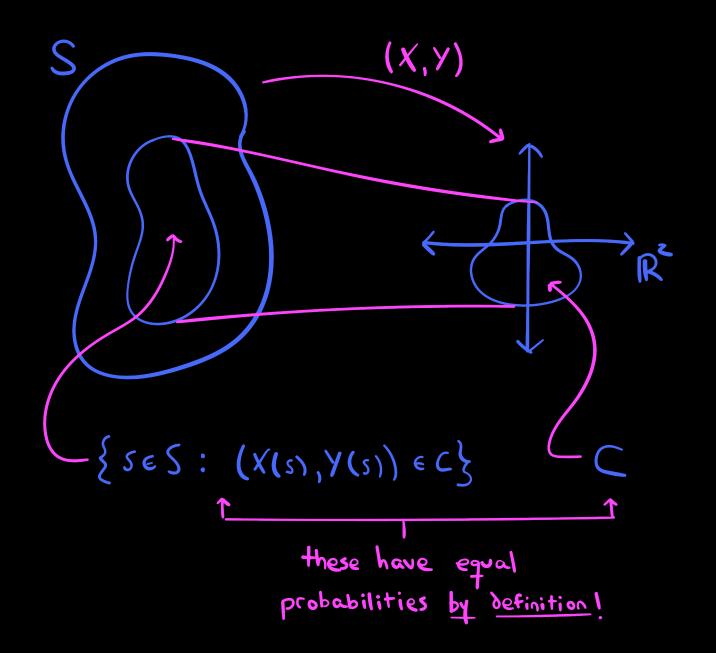
Pmeasures P_{xy} measures probability here probability here (X,Y)

Definition 7.2

Let $(X,Y):S o \mathbb{R}^2$ be a 2-dimensional random vector on a probability space S with probability measure P. We define the *probability measure* of (X,Y), denoted P_{XY} , via the formula

$$P_{XY}(C) = P(\{s \in S : (X(s), Y(s)) \in C\}),$$
 (7.1)

for all events $C\subset \mathbb{R}^2$. The probability measure P_{XY} is also called the *joint distribution* or the *bivariate distribution* of X and Y.





Do problem 1 on the worksheet.

Definition 7.3

Let (X, Y) be a 2-dimensional random vector.

ullet We shall say (X,Y) is discrete, or that X and Y are jointly discrete, if the joint probability distribution P_{XY} is discrete. In other words, we require that there exists a joint probability mass function p(x,y) such that

$$P((X,Y) \in C) = \sum_{(x,y) \in C} p(x,y)$$

for all events $C \subset \mathbb{R}^2$.

ullet We shall say (X,Y) is continuous, or that X and Y are jointly continuous, if the joint probability distribution P_{XY} is continuous. In other words, we require that there exists a joint probability density function f(x,y) such that

$$P\left((X,Y)\in C
ight)=\iint_C f(x,y)\,\mathrm{d}x\mathrm{d}y$$

for all events $C \subset \mathbb{R}^2$.



Theorem 7.1

Let (X, Y) be a 2-dimensional random vector.

- 1. The random vector (X,Y) is discrete if and only if both X and Y are discrete.
- 2. If (X,Y) is continuous, then X and Y are both continuous. However, it does not necessarily follow that if both X and Y are continuous, then so too is (X,Y).



Do problems 2-4 on the worksheet.

7.3. Bivariate distribution functions

Let (X,Y) be a 2-dimensional random vector. The *distribution function* of (X,Y) is the function $F:\mathbb{R}^2 \to \mathbb{R}$ defined by

$$F(x,y) = P(X \le x, Y \le y).$$

In particular:

1. If (X,Y) is discrete with probability mass function p(x,y), then

$$F(x,y) = \sum_{x^\star \leq x, \ y^\star \leq y} p(x^\star, y^\star).$$

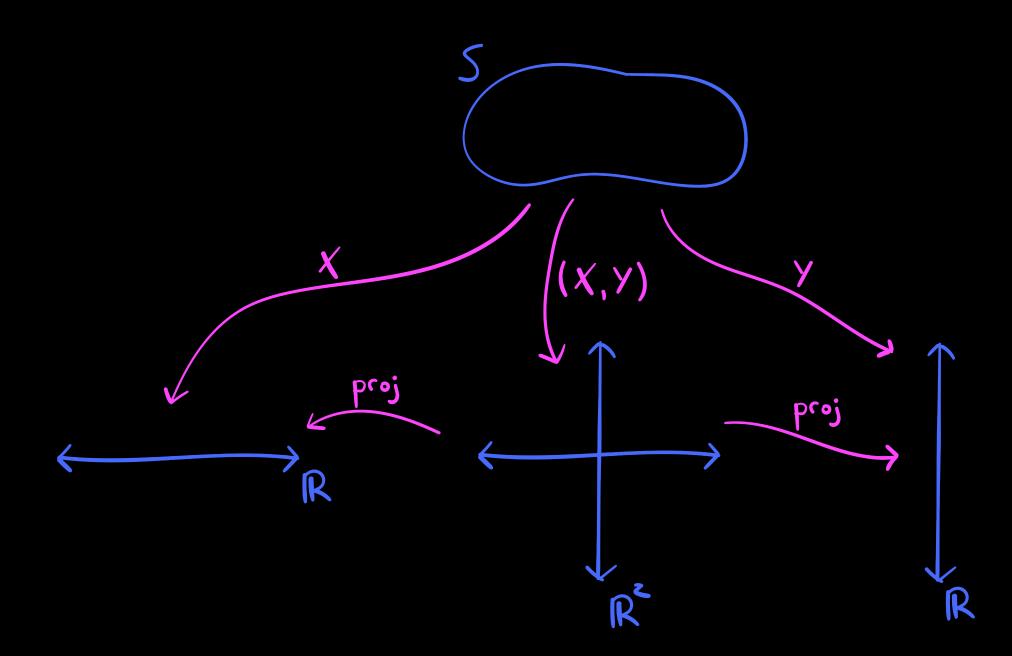
2. If (X,Y) is continuous with probability density function f(x,y), then

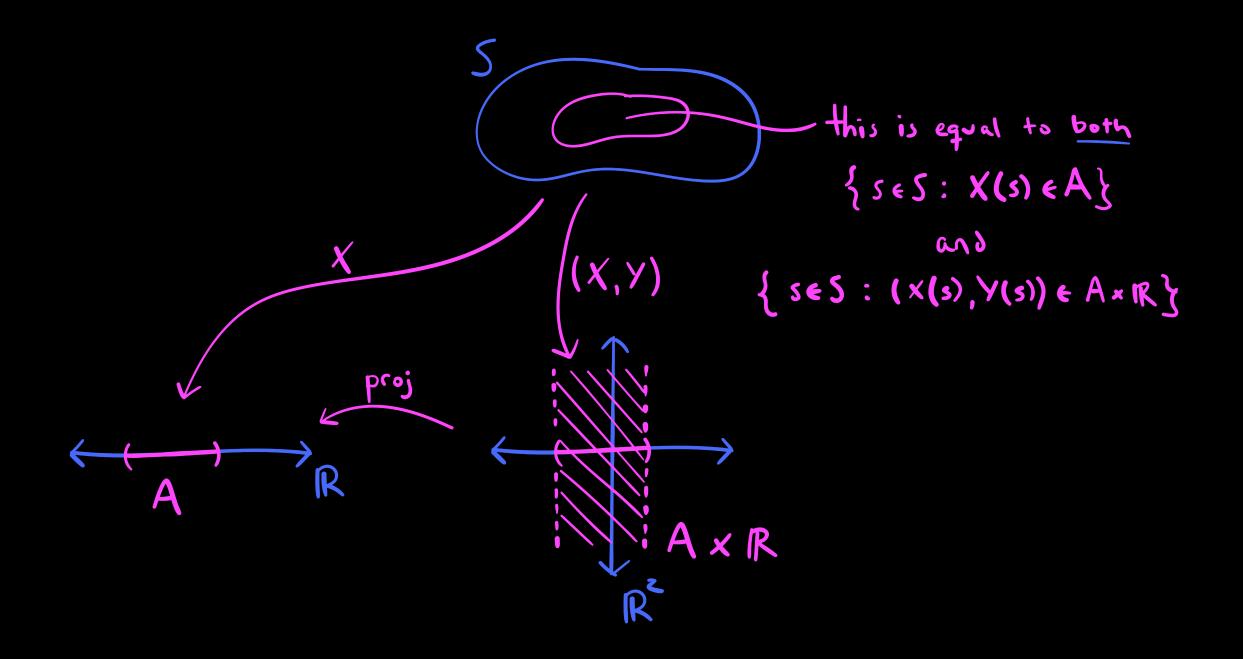
$$F(x,y) = \int_{-\infty}^y \int_{-\infty}^x f(x^\star,y^\star) \; \mathrm{d}x^\star \mathrm{d}y^\star.$$



Do problem 5 on the worksheet.

7.4. Marginal distributions





Let (X,Y) be a 2-dimensional random vector with induced probability measure P_{XY} . Then the measures P_X and P_Y may be obtained via the formulas

$$P_X(A) = P_{XY}(A imes \mathbb{R}) \quad ext{and} \quad P_Y(B) = P_{XY}(\mathbb{R} imes B)$$

for all events $A,B\subset\mathbb{R}$. In particular:

1. If (X,Y) is discrete with probability mass function p(x,y), then

$$P(X \in A) = \sum_{x \in A} \sum_{y \in \mathbb{R}} p(x,y) \quad ext{and} \quad P(Y \in B) = \sum_{y \in B} \sum_{x \in \mathbb{R}} p(x,y).$$

2. If (X,Y) is continuous with probability density function f(x,y), then

$$P(X \in A) = \int_A \int_{-\infty}^\infty f(x,y) \; \mathrm{d}y \mathrm{d}x$$

and

$$P(Y \in B) = \int_{B} \int_{-\infty}^{\infty} f(x,y) \; \mathrm{d}x \mathrm{d}y.$$



Definition 7.5

Let (X,Y) be a 2-dimensional random vector. Then the distributions P_X and P_Y are called the *marginal distributions* of (X,Y).

Theorem 7.3

Let (X,Y) be a 2-dimensional random vector.

1. If (X,Y) is discrete with probability mass function p(x,y), then both X and Y are discrete with probability mass functions given by

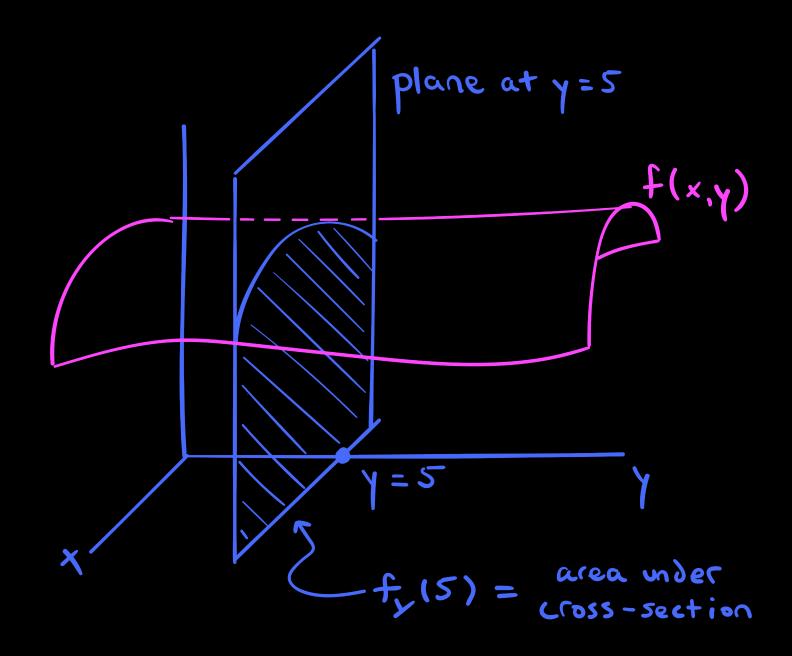
$$p_X(x) = \sum_{y \in \mathbb{R}} p(x,y) \quad ext{and} \quad p_Y(y) = \sum_{x \in \mathbb{R}} p(x,y).$$

2. If (X,Y) is continuous with probability density function f(x,y), then both X and Y are continuous with probability density functions given by

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y) \, \mathrm{d}y \quad ext{and} \quad f_Y(y) = \int_{-\infty}^{\infty} f(x,y) \, \mathrm{d}x.$$
 (7.5)

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- 1. To obtain the marginal mass $p_X(x)$ from the joint mass p(x,y), we "sum out" the dependence of p(x,y) on y. Likewise for obtaining $p_Y(y)$ from p(x,y).
- 2. To obtain the marginal density $f_X(x)$ from the joint density f(x,y), we "integrate out" the dependence of f(x,y) on y. Likewise for obtaining $f_Y(y)$ from f(x,y).





Do problems 6 and 7 on the worksheet.

7.5. Bivariate empirical distributions

Definition 7.6

Let $(X_1, Y_1), (X_2, Y_2), \ldots, (X_m, Y_m)$ be a sequence of 2-dimensional random vectors, all defined on the same probability space.

• The random vectors are called a *bivariate random sample* if they are *independent* and *identically distributed* (IID).

Provided that the sequence is a bivariate random sample, an observed bivariate random sample, or a bivariate dataset, is a sequence of pairs of real numbers

$$(x_1,y_1),(x_2,y_2),\ldots,(x_m,y_m)$$

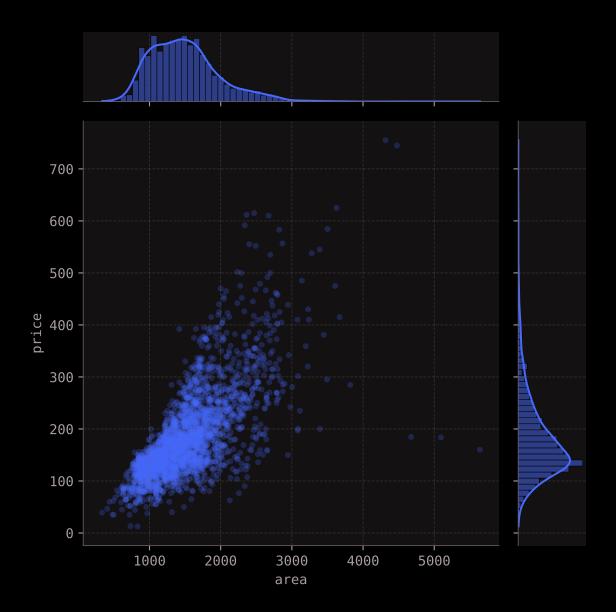
where (x_i, y_i) is an observation of (X_i, Y_i) .

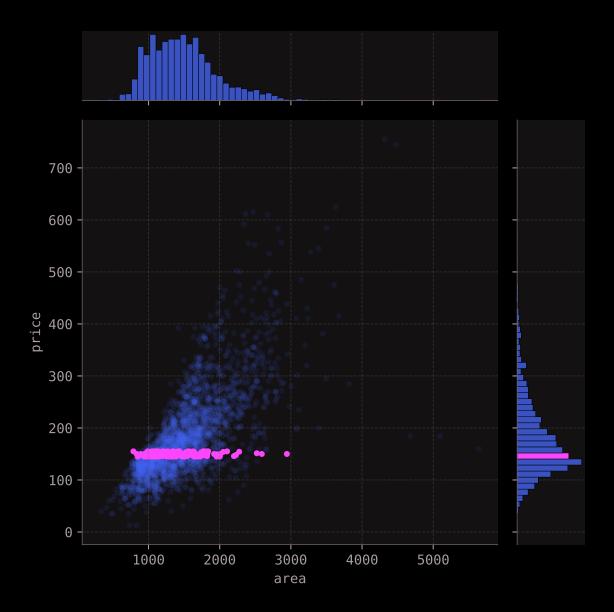
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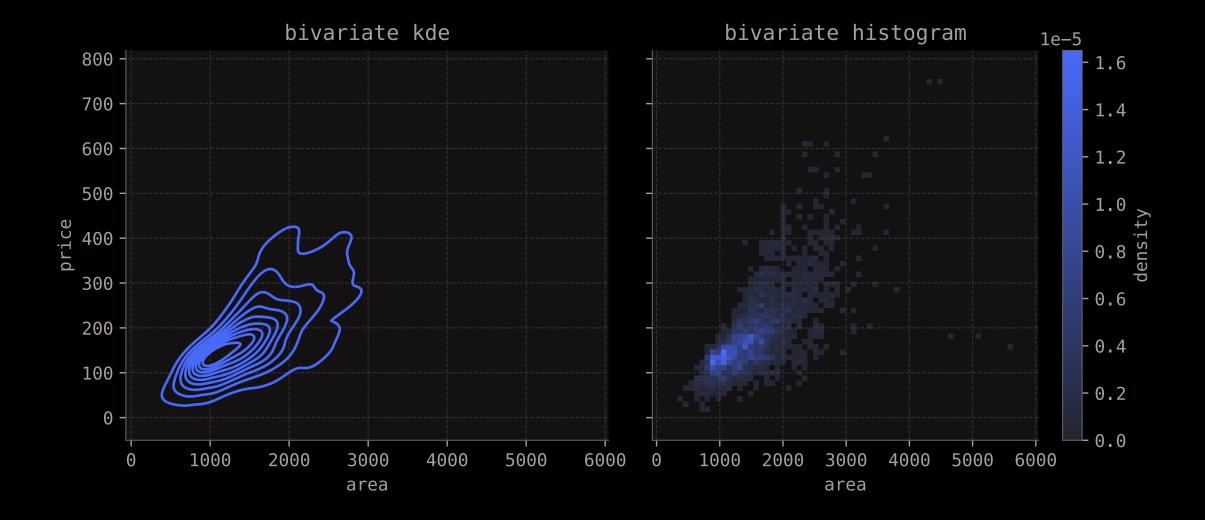
Definition 7.7

Let $(x_1,y_1),(x_2,y_2),\ldots,(x_m,y_m)$ be an observed bivariate random sample, i.e., a bivariate dataset. The *empirical distribution* of the dataset is the discrete probability measure on \mathbb{R}^2 with joint probability mass function

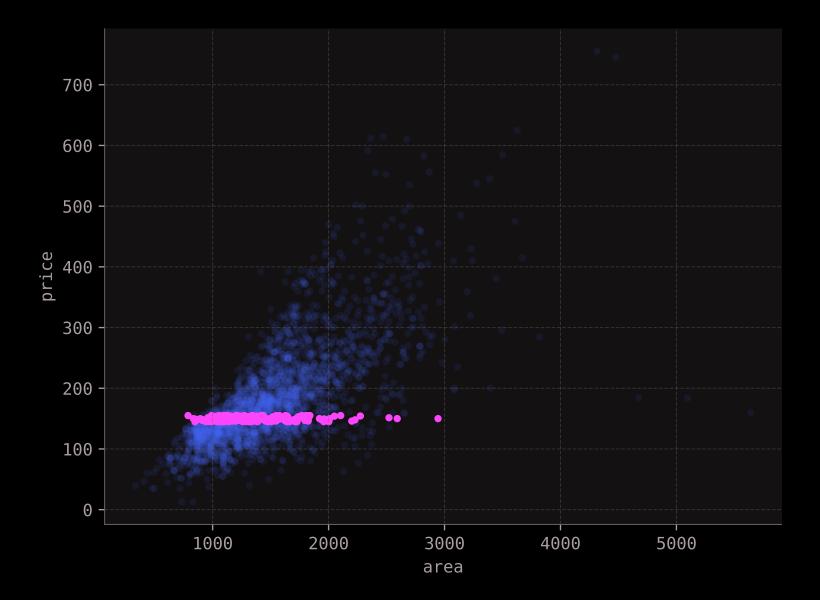
$$p(x,y) = rac{ ext{number of data points } (x_i,y_i) ext{ that match } (x,y)}{m}$$

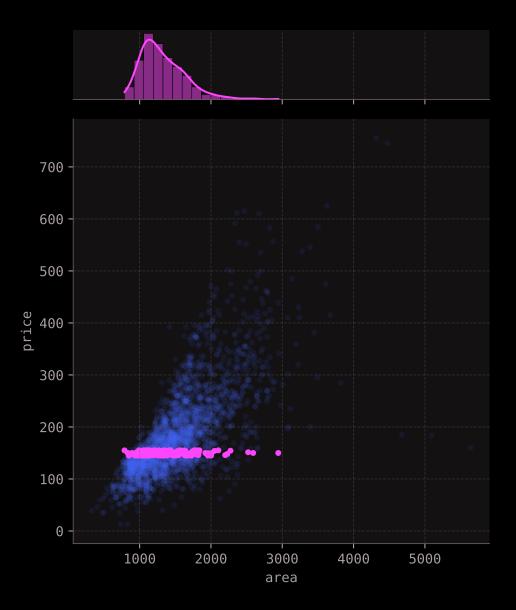






7.6. Conditional distributions





Let X and Y be random variables.

ullet Suppose (X,Y) is discrete, so that both X and Y are discrete as well. The conditional probability mass function of X given Y is the function

$$p_{X|Y}(x|y) = rac{p_{XY}(x,y)}{p_Y(y)},$$

defined for all those y such that $p_Y(y) \neq 0$.

ullet Suppose (X,Y) is continuous, so that both X and Y are continuous as well. The conditional probability density function of X given Y is the function

$$f_{X|Y}(x|y) = rac{f_{XY}(x,y)}{f_{Y}(y)},$$

defined for all those y such that $f_Y(y) \neq 0$.



Do problems 8 and 9 on the worksheet.

Theorem 7.4

Let X and Y be random variables.

• In the case that (X,Y) is discrete, for fixed y with $p_Y(y) \neq 0$, the function $p_{X|Y}(x|y)$ is a probability mass function in the variable x. In particular, we have

$$P(X \in A|Y = y) = \sum_{x \in A} p_{X|Y}(x|y),$$
 (7.6)

for all events $A \subset \mathbb{R}$.

• In the case that (X,Y) is continuous, for fixed y with $f_Y(y) \neq 0$, the function $f_{X|Y}(x|y)$ is a probability density function in the variable x.



Do problems 10 and 11 on the worksheet.