Problem 1: Suppose that X and Y are random variables with the joint density function

$$f(x,y) = \begin{cases} 2xy + 0.5 & : 0 \le x, y \le 1, \\ 0 & : \text{otherwise.} \end{cases}$$

Compute the covariance of X and Y.

Using the Shortcut Formula for Covariance, we compute:

$$\sigma_{XY} = E(XY) - E(X)E(Y).$$

But first, let's grab the expectations of X and Y. To do this, we integrate out y to get the density of x:

$$f(x) = \int_{\mathbb{R}} f(x,y)\ \mathrm{d}y = \int_0^1 (2xy + 0.5)\ \mathrm{d}y = x + 0.5$$

for $0 \le x \le 1$, and $f(x) = 0$ otherwise. Then, we compute:

$$E(X) = \int_{\mathbb{R}} x f(x)\ \mathrm{d}x = \int_0^1 (x^2 + 0.5x)\ \mathrm{d}x = \frac{7}{12}.$$

Now, if you look at the joint density function, you'll notice that it is symmetric in x and y. This means that $E(Y) = 7/12$, as well. Finally, we compute the covariance from the shortcut formula:

$$\begin{aligned} \sigma_{XY} &= E(XY) - \frac{7^2}{12^2} \\ &= \iint_{\mathbb{R}^2} xy f(x,y)\ \mathrm{d}y\mathrm{d}x - \frac{7^2}{12^2} \\ &= \int_0^1 \int_0^1 (2x^2y^2 + 0.5xy)\ \mathrm{d}y\mathrm{d}x - \frac{7^2}{12^2} \\ &= \frac{25}{72} - \frac{7^2}{12^2} \\ &= \frac{1}{144} \\ &\approx 0.007. \end{aligned}$$

Problem 2: Suppose that X and Y are random variables with the joint density function

$$f(x,y) = \begin{cases} 3x & : 0 \le y \le x \le 1, \\ 0 & : \text{otherwise.} \end{cases}$$

Compute the covariance of X and Y.

We follow the same strategy as the previous problem. First, we get the marginal densities:

$$f(x) = \int_{\mathbb{R}} f(x,y)\ \mathrm{d}y = \int_0^x 3x\ \mathrm{d}y = 3x^2$$

for $0 \le x \le 1$ and $f(x) = 0$ otherwise; also:

$$f(y) = \int_{\mathbb{R}} f(x,y)\ \mathrm{d}x = \int_y^1 3x\ \mathrm{d}x = \frac{3}{2}(1 - y^2)$$

for $0 \le y \le 1$. Then, we compute:

$$E(X) = \int_{\mathbb{R}} x f(x)\ \mathrm{d}x = \int_0^1 3x^3\ \mathrm{d}x = \frac{3}{4}$$

and
$$E(Y) = \int_{\mathbb{R}} y f(y)\ \mathrm{d}y = \frac{3}{2} \int_0^1 y(1 - y^2)\ \mathrm{d}y = \frac{3}{8}.$$

Finally, we compute
$$E(XY) = \iint_{\mathbb{R}^2} xy f(x, y)\ \mathrm{d}y\mathrm{d}x = \int_0^1 \int_0^x 3x^2 y\ \mathrm{d}y\mathrm{d}x = \frac{3}{10}$$

and hence
$$\sigma_{XY} = E(XY) - E(X)E(Y) = \frac{3}{10} - \frac{3}{4} \cdot \frac{3}{8} = \frac{3}{160} \approx 0.019.$$