

**Problem 1:** Suppose that  $X$  and  $Y$  are jointly continuous random variables with density

$$f(x, y) = \begin{cases} 24xy & : 0 \leq x \leq 1, 0 \leq y \leq 1, x + y \leq 1, \\ 0 & : \text{otherwise.} \end{cases}$$

Compute the expectation  $E(XY)$ .

Using the bivariate LotUS, we get:

$$E(XY) = \iint_{\mathbb{R}^2} xyf(x, y) \, dydx = 24 \int_0^1 \int_0^{1-x} x^2y^2 \, dydx = \frac{2}{15}.$$

**Problem 2:** Suppose  $X$  and  $Y$  are jointly continuous random variables with the same density from Problem 1. Compute a formula for the conditional expectation  $E(Y \mid X = x)$ . Take care to precisely state the domain of this function.

We first get the marginal density for  $X$ :

$$f(x) = \int_{\mathbb{R}} f(x, y) \, dy = 24 \int_0^{1-x} xy \, dy = 12x(1-x)^2,$$

where the second inequality holds for all  $0 \leq x \leq 1$ ; otherwise, we have  $f(x) = 0$ . Note that the marginal  $f(x)$  is nonzero for  $0 < x < 1$ . For these particular  $x$ -values, the conditional density is defined and is given by

$$f(y|x) = \frac{f(x, y)}{f(x)} = \begin{cases} \frac{2y}{(1-x)^2} & : 0 \leq x \leq 1, 0 \leq y \leq 1, x + y \leq 1, \\ 0 & : \text{otherwise.} \end{cases}$$

Therefore, for all  $x$  with  $0 < x < 1$ , we have

$$E(Y \mid X = x) = \int_{\mathbb{R}} yf(y|x) \, dy = \frac{2}{(1-x)^2} \int_0^{1-x} y^2 \, dy = \frac{2}{3}(1-x).$$

**Problem 3:** Let  $X$  and  $Y$  be two random variables on the probability space  $S = \{a, b, c\}$ . Suppose that the probability distribution  $P$  on  $S$  has mass function  $p(s)$  and that  $X$  and  $Y$  are defined according to the following table:

$s$	$p(s)$	$X(s)$	$Y(s)$
$a$	0.2	1	2
$b$	0.5	2	1
$c$	0.3	1	1

Compute the random variable  $E(Y \mid X)$ .

Our first goal is to get the joint mass function  $p(x, y)$ . We compute:

$p(x, y)$	$y = 1$	$y = 2$
$x = 1$	0.3	0.2
$x = 2$	0.5	0

By adding across the rows, we get the marginal:

$x$	$p(x)$
1	0.5
2	0.5

Then, from the formula  $p(y|x) = p(x, y)/p(x)$ , we get

$p(y x)$	$y = 1$	$y = 2$
$x = 1$	0.6	0.4
$x = 2$	1	0

If  $x \neq 1$  or  $2$ , then the conditional mass  $p(y|x)$  is undefined. For  $x = 1$  or  $2$ , we have

$$E(Y | X = x) = \sum_{y=1}^2 yp(y|x) = \begin{cases} 1 \cdot 0.6 + 2 \cdot 0.4 = 1.4 & : x = 1, \\ 1 \cdot 1 + 2 \cdot 0 = 1 & : x = 2. \end{cases}$$

Finally, recall that the random variable  $E(Y | X)$  is defined as the composite  $h(X) = h \circ X$ , where  $h(x) = E(Y | X = x)$  for  $x = 1, 2$ . Hence, we conclude the problem with:

$s$	$E(Y   X)(s)$
$a$	1.4
$b$	1
$c$	1.4

**Problem 4:** Suppose that a point  $X = x$  is chosen uniformly in the interval  $(0, 1)$ . After  $x$  has been chosen, suppose that a second point  $Y = y$  is chosen uniformly in the interval  $[x, 1]$ . Compute the expectation  $E(Y)$ .

We will compute the expectation via the Law of Total Expectation:

$$E(Y) = E[E(Y | X)].$$

So, we begin by noting that  $E(Y | X = x) = (x + 1)/2 = x/2 + 1/2$ , since the mean of a uniform distribution over an interval is the midpoint between its boundary points. Therefore  $E(Y | X) = X/2 + 1/2$ , and so

$$E(Y) = E(X/2 + 1/2) = E(X)/2 + 1/2 = 1/4 + 1/2 = 3/4.$$

**Problem 5:** The waiting time  $X$  in minutes between calls to a 911 center is exponentially distributed with mean  $\mu = 2$  minutes. Compute the distribution of the transformed random variable  $Y = 60X$  that measures the waiting time in seconds.

Since the mean of  $X \sim \text{Exp}(\lambda)$  is  $1/\lambda$ , we see that  $\lambda = 1/2$  and so

$$f_X(x) = \begin{cases} \frac{1}{2}e^{-x/2} & : x > 0, \\ 0 & : \text{otherwise.} \end{cases}$$

In the notation of the Density Transformation Theorem, the support of the density is  $T = (0, \infty)$ . If we define  $r : T \rightarrow \mathbb{R}$  by  $r(x) = 60x$ , then the range  $U$  of  $r$  is the open interval  $(0, \infty)$ . Note that the inverse function  $s : U \rightarrow \mathbb{R}$  is given by  $s(y) = y/60$ , which is continuously differentiable. Therefore, we have everything that we need to apply the Density Transformation Theorem. We get:

$$f_Y(y) = \begin{cases} \frac{1}{120}e^{-y/120} & : y > 0, \\ 0 & : \text{otherwise,} \end{cases}$$

and thus  $Y \sim \text{Exp}(1/120)$ .

**Problem 6:** Suppose that  $X$  and  $Y$  are two random variables such that  $Y = e^X$  and  $X \sim \mathcal{N}(\mu, \sigma^2)$ . Compute the density of  $Y$ .

The support of the density of  $X$  is all of  $\mathbb{R}$ ; so, in the notation of the Density Transformation Theorem we have  $T = \mathbb{R}$ . Define  $r : T \rightarrow \mathbb{R}$  by setting  $r(x) = e^x$ , and note that the image  $U$  of  $r$  is the open interval  $(0, \infty)$ . The inverse function  $s : U \rightarrow \mathbb{R}$  is given by  $s(y) = \log y$ , which is continuously differentiable. Therefore, we have everything that we need in order to use the Density Transformation Theorem. We compute:

$$f_Y(y) = \begin{cases} \frac{1}{y\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2\sigma^2}(\log y - \mu)^2\right] & : y > 0, \\ 0 & : \text{otherwise.} \end{cases}$$

Hence,  $Y$  is a *lognormal* random variable.

**Problem 7:** Suppose that  $\mathbf{X} = (X_1, X_2)$  is a two-dimensional continuous random vector with density

$$f(x_1, x_2) = \begin{cases} 4x_1x_2 & : 0 < x_1 < 1, \ 0 < x_2 < 1, \\ 0 & : \text{otherwise.} \end{cases}$$

Letting  $T$  be the support of the density, define  $r : T \rightarrow \mathbb{R}^2$  by setting

$$r(x_1, x_2) = \left(\frac{x_1}{x_2}, x_1x_2\right)$$

for  $(x_1, x_2) \in \mathbb{R}^2$ . Compute the density of the random vector  $\mathbf{Y} = r(\mathbf{X})$ .

As I will explain during class, the image  $U$  of  $r$  is the open region in the  $(y_1, y_2)$ -plane bounded by the curves

$$y_1 = y_2, \quad y_2 = 0, \quad y_2 = \frac{1}{y_1}.$$

To compute an inverse  $s : U \rightarrow \mathbb{R}^2$  of  $r$ , notice that the equations

$$y_1 = \frac{x_1}{x_2} \quad \text{and} \quad y_2 = x_1x_2$$

can be solved uniquely for  $x_1$  and  $x_2$  yielding the solutions

$$x_1 = \sqrt{y_1y_2} \quad \text{and} \quad x_2 = \sqrt{\frac{y_2}{y_1}}.$$

Thus we may define  $s$  by setting

$$s(y_1, y_2) = \left(\sqrt{y_1y_2}, \sqrt{\frac{y_2}{y_1}}\right)$$

for all  $(y_1, y_2) \in U$ . To verify that  $s$  is continuously differentiable, we compute its Jacobian matrix:

$$\frac{\partial(s_1, s_2)}{\partial(y_1, y_2)}(y_1, y_2) = \begin{bmatrix} \frac{1}{2}\sqrt{\frac{y_2}{y_1}} & \frac{1}{2}\sqrt{\frac{y_1}{y_2}} \\ -\frac{1}{2}\sqrt{\frac{y_2}{y_1^3}} & \frac{1}{2}\sqrt{\frac{1}{y_1y_2}} \end{bmatrix}.$$

The four partial derivatives are all continuous on  $U$ , and thus  $s$  is continuously differentiable. We therefore have everything that we need to apply the Density Transformation Theorem. First, we compute the determinant of the Jacobian matrix:

$$\det \frac{\partial(s_1, s_2)}{\partial(y_1, y_2)}(y_1, y_2) = \frac{1}{2y_1}.$$

Then, we have

$$f\left(\sqrt{y_1 y_2}, \sqrt{\frac{y_2}{y_1}}\right) \cdot \frac{1}{2y_1} = 4\sqrt{y_1 y_2} \cdot \sqrt{\frac{y_2}{y_1}} \cdot \frac{1}{2y_1} = \frac{2y_2}{y_1}$$

for  $(y_1, y_2) \in U$ , and thus

$$f(y_1, y_2) = \begin{cases} \frac{2y_2}{y_1} & : (y_1, y_2) \in U, \\ 0 & : \text{otherwise.} \end{cases}$$

**Problem 8:** Suppose that  $X$  is a continuous random variable with uniform distribution on  $[a, b]$ . Compute its moment generating function  $\psi(t)$ , and then find all moments  $E(X^k)$ , for  $k \geq 1$ .

We have

$$f(x) = \frac{1}{b-a}$$

for all  $x \in [a, b]$  and  $f(x) = 0$  otherwise. Thus, by the LotUS, we have

$$\psi(t) = E(e^{tX}) = \frac{1}{b-a} \int_a^b e^{tx} \, dx = \frac{e^{tb} - e^{ta}}{t(b-a)}.$$

If we expand the exponentials, we get

$$\begin{aligned} \psi(t) &= \frac{1}{t(b-a)} \left[ \sum_{k=0}^{\infty} \frac{t^k b^k}{k!} - \sum_{k=0}^{\infty} \frac{t^k a^k}{k!} \right] \\ &= \frac{1}{b-a} \sum_{k=0}^{\infty} \frac{b^{k+1} - a^{k+1}}{(k+1)!} t^k \end{aligned}$$

for all  $t \in \mathbb{R}$ . Thus,

$$E(X^k) = \psi^{(k)}(0) = \frac{b^{k+1} - a^{k+1}}{(k+1)(b-a)} = \frac{a^k + a^{k-1}b + \cdots + ab^{k-1} + b^k}{k+1}.$$

**Problem 9:** Use moment generating functions to confirm that the mean and variance of a random variable  $X \sim \mathcal{N}(\mu, \sigma^2)$  are indeed  $\mu$  and  $\sigma^2$ .

Letting  $\psi(t)$  be the moment generating function of  $X$ , we have

$$\psi'(t) = \psi(t)(\mu + \sigma^2 t),$$

and so

$$E(X) = \psi'(0) = \mu.$$

To compute the variance, we first compute

$$\psi''(t) = \psi(t)(\mu + \sigma^2 t)^2 + \psi(t)\sigma^2.$$

Then

$$E(X^2) = \psi''(0) = \mu^2 + \sigma^2,$$

and so

$$V(X) = E(X^2) - E(X)^2 = \mu^2 + \sigma^2 - \mu^2 = \sigma^2.$$

**Problem 10:** Suppose that  $X$  and  $Y$  are random variables with the joint density function

$$f(x, y) = \begin{cases} 2xy + 0.5 & : 0 \leq x \leq 1, 0 \leq y \leq 1, \\ 0 & : \text{otherwise.} \end{cases}$$

Compute the covariance of  $X$  and  $Y$ .

Using the Shortcut Formula for Covariance, we compute:

$$\sigma_{XY} = E(XY) - E(X)E(Y).$$

But first, let's grab the expectations of  $X$  and  $Y$ . To do this, we integrate out  $y$  to get the density of  $x$ :

$$f(x) = \int_{\mathbb{R}} f(x, y) \, dy = \int_0^1 (2xy + 0.5) \, dy = x + 0.5$$

for  $0 \leq x \leq 1$ , and  $f(x) = 0$  otherwise. Then, we compute:

$$E(X) = \int_{\mathbb{R}} xf(x) \, dx = \int_0^1 (x^2 + 0.5x) \, dx = \frac{7}{12}.$$

Now, if you look at the joint density function, you'll notice that it is symmetric in  $x$  and  $y$ . This means that  $E(Y) = 7/12$ , as well. Finally, we compute the covariance from the shortcut formula:

$$\begin{aligned} \sigma_{XY} &= E(XY) - \frac{7^2}{12^2} \\ &= \iint_{\mathbb{R}^2} xyf(x, y) \, dydx - \frac{7^2}{12^2} \\ &= \int_0^1 \int_0^1 (2x^2y^2 + 0.5xy) \, dydx - \frac{7^2}{12^2} \\ &= \frac{25}{72} - \frac{7^2}{12^2} \\ &= \frac{1}{144} \\ &\approx 0.007. \end{aligned}$$

**Problem 11:** Compute the correlation  $\rho_{XY}$  of the random variables in the previous problem.

First, we compute the variances:

$$\sigma_X^2 = \int_{\mathbb{R}} x^2 f(x) \, dx = \int_0^1 x^2(x + 0.5) \, dx = \frac{5}{12}.$$

By symmetry, we also have  $\sigma_Y^2 = 5/12$ . Thus,

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{1/144}{(5/12)^2} = \frac{1}{60} \approx 0.017.$$