**Problem 1:** Suppose that we flip a loaded coin twice, with probability $p$ of landing heads. Let $X = 1$ if the first flip lands heads, and $X = 0$ if it lands tails. Similarly, let $Y = 1$ if the second flip lands heads, and $Y = 0$ if it lands tails. Compute the joint distribution of $(X, Y)$. Verify that your answer is correct by checking that all probabilities sum to 1.

**Problem 2:** Let $(X, Y)$ be discrete with probability mass function $p(x, y)$ given in the following table:

| $x \diagdown y$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0.08 | 0.07 | 0.06 | 0.01 | 0.01 |
| 1 | 0.06 | 0.10 | 0.12 | 0.05 | 0.02 |
| 2 | 0.05 | 0.06 | 0.09 | 0.04 | 0.03 |
| 3 | 0.02 | 0.03 | 0.03 | 0.03 | 0.04 |

(a) Compute $P(X \leq 2, Y \leq 2)$.

(b) Compute $P(X = Y)$.

(c) Compute $P(X > Y)$.

**Problem 3:** Suppose that $(X, Y)$ is continuous with probability density function

$$f(x,y) = \begin{cases} cx^2y & : x^2 \le y \le 1, \\ 0 & : \text{otherwise.} \end{cases}$$

(a) Determine the value of $c$ that makes $f$ a valid density.

(b) Compute $P(X \ge Y)$.

**Problem 4:** Suppose that the continuous random vector $(X, Y)$ is uniformly distributed over the triangle in $\mathbb{R}^2$ with vertices $(-1, 0)$, $(1, 0)$, and $(0, 1)$.

(a) Compute the density function of $(X, Y)$.

(b) Compute $P(X \le 3/4, Y \le 3/4)$.

**Problem 5:** Suppose that $(X, Y)$ is continuous with probability density function

$$f(x,y) = \begin{cases} 30xy^2 & : x - 1 \le y \le 1 - x,\ 0 \le x \le 1, \\ 0 & : \text{otherwise.} \end{cases}$$

Compute $F(1/2, 1/2)$.

**Problem 6:** Compute the marginal probability mass function distribution $p_X(x)$ of the discrete random vector $(X, Y)$ with probability mass function

| $x \diagdown y$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0.08 | 0.07 | 0.06 | 0.01 | 0.01 |
| 1 | 0.06 | 0.10 | 0.12 | 0.05 | 0.02 |
| 2 | 0.05 | 0.06 | 0.09 | 0.04 | 0.03 |
| 3 | 0.02 | 0.03 | 0.03 | 0.03 | 0.04 |

Verify that your computations are correct by making sure all marginal probabilities sum to 1. How would you compute the other marginal mass function $p_Y(y)$?

**Problem 7:** Suppose that $(X, Y)$ is continuous with probability density function

$$f(x, y) = \begin{cases} \frac{21}{4}x^2y & : x^2 \le y \le 1, \\ 0 & : \text{otherwise.} \end{cases}$$

Compute the marginal density functions $f_X(x)$ and $f_Y(y)$.

**Problem 8:** Suppose the joint PMF of two discrete random variables $X$ and $Y$ is given by

| $x \diagdown y$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 0.1 | 0 | 0.1 | 0 |
| 2 | 0.3 | 0 | 0.1 | 0.2 |
| 3 | 0 | 0.2 | 0 | 0 |

Determine the conditional mass function $p_{X|Y}(x|1)$.

**Problem 9:** Suppose that $(X, Y)$ is continuous with probability density function

$$f(x,y) = \begin{cases} \frac{21}{4}x^2y & : x^2 \leq y \leq 1, \\ 0 & : \text{otherwise.} \end{cases}$$

Compute the conditional density function $f_{Y|X}(y|x)$.

**Problem 10:** A soft-drink machine has a random amount $Y$ in supply at the beginning of a given day and dispenses a random amount $X$ during the day (with measurements in gallons). It is not resupplied during the day, hence $X \leq Y$. It has been observed that $X$ and $Y$ have joint density given by

$$f(x,y) = \begin{cases} 1/2, & : 0 \leq x \leq y \leq 2, \\ 0 & : \text{otherwise.} \end{cases}$$

Find the conditional density $f(x|y)$ and evaluate the probability that less than 1/2 gallon will be sold, given that the machine contains 1.5 gallons at the start of the day.

**Problem 11:** Suppose that a person's score $X$ on a mathematics aptitude test is a number between 0 and 1, and that their score $Y$ on a music aptitude test is also a number between 0 and 1. Suppose further that in the population of all college students in the United States, the scores $X$ and $Y$ are distributed according to the following joint PDF

$$f(x,y) = \begin{cases} \frac{2}{5}(2x+3y) & : 0 \leq x, y \leq 1, \\ 0 & : \text{otherwise.} \end{cases}$$

(a) What proportion of college students obtain a score greater than 0.8 on the mathematics test?

(b) If a student's score on the music test is 0.3, what is the probability that their score on the mathematics test will be greater than 0.8?

(c) If a student's score on the mathematics test is 0.3, what is the probability that their score on the music test will be greater than 0.8?

**Problem 12:** Let $X$ be the number of heads obtained from a single flip of a coin, so that $X \sim \mathcal{B}er(\theta)$ for some unknown probability $\theta$. Suppose further that $\theta$ is an observed value of a $\mathcal{B}eta(2,2)$ random variable. If we flip the coin and obtain $x = 1$, how should we "update" the distribution of $\theta$?

**Problem 13:** Suppose that three random vectors $X$, $Y$, and $Z$ are jointly continuous with density function

$$f(x,y,z) = \begin{cases} c(x+2y+3z) & : 0 \le x,y,z \le 1, \\ 0 & : \text{otherwise.} \end{cases}$$

(a) Determine the value of $c$ that makes $f(x,y,z)$ a valid density function.

(b) Compute the marginal density $f_{XY}(x,y)$.

(c) Compute the probability $P\left(Z < 1/2 \mid X = 1/4, Y = 3/4\right)$.

**Problem 14:** Suppose that $X$, $Y$, and $Z$ have joint "mixed density" function

$$f(x,y,z) = \begin{cases} cx^{1+y+z}(1-x)^{3-y-z} & : 0 < x < 1,\ y,z \in \{0,1\}, \\ 0 & : \text{otherwise.} \end{cases}$$

(a) Determine the value of $c$.

(b) Compute the marginal "density" $f_{XY}(x,y)$.

(c) Compute the conditional "density" $f_{Z|XY}(z|1/4,1)$.

**Problem 15:** Suppose that two measurements $X$ and $Y$ are made of the rainfall at a certain location on May 1 of two consecutive years. Supposing that $X$ and $Y$ are independent and that their marginal density functions are each given by

$$f_X(x) = \begin{cases} 2x & : 0 \leq x \leq 1, \\ 0 & : \text{otherwise}, \end{cases} \quad \text{and} \quad f_Y(y) = \begin{cases} 2y & : 0 \leq y \leq 1, \\ 0 & : \text{otherwise}, \end{cases}$$

determine their joint density and compute the probability $P(X + Y \leq 1)$.

**Problem 16:** Suppose that the joint density function of two continuous random variables $X$ and $Y$ is given by

$$f(x, y) = \begin{cases} kx^2y^2 & : x^2 + y^2 \leq 1, \\ 0 & : \text{otherwise}, \end{cases}$$

for some constant $k$. Prove that $X$ and $Y$ are dependent.

**Problem 17:** Suppose that a point $(X, Y)$ is chosen at random from the rectangle $R$ defined as follows:

$$R = \{(x, y) : 0 \leq x \leq 2,\ 1 \leq y \leq 4\}.$$

(a) Determine the joint density of $X$ and $Y$, the marginal density of $X$, and the marginal density of $Y$.

(b) Are $X$ and $Y$ independent?

**Problem 18:** Let $n \geq 1$ be an integer and suppose $X \sim \Gamma(n+1, 1)$. Suppose that $Y_1, Y_2, \ldots, Y_n$ is an IID random sample such that the conditional distributions of each $Y_i$ given $X$ have densities

$$f(y_i|x) = \begin{cases} \frac{1}{x} & : 0 < y_i < x, \\ 0 & : \text{otherwise.} \end{cases}$$

(a) Determine the joint density of the random sample.

(b) Determine the conditional density of $X$ for any given observed values of the random sample.