**Problem 1:** Suppose $X \sim \mathcal{N}(\mu, 1)$ and that we define two estimators by

$$\widehat{\mu}_1 = X \quad \text{and} \quad \widehat{\mu}_2 = 3.$$

Compute the mean squared errors of $\widehat{\mu}_1$ and $\widehat{\mu}_2$.

Since $\widehat{\mu}_2 = 3$ is constant, we have

$$R(\mu, \widehat{\mu}_2) = E((\mu - 3)^2) = (\mu - 3)^2.$$

By linearity of expectation, we have

$$R(\mu, \widehat{\mu}_1) = E((\mu - \widehat{\mu}_1)^2) = \mu^2 - 2\mu E(\widehat{\mu}_1) + E(\widehat{\mu}_1^2).$$

But we have $E(\widehat{\mu}_1) = \mu$, and from the short-cut formula for variance, we have

$$E(\widehat{\mu}_1^2) = 1 + E(\widehat{\mu}_1)^2 = 1 + \mu^2.$$

Thus,

$$R(\mu, \widehat{\mu}_1) = \mu^2 - 2\mu^2 + 1 + \mu^2 = 1.$$

**Problem 2:** Suppose that the number $X$ of defects on a roll of magnetic recording tape has a Poisson distribution for which the mean $\mu$ is either 1.0 or 1.5. Supposing that the prior probability function on $\mu$ is given by

$$p(\mu) = \begin{cases} 0.4 & : \mu = 1.0, \\ 0.6 & : \mu = 1.5, \\ 0 & : \text{otherwise}, \end{cases}$$

and that a selected roll of tape has $X = 3$ defects, what is the posterior probability function of $\mu$? What is $\widehat{\mu}_{\text{MAP}}$?

From Bayes' Theorem and the Law of Total Probability, we have

$$\begin{aligned} p(\mu | x = 3) &= \frac{p(x = 3|\mu)p(\mu)}{p(x = 3|\mu = 1.0)p(\mu = 1.0) + p(x = 3|\mu = 1.5)p(\mu = 1.5)} \\ &= \frac{\frac{1}{3!}\mu^3 e^{-\mu} p(\mu)}{\frac{1}{3!}e^{-1} \cdot 0.4 + \frac{1}{3!}(1.5)^3 e^{-1.5} \cdot 0.6} \\ &\approx \begin{cases} 0.246 & : \mu = 1.0, \\ 0.754 & : \mu = 1.5, \\ 0 & : \text{otherwise}. \end{cases} \end{aligned}$$

From this, we see that $\widehat{\mu}_{\text{MAP}} = 1.5$.