5.7. Exponential distributions

Definition 5.8

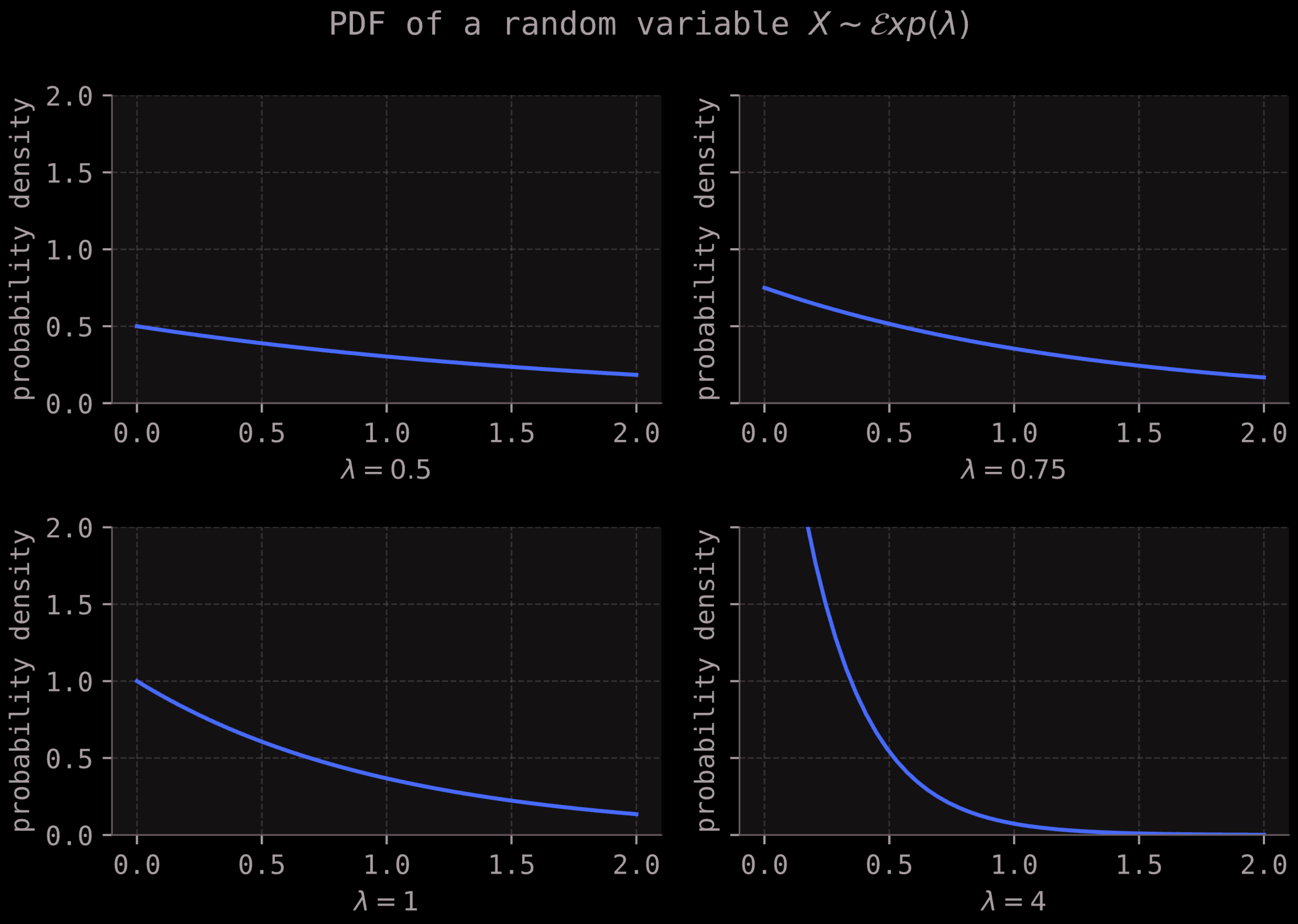
Let $\lambda > 0$ be a real number. A continuous random variable X is said to have an *exponential distribution* with parameter λ, denoted

$$X \sim \mathcal{E}xp(\lambda),$$

if its probability density function is given by

$$f(x; \lambda) = \lambda e^{-\lambda x}$$

with support $(0, \infty)$.


PDF of a random variable $X \sim \mathcal{E}xp(\lambda)$
probability density
$\lambda = 0.5$
$\lambda = 0.75$
$\lambda = 1$
$\lambda = 4$
0.0
0.5
1.0
1.5
2.0

An archetypical exponential scenario (email scenario continued)

Q: Let $T_1, T_2, \ldots$ be the arrival times of the emails in your inbox, and suppose that they arrive at a mean rate λ (measured in reciprocal hours). We set $I_1 = T_1$, the time of *first arrival*, and for each $k = 2, 3, \ldots$ the differences

$$I_k = T_k - T_{k-1}$$

are called *interarrival times*.

What are the distributions of the random variables $I_1, I_2, \ldots$?

A: $I_k \sim \mathcal{E}xp(\lambda)$ for all $k = 1, 2, \ldots$.

Theorem 5.9 (Expectations and variances of exponential variables)

If $X \sim \mathcal{E}xp(\lambda)$, then

$$E(X) = \frac{1}{\lambda} \quad \text{and} \quad V(X) = \frac{1}{\lambda^2}.$$

Problem Prompt

Do problem 14 on the worksheet.

5.8. Gamma distributions

Definition 5.9

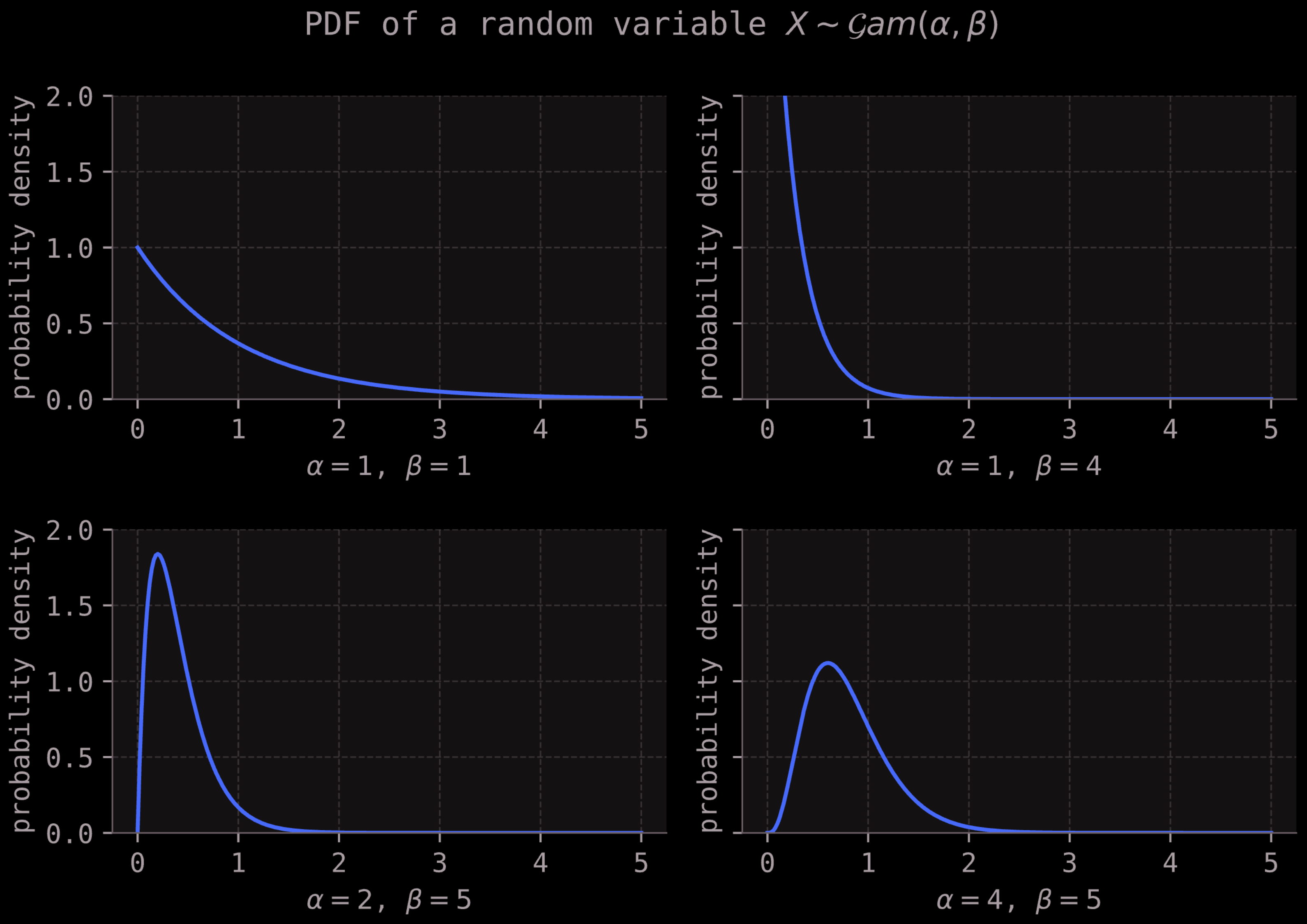
Let $\alpha, \beta > 0$ be real numbers. A continuous random variable X is said to have a *gamma distribution* with parameters α and β, denoted

$$X \sim \mathcal{G}am(\alpha, \beta),$$

if its probability density function is given by

$$f(x; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}. \tag{5.13}$$

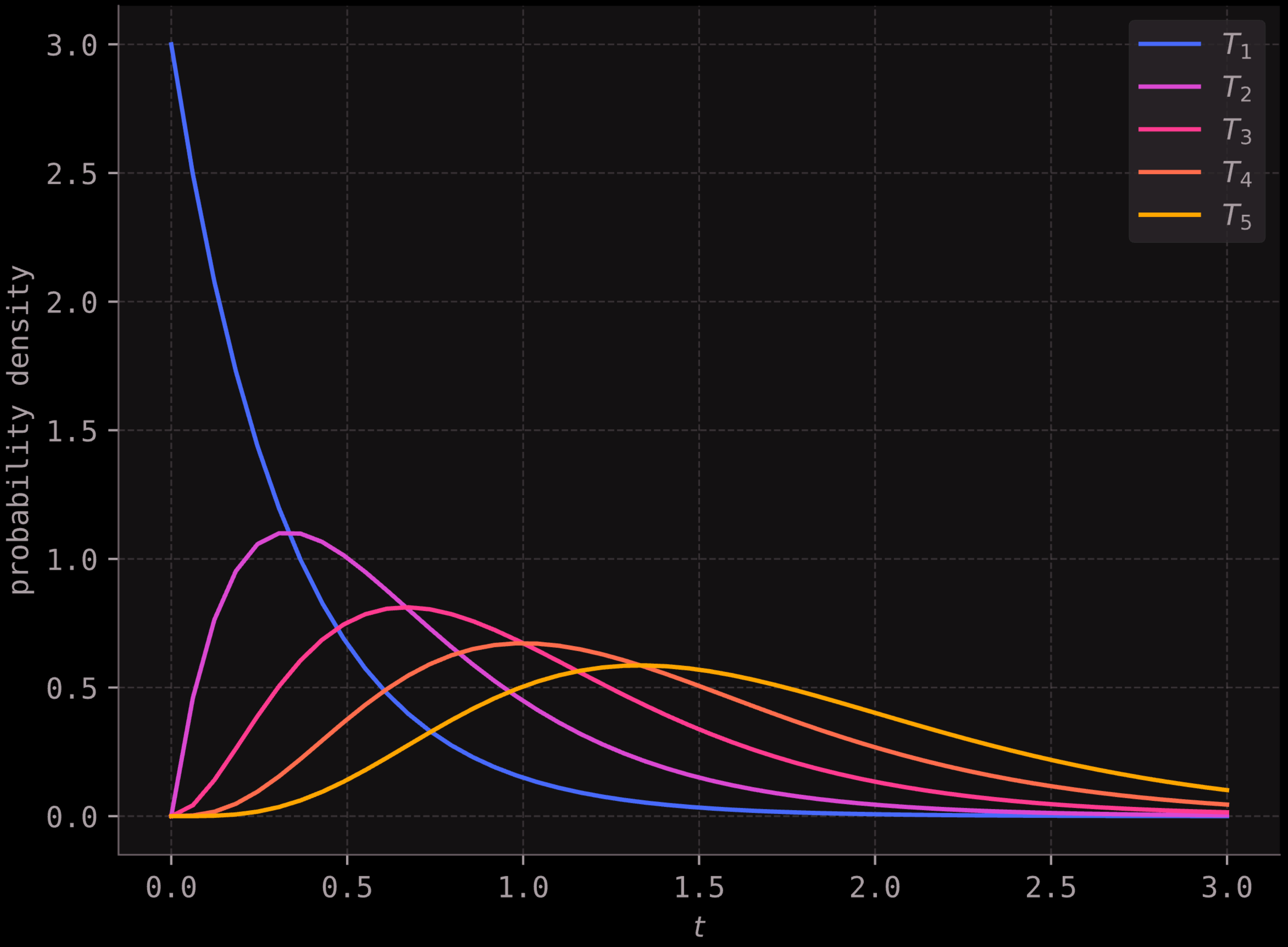
with support $(0, \infty)$.


PDF of a random variable $X \sim \mathcal{G}am(\alpha, \beta)$
probability density
2.0
1.5
1.0
0.5
0.0
0
1
2
3
4
5
$\alpha = 1, \ \beta = 1$
$\alpha = 1, \ \beta = 4$
$\alpha = 2, \ \beta = 5$
$\alpha = 4, \ \beta = 5$

An archetypical gamma scenario (email scenario continued, again)

Q: Remember that T_k (for $k = 1, 2, \ldots$) denotes the (random) arrival time of the k-th email in your inbox, and that the mean rate at which they arrive is λ (measured in reciprocal hours). What are the distributions of the random variables T_k?

A: We have $T_k \sim \mathcal{G}am(k, \lambda)$ for all $k = 1, 2, \ldots$.



T_1
T_2
T_3
T_4
T_5
probability density
t
0.0
0.5
1.0
1.5
2.0
2.5
3.0

Theorem 5.10

If $\alpha > 1$, then $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$.

Theorem 5.11 (Expectations and variances of gamma variables)

If $X \sim \mathcal{G}am(\alpha, \beta)$, then

$$E(X) = \frac{\alpha}{\beta} \quad \text{and} \quad V(X) = \frac{\alpha}{\beta^2}.$$

Problem Prompt

Do problem 15 on the worksheet.

5.9. Beta distributions

Definition 5.10

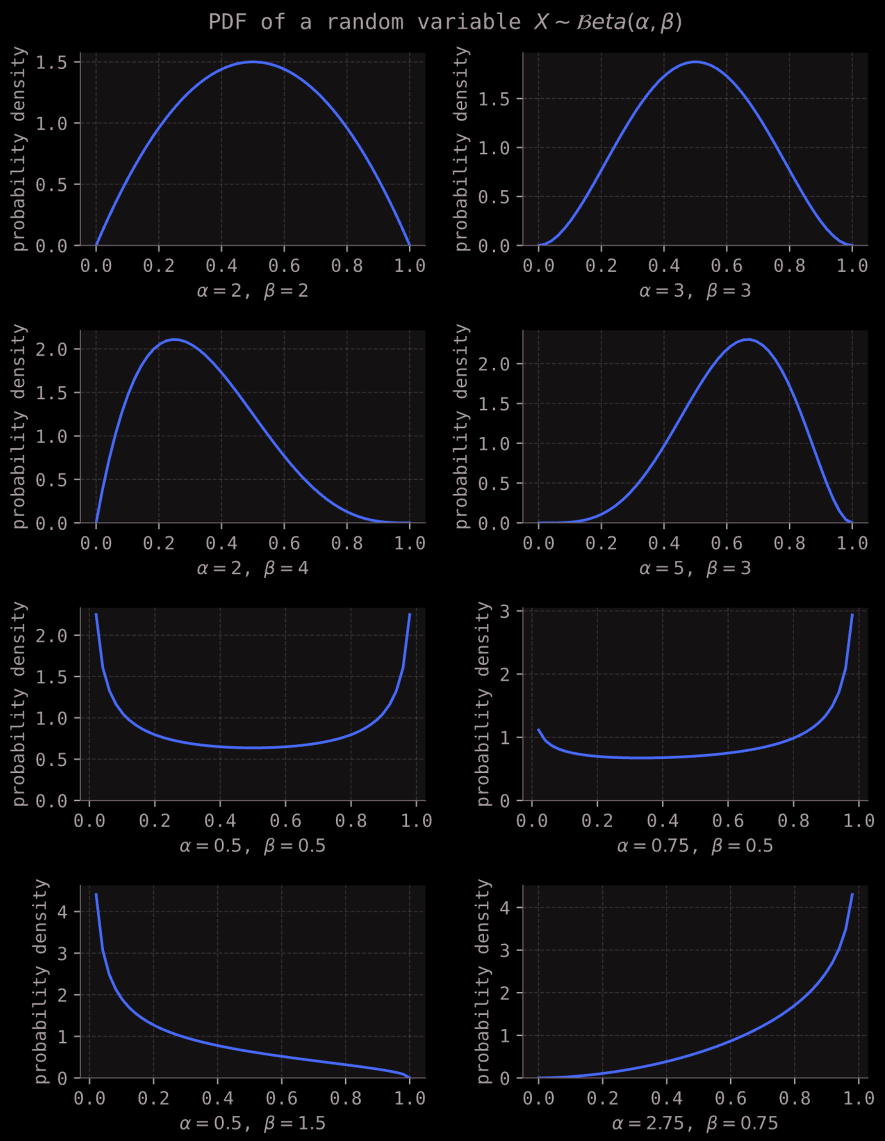
Let $\alpha, \beta > 0$ be real numbers. A continuous random variable X is said to have a *beta distribution* with parameters α and β, denoted

$$X \sim \mathcal{B}eta(\alpha, \beta),$$

if its probability density function is given by

$$f(x; \alpha, \beta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}.$$

with support $(0, 1)$.


PDF of a random variable $X \sim \mathcal{B}eta(\alpha, \beta)$
probability density
$\alpha = 2, \ \beta = 2$
$\alpha = 3, \ \beta = 3$
$\alpha = 2, \ \beta = 4$
$\alpha = 5, \ \beta = 3$
$\alpha = 0.5, \ \beta = 0.5$
$\alpha = 0.75, \ \beta = 0.5$
$\alpha = 0.5, \ \beta = 1.5$
$\alpha = 2.75, \ \beta = 0.75$

Theorem 5.12 (Expectations and variances of beta variables)

If $X \sim \mathcal{B}eta(\alpha, \beta)$, then

$$E(X) = \frac{\alpha}{\alpha + \beta} \quad \text{and} \quad V(X) = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}.$$

Problem Prompt

Do problem 16 on the worksheet.