5.6. Normal distributions

Definition 5.6

Let μ and σ be real numbers with $\sigma > 0$. A continuous random variable X is said to have a *normal distribution* (or *Gaussian distribution*) with parameters μ and σ, denoted

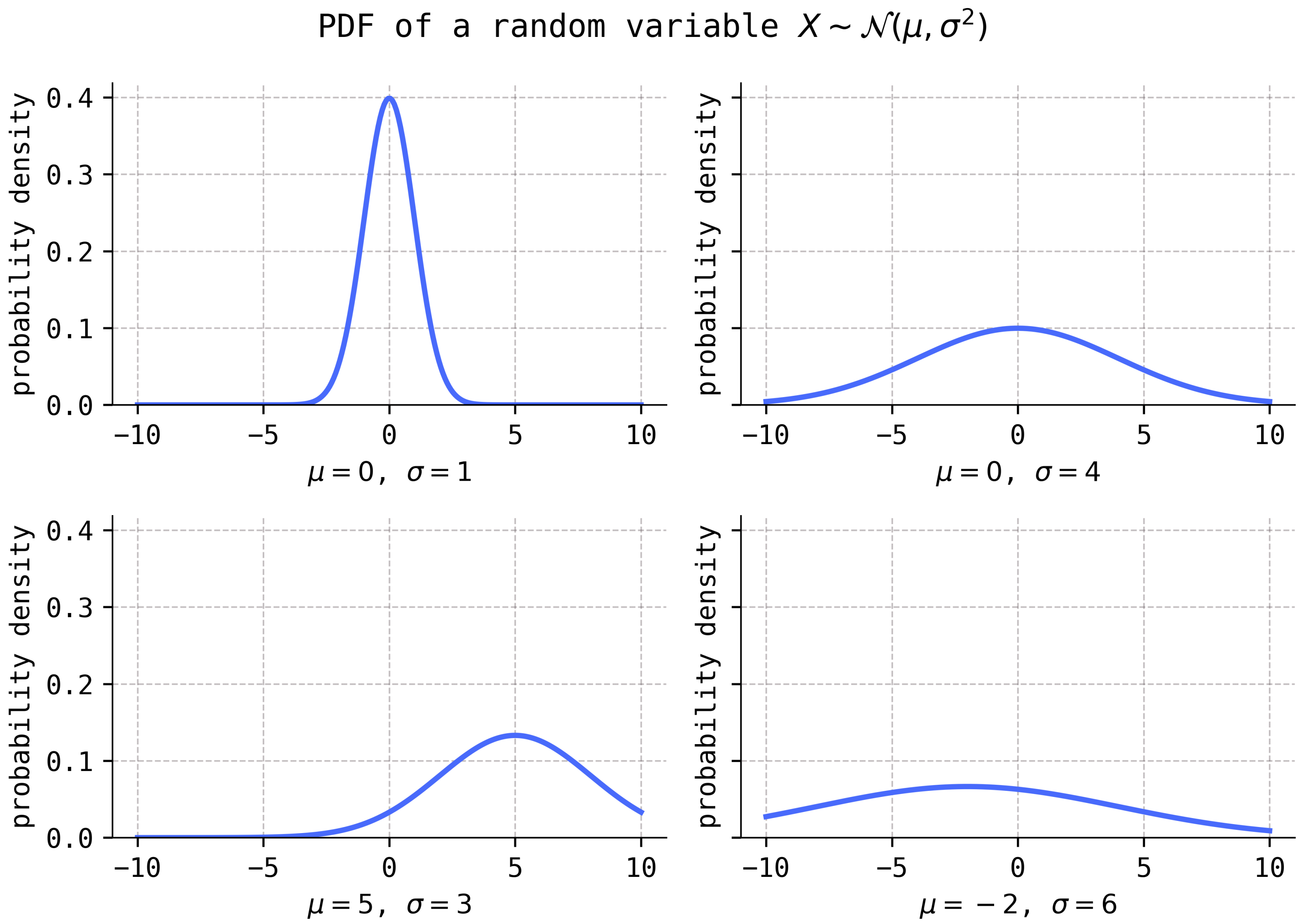
$$X \sim \mathcal{N}(\mu, \sigma^2),$$

if its probability density function is given by

$$f(x; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$$

with support $\mathbb{R}$.

If $\mu = 0$ and $\sigma = 1$, then X is called a *standard normal variable* (or *standard Gaussian variable*) and is often represented as Z.


PDF of a random variable $X \sim \mathcal{N}(\mu, \sigma^2)$
probability density
0.0
0.1
0.2
0.3
0.4
−10
−5
0
5
10
$\mu = 0$, $\sigma = 1$
$\mu = 0$, $\sigma = 4$
$\mu = 5$, $\sigma = 3$
$\mu = -2$, $\sigma = 6$

Theorem 5.7 (Affine transformations of normal variables)

Let $X \sim \mathcal{N}(\mu, \sigma^2)$ and a and b be two constants with $a \neq 0$. Then $Y = aX + b$ is a normal random variable with $\mu_Y = a\mu + b$ and $\sigma_Y = a\sigma$.

Corollary 5.1 (Standardization of normal variables)

If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $Z = (X - \mu)/\sigma$ has a standard normal distribution.

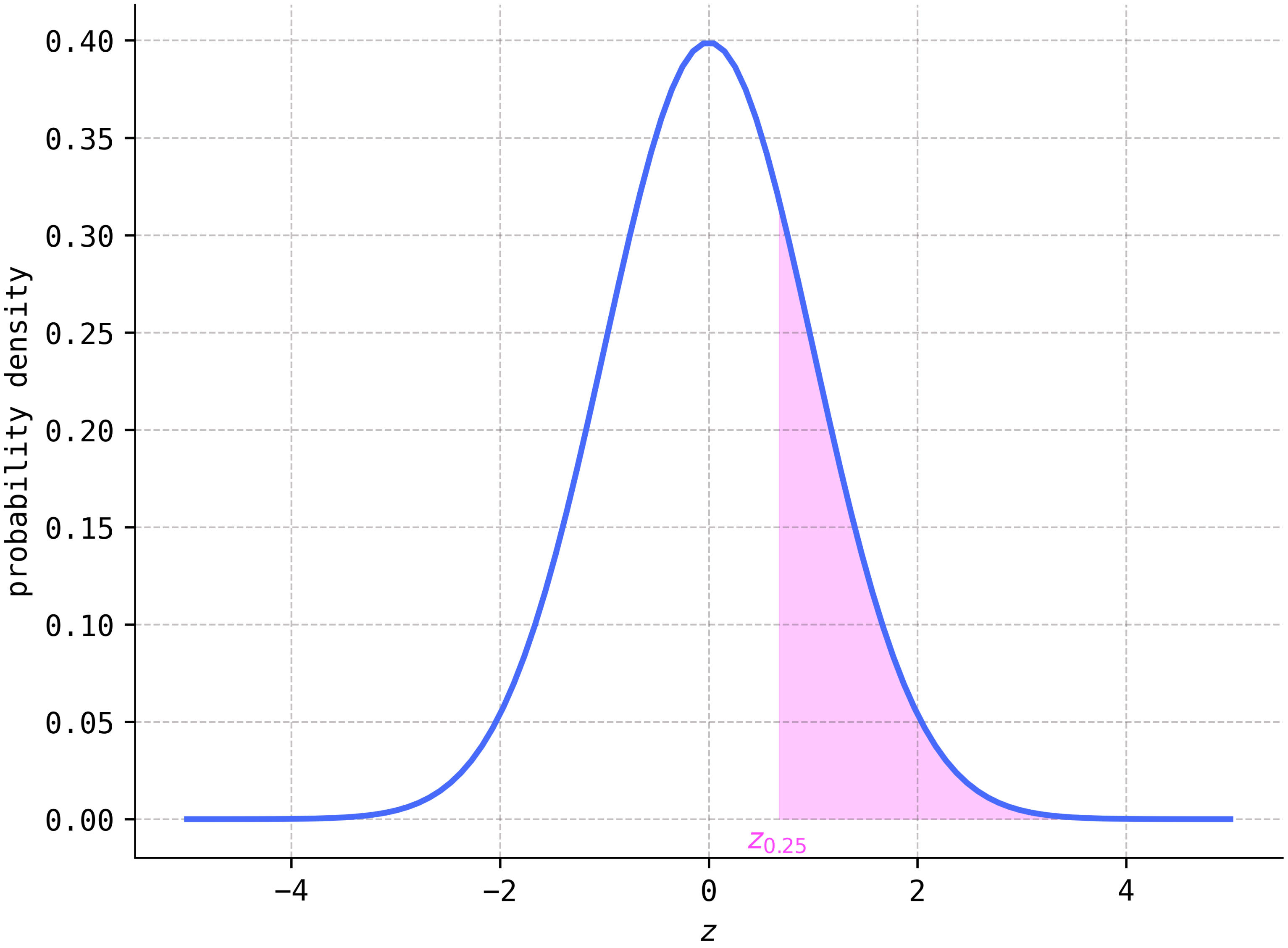
Definition 5.7

Let $Z \sim \mathcal{N}(0, 1)$ be a standard normal random variable.

- The cumulative distribution function of Z is denoted $\Phi(z)$.
- For each real number α, we define z_α to be the number for which the area under the density curve of Z to the *right* is exactly α. In other words, we have

$$z_\alpha = \Phi^{-1}(1 - \alpha).$$

The number z_α is called a *critical value*.



0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
probability density
$z_{0.25}$
−4
−2
0
2
4
z

Theorem 5.8 (Expectations and variances of normal variables)

If $X \sim \mathcal{N}(\mu, \sigma^2)$, then

$$E(X) = \mu \quad \text{and} \quad V(X) = \sigma^2.$$

Problem Prompt

Do problems 10-13 on the worksheet.