Problem 1: Suppose that X and Y are jointly continuous random variables with density

$$f(x,y) = \begin{cases} 24xy & : 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ x+y \leq 1, \\ 0 & : \text{otherwise.} \end{cases}$$

Compute the expectation $E(XY)$.

Problem 2: Suppose X and Y are jointly continuous random variables with the same density from Problem 1. Compute a formula for the conditional expectation $E(Y \mid X = x)$. Take care to precisely state the domain of this function.

Problem 3: Let X and Y be two random variables on the probability space $S = \{a, b, c\}$. Suppose that the probability distribution P on S has mass function $p(s)$ and that X and Y are defined according to the following table:

s	$p(s)$	$X(s)$	$Y(s)$
a	0.2	1	2
b	0.5	2	1
c	0.3	1	1

Compute the random variable $E(Y \mid X)$.

Problem 4: Suppose that a point $X = x$ is chosen uniformly in the interval $(0, 1)$. After x has been chosen, suppose that a second point $Y = y$ is chosen uniformly in the interval $[x, 1]$. Compute the expectation $E(Y)$.

Problem 5: The waiting time X in minutes between calls to a 911 center is exponentially distributed with mean $\mu = 2$ minutes. Compute the distribution of the transformed random variable $Y = 60X$ that measures the waiting time in seconds.

Problem 6: Suppose that X and Y are two random variables such that $Y = e^X$ and $X \sim \mathcal{N}(\mu, \sigma^2)$. Compute the density of Y.

Problem 7: Suppose that $\mathbf{X} = (X_1, X_2)$ is a two-dimensional continuous random vector with density

$$f(x_1, x_2) = \begin{cases} 4x_1x_2 & : 0 < x_1 < 1,\ 0 < x_2 < 1, \\ 0 & : \text{otherwise.} \end{cases}$$

Letting T be the support of the density, define $r : T \to \mathbb{R}^2$ by setting

$$r(x_1, x_2) = \left(\frac{x_1}{x_2}, x_1 x_2 \right)$$

for $(x_1, x_2) \in \mathbb{R}^2$. Compute the density of the random vector $\mathbf{Y} = r(\mathbf{X})$.

Problem 8: Suppose that X is a continuous random variable with uniform distribution on $[a, b]$. Compute its moment generating function $\psi(t)$, and then find all moments $E(X^k)$, for $k \geq 1$.

Problem 9: Use moment generating functions to confirm that the mean and variance of a random variable $X \sim \mathcal{N}(\mu, \sigma^2)$ are indeed μ and σ^2.

Problem 10: Suppose that X and Y are random variables with the joint density function

$$f(x,y) = \begin{cases} 2xy + 0.5 & : 0 \leq x \leq 1,\ 0 \leq y \leq 1, \\ 0 & : \text{otherwise.} \end{cases}$$

Compute the covariance of X and Y.

Problem 11: Compute the correlation ρ_{XY} of the random variables in the previous problem.

Problem 12: Many students applying for college take the SAT, which consists of math and verbal components (the latter is currently called evidence-based reading and writing). Let X and Y denote the math and verbal scores, respectively, for a randomly selected student. According to the College Board, the population of students taking the exam in 2017 had the following results:

$$\mu_X = 527, \quad \sigma_X = 107, \quad \mu_Y = 533, \quad \sigma_Y = 100, \quad \rho_{XY} = 0.77.$$

Supposing that $(X, Y) \sim \mathcal{N}_2(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, determine the probability that a student's total score $X + Y$ exceeds 1250, the minimum admission score for a particular university.