Problem 1: Suppose that X and Y are jointly continuous random variables with density

$$f(x,y) = \begin{cases} 24xy & : 0 \le x \le 1,\ 0 \le y \le 1,\ x+y \le 1, \\ 0 & : \text{otherwise.} \end{cases}$$

Compute the expectation $E(XY)$.

Using the bivariate LoTUS, we get:

$$E(XY) = \iint_{\mathbb{R}^2} xyf(x,y)\ \mathrm{d}y\mathrm{d}x = 24\int_0^1 \int_0^{1-x} x^2y^2\ \mathrm{d}y\mathrm{d}x = \frac{2}{15}.$$

Problem 2: Suppose X and Y are jointly continuous random variables with the same density from Problem 1. Compute a formula for the conditional expectation $E(Y \mid X = x)$. Take care to precisely state the domain of this function.

We first get the marginal density for X:

$$f(x) = \int_{\mathbb{R}} f(x,y)\ \mathrm{d}y = 24\int_0^{1-x} xy\ \mathrm{d}y = 12x(1-x)^2,$$

where the second inequality holds for all $0 \le x \le 1$; otherwise, we have $f(x) = 0$. Note that the marginal $f(x)$ is nonzero for $0 < x < 1$. For these particular x-values, the conditional density is defined and is given by

$$f(y|x) = \frac{f(x,y)}{f(x)} = \begin{cases} \frac{2y}{(1-x)^2} & : 0 \le x \le 1,\ 0 \le y \le 1,\ x+y \le 1, \\ 0 & : \text{otherwise.} \end{cases}$$

Therefore, for all x with $0 < x < 1$, we have

$$E(Y \mid X = x) = \int_{\mathbb{R}} yf(y|x)\ \mathrm{d}y = \frac{2}{(1-x)^2}\int_0^{1-x} y^2\ \mathrm{d}y = \frac{2}{3}(1-x).$$

Problem 3: Let X and Y be two random variables on the probability space $S = \{a,b,c\}$. Suppose that the probability distribution P on S has mass function $p(s)$ and that X and Y are defined according to the following table:

s	$p(s)$	$X(s)$	$Y(s)$
a	0.2	1	2
b	0.5	2	1
c	0.3	1	1

Compute the random variable $E(Y \mid X)$.

Our first goal is to get the joint mass function $p(x,y)$. We compute:

$p(x,y)$	$y=1$	$y=2$
$x=1$	0.3	0.2
$x=2$	0.5	0

By adding across the rows, we get the marginal:

x	$p(x)$
1	0.5
2	0.5

Then, from the formula $p(y|x) = p(x,y)/p(x)$, we get

$p(y\|x)$	$y=1$	$y=2$
$x=1$	0.6	0.4
$x=2$	1	0

If $x \neq 1$ or 2, then the conditional mass $p(y|x)$ is undefined. For $x = 1$ or 2, we have

$$E(Y \mid X = x) = \sum_{y=1}^{2} y p(y|x) = \begin{cases} 1 \cdot 0.6 + 2 \cdot 0.4 = 1.4 & : x = 1, \\ 1 \cdot 1 + 2 \cdot 0 = 1 & : x = 2. \end{cases}$$

Finally, recall that the random variable $E(Y \mid X)$ is defined as the composite $h(X) = h \circ X$, where $h(x) = E(Y \mid X = x)$ for $x = 1, 2$. Hence, we conclude the problem with:

s	$E(Y \mid X)(s)$
a	1.4
b	1
c	1.4

Problem 4: Suppose that a point $X = x$ is chosen uniformly in the interval $(0, 1)$. After x has been chosen, suppose that a second point $Y = y$ is chosen uniformly in the interval $[x, 1]$. Compute the expectation $E(Y)$.

We will compute the expectation via the Law of Total Expectation:

$$E(Y) = E\Big[E(Y \mid X)\Big].$$

So, we begin by noting that $E(Y \mid X = x) = (x + 1)/2 = x/2 + 1/2$, since the mean of a uniform distribution over an interval is the midpoint between its boundary points. Therefore $E(Y \mid X) = X/2 + 1/2$, and so

$$E(Y) = E\Big(X/2 + 1/2\Big) = E(X)/2 + 1/2 = 1/4 + 1/2 = 3/4.$$

Problem 5: Suppose that X is a continuous random variable with uniform distribution on $[a, b]$. Compute its moment generating function $\psi(t)$, and then find all moments $E(X^k)$, for $k \geq 1$.

We have

$$f(x) = \frac{1}{b-a}$$

for all $x \in [a, b]$ and $f(x) = 0$ otherwise. Thus, by the LoTUS, we have

$$\psi(t) = E(e^{tX}) = \frac{1}{b-a} \int_a^b e^{tx} \, \mathrm{d}x = \frac{e^{tb} - e^{ta}}{t(b-a)}.$$

If we expand the exponentials, we get

$$\begin{aligned} \psi(t) &= \frac{1}{t(b-a)} \left[\sum_{k=0}^{\infty} \frac{t^k b^k}{k!} - \sum_{k=0}^{\infty} \frac{t^k a^k}{k!} \right] \\ &= \frac{1}{b-a} \sum_{k=0}^{\infty} \frac{b^{k+1} - a^{k+1}}{(k+1)!} t^k \end{aligned}$$

for all $t \in \mathbb{R}$. Thus,

$$E(X^k) = \psi^{(k)}(0) = \frac{b^{k+1} - a^{k+1}}{(k+1)(b-a)} = \frac{a^k + a^{k-1}b + \cdots + ab^{k-1} + b^k}{k+1}.$$

Problem 6: Use moment generating functions to confirm that the mean and variance of a random variable $X \sim \mathcal{N}(\mu, \sigma^2)$ are indeed μ and σ^2.

Letting $\psi(t)$ be the moment generating function of X, we have

$$\psi'(t) = \psi(t)(\mu + \sigma^2 t),$$

and so

$$E(X) = \psi'(0) = \mu.$$

To compute the variance, we first compute

$$\psi''(t) = \psi(t)(\mu + \sigma^2 t)^2 + \psi(t)\sigma^2.$$

Then

$$E(X^2) = \psi''(0) = \mu^2 + \sigma^2,$$

and so

$$V(X) = E(X^2) - E(X)^2 = \mu^2 + \sigma^2 - \mu^2 = \sigma^2.$$

Problem 7: Suppose that X and Y are random variables with the joint density function

$$f(x,y) = \begin{cases} 2xy + 0.5 & : 0 \le x, y \le 1, \\ 0 & : \text{otherwise.} \end{cases}$$

Compute the covariance of X and Y.

Using the Shortcut Formula for Covariance, we compute:

$$\sigma_{XY} = E(XY) - E(X)E(Y).$$

But first, let's grab the expectations of X and Y. To do this, we integrate out y to get the density of x:

$$f(x) = \int_{\mathbb{R}} f(x,y) \; \mathrm{d}y = \int_0^1 (2xy + 0.5) \; \mathrm{d}y = x + 0.5$$

for $0 \le x \le 1$, and $f(x) = 0$ otherwise. Then, we compute:

$$E(X) = \int_{\mathbb{R}} x f(x) \; \mathrm{d}x = \int_0^1 (x^2 + 0.5x) \; \mathrm{d}x = \frac{7}{12}.$$

Now, if you look at the joint density function, you'll notice that it is symmetric in x and y. This means that $E(Y) = 7/12$, as well. Finally, we compute the covariance from the shortcut formula:

$$\begin{aligned} \sigma_{XY} &= E(XY) - \frac{7^2}{12^2} \\ &= \iint_{\mathbb{R}^2} xy f(x,y) \; \mathrm{d}y\mathrm{d}x - \frac{7^2}{12^2} \\ &= \int_0^1 \int_0^1 (2x^2y^2 + 0.5xy) \; \mathrm{d}y\mathrm{d}x - \frac{7^2}{12^2} \\ &= \frac{25}{72} - \frac{7^2}{12^2} \\ &= \frac{1}{144} \\ &\approx 0.007. \end{aligned}$$

Problem 8: Suppose that X and Y are random variables with the joint density function

$$f(x,y) = \begin{cases} 3x & : 0 \le y \le x \le 1, \\ 0 & : \text{otherwise.} \end{cases}$$

Compute the covariance of X and Y.

We follow the same strategy as the previous problem. First, we get the marginal densities:

$$f(x) = \int_{\mathbb{R}} f(x,y) \; \mathrm{d}y = \int_0^x 3x \; \mathrm{d}y = 3x^2$$

for $0 \le x \le 1$ and $f(x) = 0$ otherwise; also:

$$f(y) = \int_{\mathbb{R}} f(x,y) \; \mathrm{d}x = \int_y^1 3x \; \mathrm{d}x = \frac{3}{2}(1 - y^2)$$

for $0 \le y \le 1$. Then, we compute:

$$E(X) = \int_{\mathbb{R}} x f(x) \; \mathrm{d}x = \int_0^1 3x^3 \; \mathrm{d}x = \frac{3}{4}$$

and

$$E(Y) = \int_{\mathbb{R}} y f(y) \ \mathrm{d}y = \frac{3}{2} \int_0^1 y(1 - y^2) \ \mathrm{d}y = \frac{3}{8}.$$

Finally, we compute

$$E(XY) = \iint_{\mathbb{R}^2} xy f(x, y) \ \mathrm{d}y \mathrm{d}x = \int_0^1 \int_0^x 3x^2 y \ \mathrm{d}y \mathrm{d}x = \frac{3}{10}$$

and hence

$$\sigma_{XY} = E(XY) - E(X)E(Y) = \frac{3}{10} - \frac{3}{4} \cdot \frac{3}{8} = \frac{3}{160} \approx 0.019.$$