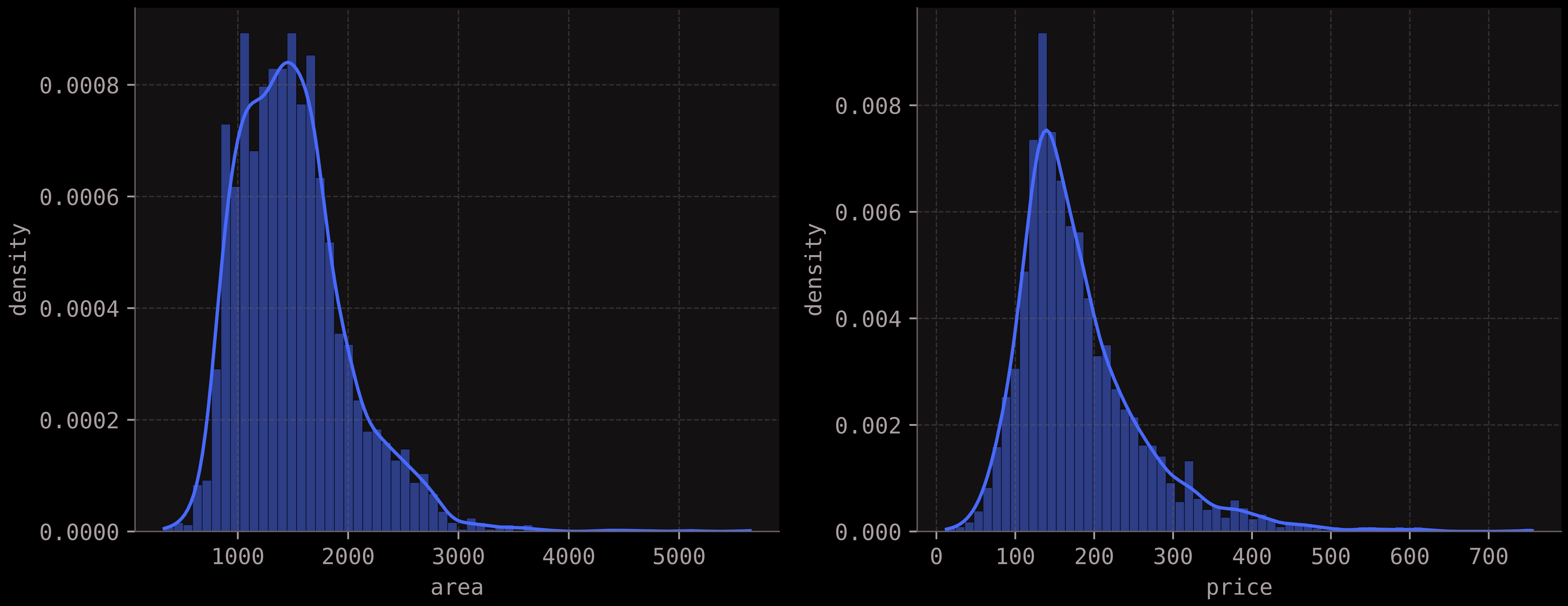
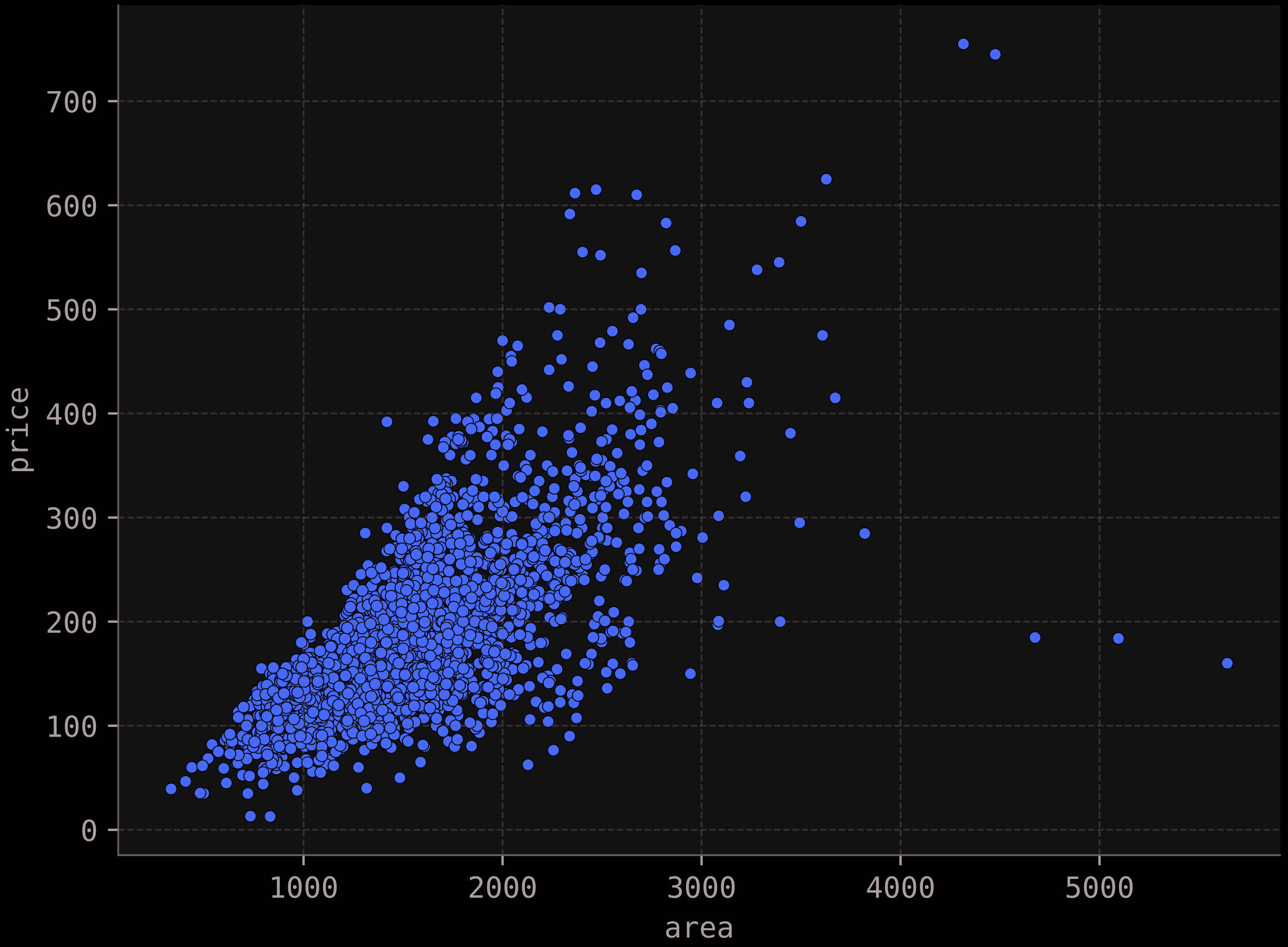
7. Random vectors

7.1. Motivation


0.0008
0.0006
0.0004
0.0002
0.0000
density
1000
2000
3000
4000
5000
area
0.008
0.006
0.004
0.002
0.000
density
0
100
200
300
400
500
600
700
price



700
600
500
400
300
200
100
0
price
1000
2000
3000
4000
5000
area

7.2. 2-dimensional random vectors

Definition 7.1

Let S be a probability space. A 2-*dimensional random vector* is a function

$$\mathbf{X} : S \to \mathbb{R}^2.$$

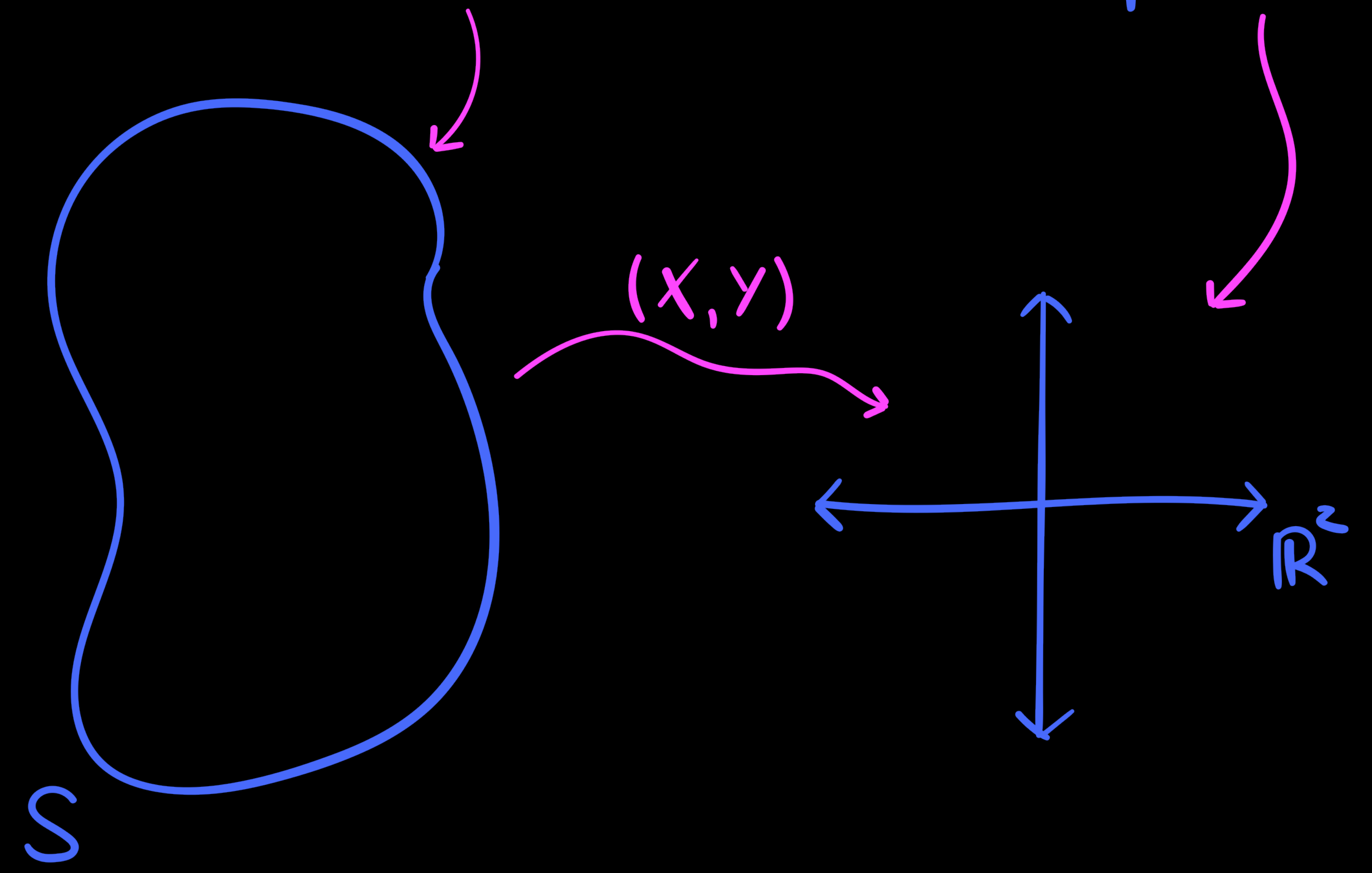
Thus, we may write $\mathbf{X}(s) = (X_1(s), X_2(s))$ for each sample point $s \in S$, where

$$X_1 : S \to \mathbb{R} \quad \text{and} \quad X_2 : S \to \mathbb{R}$$

are random variables. When we do so, the random variables X_1 and X_2 are called the *components* of the random vector $\mathbf{X}$.

P measures probability here

P_{xy} measures probability here

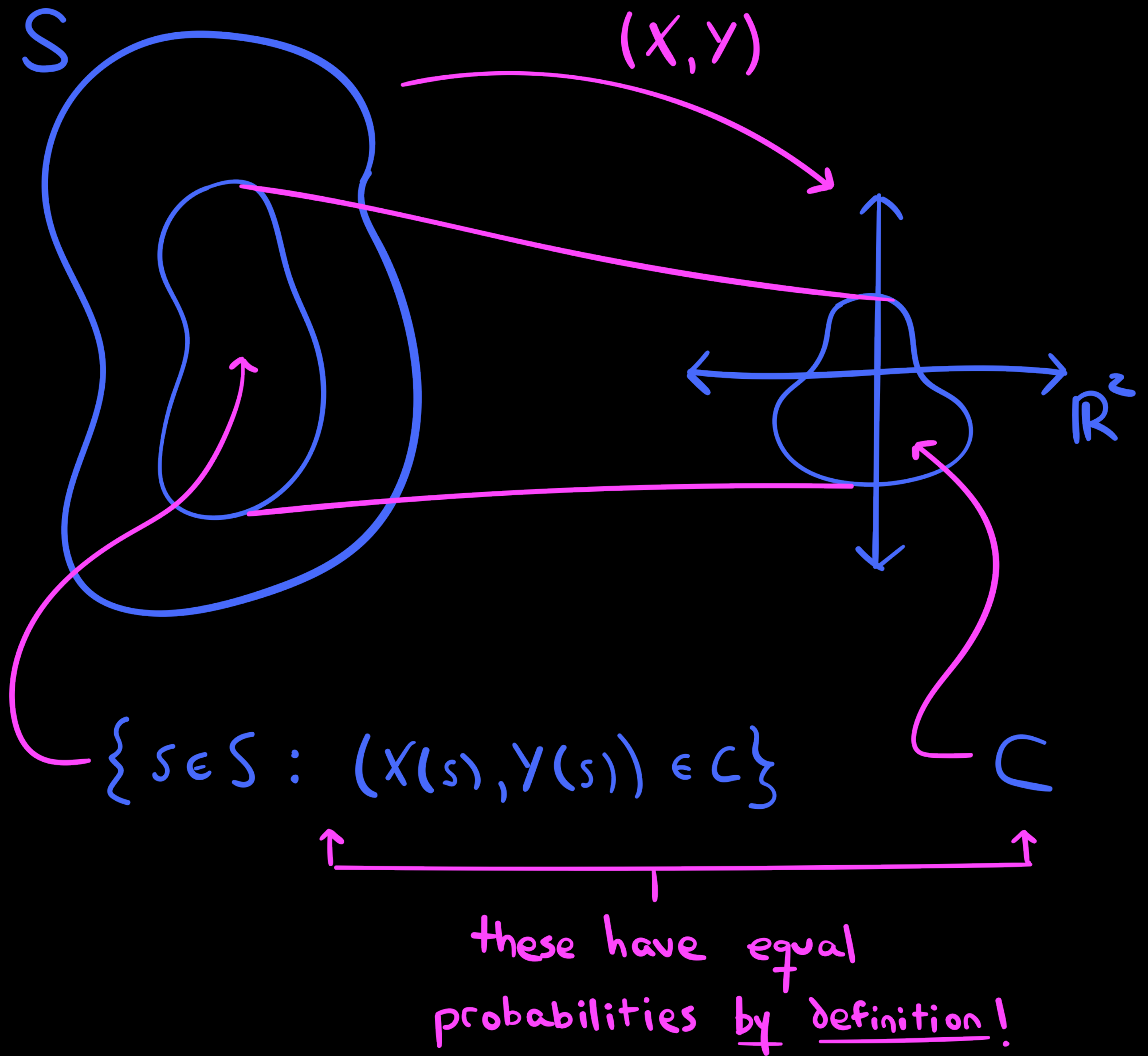


Definition 7.2

Let $(X, Y) : S \to \mathbb{R}^2$ be a 2-dimensional random vector on a probability space S with probability measure P. We define the *probability measure* of (X, Y), denoted P_{XY}, via the formula

$$P_{XY}(C) = P\left(\{s \in S : (X(s), Y(s)) \in C\}\right), \tag{7.1}$$

for all events $C \subset \mathbb{R}^2$. The probability measure P_{XY} is also called the *joint distribution* or the *bivariate distribution* of X and Y.


S
(X, Y)
$\mathbb{R}^2$
C
$\{s \in S : (X(s), Y(s)) \in C\}$
these have equal probabilities by definition!

Problem Prompt

Do problem 1 on the worksheet.

Definition 7.3

Let (X, Y) be a 2-dimensional random vector.

- We shall say (X, Y) is *discrete*, or that X and Y are *jointly discrete*, if the joint probability distribution P_{XY} is discrete. In other words, we require that there exists a *joint probability mass function* $p(x, y)$ such that

  $$P((X, Y) \in C) = \sum_{(x,y)\in C} p(x, y)$$

  for all events $C \subset \mathbb{R}^2$.
- We shall say (X, Y) is *continuous*, or that X and Y are *jointly continuous*, if the joint probability distribution P_{XY} is continuous. In other words, we require that there exists a *joint probability density function* $f(x, y)$ such that

  $$P\left((X, Y) \in C\right) = \iint_C f(x, y)\, \mathrm{d}x\mathrm{d}y$$

  for all events $C \subset \mathbb{R}^2$.

Theorem 7.1

Let (X, Y) be a 2-dimensional random vector.

1. The random vector (X, Y) is discrete if and only if both X and Y are discrete.
2. If (X, Y) is continuous, then X and Y are both continuous. However, it does *not* nececessarily follow that if both X and Y are continuous, then so too is (X, Y).

Problem Prompt

Do problems 2-4 on the worksheet.

7.3. Bivariate distribution functions

Definition 7.4

Let (X, Y) be a 2-dimensional random vector. The *distribution function* of (X, Y) is the function $F : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$F(x, y) = P(X \leq x, Y \leq y).$$

In particular:

1. If (X, Y) is discrete with probability mass function $p(x, y)$, then

$$F(x, y) = \sum_{x^\star \leq x,\, y^\star \leq y} p(x^\star, y^\star).$$

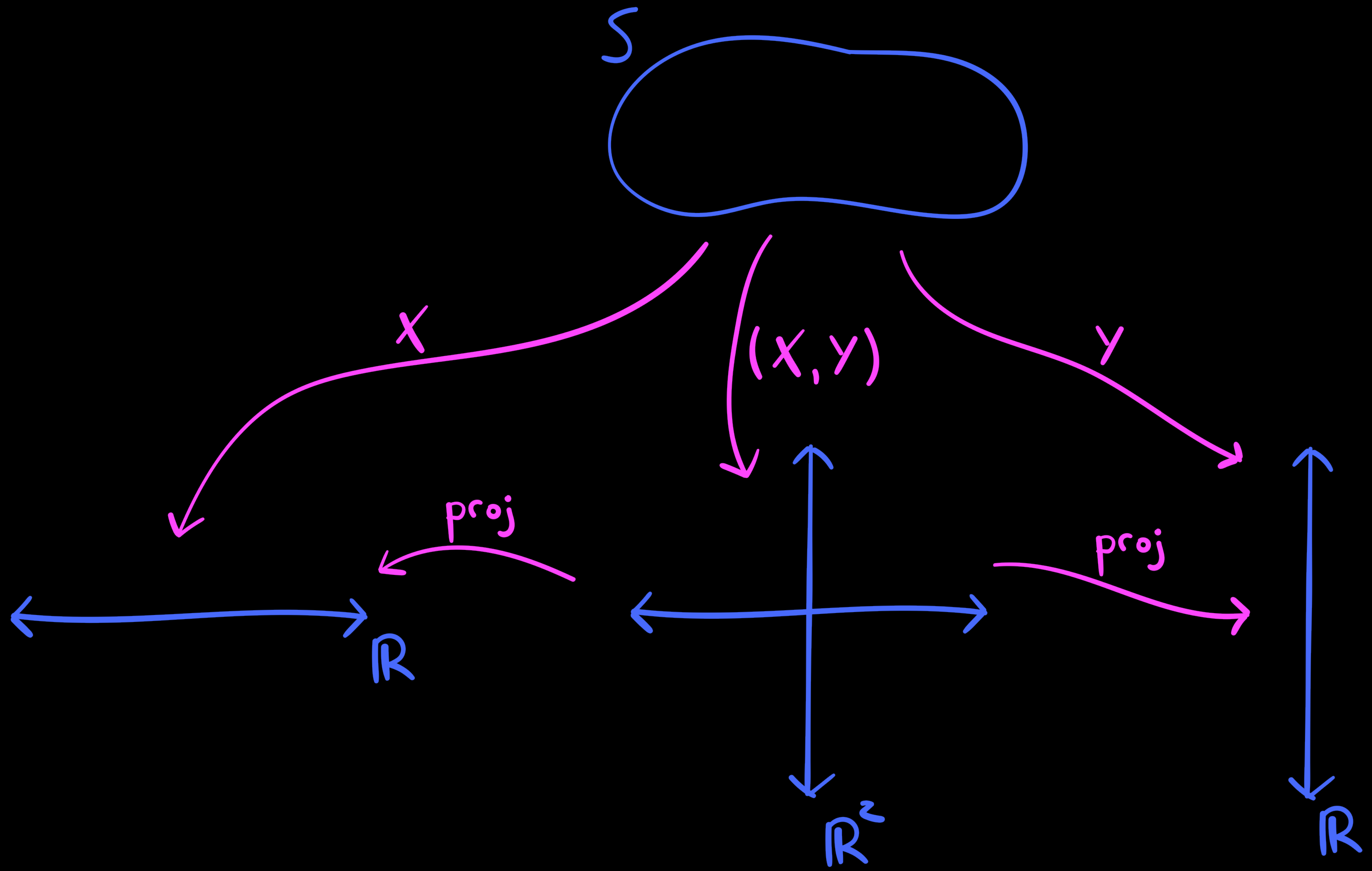
2. If (X, Y) is continuous with probability density function $f(x, y)$, then

$$F(x, y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(x^\star, y^\star)\, \mathrm{d}x^\star \mathrm{d}y^\star.$$

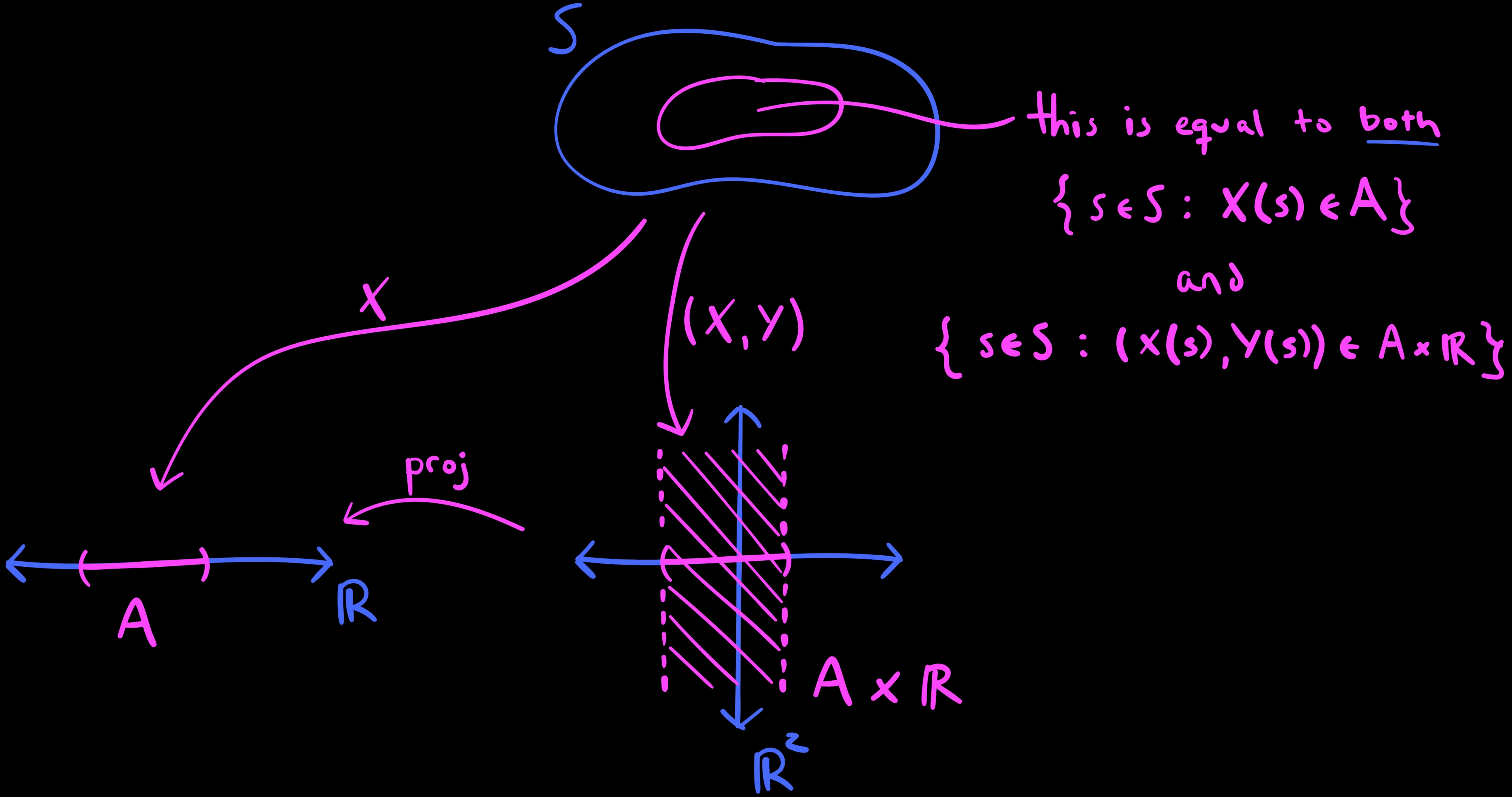
Problem Prompt

Do problem 5 on the worksheet.

7.4. Marginal distributions



S
X
(X, Y)
Y
proj
proj
R
R²
R



S
this is equal to both
$\{ s \in S : X(s) \in A \}$
and
$\{ s \in S : (X(s), Y(s)) \in A \times \mathbb{R} \}$
X
(X, Y)
proj
A
$\mathbb{R}$
$A \times \mathbb{R}$
$\mathbb{R}^2$

Theorem 7.2

Let (X, Y) be a 2-dimensional random vector with induced probability measure P_{XY}. Then the measures P_X and P_Y may be obtained via the formulas

$$P_X(A) = P_{XY}(A \times \mathbb{R}) \quad \text{and} \quad P_Y(B) = P_{XY}(\mathbb{R} \times B)$$

for all events $A, B \subset \mathbb{R}$. In particular:

1. If (X, Y) is discrete with probability mass function $p(x, y)$, then

$$P(X \in A) = \sum_{x \in A} \sum_{y \in \mathbb{R}} p(x, y) \quad \text{and} \quad P(Y \in B) = \sum_{y \in B} \sum_{x \in \mathbb{R}} p(x, y).$$

2. If (X, Y) is continuous with probability density function $f(x, y)$, then

$$P(X \in A) = \int_A \int_{-\infty}^{\infty} f(x, y) \, \mathrm{d}y \mathrm{d}x$$

and

$$P(Y \in B) = \int_B \int_{-\infty}^{\infty} f(x, y) \, \mathrm{d}x \mathrm{d}y.$$

Definition 7.5

Let (X, Y) be a 2-dimensional random vector. Then the distributions P_X and P_Y are called the *marginal distributions* of (X, Y).

Theorem 7.3

Let (X, Y) be a 2-dimensional random vector.

1. If (X, Y) is discrete with probability mass function $p(x, y)$, then both X and Y are discrete with probability mass functions given by

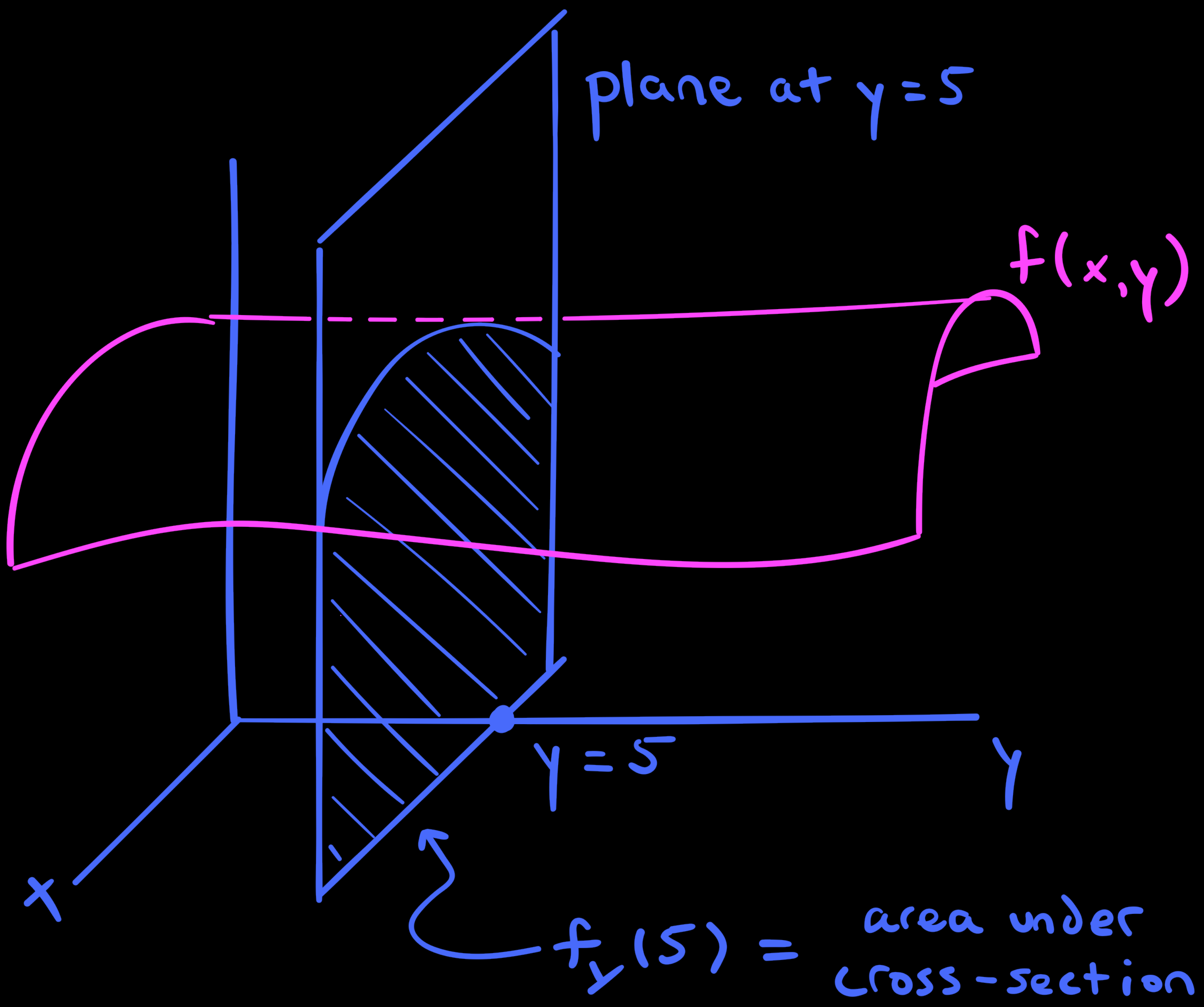
$$p_X(x) = \sum_{y \in \mathbb{R}} p(x, y) \quad \text{and} \quad p_Y(y) = \sum_{x \in \mathbb{R}} p(x, y).$$

2. If (X, Y) is continuous with probability density function $f(x, y)$, then both X and Y are continuous with probability density functions given by

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, \mathrm{d}y \quad \text{and} \quad f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\, \mathrm{d}x. \tag{7.5}$$

Tip

1. To obtain the marginal mass $p_X(x)$ from the joint mass $p(x, y)$, we "sum out" the dependence of $p(x, y)$ on y. Likewise for obtaining $p_Y(y)$ from $p(x, y)$.
2. To obtain the marginal density $f_X(x)$ from the joint density $f(x, y)$, we "integrate out" the dependence of $f(x, y)$ on y. Likewise for obtaining $f_Y(y)$ from $f(x, y)$.


plane at y = 5
f(x,y)
x
y
y = 5
$f_y(5)$ = area under cross-section

Problem Prompt

Do problems 6 and 7 on the worksheet.

7.5. Bivariate empirical distributions

Definition 7.6

Let $(X_1, Y_1), (X_2, Y_2), \ldots, (X_m, Y_m)$ be a sequence of 2-dimensional random vectors, all defined on the same probability space.

- The random vectors are called a *bivariate random sample* if they are *independent* and *identically distributed* (IID).

Provided that the sequence is a bivariate random sample, an *observed bivariate random sample*, or a *bivariate dataset*, is a sequence of pairs of real numbers

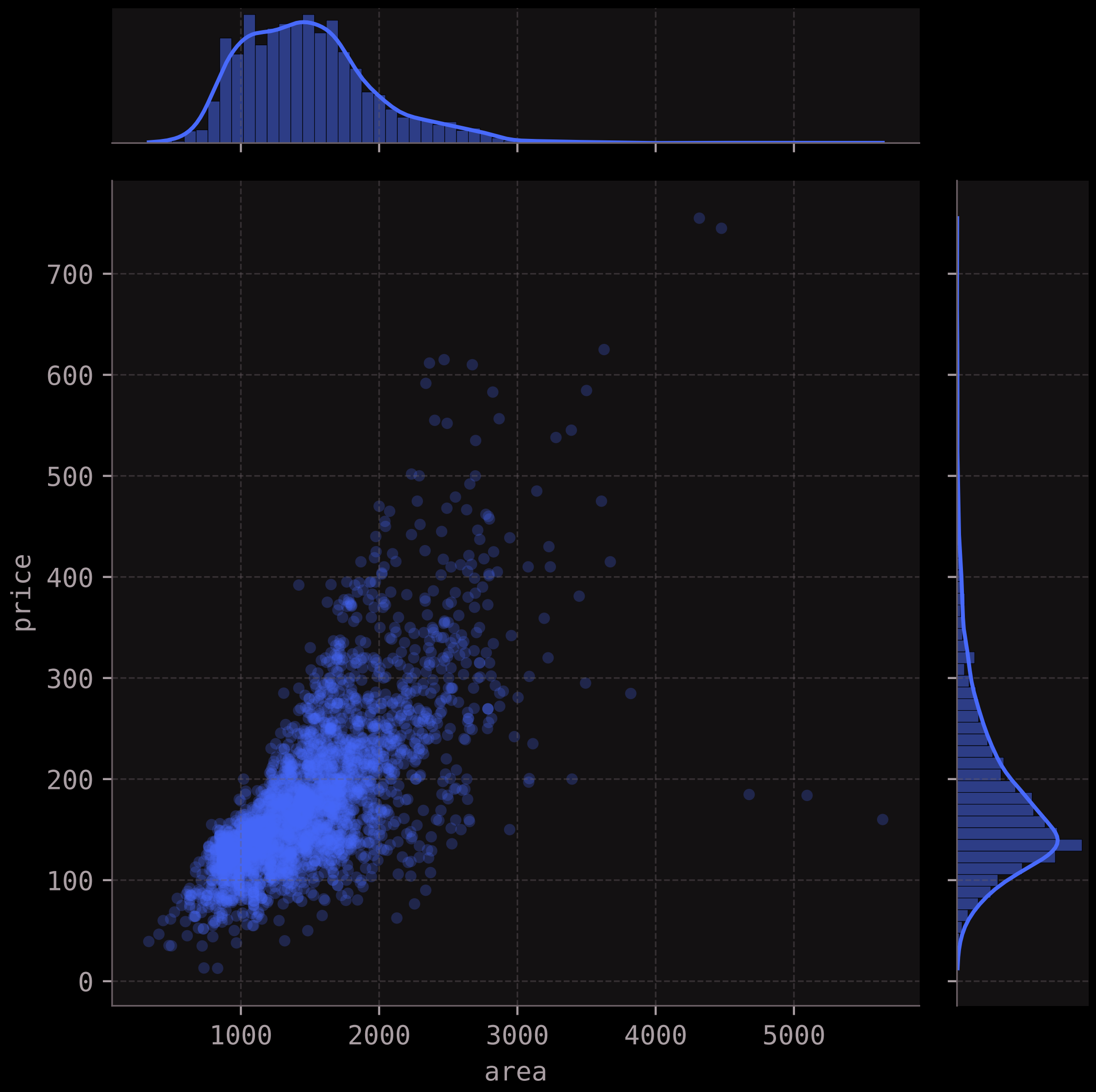
$$(x_1, y_1), (x_2, y_2), \ldots, (x_m, y_m)$$

where (x_i, y_i) is an observation of (X_i, Y_i).

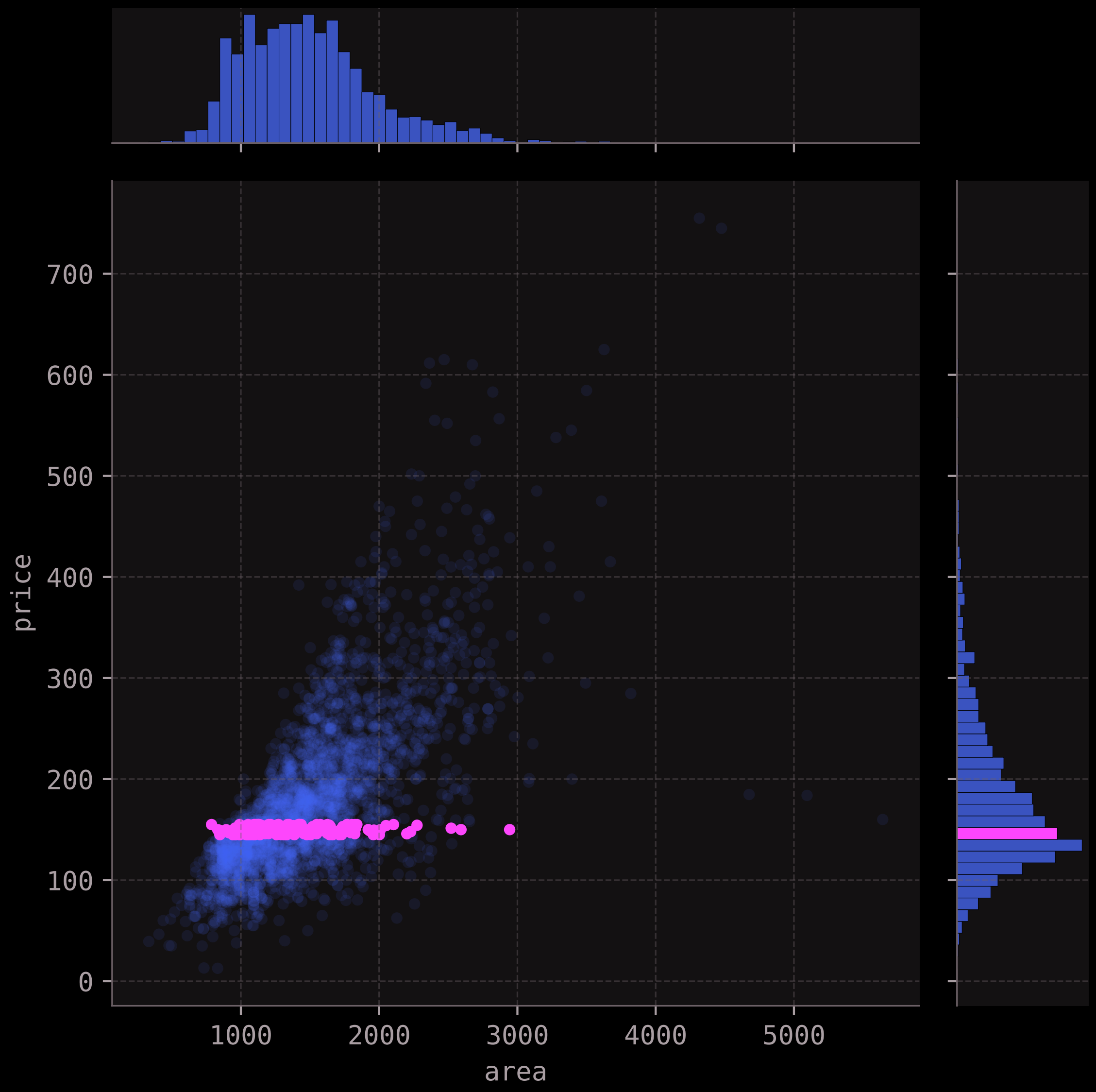
Definition 7.7

Let $(x_1, y_1), (x_2, y_2), \ldots, (x_m, y_m)$ be an observed bivariate random sample, i.e., a bivariate dataset. The *empirical distribution* of the dataset is the discrete probability measure on $\mathbb{R}^2$ with joint probability mass function

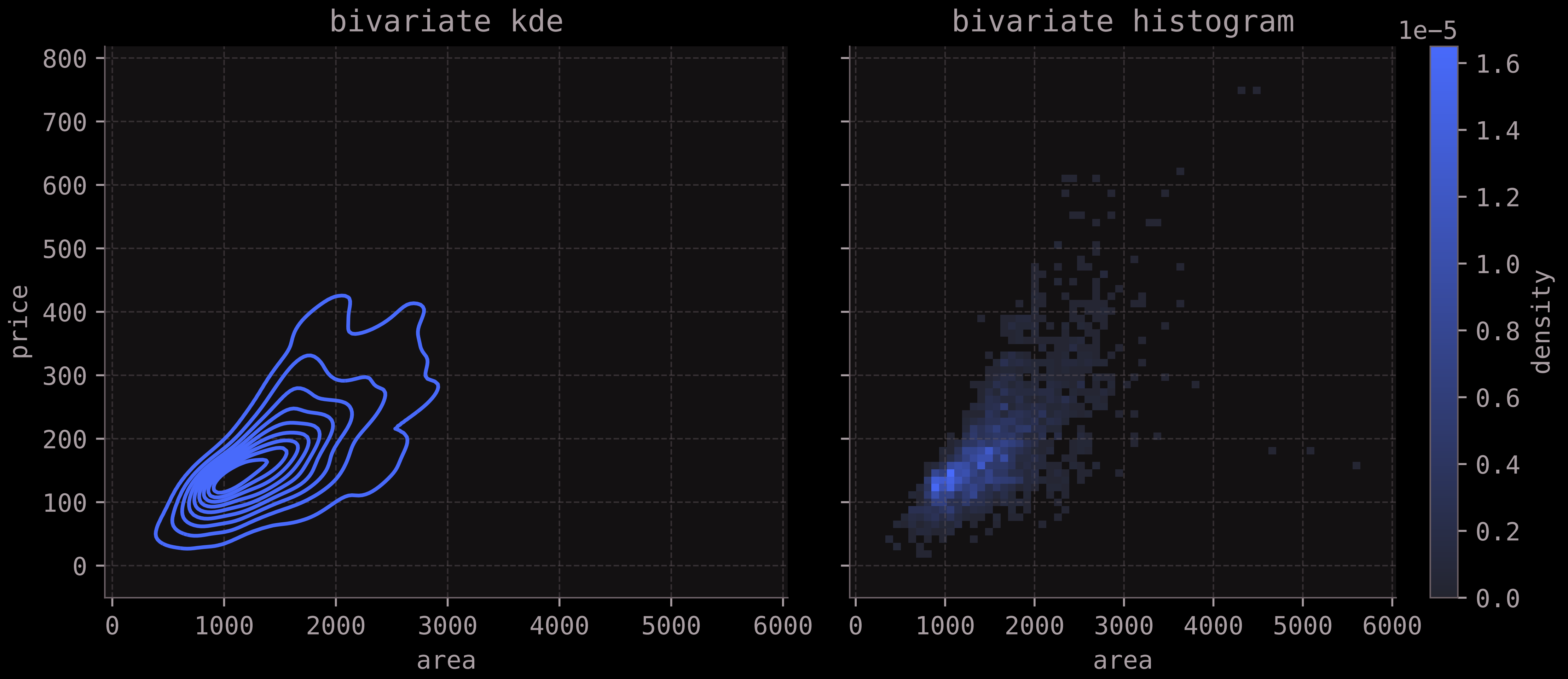
$$p(x,y) = \frac{\text{number of data points } (x_i, y_i) \text{ that match } (x,y)}{m}.$$



700
600
500
400
300
200
100
0
price
1000
2000
3000
4000
5000
area

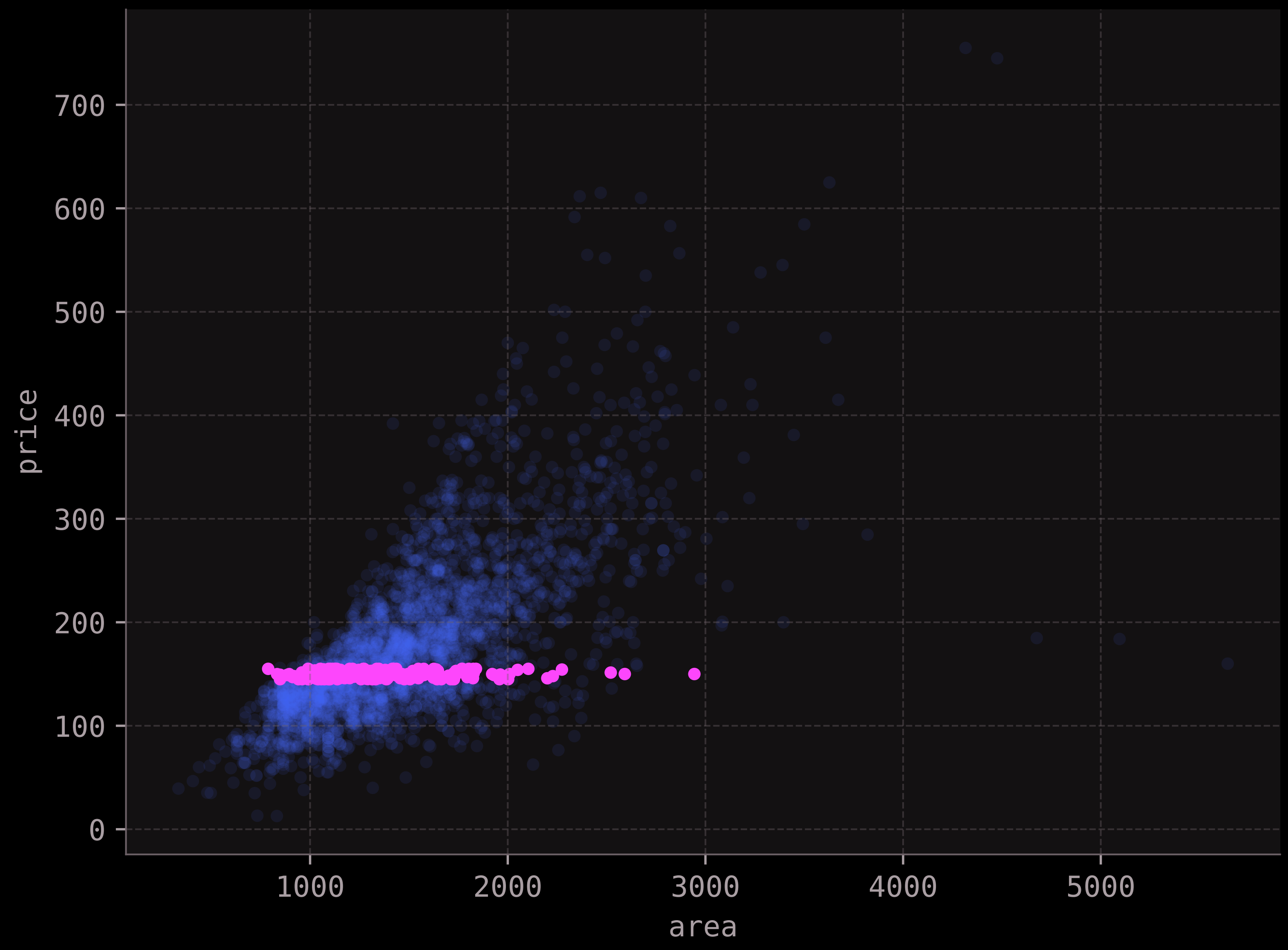


700
600
500
400
300
200
100
0
price
1000
2000
3000
4000
5000
area

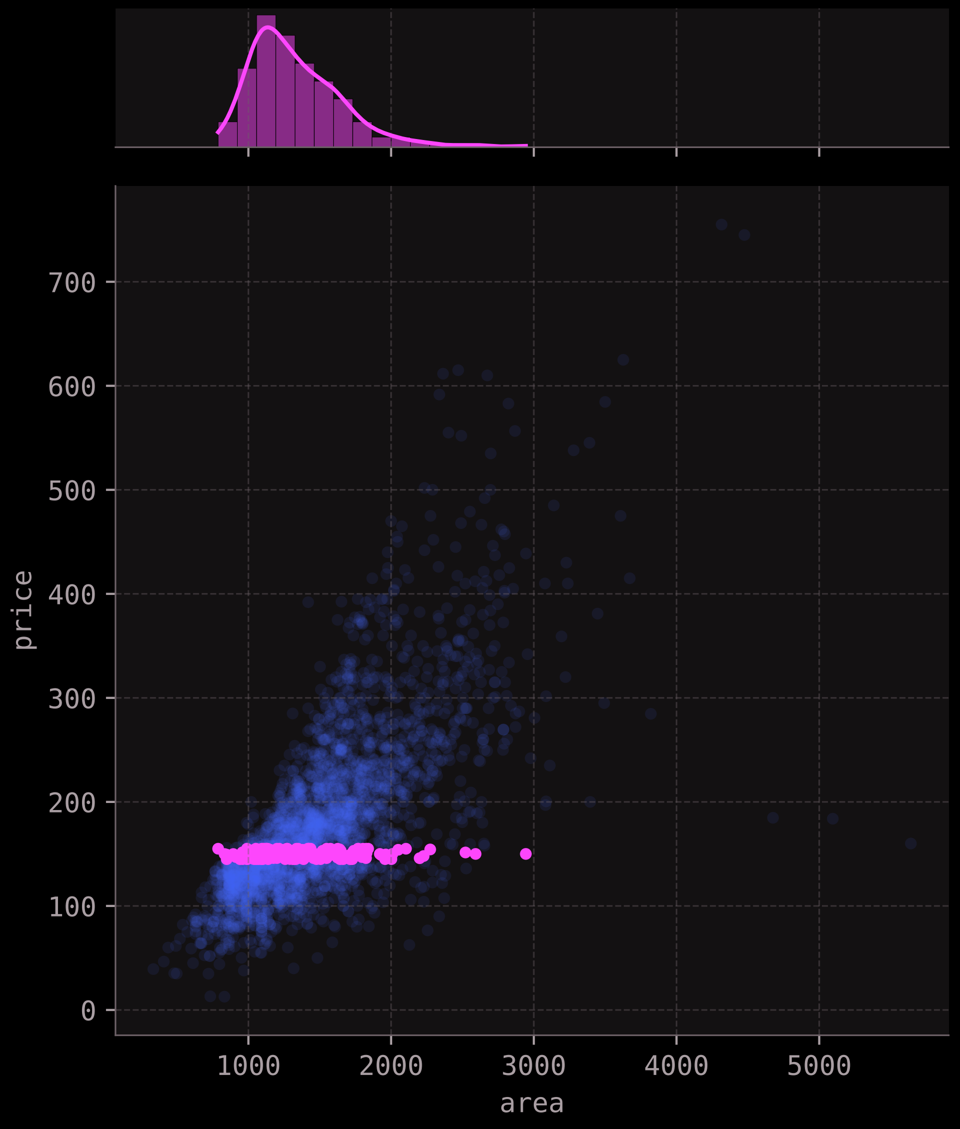


bivariate kde
bivariate histogram
price
area
area
density
1e−5
0
100
200
300
400
500
600
700
800
0
1000
2000
3000
4000
5000
6000
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6

7.6. Conditional distributions



700
600
500
400
300
200
100
0
price
1000
2000
3000
4000
5000
area



price
700
600
500
400
300
200
100
0
1000
2000
3000
4000
5000
area

Definition 7.8

Let X and Y be random variables.

- Suppose (X, Y) is discrete, so that both X and Y are discrete as well. The *conditional probability mass function of X given Y* is the function

  $$p_{X|Y}(x|y) = \frac{p_{XY}(x, y)}{p_Y(y)},$$

  defined for all those y such that $p_Y(y) \neq 0$.
- Suppose (X, Y) is continuous, so that both X and Y are continuous as well. The *conditional probability density function of X given Y* is the function

  $$f_{X|Y}(x|y) = \frac{f_{XY}(x, y)}{f_Y(y)},$$

  defined for all those y such that $f_Y(y) \neq 0$.

Problem Prompt

Do problems 8 and 9 on the worksheet.

Theorem 7.4

Let X and Y be random variables.

- In the case that (X, Y) is discrete, for fixed y with $p_Y(y) \neq 0$, the function $p_{X|Y}(x|y)$ is a probability mass function in the variable x. In particular, we have

$$P(X \in A|Y = y) = \sum_{x \in A} p_{X|Y}(x|y), \tag{7.6}$$

  for all events $A \subset \mathbb{R}$.
- In the case that (X, Y) is continuous, for fixed y with $f_Y(y) \neq 0$, the function $f_{X|Y}(x|y)$ is a probability density function in the variable x.

Problem Prompt

Do problems 10 and 11 on the worksheet.

7.7. The Law of Total Probability and Bayes' Theorem for random variables

Theorem 7.5 (The Law of Total Probability (for random variables))

Let X and Y be random variables.

- If X and Y are jointly discrete, then

$$p_X(x) = \sum_{y \in \mathbb{R}} p_{X|Y}(x|y) p_Y(y) \tag{7.7}$$

  for each $x \in \mathbb{R}$.

- If X and Y are jointly continuous, then

$$f_X(x) = \int_{\mathbb{R}} f_{X|Y}(x|y) f_Y(y) \, \mathrm{d}y. \tag{7.8}$$

  for each $x \in \mathbb{R}$.

Theorem 7.6 (Bayes' Theorem (for random variables))

Let X and Y be random variables.

- If X and Y are jointly discrete, then

$$p_{X|Y}(x|y) = \frac{p_{Y|X}(y|x)p_X(x)}{p_Y(y)}. \tag{7.9}$$

- If X and Y are jointly continuous, then

$$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x)f_X(x)}{f_Y(y)}. \tag{7.10}$$

Problem Prompt

Do problem 12 on the worksheet.

7.8. Random vectors in arbitrary dimensions

Definition 7.9

Let S be a probability space and $n \geq 1$ an integer. An *n-dimensional random vector* is a function

$$\mathbf{X} : S \to \mathbb{R}^n.$$

Thus, we may write

$$\mathbf{X}(s) = (X_1(s), X_2(s), \ldots, X_n(s))$$

for each sample point $s \in S$. When we do so, the functions $X_1, X_2, \ldots, X_n$ are ordinary random variables that are called the *components* of the random vector $\mathbf{X}$.

Definition 7.10

Let $(X_1, X_2, \ldots, X_n) : S \to \mathbb{R}^n$ be an n-dimensional random vector on a probability space S with probability measure P. We define the *probability measure* of the random vector, denoted $P_{X_1 X_2 \cdots X_n}$, via the formula

$$P_{X_1 X_2 \cdots X_n}(C) = P\left(\{s \in S : (X_1(s), X_2(s), \ldots, X_n(s)) \in C\}\right), \tag{7.11}$$

for all events $C \subset \mathbb{R}^n$. The probability measure $P_{X_1 X_2 \cdots X_n}$ is also called the *joint distribution* of the component random variables $X_1, X_2, \ldots, X_n$.

Problem Prompt

Do problems 13 and 14 on the worksheet.

7.9. Independence

Definition 7.11

Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_m$ be random vectors, all defined on the same probability space, but possibly of different dimensions. Then these random vectors are said to be *independent* if

$$P(\mathbf{X}_1 \in C_1, \mathbf{X}_2 \in C_2, \ldots, \mathbf{X}_m \in C_m) = P(\mathbf{X}_1 \in C_1)P(\mathbf{X}_2 \in C_2) \cdots P(\mathbf{X}_m \in C_m)$$

for all events $C_1, C_2, \ldots, C_m$. If the vectors are not independent, they are called *dependent*.

Theorem 7.7 (Mass/Density Criteria for Independence)

Let $X_1, X_2, \ldots, X_m$ be random variables.

- Suppose that the random variables are jointly discrete. Then they are independent if and only if

  $$p_{X_1 X_2 \cdots X_m}(x_1, x_2, \ldots, x_m) = p_{X_1}(x_1) p_{X_2}(x_2) \cdots p_{X_m}(x_m)$$

  for all $x_1, x_2, \ldots, x_m \in \mathbb{R}$.

- Suppose that the random variables are jointly continuous. Then they are independent if and only if

  $$f_{X_1 X_2 \cdots X_m}(x_1, x_2, \ldots, x_m) = f_{X_1}(x_1) f_{X_2}(x_2) \cdots f_{X_m}(x_m) \tag{7.15}$$

  for all $x_1, x_2, \ldots, x_m \in \mathbb{R}$.

Theorem 7.8 (Invariance of Independence)

Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_m$ be independent random vectors and let $g_1, \ldots, g_m$ be vector-valued functions for which the transformed random vectors

$$g_1(\mathbf{X}_1), g_2(\mathbf{X}_2), \ldots, g_m(\mathbf{X}_m) \tag{7.18}$$

are all defined. Then the random vectors (7.18) are independent.

Corollary 7.1 (Independence of Components)

Suppose $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_m$ are independent random vectors. Then all sequences

$$X_{1,j_1}, X_{2,j_2}, \ldots, X_{m,j_m} \tag{7.19}$$

of component random variables are independent, where X_{ij} is the j-th component random variable of $\mathbf{X}_i$.

Theorem 7.9 (Conditional Criteria for Independence)

Let X and Y be two random variables.

- Suppose X and Y are jointly discrete. Then they are independent if and only if

  $$p_{X|Y}(x|y) = p_X(x)$$

  for all $x \in \mathbb{R}$ and all $y \in \mathbb{R}$ such that $p_Y(y) > 0$.
- Suppose X and Y are jointly continuous. Then they are independent if and only if

  $$f_{X|Y}(x|y) = f_X(x)$$

  for all $x \in \mathbb{R}$ and all $y \in \mathbb{R}$ such that $f_Y(y) > 0$.

Problem Prompt

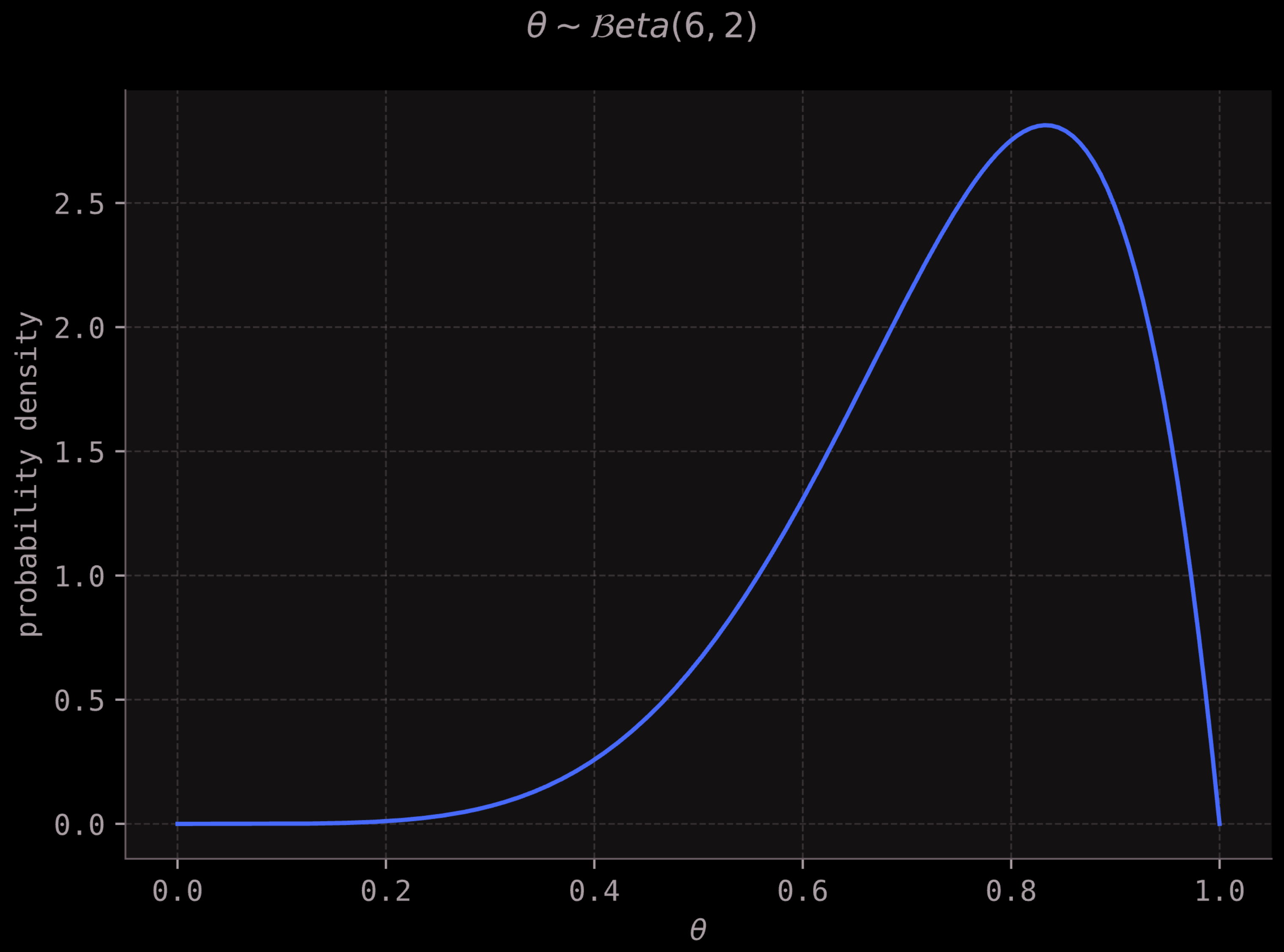
Do problems 15-18 on the worksheet.

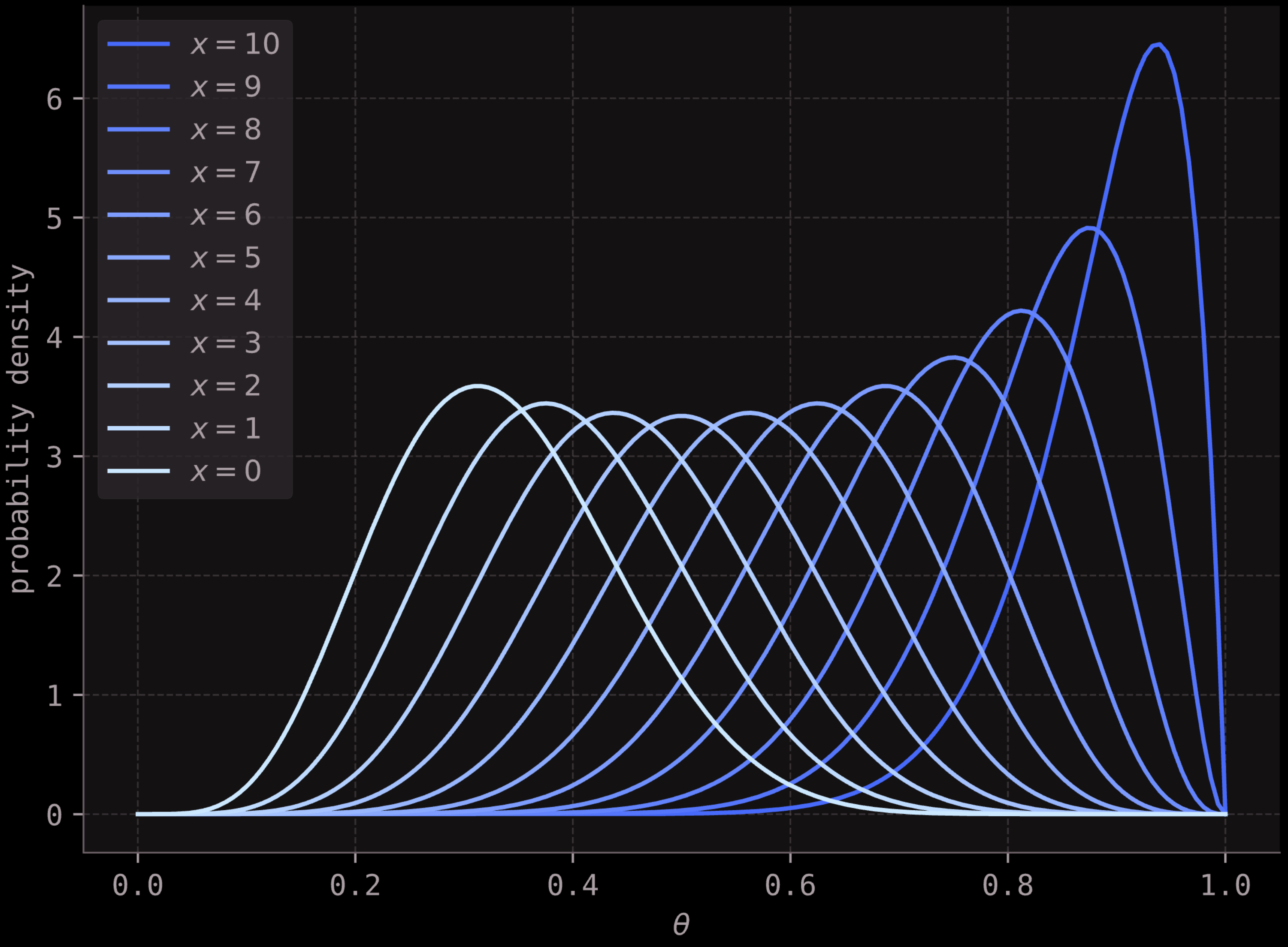
7.10. Case study: an untrustworthy friend

The Canonical Bayesian Coin-Flipping Scenario

Our friend suggests that we play a game, betting money on whether a coin flip lands heads or tails. If the coin lands heads, our friend wins; if the coin lands tails, we win.

However, our friend has proven to be untrustworthy in the past. We suspect that the coin might be unfair, with a probability of $\theta = 0.75$ of landing heads. So, before we play the game, we collect data and flip the coin ten times and count the number of heads. Depending on our results, how might we alter our prior estimate of $\theta = 0.75$ for the probability of landing heads?


$\theta \sim \mathcal{B}eta(6, 2)$
probability density
0.0
0.5
1.0
1.5
2.0
2.5
0.0
0.2
0.4
0.6
0.8
1.0
θ



x = 10
x = 9
x = 8
x = 7
x = 6
x = 5
x = 4
x = 3
x = 2
x = 1
x = 0
probability density
0
1
2
3
4
5
6
0.0
0.2
0.4
0.6
0.8
1.0
θ