Sum-Product Algorithm: Chain Factor Graph Example

Problem Setup

Consider a Markov chain over three binary variables $x_1, x_2, x_3 \in \{0, 1\}$, with joint distribution:

$$p(x_1, x_2, x_3) = f_1(x_1) \cdot f_2(x_1, x_2) \cdot f_3(x_2, x_3)$$

Our goal is to compute the marginal $p(x_2)$ using the **sum-product algorithm**.

Factor Values

Let the factors be defined as:

$$f_1(x_1) = [0.6, 0.4]$$

$$f_2(x_1, x_2) = \begin{bmatrix} 0.3 & 0.7 \\ 0.9 & 0.1 \end{bmatrix}, \quad f_3(x_2, x_3) = \begin{bmatrix} 0.2 & 0.8 \\ 0.6 & 0.4 \end{bmatrix}$$

Message Passing Steps

Step 1: Message $\mu_{f_1 \to x_1}(x_1)$

$$\mu_{f_1 \to x_1}(x_1) = f_1(x_1) = [0.6, 0.4]$$

Step 2: Message $\mu_{x_1 \to f_2}(x_1)$

$$\mu_{x_1 \to f_2}(x_1) = \mu_{f_1 \to x_1}(x_1) = [0.6, 0.4]$$

Step 3: Message $\mu_{f_2 \to x_2}(x_2)$

$$\mu_{f_2 \to x_2}(x_2) = \sum_{x_1} f_2(x_1, x_2) \cdot \mu_{x_1 \to f_2}(x_1)$$

$$\mu_{f_2 \to x_2}(0) = 0.3 \cdot 0.6 + 0.9 \cdot 0.4 = 0.18 + 0.36 = 0.54$$
$$\mu_{f_2 \to x_2}(1) = 0.7 \cdot 0.6 + 0.1 \cdot 0.4 = 0.42 + 0.04 = 0.46$$

Step 4: Message $\mu_{f_3 \to x_2}(x_2)$

Assume no information from x_3, i.e. $\mu_{x_3 \to f_3}(x_3) = 1$:

$$\mu_{f_3 \to x_2}(x_2) = \sum_{x_3} f_3(x_2, x_3) \Rightarrow [1.0, 1.0]$$

Step 5: Final Marginal for x_2

$$p(x_2) \propto \mu_{f_2 \to x_2}(x_2) \cdot \mu_{f_3 \to x_2}(x_2)$$

$$p(x_2 = 0) \propto 0.54, \quad p(x_2 = 1) \propto 0.46$$

Normalize:

$$p(x_2 = 0) = \frac{0.54}{0.54 + 0.46} = 0.54, \quad p(x_2 = 1) = 0.46$$

Final Answer

$$\boxed{p(x_2) = [0.54,\ 0.46]}$$