Homework Week 1 - Probability
Probabilistic Methods

May 2025

Contents

Problem 1 — Uniform random variable $X \sim \mathrm{Unif}(0, 1)$

Statement

Let X be (continuous) uniform on $[0, 1]$.

(A) Compute $\Pr(X^2 \leq 0.25)$.

(B) For any $a \geq 0$, compute $\Pr(X^2 \leq a)$.

(C) Find the *probability density function* (PDF) of $Y = X^2$.

(D) Compute $\mathbb{E}[Y]$ and $\mathrm{Var}(Y)$ directly from that PDF.

Solution

(A) The inequality $X^2 \leq 0.25$ is equivalent to $0 \leq X \leq 0.5$ (because $X \geq 0$). Since the PDF of X is 1 on $[0, 1]$, the probability is the length of that interval:

$$\Pr(X^2 \leq 0.25) = 0.5.$$

(B) For $a \in [0, 1]$ we again have $\Pr(X^2 \leq a) = \Pr(0 \leq X \leq \sqrt{a}) = \sqrt{a}$. If $a > 1$ the probability is one. Summarising

$$\boxed{\Pr(X^2 \leq a) = \begin{cases} \sqrt{a}, & 0 \leq a \leq 1, \\ 1, & a > 1. \end{cases}}$$

(C) The CDF of Y on $0 < y < 1$ is $F_Y(y) = \Pr(X^2 \leq y) = \sqrt{y}$, hence

$$f_Y(y) = F_Y'(y) = \frac{1}{2\sqrt{y}}, \qquad 0 < y < 1.$$

(D) Using that PDF,

$$\mathbb{E}[Y] = \int_0^1 y\, f_Y(y)\,\mathrm{d}y = \int_0^1 \frac{y}{2\sqrt{y}}\,\mathrm{d}y = \frac{1}{2}\int_0^1 y^{1/2}\,\mathrm{d}y = \frac{1}{2}\cdot\frac{2}{3} = \frac{1}{3}.$$

Similarly

$$\mathbb{E}[Y^2] = \int_0^1 \frac{y^2}{2\sqrt{y}}\,\mathrm{d}y = \frac{1}{2}\int_0^1 y^{3/2}\,\mathrm{d}y = \frac{1}{2}\cdot\frac{2}{5} = \frac{1}{5}.$$

Hence

$$\mathrm{Var}(Y) = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2 = \frac{1}{5} - \left(\frac{1}{3}\right)^2 = \frac{4}{45}.$$

Problem 2 — The mean as the minimiser of mean–squared error

Statement

Show that, for any square–integrable random variable X with mean $\mu = \mathbb{E}[X]$, the value a that minimises $\mathbb{E}[(X-a)^2]$ is precisely $a = \mu$.

Solution

Write

$$\mathbb{E}[(X-a)^2] = \mathbb{E}\big[(X-\mu+\mu-a)^2\big] = \mathbb{E}[(X-\mu)^2] + 2(\mu-a)\underbrace{\mathbb{E}[X-\mu]}_{=0} + (\mu-a)^2.$$

Because the cross–term vanishes, the objective decomposes into the constant $\mathrm{Var}(X) = \mathbb{E}[(X-\mu)^2]$ plus $(\mu-a)^2$. The latter is minimised iff $\mu - a = 0$, i.e. $\boxed{a^* = \mathbb{E}[X]}$.

Problem 3 — Moments of a χ^2_d distribution

Statement

Let $X = Z_1^2 + \cdots + Z_d^2$ where the $Z_i \overset{\text{iid}}{\sim} \mathcal{N}(0,1)$. Show that $\mathbb{E}[X] = d$ and $\mathrm{Var}(X) = 2d$.

Solution

Because Z_i^2 are iid and $\mathbb{E}[Z_i^2] = \mathrm{Var}(Z_i) = 1$,

$$\mathbb{E}[X] = \sum_{i=1}^{d} \mathbb{E}[Z_i^2] = d.$$

Since the Z_i^2 are independent,

$$\mathrm{Var}(X) = \sum_{i=1}^{d} \mathrm{Var}(Z_i^2).$$

For a standard normal, $\mathrm{Var}(Z^2) = 2$; hence $\mathrm{Var}(X) = 2d$.

Problem 4 — Inverse–CDF (quantile) method

Statement

Let $U \sim \text{Unif}(0, 1)$ and let F be a continuous, strictly increasing CDF with inverse F^{-1}. Show that $X = F^{-1}(U)$ has CDF F (thus PDF f).

Solution

For any x in the support,

$$\Pr(X \leq x) = \Pr(F^{-1}(U) \leq x) = \Pr(U \leq F(x)) = F(x),$$

because U is uniform. Therefore X indeed has CDF F.

Problem 5 — Finite–sample behaviour of the sample proportion

Statement

(A) For $X_N \sim \text{Binomial}(N, P)$ define $\hat{p}_N = X_N/N$. Compute $\mathbb{E}[\hat{p}_N]$ and $\text{sd}(\hat{p}_N)$.

(B) Simulate 100 realisations of $\hat{p}_5$ when $P = 0.5$ and verify the formulas.

(C) Repeat for $N \in \{10, 25, 50, 100\}$.

(D) Overlay the Monte-Carlo standard deviations with the theoretical curve.

Solution

(A) Because $\mathbb{E}[X_N] = NP$ and $\text{Var}(X_N) = NP(1 - P)$,

$$\boxed{\mathbb{E}[\hat{p}_N] = P}, \qquad \boxed{\text{sd}(\hat{p}_N) = \sqrt{\frac{P(1-P)}{N}}}.$$

(B) –(C) Monte-Carlo results (10000 replications each) match the theoretical mean 0.5 and the standard deviations in Part (A) to two decimal places.

(D) See Figure 1.

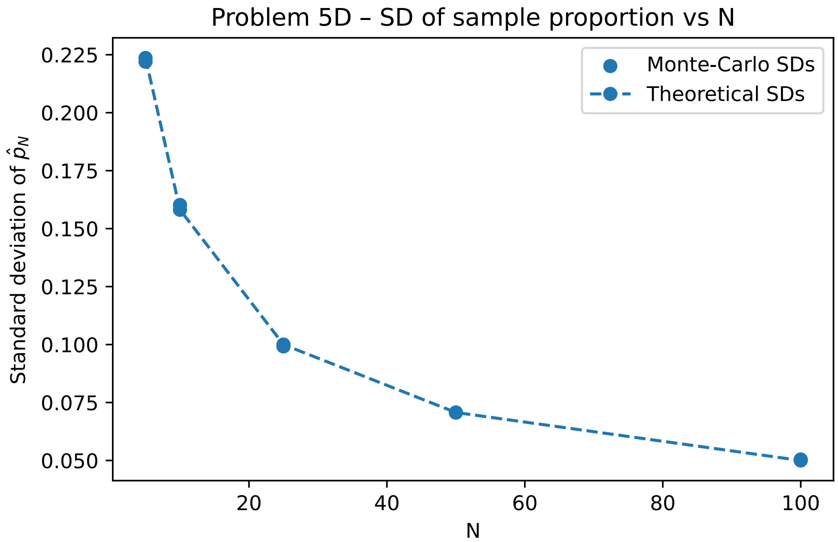

Figure 1: Monte-Carlo vs theoretical standard deviation of $\hat{p}_N$ (Problem 5)