Week	Topic	Reading
1	• Data, Models, and Information • Elementary statistics: Definitions • Overview of R	OIS 1 (ISL 1)
2	• Elementary statistics: Applications & Plots	OIS 1 (ISL 1)
3	• Introduction to data analysis with R • Review of tabular and graphical displays of data	ITR 1, 2, 5, 6, 7, 12
4	• Random variables: expectation and variance • Joint and conditional probability • Bayes rule	OIS 2
5	• Random variables: distributions (normal, binomial, poisson)	OIS 3

Definitions, basic concepts, R practice

Lots of definitions and pen-and-paper practice

Intro to Probability Theory: A bunch of definitions & problems

(lots of definitions & equations, followed by some playing of online games)

Intro to Probability Theory: A bunch of definitions & problems

Why bother?

- Good background for all of the rest of this class
- Necessary for further statistics classes (Bayesian)

Probability forms the foundation for *inference*

- A certain county has a population that is 50% women and 50% men. A jury is supposedly randomly selected. The jury ends up having a composition that is 40% women. Was there selection bias, or was this just due to random chance?

- In a randomized double-blind controlled experiment, a new surgery saved lives 60% of the time, while the old surgery saved lives only 55% of the time. Is this a big enough difference to replace the old surgery with the new one?

- A pack of potato chips is supposed to be manufactured to have an average weight of 10 ounces. 30 random bags of chips are weighed, and have an average weight of 9.6 ounces. Is the manufacturer cheating?

  If the bags really have an average of 10 ounces, what is the probability we would get a sample average this low?

Basic probability

Roll a fair die once.

Notation: P(event)

P(roll a 1) =

P(roll at least a 4) =

P(roll at most 2) =

Basic probability

Roll a fair die once.

Notation: P(event)

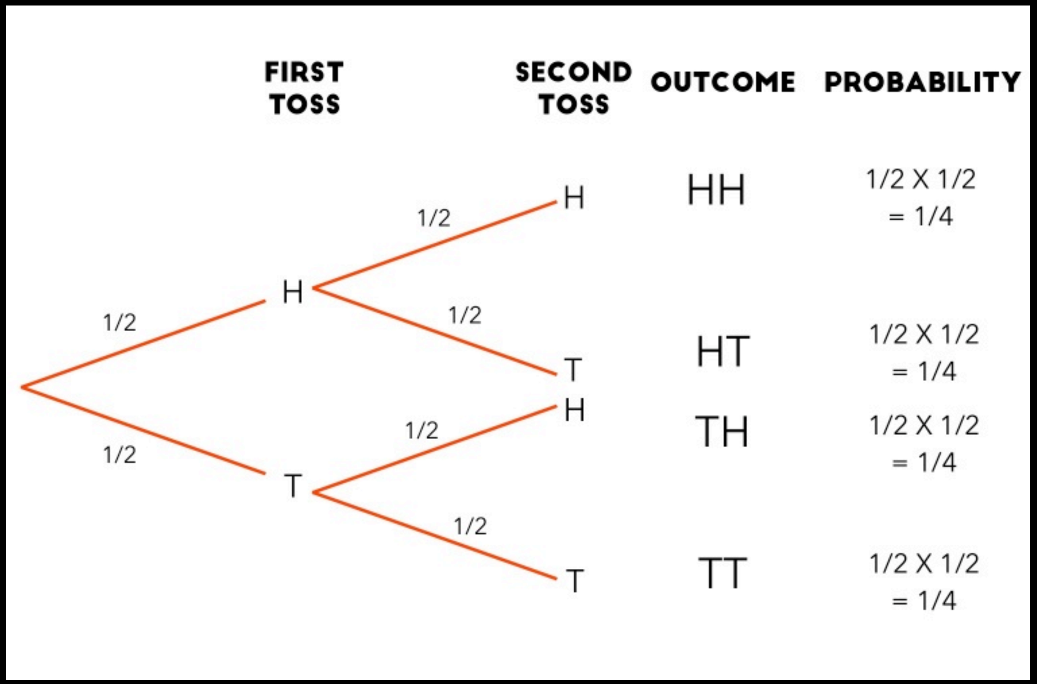
P(roll a 1) = 1/6

P(roll < 4) = 3/6

P(roll <= 2) = 2/6

The classical theory of probability

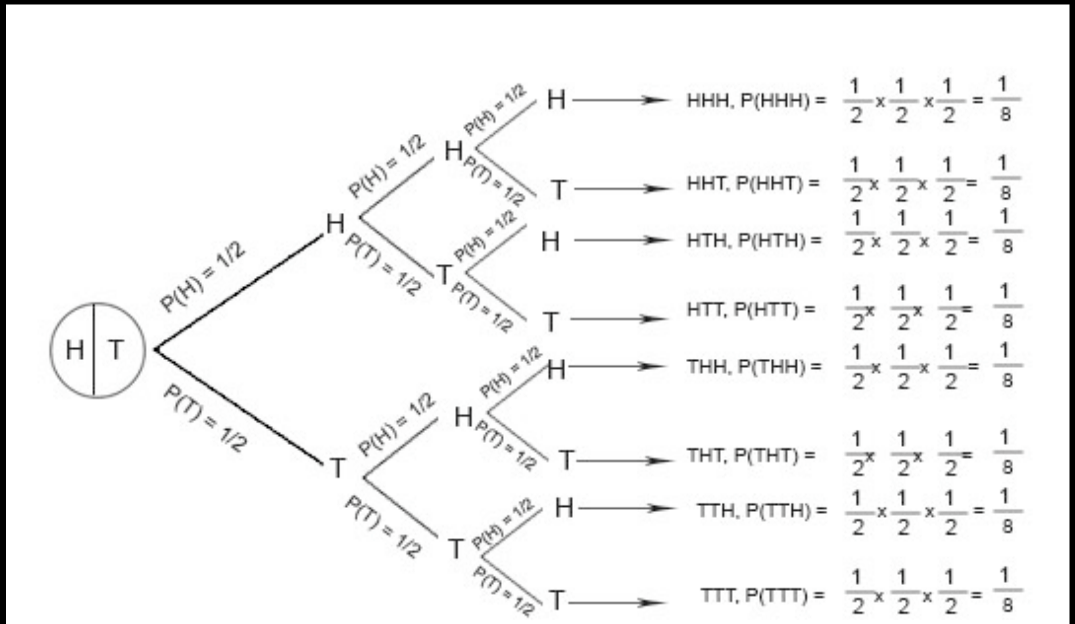
P(event) = $\dfrac{\text{\# of ways event can happen}}{\text{\# of total possible outcomes}}$



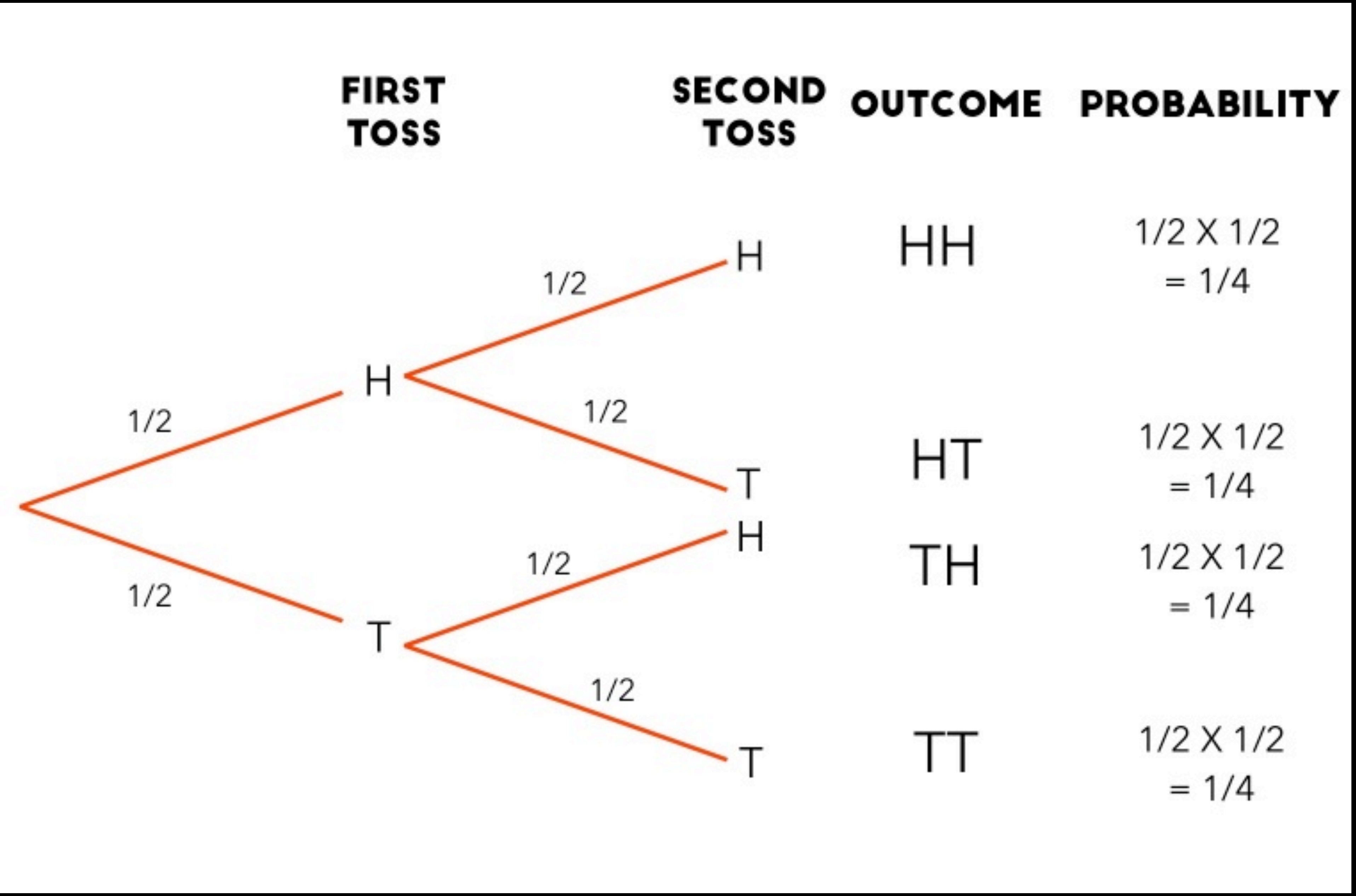
P(you get HH in two tosses of a fair coin) = ?

Make a tree diagram

P(you get HHH in three tosses of a fair coin) = ?



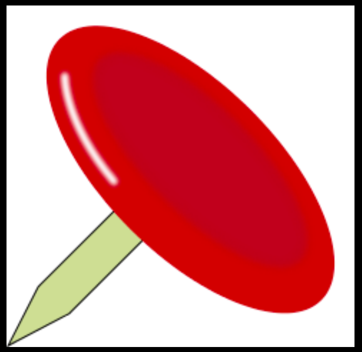
The classical theory of probability



But wait...

P(tack lands face down) = ?

P(tack lands face down) = ½ ?



P(comet hits earth tomorrow) = ?

P(comet hits earth tomorrow) = ½ ?

The relative frequency theory

P(event occurs) = the proportion of times the outcome would occur if we observed the random process an infinite number of times

P(tack lands face down) = <u># of times it lands face down</u>
\# of times we toss it

as # of times we tosses gets very large

Let n be the number of times we *repeat* the random process

As n gets larger, the approximation tends to get better,

More technically, the **Law of Large Numbers** says that as more observations are collected, the observed proportion of occurrences with a particular outcome after n trials converges to the true probability p of that outcome. i.e., The more times you repeat experiment, the better estimate of probability tends to get.

But wait...

P(comet hits earth tomorrow) = ?

We just repeat this experiment over and over, right?

Subjective theory

P(comet hits earth tomorrow) = ?

The subjective theory of probability says that probability is a statement about our knowledge or lack of knowledge. Probability is therefore expressed as degrees of rational belief.

Probability is a property of us, not of the world.

Probability: Practically

What is the probability of event #1 **<u>or</u>** event #2 occurring?

- $P(E_1 \text{ or } E_2)$ - General Addition Rule

What is the probability of event #2 **<u>and</u>** event #2 occurring?

- $P(E_1 \text{ and } E_2)$ - General Multiplication Rule

What is the probability of event #1 **<u>given</u>** event #2 (if event #1 depends on event #2)?

- $P(E_1 \mid E_2)$ - Conditional probability - marginal & joint probabilities; tree diagrams; Bayes' Theorem

How do these relate $P(E_1)$ and $P(E_2)$?

2 rules of probability

$0 \leq P(E) \leq 1$ (or $0\% \leq P(E) \leq 100\%$)

0 implies the event is impossible and
1 implies the event is certain

Also, $P(E) + P(\text{not } E) = 1$ and so

$P(\text{not } E) = 1 - P(E)$

e.g. P(not getting a Queen in a deck of 52 cards)
=1 – P(getting a Queen in a deck of 52 cards)
$= 1 - 4/52 = 48/52$

Disjoint and non-disjoint outcomes

Disjoint (mutually exclusive) outcomes: Cannot happen at the same time.

- The outcome of a single coin toss cannot be a head and a tail.
- A student both cannot fail and pass a class.
- A single card drawn from a deck cannot be an ace and a queen.

Non-disjoint outcomes: Can happen at the same time.

- A student can get an A in Stats and A in Programming in the same semester.

Addition Rule of disjoint outcomes

If A_1 and A_2 represent two <u>disjoint</u> outcomes, then the probability that one of them occurs, i.e. one or the other occurs, is:

$P(A_1 \text{ or } A_2) = P(A_1) + P(A_2)$

Addition Rule of disjoint outcomes

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$P(A_1 \text{ or } A_2) = P(A_1) + P(A_2)$

Draw 1 card

P(Jack or King) = 4/52 + 4/52

Addition Rule of disjoint outcomes

If A_1 and A_2 represent two disjoint outcomes, then the probability that one of them occurs, i.e. one or the other occurs, is:

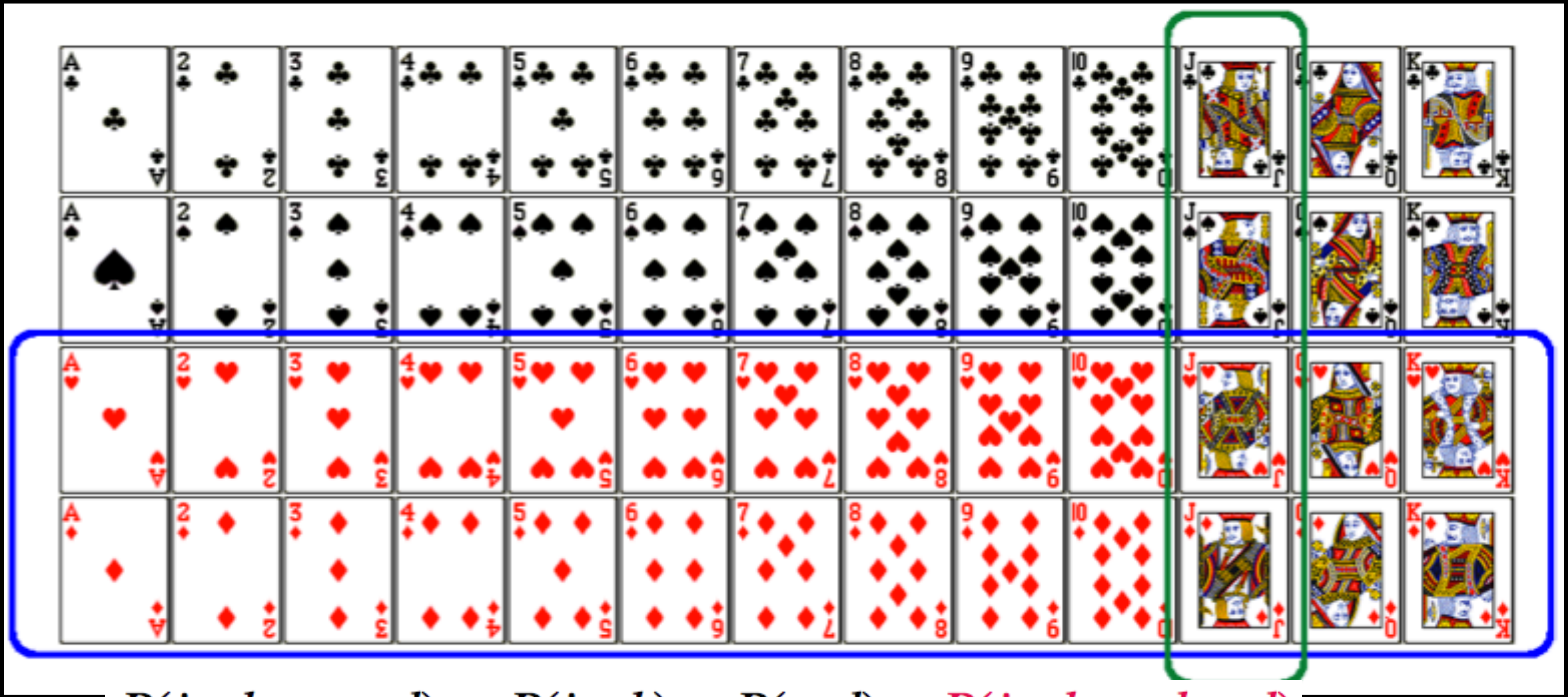
$P(A_1 \text{ or } A_2) = P(A_1) + P(A_2) + \ldots$

Draw 1 card
P(Jack or King) = 4/52 + 4/52

Draw 1 card
P(Jack or Red) = 4/52 + 13/52 ?

Union of non-disjoint events

What is the probability of drawing a jack or a red card from a well shuffled full deck?



$$P(\text{jack or red}) = P(\text{jack}) + P(\text{red}) - P(\text{jack and red})$$

$$= \frac{4}{52} + \frac{26}{52} - \frac{2}{52} = \frac{28}{52}$$

http://www.milefoot.com/math/discrete/counting/cardfreq.htm

Practice

What is the probability that a randomly sampled student thinks marijuana should be legalized or they agree with their parents' political views?

	Share Parents' Politics		
Legalize MJ	No	Yes	Total
No	11	40	51
Yes	36	78	114
Total	47	118	165

(1) (40 + 36 - 78) / 165
(2) (114 + 118 - 78) / 165
(3) 78 / 165
(4) 78 / 188
(5) 11 / 47

Recap

General addition rule

$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

Note: For disjoint/mutually exclusive events, $P(A \text{ and } B) = 0$, so the above formula simplifies to $P(A \text{ or } B) = P(A) + P(B)$

Probability distributions

A probability distribution lists all possible events and the probabilities with which they occur.

- The probability distribution for the sex of one kid:

Event	Male	Female
Probability	0.5	0.5

(note this example assumes intersex contribution is negligible)

Probability distributions

A probability distribution lists all possible events and the probabilities with which they occur.

- The probability distribution for the sex of one kid:

Event	Male	Female
Probability	0.5	0.5

- Rules for probability distributions:
  1. The events listed must be disjoint
  2. Each probability must be between 0 and 1
  3. The probabilities must total 1

- The probability distribution for the sexes of two kids:

Event	MM	FF	MF	FM
Probability	0.25	0.25	0.25	0.25

Practice

Sample space is the collection of all possible outcomes of a trial.

- A couple has one kid, what is the sample space for the sex of this kid? S = {M, F}
- A couple has two kids, what is the sample space for the sex of these kids?

Complementary events are two mutually exclusive events whose probabilities that add up to 1.

- A couple has one kid. If we know that the kid is not a boy, what is sex of this kid? { ~~M~~, *F* } Boy and girl are complementary outcomes.
- A couple has two kids, if we know that they are not both girls, what are the possible sex combinations for these kids?

S = { MM, ~~FF~~, FM, MF }

Independence

Two processes are independent if knowing the outcome of one provides no useful information about the outcome of the other.

- Knowing that the coin landed on a head on the first toss does not provide any useful information for determining what the coin will land on in the second toss.
  >> Outcomes of two tosses of a coin are independent.

Practice

Between January 9-12, 2013, SurveyUSA interviewed a random sample of 500 NC residents asking them whether they think widespread gun ownership protects law abiding citizens from crime, or makes society more dangerous. 58% of all respondents said it protects citizens. 67% of White respondents, 28% of Black respondents, and 64% of Hispanic respondents shared this view. Which of the below is true?

Opinion on gun ownership and race ethnicity are most likely
(a) complementary
(b) mutually exclusive
(c) independent
(d) dependent
(e) disjoint

Checking for independence

If P(A occurs, given that B is true) = P(A | B) = P(A),
then A and B are independent.

P(protects citizens) = 0.58 **P(A)**

P(randomly selected NC resident says gun ownership protects citizens, given that the resident is white)
= P(protects citizens | White) = 0.67
P(protects citizens | Black) = 0.28
P(protects citizens | Hispanic) = 0.64

P(A|B) != P(A)

P(protects citizens) varies by race/ethnicity, therefore opinion on gun ownership and race ethnicity are *most likely dependent*.

Practice

Between January 9-12, 2013, SurveyUSA interviewed a random sample of 500 NC residents asking them whether they think widespread gun ownership protects law abiding citizens from crime, or makes society more dangerous. 58% of all respondents said it protects citizens. 67% of White respondents, 28% of Black respondents, and 64% of Hispanic respondents shared this view. Which of the below is true?

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(b) mutually exclusive
(c) independent
(d) dependent
(e) disjoint

Product rule for independent events

P(A and B) = P(A) x P(B)
Or more generally, $P(A_1$ and … and $A_k) = P(A_1) \times \ldots \times P(A_k)$

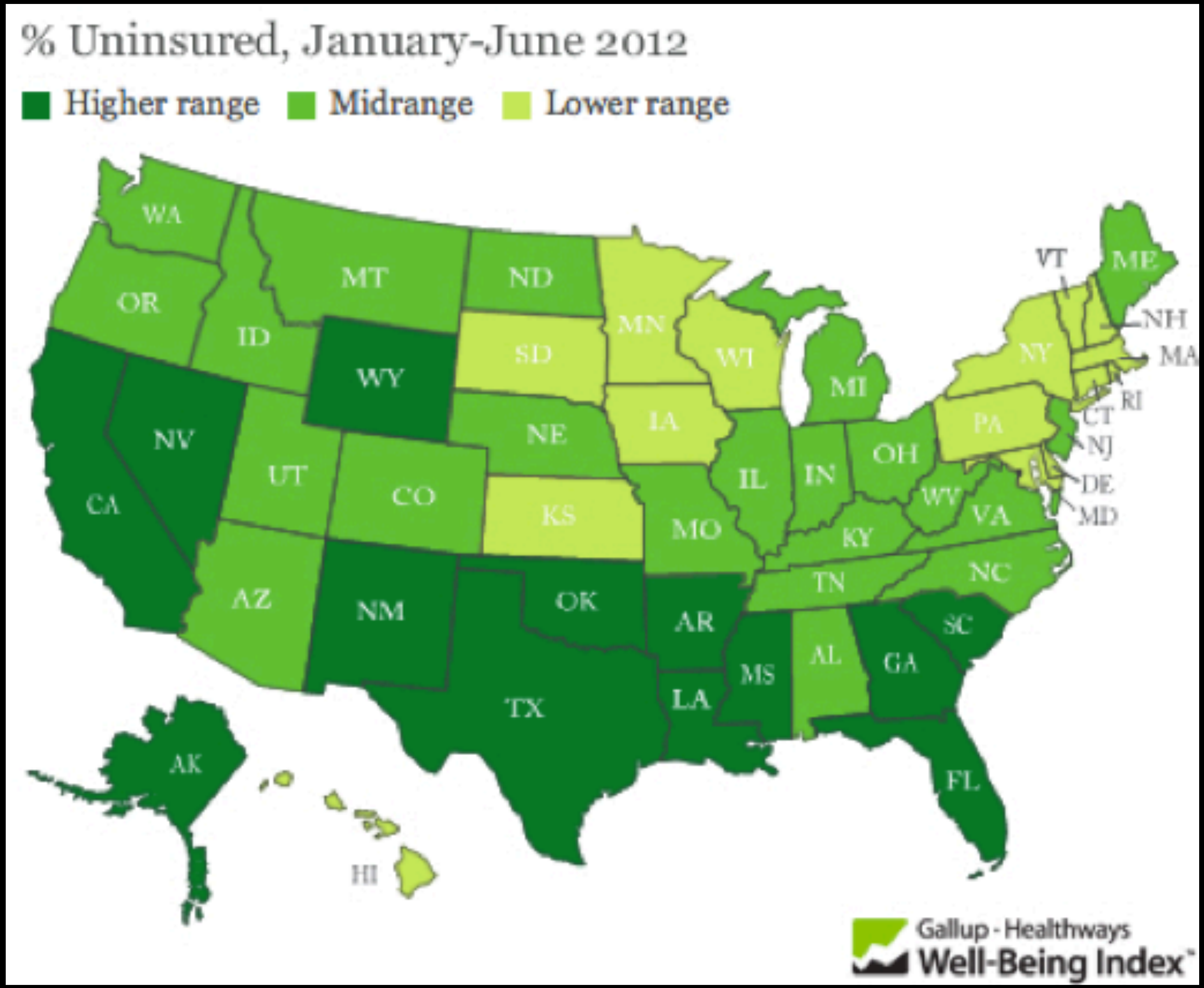
You toss a coin twice, what is the probability of getting two tails in a row?

P(T on the first toss) x P(T on the second toss)
= (1 / 2) x (1 / 2) = 1 / 4

Practice

A recent Gallup poll suggests that 25.5% of Texans do not have health insurance as of June 2012. Assuming that the uninsured rate stayed constant, what is the probability that two randomly selected Texans are both uninsured?

(1) 25.5^2
(2) 0.255^2
(3) 0.255×2
(4) $(1 - 0.255)^2$



http://www.gallup.com/poll/156851/uninsured-rate-stable-across-states-far-2012.aspx

Disjoint vs. complementary

Do the sum of probabilities of two disjoint events always add up to 1?

Not necessarily, there may be more than 2 events in the sample space, e.g. party affiliation.

Do the sum of probabilities of two complementary events always add up to 1?

Yes, that's the definition of complementary, e.g. heads and tails.

Practice

Roughly 20% of undergraduates at a university are vegetarian or vegan. What is the probability that, among a random sample of 3 undergraduates, at least one is vegetarian or vegan?

(1) $1 - 0.2 \times 3$
(2) $1 - 0.2^3$
(3) 0.8^3
(4) $1 - 0.8 \times 3$
(5) $1 - 0.8^3$

Relapse

Researchers randomly assigned 72 chronic users of cocaine into three groups: desipramine (antidepressant), lithium (standard treatment for cocaine) and placebo. Results of the study are summarized below.

	relapse	no relapse	total
desipramine	10	14	24
lithium	18	6	24
placebo	20	4	24
total	48	24	72

http://www.oswego.edu/~srp/stats/2_way_tbl_1.htm

Marginal probability

What is the probability that a patient relapsed?

	relapse	no relapse	total
desipramine	10	14	24
lithium	18	6	24
placebo	20	4	24
total	48	24	72

Marginal probability

What is the probability that a patient relapsed?

	relapse	no relapse	total
desipramine	10	14	24
lithium	18	6	24
placebo	20	4	24
total	48	24	72

P(relapsed) = 48 / 72 ~ 0.67

Joint probability

What is the probability that a patient received the antidepressant (desipramine) <u>and</u> relapsed?

	relapse	no relapse	total
desipramine	10	14	24
lithium	18	6	24
placebo	20	4	24
total	48	24	72

Joint probability

What is the probability that a patient received the antidepressant (desipramine) <u>and</u> relapsed?

	relapse	no relapse	total
desipramine	10	14	24
lithium	18	6	24
placebo	20	4	24
total	48	24	72

P(relapsed and desipramine) = 10 / 72 ~ 0.14

Conditional probability

The conditional probability of the outcome of interest A given condition B is calculated as

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

"Probability of event A, <u>given</u> (|) event B"

	relapse	no relapse	total
desipramine	10	14	24
lithium	18	6	24
placebo	20	4	24
total	48	24	72

$$P(relapse|desipramine) = \frac{P(relapse \text{ and } desipramine)}{P(desipramine)}$$

$$= \frac{10/72}{24/72}$$

$$= \frac{10}{24}$$

$$= 0.42$$

Conditional probability (cont.)

If we know that a patient received the antidepressant (desipramine), what is the probability that they relapsed?

	relapse	no relapse	total
desipramine	10	14	24
lithium	18	6	24
placebo	20	4	24
total	48	24	72

P(relapse | desipramine) = 10 / 24 ~ 0.42

Conditional probability (cont.)

If we know that a patient received the antidepressant (desipramine), what is the probability that they relapsed?

	relapse	no relapse	total
desipramine	10	14	24
lithium	18	6	24
placebo	20	4	24
total	48	24	72

P(relapse | desipramine) = 10 / 24 ~ 0.42

P(relapse | lithium) = 18 / 24 ~ 0.75

P(relapse | placebo) = 20 / 24 ~ 0.83

Conditional probability (cont.)

If we know that a patient relapsed, what is the probability that they received the antidepressant (desipramine)?

	relapse	no relapse	total
desipramine	10	14	24
lithium	18	6	24
placebo	20	4	24
total	48	24	72

P(desipramine | relapse) = 10 / 48 ~ 0.21

P(lithium | relapse) = 18 / 48 ~ 0.38

P(placebo | relapse) = 20 / 48 ~ 0.42

General multiplication rule

- Earlier we saw that if two events are independent, their joint probability is simply the product of their probabilities. If the events are not believed to be independent, the joint probability is calculated slightly differently.

- If A and B represent two outcomes or events, then

$$
\begin{aligned}
P(A \text{ and } B) &= P(A) \times P(B \mid A) \\
&= P(B) \times P(A \mid B)
\end{aligned}
$$

Note that this formula is simply the conditional probability formula, rearranged.

Bayes' Theorem

The conditional probability formula we have seen so far is a special case of the Bayes' Theorem, which is applicable even when events have more than just two outcomes.

Bayes' Theorem

$$P(outcome\ A\ of\ variable\ 1 \mid outcome\ B\ of\ variable\ 2)$$

$$= \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A_2)P(A_2) + \cdots + P(B|A_k)P(A_k)}$$

where $A_2, \cdots, A_k$ represent all other possible outcomes of variable 1.

NOTE: likely to come across this in other classes

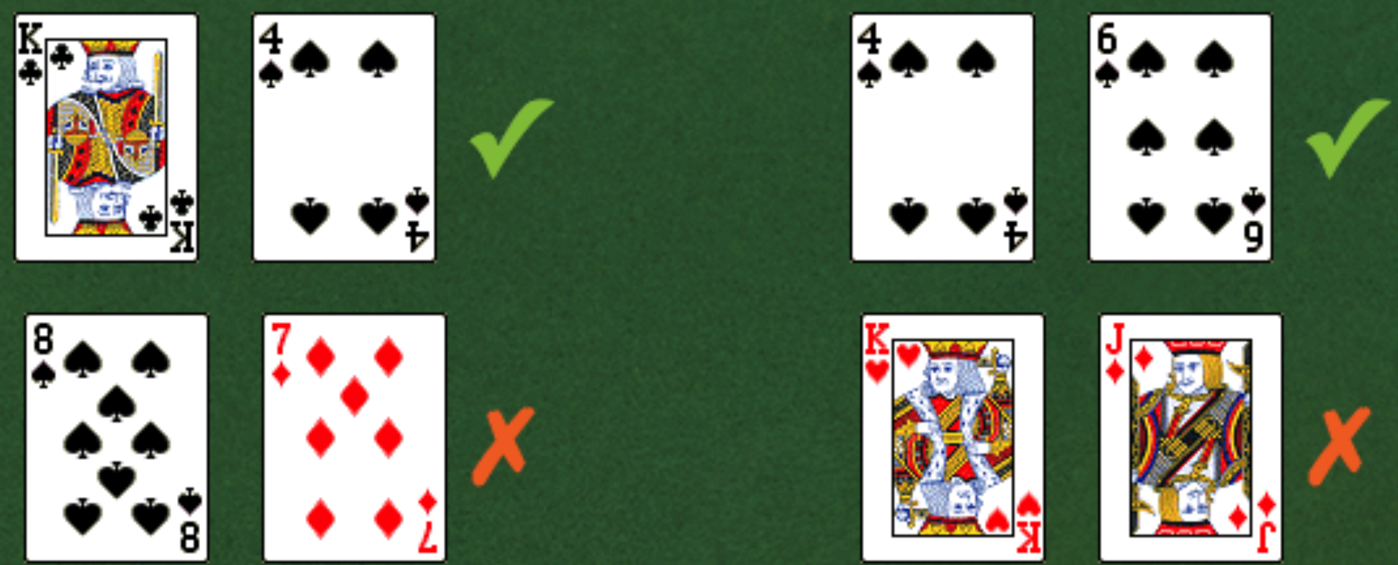
Bank: $171
Level 3

Beat the Odds

Training

You draw 2 cards from a deck. What's the probability that they are both black?

EXAMPLES



Go to the Lab »

YOUR ANSWER

$p =$

Submit

« Back to menu

Skip this problem

Bank: $174
Level 3

Beat the Odds

Training

You flip 3 coins. What's the probability that none are tails?

EXAMPLES

Go to the Lab »

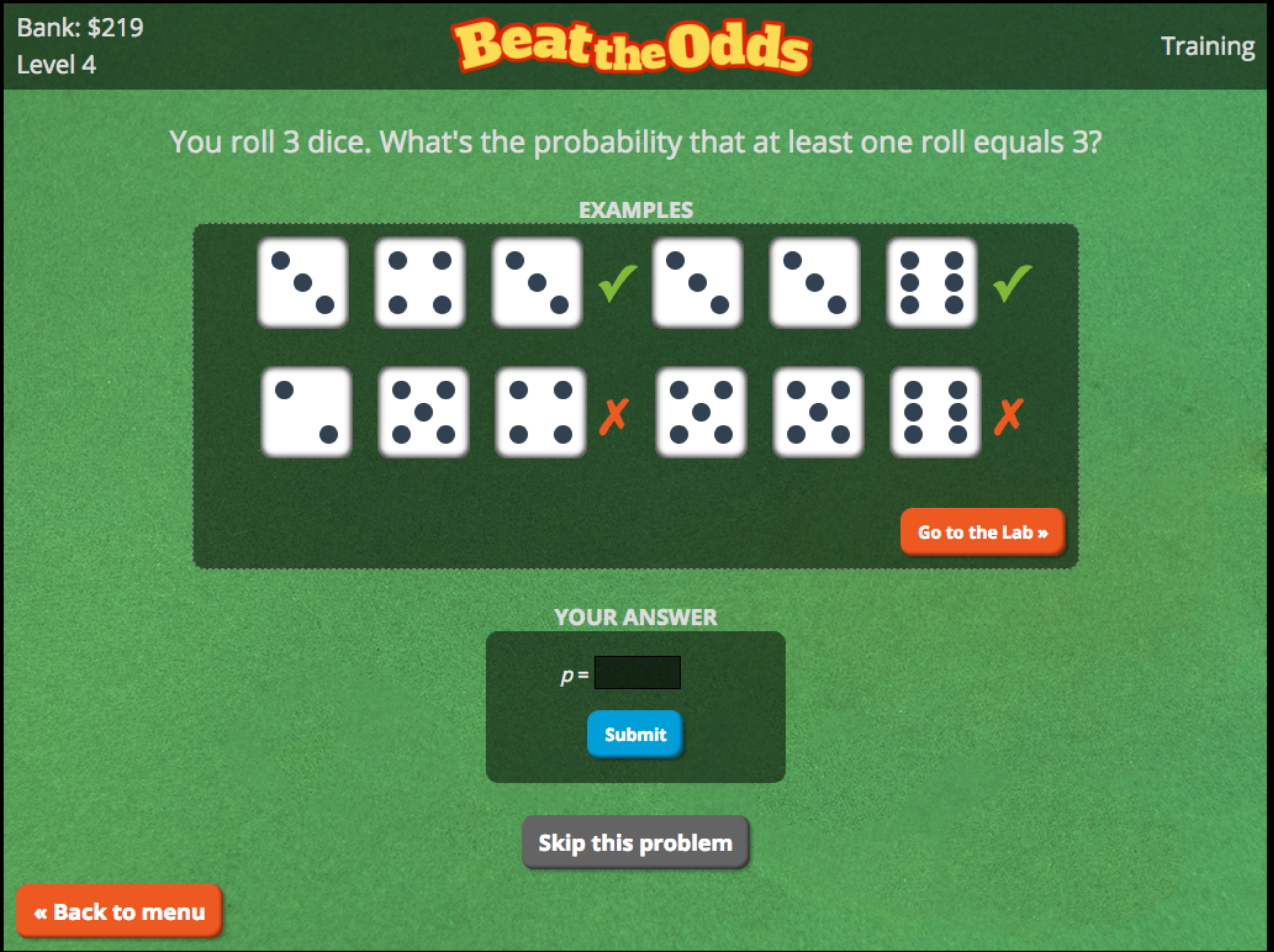
YOUR ANSWER

$p =$

Submit

Skip this problem

« Back to menu


Bank: $219
Level 4
Beat the Odds
Training
You roll 3 dice. What's the probability that at least one roll equals 3?
EXAMPLES
Go to the Lab »
YOUR ANSWER
p =
Submit
Skip this problem
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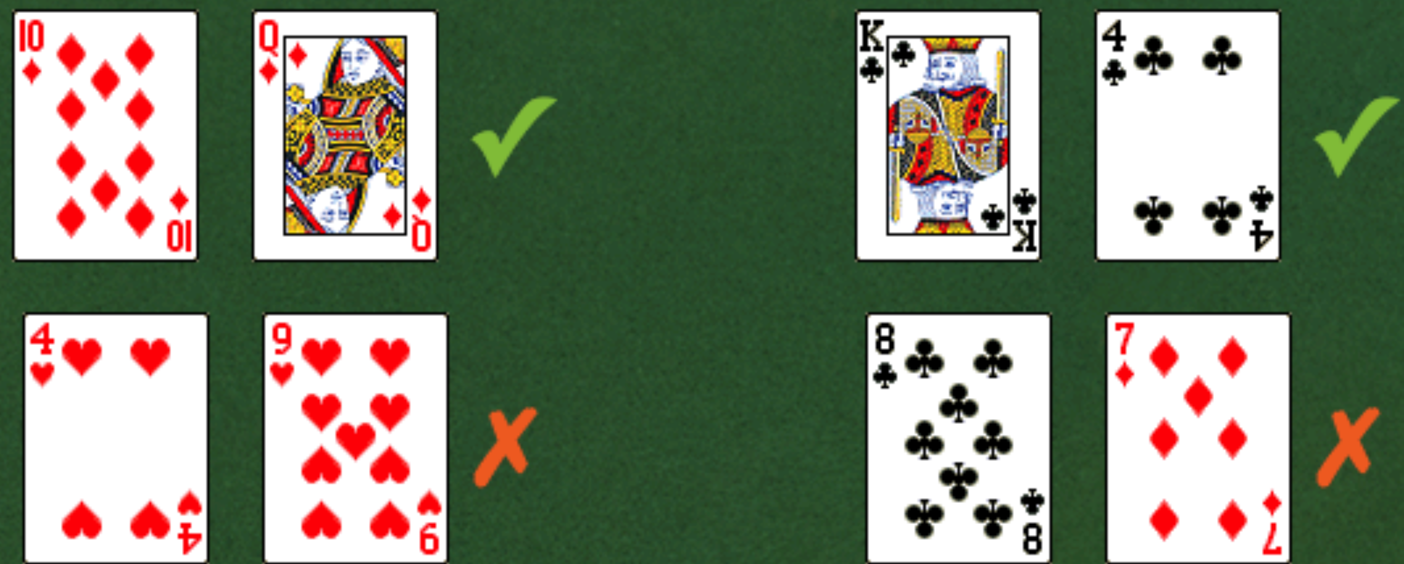
Bank: $260
Level 4

Beat the Odds

Training

You draw 2 cards from a deck. What's the probability that exactly one is a face card (Jack, Queen, or King)?

EXAMPLES



Go to the Lab »

YOUR ANSWER

$p =$

Submit

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Skip this problem

Beat the Odds Game:
http://d3tt741pwxqwm0.cloudfront.net/WGBH/mgbh/mgbh_int_beatodds/index.html