4.13.34

Jnanesh Sathisha Karmar- EE25BTECH11029

Question:
The equations to a pair of opposite sides of a parallelogram are $x^2 - 5x + 6 = 0$ and $y^2 - 6y + 5 = 0$. Find the equations to its diagonals.

1) $x + 4y = 13, y = 4x - 7$
2) $4x + y = 13, y = 4x - 7$
3) $4x + y = 13, 4y = x - 7$
4) $y - 4x = 13, y + 4x = 7$

Solution:
We can solve this problem by treating a pair of parallel lines as a degenerate conic section and finding where a line (the diagonal) intersects it. The general equation for a conic is given by $g(\mathbf{x}) = \mathbf{x}^\top \left(V\right) \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0$, and a parametric line is given by $\mathbf{x} = \mathbf{h} + \kappa \mathbf{m}$. The intersection points are found using the formula:

$$\kappa_{1,2} = \frac{-\mathbf{m}^\top \left(\left(V\right)\mathbf{h} + \mathbf{u}\right) \pm \sqrt{\left(\mathbf{m}^\top \left(\left(V\right)\mathbf{h} + \mathbf{u}\right)\right)^2 - \left(\mathbf{m}^\top \left(V\right)\mathbf{m}\right) g(\mathbf{h})}}{\mathbf{m}^\top \left(V\right)\mathbf{m}} \tag{1}$$

Let's represent the pair of vertical lines $x^2 - (x_1 + x_2)x + x_1x_2 = 0$ as our conic section.

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} -\frac{x_1+x_2}{2} \\ 0 \end{pmatrix} \ f = x_1x_2 \tag{2}$$

The diagonal is a line starting from the parallelogram's center $\mathbf{h}$ with a direction vector $\mathbf{m}$.

$$\mathbf{h} = \begin{pmatrix} \frac{x_1+x_2}{2} \\ \frac{y_1+y_2}{2} \end{pmatrix}, \ \mathbf{m} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix} \tag{3}$$

First, we evaluate the term $\mathbf{Vh} + \mathbf{u}$:

$$\mathbf{Vh} + \mathbf{u} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{x_1+x_2}{2} \\ \frac{y_1+y_2}{2} \end{pmatrix} + \begin{pmatrix} -\frac{x_1+x_2}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{x_1+x_2}{2} \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{x_1+x_2}{2} \\ 0 \end{pmatrix} = \mathbf{0} \tag{4}$$

Since $\mathbf{Vh} + \mathbf{u} = \mathbf{0}$, the formula for κ simplifies dramatically:

$$\kappa_{1,2} = \frac{\pm \sqrt{-(\mathbf{m}^\top \left(V\right)\mathbf{m}) g(\mathbf{h})}}{\mathbf{m}^\top \left(V\right)\mathbf{m}} \tag{5}$$

Next, we evaluate the remaining terms in the general case:

$$\mathbf{m}^\top \left(V\right)\mathbf{m} = (x_2 - x_1)^2 \tag{6}$$

$$g(\mathbf{h}) = \left(\frac{x_1 + x_2}{2}\right)^2 - (x_1 + x_2)\left(\frac{x_1 + x_2}{2}\right) + x_1x_2 = -\frac{(x_2 - x_1)^2}{4} \tag{7}$$

Substituting these back into the simplified formula for κ:

$$\kappa_{1,2} = \frac{\pm \sqrt{-(x_2 - x_1)^2 \left(-\frac{(x_2-x_1)^2}{4}\right)}}{(x_2 - x_1)^2} = \frac{\pm \sqrt{\frac{(x_2-x_1)^4}{4}}}{(x_2 - x_1)^2} = \frac{\pm \frac{(x_2-x_1)^2}{2}}{(x_2 - x_1)^2} = \pm \frac{1}{2} \tag{8}$$

This shows that the vertices are located at $\kappa = \pm 1/2$ from the center along the direction vector $\mathbf{m}$.

Applying to the specific problem:

From $x^2 - 5x + 6 = 0$, we have $x_1 = 2, x_2 = 3$.
From $y^2 - 6y + 5 = 0$, we have $y_1 = 1, y_2 = 5$.
The center point $\mathbf{h}$ and diagonal direction vectors $\mathbf{m}_1, \mathbf{m}_2$ are:

$$\mathbf{h} = \begin{pmatrix} 2.5 \\ 3 \end{pmatrix}, \ \mathbf{m}_1 = \begin{pmatrix} 3-2 \\ 5-1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \ \mathbf{m}_2 = \begin{pmatrix} 2-3 \\ 5-1 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix} \tag{9}$$

The vertices are $\mathbf{v} = \mathbf{h} \pm \frac{1}{2}\mathbf{m}$.

Diagonal 1 (using $\mathbf{m}_1$):

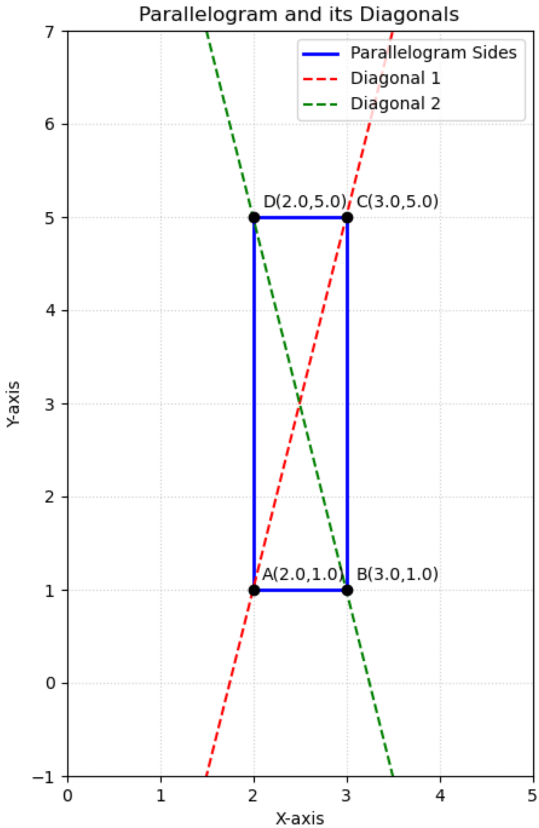
$$\mathbf{v}_C = \begin{pmatrix} 2.5 \\ 3 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \text{ and } \mathbf{v}_A = \begin{pmatrix} 2.5 \\ 3 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

The line passing through $\mathbf{A}(2,1)$ and $\mathbf{C}(3,5)$ is $y - 1 = 4(x-2) \implies \boldsymbol{y = 4x - 7}$.

Diagonal 2 (using $\mathbf{m}_2$):

$$\mathbf{v}_D = \begin{pmatrix} 2.5 \\ 3 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \text{ and } \mathbf{v}_B = \begin{pmatrix} 2.5 \\ 3 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

The line passing through $\mathbf{B}(3,1)$ and $\mathbf{D}(2,5)$ is $y - 1 = -4(x-3) \implies \boldsymbol{4x + y = 13}$.


Parallelogram and its Diagonals
Parallelogram Sides
Diagonal 1
Diagonal 2
D(2.0,5.0)
C(3.0,5.0)
A(2.0,1.0)
B(3.0,1.0)
Y-axis
X-axis
7
6
5
4
3
2
1
0
−1
0
1
2
3
4
5