2.8.10

Jnanesh Sathisha Karmar- EE25BTECH11029

Question If with reference to the right handed system of mutually perpendicular unit vectors $\hat{i}, \hat{j}$ and $\hat{k}, \alpha = 3\hat{i} - \hat{j}, \beta = 2\hat{i} + \hat{j} - 3\hat{k}$, then express β in the form $\beta = \beta_1 + \beta_2$ where β_1 is parallel to α and β_2 is perpendicular to α .

Solution Given details:

$$\alpha = 3\hat{i} - \hat{j} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} \tag{1}$$

$$\beta = 2\hat{i} + \hat{j} - 3\hat{k} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \tag{2}$$

β_1 is a projection of β on α
The projection formula for projection is:

$$\beta_1 = \frac{\beta^{\mathbf{T}}\alpha}{\left\|\alpha^{\mathbf{2}}\right\|}\alpha \tag{3}$$

$$= \frac{\begin{pmatrix} 2 & 1 & -3 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}}{(3)^2 + (-1)^2 + (0)^2}\begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} \tag{4}$$

$$= \frac{5}{10}\begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} \tag{5}$$

$$= \begin{pmatrix} \frac{3}{2} \\ \frac{-1}{2} \\ 0 \end{pmatrix} \tag{6}$$

Now according to the given equation :

$$\beta = \beta_1 + \beta_2 \tag{7}$$

$$\beta_2 = \beta - \beta_1 \tag{8}$$

$$\beta_2 = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} - \begin{pmatrix} \frac{3}{2} \\ \frac{-1}{2} \\ 0 \end{pmatrix} \tag{9}$$

$$\beta_2 = \begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \\ -3 \end{pmatrix} \tag{10}$$

Lets verify wheter β_2 is perpendicular to α
For that:

$$\alpha^T.\beta_2 = 0 \tag{11}$$

$$\begin{pmatrix} 3 & -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix} \tag{12}$$

Therefore β_2 is perpendicular to α
Therefore β is:

$$\beta = \beta_1 + \beta_2 \tag{13}$$

$$\beta = \begin{pmatrix} \frac{3}{2} \\ \frac{-1}{2} \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \\ -3 \end{pmatrix} \tag{14}$$

Performing QR decomposition on the matrix $\begin{pmatrix} \alpha & \beta \end{pmatrix}$

$$\mathbf{A} = \begin{pmatrix} \alpha & \beta \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ -1 & 1 \\ 0 & -3 \end{pmatrix} \tag{15}$$

Finding $\mathbf{q_1}$ (normalized α)

$$||\alpha|| = \sqrt{\left(\alpha\right)\left(\alpha\right)^T} = \sqrt{9+1} = \sqrt{10} \tag{16}$$

$$\mathbf{q_1} = \frac{\alpha}{||\alpha||} = \begin{pmatrix} \frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}} & 0 \end{pmatrix} \tag{17}$$

The projection of β on $\mathbf{q_1}$
projection coefficient:

$$\mathbf{r_{12}} = \mathbf{q_1^T}\beta = \frac{3}{\sqrt{10}}.2 + \frac{-1}{\sqrt{10}} + 0 = \frac{5}{\sqrt{10}} \tag{18}$$

projection:

$$proj(\beta)_{q_1} = \mathbf{r_{12}}.\mathbf{q_1} = \frac{5}{\sqrt{10}}.\begin{pmatrix} \frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}} & 0 \end{pmatrix} = \begin{pmatrix} 1.5 & -0.5 & -3 \end{pmatrix} \tag{19}$$

subtract projection from β:

$$\mathbf{u_1} = \beta - proj(\beta)_{q_1} = \begin{pmatrix} 0.5 & 1.5 & -3 \end{pmatrix} \tag{20}$$

we normalize $\mathbf{u_2}$ to get $\mathbf{q_2}$
compute $||\mathbf{u_2}||$

$$||u_2|| = \sqrt{\left(u_2\right)\left(u_2\right)^T} = \sqrt{11.5} \tag{21}$$

$$\mathbf{q_2} = \frac{\mathbf{u_2}}{||u_2||} = \begin{pmatrix} \frac{0.5}{\sqrt{11.5}} & \frac{1.5}{\sqrt{11.5}} & \frac{-3}{\sqrt{11.5}} \end{pmatrix} \tag{22}$$

$\mathbf{Q}$'s columns are $\mathbf{q_1}$ and $\mathbf{q_2}$

$$\mathbf{Q} = \begin{pmatrix} 3 & 0.5 \\ -1 & 1.5 \\ 0 & -3 \end{pmatrix} \tag{23}$$

$\mathbf{R}$ is:

$$\mathbf{R} = \begin{pmatrix} \sqrt{10} & \frac{5}{\sqrt{10}} \\ 0 & \sqrt{11.5} \end{pmatrix} \tag{24}$$

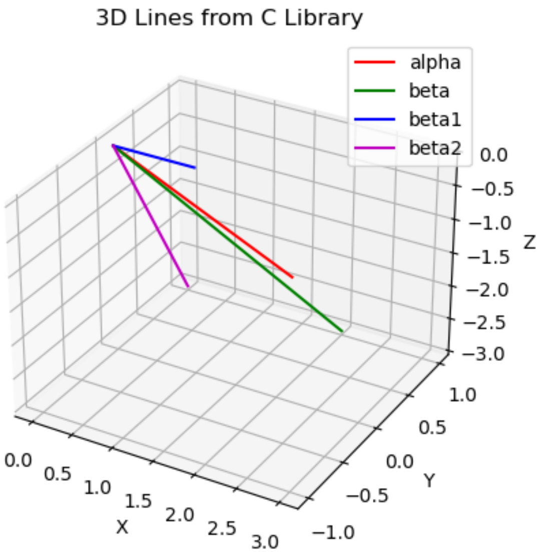

Fig. 0. vectors