Alternating Homology

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Motivation

$$f : X \longrightarrow Y$$

$$D^k(f) = \text{closure}\{(x_1, \ldots, x_k) \in X^k \mid f(x_1) = \ldots = f(x_k),\ x_i \neq x_j \text{ if } i \neq j\}$$

$$\pi_k : D^k(f) \longrightarrow D^{k-1}(f)$$

[1] Victor V. Goryunov. Semi-simplicial resolutions and homology of images and discriminants of mappings. *Proceedings of the London Mathematical Society*, s3-70(2):363–385, 1995.

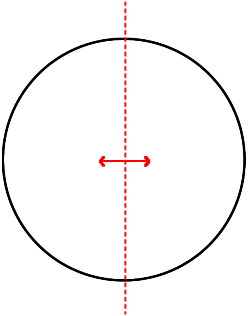
[2] Kevin Houston. Local topology of images of finite complex analytic maps. *Topology*, 5:1077–1121, 1997.

Image Computing Spectral Sequence

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\dots \arrow[r, "\partial"] & C_{n+1}^{alt}(D^{k+1}(f)) \arrow[r, "\partial"] \arrow[d, "\pi_{k+1\#}"] & C_{n}^{alt}(D^{k+1}(f)) \arrow[r, "\partial"] \arrow[d, "\pi_{k+1\#}"] & C_{n-1}^{alt}(D^{k+1}(f)) \arrow[r, "\partial"] \arrow[d, "\pi_{k+1\#}"] & \dots \\
\dots \arrow[r, "\partial"] & C_{n+1}^{alt}(D^{k}(f)) \arrow[r, "\partial"] \arrow[d, "\pi_{k\#}"] & C_{n}^{alt}(D^{k}(f)) \arrow[r, "\partial"] \arrow[d, "\pi_{k\#}"] & C_{n-1}^{alt}(D^{k}(f)) \arrow[r, "\partial"] \arrow[d, "\pi_{k\#}"] & \dots \\
\dots \arrow[r, "\partial"] & C_{n+1}^{alt}(D^{k-1}(f)) \arrow[r, "\partial"] \arrow[d] & C_{n}^{alt}(D^{k-1}(f)) \arrow[r, "\partial"] \arrow[d] & C_{n-1}^{alt}(D^{k-1}(f)) \arrow[r, "\partial"] \arrow[d] & \dots \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

S_k-space

A topological space X acted on by symmetric group S_k.

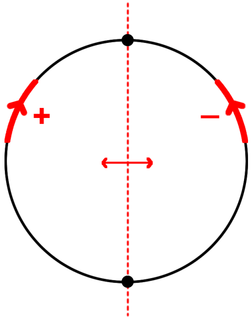


$k = 2$

Alternating Singular Homology

$$C_n^{alt}(X) = \{c \in C_n(X) \,|\, \sigma_{\#}(c) = \mathrm{sgn}(\sigma)c \quad \forall \sigma \in S_k\}$$

$$H_n^{alt}(X) = \frac{\operatorname{Ker} \partial}{\operatorname{Im} \partial}$$



Trivial space

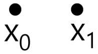
$$\forall \sigma \in S_k \quad \sigma x = x$$

$$k > 1 \implies C_n^{alt}(\{x\}) = 0 \quad \forall n$$

Connected components

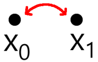
Homology

$$H_n(\{x_0, x_1\}) \cong H_n(\{x_0\}) \oplus H_n(\{x_1\})$$

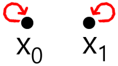


Alternating Homology

$$H_0^{alt}(\{x_0, x_1\}) \cong \mathbb{Z}$$



$$H_0^{alt}(\{x_0\}) \oplus H_0^{alt}(\{x_1\}) \cong 0 \oplus 0$$



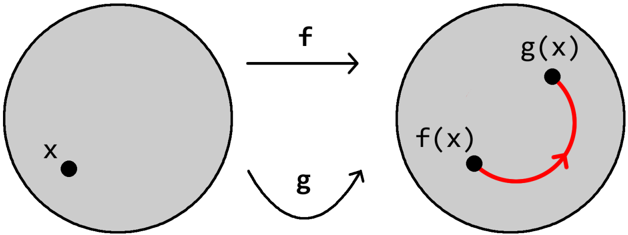
Reduced Homology

$$\ldots \longrightarrow C_2(X) \xrightarrow{\partial} C_1(X) \xrightarrow{\partial} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

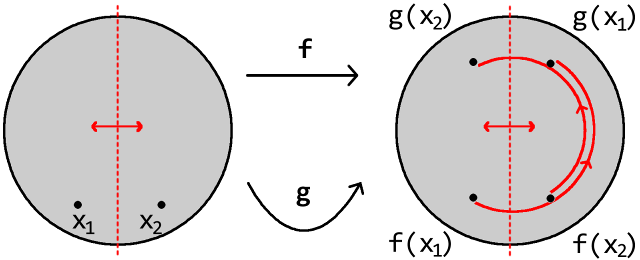
$$\varepsilon(\Sigma\alpha_i\sigma_i) = \Sigma\alpha_i$$

$$c = \Sigma\alpha_i c_i \in C_n^{alt}(X), \quad \sigma_{\#}(\Sigma\alpha_i c_i) = \Sigma(-\alpha_i)c_i$$

S_k-homotopy invariance



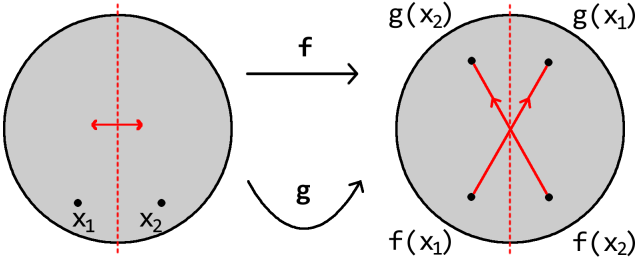
S_k-homotopy invariance



S_k-homotopy invariance

Definition

For $f, g \colon X \to Y$ an S_k-homotopy $F \colon X \times I \longrightarrow Y$ is a homotopy from f to g that is equivariant in the first argument.
$F(\sigma x, t) = \sigma F(x, t) \quad \forall t \in I, \forall \sigma \in S_k$.



Excision

Relative Alternating Chain Groups

For $A \subset X$ an S_k-subspace, $C_n^{alt}(X, A) := \frac{C_n^{alt}(X)}{C_n^{alt}(A)}$

Theorem

For S_k-space X and S_k-subspaces $Z \subset A \subset X$ where $\overline{Z} \subset \mathring{A}$, the inclusion $(X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ induces isomorphisms $H_n^{alt}(X \setminus Z, A \setminus Z) \cong H_n^{alt}(X, A)$.

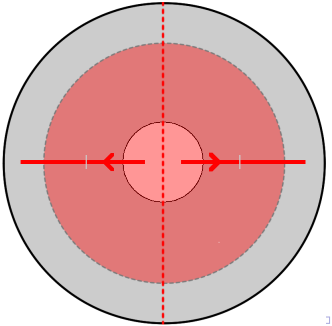
Theorem

For S_k-subspaces $A, B \subset X$, where $\mathring{A} \cup \mathring{B} = X$, the inclusion $(B, B \cap A) \hookrightarrow (X, A)$ induces isomorphisms $H_n^{alt}(B, B \cap A) \cong H_n^{alt}(X, A)$.

Excision

$$\mathfrak{i} : C_n^{alt}(A+B) \hookrightarrow C_n^{alt}(X) \qquad \rho : C_n^{alt}(X) \to C_n^{alt}(A+B)$$

$$\partial D + D\partial = \mathbb{1} - \mathfrak{i}\rho \text{ and } \rho\mathfrak{i} = \mathbb{1}$$



Excision

$$H_n^{alt}\left(\frac{C_n^{alt}(X)}{C_n^{alt}(A)}\right) \overset{i}{\cong} H_n^{alt}\left(\frac{C_n^{alt}(A+B)}{C_n^{alt}(A)}\right) \cong H_n^{alt}\left(\frac{C_n^{alt}(B)}{C_n^{alt}(A\cap B)}\right)$$

$$H_n^{alt}(X,A) \cong H_n^{alt}(B, B\cap A)$$

Good S_k-pair

Definition

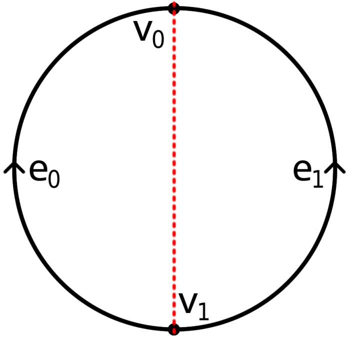
For $A \subset X$ a non-empty closed S_k-subspace, (X, A) is a Good S_k-pair if A is an equivariant deformation retract of some neighbourhood V, another S_k-subspace of X.

$$H_n^{alt}(X, A) \cong H_n^{alt}(X/A)$$

Simplicial Alternating Homology

If the action of S_k-space X induces an action of S_k on simplicial chain groups $\triangle_n(X)$ for some Δ-complex,

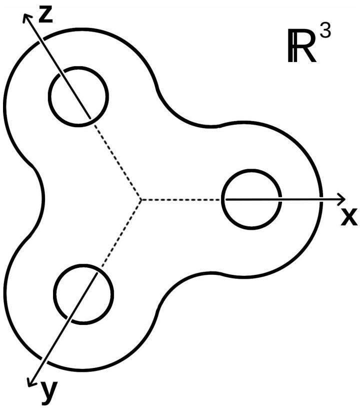
$$\triangle_n^{alt}(X) := \{c \in \triangle_n(X) \mid \sigma_\#(c) = \mathrm{sgn}(\sigma)c \quad \forall \sigma \in S_k\}$$

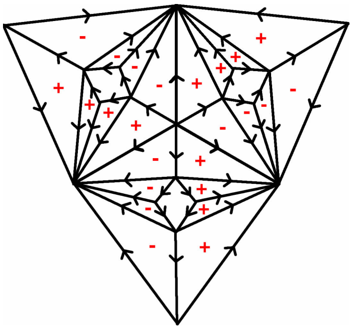


$$H_{n,\triangle}^{alt}(X) = \begin{cases} \mathbb{Z} & n = 1 \\ 0 & n \neq 1 \end{cases}$$

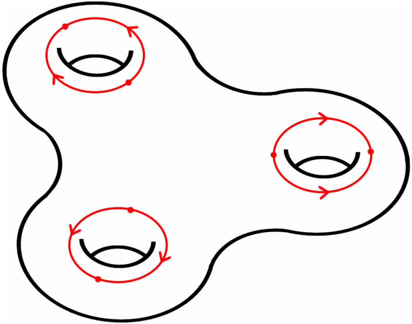
$$H_{n,\triangle}^{alt}(X) \cong H_n^{alt}(X)$$

Computation





Computation



Links to other theory

- Alternating Homology with Rational Coefficients
$$\mathrm{Alt}H_n(X;\mathbb{Q}) \cong H_n^{alt}(X;\mathbb{Q}) \cong H_n^{alt}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$$

- Alternating Homology as Homology with Local Coefficients