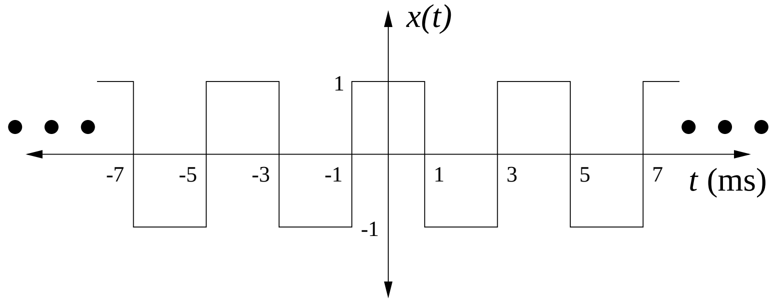
EE 1473 - Digital Communication Systems
Homework Set 2

Due: Thursday, January 24, 2019

1. Consider the waveform $x(t)$ shown below. Note that the t axis is in units of milliseconds.



This bipolar periodic square wave is frequently used as a simple model for a digital data signal. In this case, a 1 is represented by +1 Volts, and a 0 is represented by -1 Volts. Therefore, this signal corresponds to a data pattern of alternating ones and zeros.

(a) Determine the complex Fourier series coefficients c_n for this waveform, as defined by equations (2-88) and (2-89). Note that, because the waveform $x(t)$ shown above is real and even, the coefficients c_n that you obtain should be real-valued.

(b) Use MATLAB to plot the Fourier series coefficients for $x(t)$ that you determined in part (a). Use the `stem` command in MATLAB to prepare the plot. Type `help stem` in MATLAB to learn more about this command. Be sure to include both positive and negative values of n in your plot.

(c) Determine $X(f)$, the Fourier transform of $x(t)$, using the results of part (a). Sketch $X(f)$ versus f by hand.

2. In this problem, we will use MATLAB and the `fft` function to compute the spectrum of the square wave signal $x(t)$ from Problem 1.

(a) Compute samples of the signal $x(t)$ for $0 \leq t < 8$ milliseconds, and with 0.2 milliseconds between consecutive samples. This corresponds to four "bits" and ten samples per bit. Make sure that your signal is symmetric; it should consist of five +1's, followed by ten -1's, followed by ten +1's followed by ten -1's, and finally five +1's. One way to test for symmetry is to execute the following line of code (assuming that your signal samples are stored in a vector `x`),

```
max(abs(x - x(end:-1:1)))
```

If the result is not zero, then your signal is not symmetric.

Plot the resulting signal $x(t)$ versus t using the `plot` command, so that your plot resembles a continuous-time signal.

(b) Use the `fft` function in MATLAB to compute the spectrum of the signal from part (a). Remember to multiply the result by the time between samples so that the spectrum is calibrated correctly, and use the `fftshift` function to place negative frequencies to the left of $f = 0$. Plot the real and imaginary parts of the spectrum versus f.

Note: Even though the signal you created in part (a) is symmetric, and in Problem 1 this resulted in a spectrum that was purely real, you should find that the imaginary part of the spectrum is *not zero* for this case. That is because the symmetry properties of the continuous-time Fourier series and Fourier transform are different from those of the discrete Fourier transform and the FFT.

(c) The following code will force the signal stored in the vector `x` to have the appropriate symmetry properties so that its DFT is purely real.

```
N = length(x);
n = 0:N-1;
x = 0.5*(x + x(mod(-n,N)+1));
```

Add this to your code, then plot the spectrum again. Confirm that the imaginary part is negligibly small.

(d) Repeat parts (a) and (c), for $0 \le t < 128$ ms, and $\Delta t = 0.2$ ms between samples (i.e. 64 bits and 10 samples per bit). What effect does increasing the signal duration have on the spectrum as computed by the FFT?

(e) Repeat parts (a) and (c), for $0 \le t < 8$ ms, and $\Delta t = 0.02$ ms between samples (i.e. 4 bits and 100 samples per bit). What effect does increasing the sampling rate have on the spectrum as computed by the FFT?

3. Let $H(f)$ be a bandlimited lowpass spectrum that decreases smoothly with frequency according to

$$H(f) = \begin{cases} \dfrac{1}{2f_0} & |f| < (1-r)f_0 \\[2ex] \dfrac{1}{4f_0}\left\{1 + \cos\left[\dfrac{\pi\left(|f| - (1-r)f_0\right)}{2rf_0}\right]\right\} & (1-r)f_0 \le |f| \le (1+r)f_0 \\[2ex] 0 & |f| > (1+r)f_0. \end{cases}$$

This spectrum is known as a *raised cosine-rolloff spectrum*, because it transitions from the passband, where the response is constant at $1/2f_0$, to zero according to a "raised" cosine, i.e. a cosine plus a constant. The frequency f_0 is the 6-dB bandwidth of the response, and $0 \le r \le 1$ is a parameter that controls the width of the transition band.

In this problem, you will show that the corresponding time-domain signal is

$$h(t) = \left[\frac{\sin(2\pi f_0 t)}{2\pi f_0 t}\right]\left[\frac{\cos(2\pi r f_0 t)}{1 - (4rf_0t)^2}\right].$$

This is the *raised cosine-rolloff pulse*. The spectra $H_e(f)$ and signals $h_e(t)$ depicted in Figures 3-25 and 3-26 are proportional to $H(f)$ and $h(t)$, respectively, according to

$$H_e(f) = 2f_0 H(f) \qquad \text{and} \qquad h_e(t) = 2f_0 h(t).$$

In Chapter 3, we will learn that this pulse is very useful for baseband digital communications, but for in this problem you will prove that $h(t)$ and $H(f)$ form a Fourier transform pair. Doing so will allow you to apply several important properties of the Fourier transform, and other signal analysis concepts such as convolution, as detailed in the steps below. Many of the steps can be completed independently, so that if you are stuck on one, you can move on to the others and come back to it later.

(a) Begin by considering the special case where $r = 1$ and $2f_0 = 1$. Show that for this case,

$$h(t) = \mathscr{F}^{-1}\left\{H(f)\right\},$$

by following these steps.

(i) Show that

$$H(f) = \cos^2\left[\frac{\pi f}{2}\right] \cdot \Pi\left(\frac{f}{2}\right),$$

where $\Pi(\cdot)$ is the rectangular function defined in Couch eq. (2-52), and depicted in Figures 2-5 and 2-6.

(ii) Use the duality property of the Fourier transform to show that

$$\mathscr{F}^{-1}\left\{\cos\left[\frac{\pi f}{2}\right]\right\} = \frac{1}{2}\,\delta\left(t - \frac{1}{4}\right) + \frac{1}{2}\,\delta\left(t + \frac{1}{4}\right)$$

(iii) Use the convolution property of the Fourier transform to show that

$$\mathscr{F}^{-1}\left\{H(f)\right\} = \frac{1}{2}\cdot\frac{\sin(2\pi t - \pi)}{2\pi t - \pi} + \frac{\sin(2\pi t)}{2\pi t} + \frac{1}{2}\cdot\frac{\sin(2\pi t + \pi)}{2\pi t + \pi}.$$

(iv) Show that the expression from part (a)(iii) is equivalent to $h(t)$, for the case under consideration.

(b) Next, we consider the case where $2f_0 = 1$, but r can take any value in $0 \le r \le 1$.

(i) Show that

$$H(f) = \begin{cases} 1 & |f| < \dfrac{1-r}{2} \\[2ex] \dfrac{1}{2}\left\{1 + \cos\left[\pi\left(\dfrac{2|f| - 1 + r}{2r}\right)\right]\right\} & \dfrac{1-r}{2} \le |f| \le \dfrac{1+r}{2} \\[2ex] 0 & |f| > \dfrac{1+r}{2}. \end{cases}$$

(ii) Show that $H(f)$ can be expressed as the convolution of a rectangular spectrum with one half-cycle of a cosine function,

$$H(f) = \Pi(f) \,*\, \frac{\pi}{2r} \cos\left(\frac{\pi f}{r}\right) \Pi\left(\frac{f}{r}\right).$$

(iii) Show that

$$\mathscr{F}^{-1}\left\{\cos\left(\frac{\pi f}{r}\right) \Pi\left(\frac{f}{r}\right)\right\} = \frac{2r}{\pi}\left[\frac{\cos(\pi r t)}{1-(2rt)^2}\right].$$

(iv) Use the multiplication property of the Fourier transform to show that

$$\mathscr{F}^{-1}\left\{H(f)\right\} = \left[\frac{\sin(\pi t)}{\pi t}\right]\left[\frac{\cos(\pi r t)}{1-(2rt)^2}\right].$$

(v) Show that the expression from part (b)(iv) is equivalent to the expression for $h(t)$ given in the problem statement, for the case when $2f_0 = 1$.

(c) Finally, we consider the general case, where r can take any value in $0 \le r \le 1$, and f_0 can take any positive value. Use the pulse $h(t)$ and spectrum $H(f)$ from part (b), and apply the time scaling property of the Fourier transform, to show that the original $h(t)$ and $H(f)$ for the general case are also a Fourier transform pair. It is **not** necessary to go through the steps of part (b) again.

4. The frequency response of a second-order lowpass filter, with unit gain in the passband, is given by

$$H(f) = \frac{f_0^2}{f_0^2 - f^2 + j(f_0/Q)f},$$

where f_0 is the cutoff frequency and Q is the quality factor.

(a) Show that the 3-dB bandwidth is equal to f_0 if and only if $Q = 1/\sqrt{2}$, and that when this condition is met,

$$|H(f)|^2 = \frac{1}{1+(f/f_0)^4}.$$

For the remainder of this problem, assume that $Q = 1/\sqrt{2}$, so that the results of part (a) apply.

(b) Determine the 40-dB bandwidth of the filter.

(c) Determine the null bandwidth.

(d) Determine the absolute bandwidth.

(e) Determine the equivalent bandwidth.

Hint:

$$\int \frac{dx}{1+x^4} = \frac{1}{4\sqrt{2}}\left[\ln\left(\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right) + 2\tan^{-1}\left(1+2\sqrt{x}\right) - 2\tan^{-1}\left(1-2\sqrt{x}\right)\right].$$

(f) Determine the 99% power bandwidth. In doing so, you will obtain an equation that sets the value of an integral equal to a desired value, and you must find the value of the upper limit of integration so that the equation is satisfied. This equation has no closed-form solution.

Therefore, solve the problem numerically in MATLAB. The key is evaluating the integral numerically. One way to do this is to approximate the integral by a Riemann sum. Suppose that $g(x)$ is a function we wish to integrate,

$$\int_{x_1}^{x_2} g(x)\,dx.$$

Divide the interval $[x_1, x_2]$ into N bins of width

$$\Delta x = \frac{x_2 - x_1}{N}.$$

Then the integral of interest may be approximated as,

$$\int_{x_1}^{x_2} g(x)\,dx \approx \sum_{n=0}^{N-1} g(x_1 + n \cdot \Delta x) \cdot \Delta x.$$

Note that this is simply the sum of the areas of rectangles that lie under $g(x)$. As N is made larger and Δx shrinks, the approximation improves.