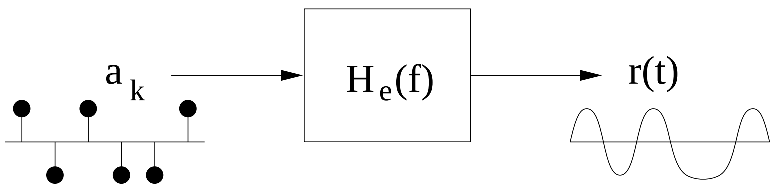
ECE 1473 - Digital Communication Systems
Homework Set 5

Due: Thursday, February 14, 2019

1. The purpose of this problem is to demonstrate the effects of intersymbol interference. Submit your MATLAB files for this problem to courseweb, using the Assignment designated for this.

   The figure below shows a baseband digital communications system, where $H_e(f)$ incorporates the combined effects of baseband signaling, transmit filter, channel, and receive filter.



   (a) Write a MATLAB function that will return the Raised Cosine Rolloff pulse, normalized to peak amplitude 1. The equation for this pulse is given by Couch Equation (3-73), except that you should replace the leading factor of $2f_0$ with 1, so that the pulse will be properly normalized. Plots of the RCRO pulse are shown in Figure 3-26(b) for $-6T_b \leq t \leq 6T_b$, where $T_b = 1/(2f_0)$ is the bit period.

   Your function must return a vector containing samples of the RCRO pulse for $-k_T T_b \leq t \leq k_T T_b$, so that $2k_T$ represents the number of bit periods over which you will compute the pulse. You will have to be careful to avoid or correct divide-by-zero errors in both terms in equation (3-73). The inputs to your function should be k_T, T_b, the number of samples per bit, and the rolloff factor r. The parameters f_0 and f_Δ should be computed within your function.

   Call your function with $k_T = 5$, $T_b = 0.2$ seconds, at least 16 samples per bit, and rolloff factors $r = 0, 0.1, 0.2, \ldots 1$. Plot the resulting pulses versus time on a single pair of axes, and compare with Figure 3-26 (b).

   (b) Write a MATLAB script that will generate random data with at least 20 bits and data values $a_n = \pm 1$. Then call the RCRO pulse function from part (a) with $T_b = 1$ second and at least 16 samples per bit. You may choose any rolloff factor other than 0, i.e., $0 < r \leq 1$. State clearly the value you have chosen for r. Form a baseband signal from the data and pulse according to the equation

$$s(t) = \sum_n a_n h_e(t - nT_b).$$

   Plot the resulting signal versus time, and show that the signal has the correct value at all bit times $t = nT_b$.

(c) Write a MATLAB function that will produce the Gaussian pulse

$$h_e(t) = e^{-t^2/T_b^2}.$$

for $-k_T T_b \leq t \leq k_T T_b$, where T_b is the bit period and $2k_T$ represents the number of bit periods over which you will compute the pulse. The inputs to your function should be k_T, T_b, and the number of samples per bit.

Call your function with $k_T = 5$, $T_b = 0.2$ seconds, and at least 16 samples per bit. Plot the resulting pulse versus time.

(d) Write a MATLAB script that will generate random data with at least 20 bits and data values $a_n = \pm 1$. Then call the Gaussian pulse function from part (c) with $T_b = 1$ second and at least 16 samples per bit. Form a baseband signal from the data and pulse according to the equation

$$s(t) = \sum_n a_n h_e(t - nT_b).$$

Plot the resulting signal versus time, and determine if the signal has the correct value at all bit times $t = nT_b$. Also determine whether intersymbol interference occurs.

For the purposes of answering this question, we will say that ISI has occurred if the baseband signal has a value at any of the bit times that is closer to zero than the correct value. In other words, if $|s(t)| > 1$ at a given bit time, then this will not be considered ISI, since a receiver that samples the waveform at the bit times and compares the sample value to a threshold will perform correctly under such conditions. On the other hand, if $|s(t)| < 0.5$ at any of the bit times, then we will say that ISI has occurred, since the sample value obtained from this signal will be closer to zero than it should be, and this will increase the likelihood of bit errors resulting from noise.

Since the answer to this question will depend on the particular data sequence, run your code several times to see the conditions when ISI does or does not occur. Under what conditions on the data pattern will ISI occur?

2. Recall from Homework Assignment 3 Problem 3 that, if we define a random variable that is the weighted sum of a set of random variables,

$$r = \sum_{i=1}^{N} h_i r_i,$$

then the mean of r is the same weighted sum applied to the individual means

$$E\{r\} = \sum_{i=1}^{N} h_i E\{r_i\}.$$

You should make use of this result in parts (b), (c) and (d) of this problem.

Let w and x be two independent random variables, with means and variances

$$\begin{array}{ll} \mu_w = E\{w\} & \sigma_w^2 = \mathrm{Var}\{w\} \\ \mu_x = E\{x\} & \sigma_x^2 = \mathrm{Var}\{x\}, \end{array}$$

respectively. Determine the mean of each of the following random variables.

(a) The random variables $y = x^2$ and $z = w^2$.

(b) The random variables $s = w + x$ and $d = w - x$.

(c) The random variables $a = s^2$, $b = sd$ and $c = d^2$.

(d) The random variable $r = 2a + 3b + 2c - y + z$.

3. The property that we applied in Problem 2 (the mean of a sum is the sum of the means) can be extended to the case of a random variable that is produced by integrating a random process. Suppose that $r(t)$ is a random process with mean function

$$\mu_r(t) = E\{r(t)\}.$$

If we compute the integral of this random process over some time interval, the result will be a random variable,

$$r = \int_{t_1}^{t_2} r(t)\,dt.$$

If we want to compute the mean of r, we can simply integrate the mean function,

$$E\{r\} = E\left\{\int_{t_1}^{t_2} r(t)\,dt\right\} = \int_{t_1}^{t_2} E\left\{r(t)\right\}\,dt = \int_{t_1}^{t_2} \mu_r(t)\,dt.$$

You should use this property in part (c) of this problem.

There are additional properties we should be aware of when dealing with both random and deterministic quantities. Suppose that c is a deterministic constant, and $c(t)$ is a deterministic signal, both of which are by definition not random. In such cases, if we are forced to evaluate the mean of a deterministic quantity, it is simply the quantity itself,

$$E\{c\} = c \qquad \text{and} \qquad E\{c(t)\} = c(t).$$

Next, suppose that r is a random variable and $r(t)$ is a random signal, and we wish to compute the expected value of the product of a deterministic quantity and a random quantity.

$$\begin{aligned} E\{cr\} &= \int_{-\infty}^{\infty} cr\, f_{\mathrm{r}}(r)\,dr \\ &= c\int_{-\infty}^{\infty} r\, f_{\mathrm{r}}(r)\,dr \\ &= cE\{r\} \end{aligned}$$

$$\begin{aligned} E\{c(t)r(t)\} &= \int_{-\infty}^{\infty} c(t)r(t)\, f_{\mathrm{r(t)}}(r)\,dr \\ &= c(t)\int_{-\infty}^{\infty} r(t)\, f_{\mathrm{r(t)}}(r)\,dr \\ &= c(t)E\{r(t)\}. \end{aligned}$$

Note that, in both cases, the deterministic quantity can be "pulled out" of the expectation operator. You should use these properties in parts (a) and (b) of this problem.

Let the random process $x(t)$ is defined as

$$x(t) = s(t) + w(t),$$

where $s(t)$ is a rectangular pulse,

$$s(t) = \begin{cases} A & 0 \le t \le T \\ 0 & \text{elsewhere,} \end{cases}$$

and $w(t)$ is a WGN process with mean $E\{w(t)\} = 0$ and autocorrelation function

$$R_w(t,\, t+\tau) = E\{w(t)w(t+\tau)\} = \sigma^2\delta(\tau).$$

(a) Determine the mean of $x(t)$.

(b) Determine the autocorrelation function for $x(t)$,

$$R_x(t,\, t+\tau) = E\{x(t)x(t+\tau)\}.$$

What can we conclude from the result? Is $x(t)$ wide-sense stationary?

(c) Suppose that $x(t)$ is processed by integrating it over $0 \le t \le T$ to produce the random variable x_T,

$$x_T = \int_0^T x(t)\, dt.$$

Compute

$$d^2 = \frac{[E\{x_T\}]^2}{\mathrm{Var}\{x_T\}}.$$

Is it possible to choose T as small as possible, while simultaneously maximizing d^2?

Backgound: Why would we want to compute x_T? Suppose that we wish to communicate data, by transmitting the pulse $s(t)$ to represent a binary 1 and the pulse $-s(t)$ to represent a binary 0. The received signal is $x(t)$, i.e. the transmitted pulse plus WGN, and we want to decide which binary value was sent. One strategy would be to integrate the received signal to compute x_T. Note that the transmitted pulse has the same polarity throughout $0 \le t \le T$, while the noise will take both positive and negative values. Therefore, integration will tend to suppress the effect of the noise, as the positive and negative values cancel each other, while preserving the energy in the transmitted pulse.

Why would we want to compute d^2? It turns out that the performance of this system, i.e. the probability of correctly determining the transmitted binary value, is a monotonically increasing function of d^2. Therefore, we would like to maximize d^2. However, we would also like to communicate data as fast as possible, which requires that T be as small as possible.

(d) Now suppose that we form a new random variable, by integrating $x(t)$ over $t_1 \leq t \leq t_2$,

$$x_{12} = \int_{t_1}^{t_2} x(t)\,dt,$$

and we wish to maximize

$$d_{12}^2 = \frac{[E\{x_{12}\}]^2}{\mathrm{Var}\{x_{12}\}}.$$

Show that d_{12}^2 will be maximized by choosing $t_1 = 0$ and $t_2 = T$. What is the significance of this result?

4. Let the random process $y(t)$ be a linear function of t, observed in AWGN,

$$y(t) = mt + w(t).$$

where m is the slope of the linear term. The noise $w(t)$ is a Gaussian random process with mean

$$\mu_w(t) = E\{w(t)\} = 0,$$

and autocorrelation function,

$$R_w(t,u) = E\{w(t)w(u)\} = \sigma^2\,\delta(t-u).$$

(a) Determine the mean of $y(t)$,

$$\mu_y(t) = E\{y(t)\}.$$

(b) Determine the autocorrelation function for $y(t)$,

$$R_y(t,u) = E\{y(t)y(u)\}.$$

Is $y(t)$ wide-sense stationary? Why or why not?

(c) Now suppose that we wish to estimate the value of m, by observing $y(t)$ for $0 \leq t \leq T$. Let $\hat{m}$ be the estimate, which is computed as follows,

$$\hat{m} = \frac{2}{T^2}\int_0^T y(t)\,dt.$$

Show that the mean of the estimate is

$$E\{\hat{m}\} = m.$$

Is this a good property for the estimate to have? Why or why not?

(d) Determine a constraint on T such that

$$\mathrm{Var}\{\hat{m}\} < k\sigma^2.$$