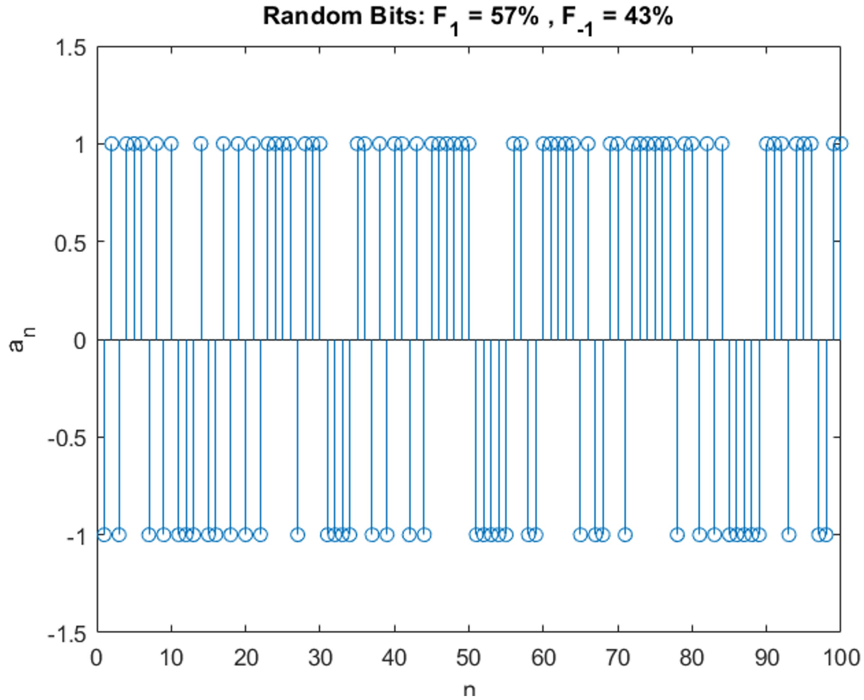
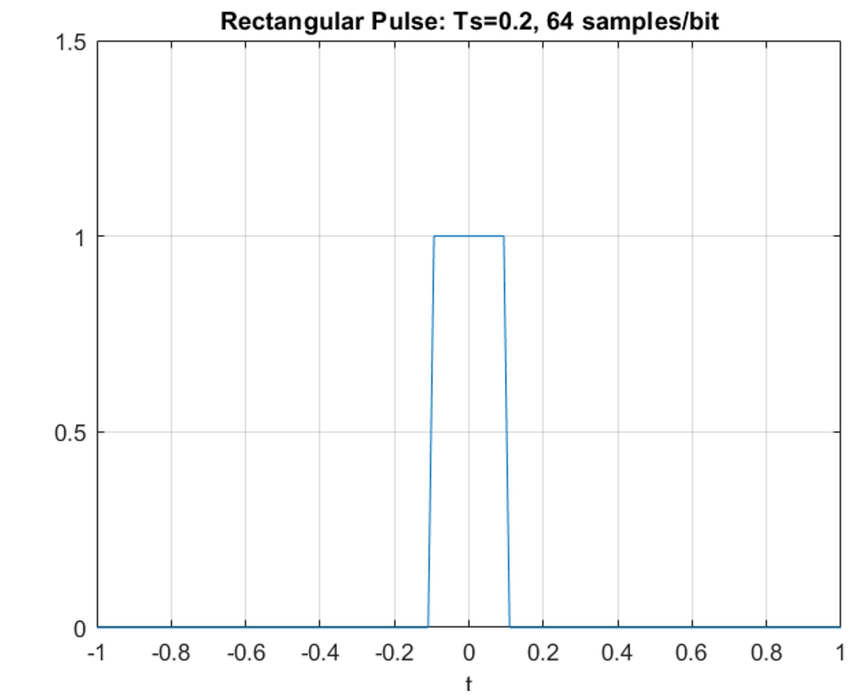
Homework Set 3 EE 1473

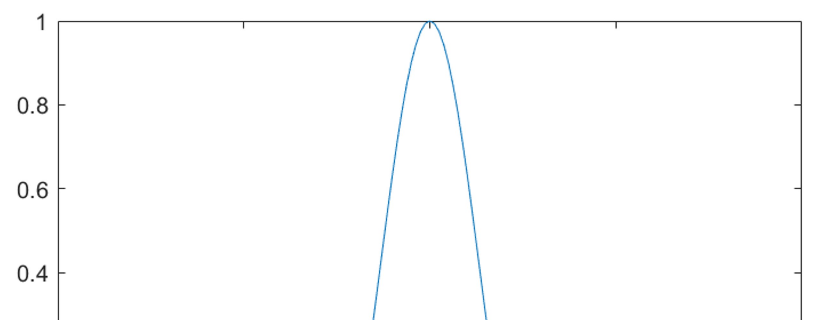
Josh Eaton

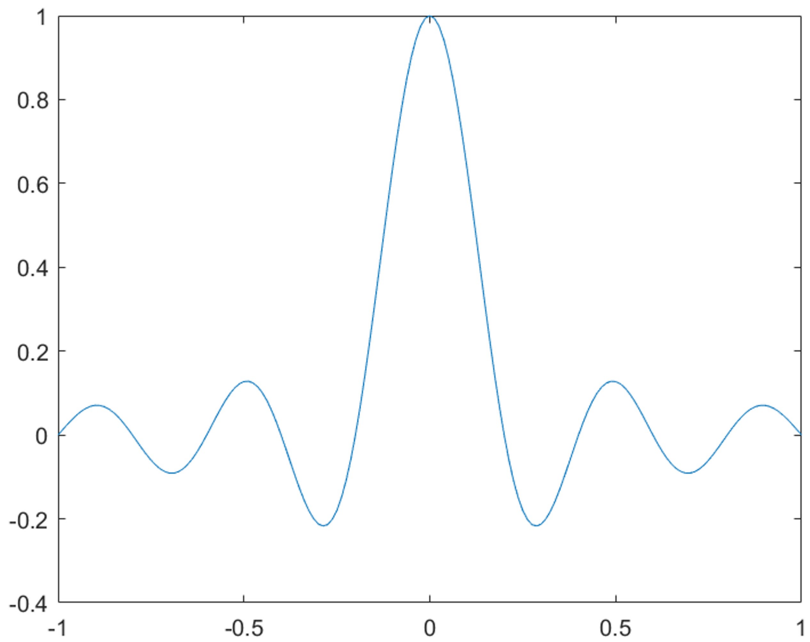
Sunday, January 27, 2019 3:08 PM

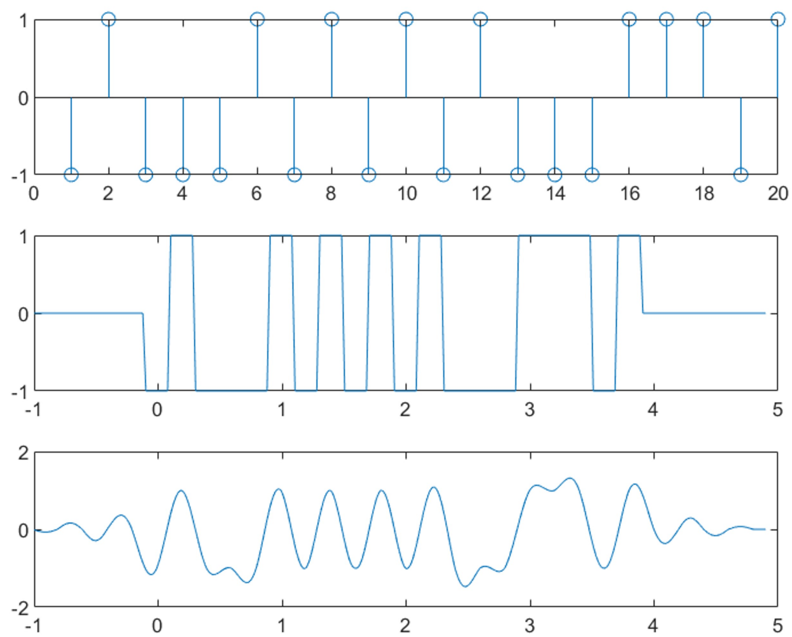
1) MATLAB











2) (a) Show that $f_x(x)$ is a valid pdf, i.e. that it is nonnegative and integrates to 1.

$$\lambda^2 x e^{-\lambda x} u(x)$$

$$\boxed{u(x) = 0 \text{ for all } x<0 \Leftrightarrow f_x(x) \geq 0 \text{ for all } x}$$

$$\lambda^2 \int_0^\infty x e^{-\lambda x} dx = -\frac{e^{-\lambda x}(\lambda x + 1)}{\lambda^2}$$

$$\lambda^2\left[-\frac{e^{-\lambda x}(\lambda x+1)}{\lambda_2}\Bigg|_0^\infty\right] = \lambda^2\left\{-e^{-\infty}+\frac{e^0}{\lambda^2}\right\}$$

$$\Rightarrow \boxed{\frac{\lambda^2}{\lambda^2}=0}$$

(b) Determine the mean.

$$\mu = E\{x\} = \int_\infty x f_x(x)\,dx$$

$$\Rightarrow \lambda^2\int_{-\infty}^{\infty} x^2 e^{-\lambda x}\,dx = -\frac{e^{-\lambda x}(\lambda^2x^2+2\lambda x+2)}{\lambda^3}\Bigg|_0^\infty$$

$$\Rightarrow \boxed{\frac{2}{\lambda}}$$

(c) Determine the second moment.

$$E\{x^2\} = \int_\infty x^2 f_x(x)\,dx$$

$$\Rightarrow \lambda^2\int_0^{\infty} x^3 e^{-\lambda x}\,dx = -\frac{e^{-\lambda x}(\lambda^3x^3+3\lambda^2x^2+6\lambda x+6)}{\lambda^4}\Bigg|_0^\infty$$

$$\Rightarrow \boxed{\frac{6}{\lambda^2}}$$

(d) Determine the variance.

$$E\{(x-\mu)^2\} = \sigma^2 = E\{x^2\}-\mu^2$$

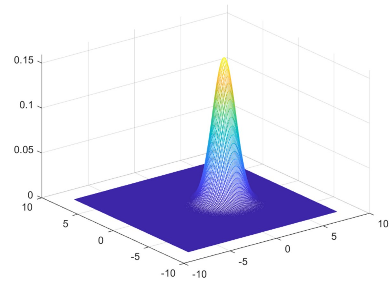
$$= \frac{6}{\lambda^2}-\frac{4}{\lambda^2} = \boxed{\frac{2}{\lambda^2}}$$

3) (a) Let x ~ N(2,1) and y ~ N(0,1) be ***independent*** random variables, and let (x,y) ∈ R^2 represent a point in the plane. What is the joint pdf for x and y? Use the mesh function in MATLAB to plot $f_{xy}(x,y)$

Independent $\rightarrow f_{xy} = (f_x)(f_y)$

$$f_{xy} = \frac{1}{2\pi} e^{-(x-2)^2/2} e^{-(y)^2/2}$$

$$\boxed{\frac{1}{2\pi} e^{-\frac{1}{2}((x-2)^2+y^2)}}$$



(b) Use the result of Example 1 to determine the probability that the point (x,y) lies in the first quadrant. Repeat for the other quadrants.

$$P(x_1 < x \le x_2) = Q\left(\frac{\mu - x_2}{\sigma}\right) - Q\left(\frac{\mu - x_1}{\sigma}\right)$$

1st: $0 < x \le \infty,\ 0 < y \le \infty \Rightarrow [Q(-\infty) - Q(2)][Q(-\infty) - Q(0)]$

2nd: $-\infty < x \le 0 \Rightarrow [Q(2) - Q(\infty)][Q(-\infty) - Q(0)]$

3rd: $-\infty < y \le 0 \Rightarrow [Q(2) - Q(\infty)][Q(0) - Q(\infty)]$

4th: $\longrightarrow [Q(-\infty) - Q(2)][Q(0) - Q(\infty)]$

$Q(0) = \frac{1}{2}$

$Q(2) = 0.0228$

$Q(-\infty) = 1$

$Q(\infty) = 0$

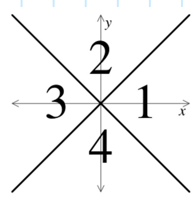
1: 0.4886

2: 0.0114

3: 0.0114

4: 0.4886

(c) The figure below shows the plane divided into four regions that one might describe as rotated quadrants.



Suppose we wish to compute the probability that (x,y) lies in Region 1. Express this probability as an integral, and explain why it is difficult to evaluate this integral without performing numerical integration.

$$P(0 < x \le \infty,\ -x < y \le x)$$

$$\Rightarrow \frac{1}{2\pi} \int_0^\infty \int_{-x}^{x} f_{xy}(x,y)\, dy\, dx$$

This is difficult to evaluate without integrating because the bounds are dependent on one of the RV's

(d) Define two new random variables,

s = y + x and d = y − x.

Because they are linear combinations of Gaussian random variables, s and d are also Gaussian. Determine the means and variances of s and d.

$$\mu_s = \mu_y + \mu_x = 2 \qquad \sigma_s^2 = \sigma_x^2 + \sigma_y^2 = 2$$

$$\mu_d = \mu_y - \mu_x = -2 \qquad \sigma_d^2 = \sigma_y^2 + \sigma_x^2 = 2$$

$$\Sigma a_i \mu_i \qquad \Sigma a_i^2 \sigma_i^2$$

(e) Use the results of Examples 3 and 4 to show that s and d are statistically independent.

let $w = s + d = y + x + y - x = y$

$\mu_w = \mu_s + \mu_d = 2 - 2 = \boxed{0}$ which we already know is μ_y

(f) Suppose that (x,y) lies in Region 1, as shown in part (c). What values must s and d take when this is true? Repeat for regions 2, 3, and 4.

1: $0 < s < \infty \quad -\infty < d < 0$ 3: $-\infty < s < 0, \quad 0 < d < \infty$

2: $0 < s < \infty \quad 0 < d < \infty$ 4: $-\infty < s < 0, \quad -\infty < d < 0$

(g) Use the results of part (f) to compute the probability that (x,y) lies in Region 1. Repeat for Regions 2, 3, and 4.

$$Q\left(\frac{\mu - x_2}{\sigma}\right) - Q\left(\frac{\mu - x_1}{\sigma}\right)$$

1: $\left[Q(-\infty) - Q(2/\sqrt{2})\right]\left[Q(-2/\sqrt{2}) - Q(\infty)\right] = 0.8489$

2: $\left[Q(-\infty) - Q(2/\sqrt{2})\right]\left[Q(-\infty) - Q(-2/\sqrt{2})\right] = 0.0725$

3: $\left[Q(2/\sqrt{2}) - Q(\infty)\right]\left[Q(-\infty) - Q(-2/\sqrt{2})\right] = 0.0062$

4: $\left[Q(2/\sqrt{2}) - Q(\infty)\right]\left[Q(-2/\sqrt{2}) - Q(\infty)\right] = 0.0725$

(h) We will use the joint pdf for x and y to model a communication system that transmits two-bit symbols. Assume that the transmitted bit-pair is 00, and this is encoded at the transmitter as (x,y) = (2,0), but noise corrupts the signal during transmission such that other bit pairs may be decoded at the receiver. In this model, the random variables x and y represent samples of noisy output signals from the receiver, and the decoding rule for choosing which bits were transmitted is as follows:

Choose 00 if (x,y) ∈ Region 1
Choose 10 if (x,y) ∈ Region 2
Choose 11 if (x,y) ∈ Region 3
Choose 01 if (x,y) ∈ Region 4.

Determine the following performance indices for this system.

(i) The probability of a symbol error, i.e. that something other than 00 will be chosen.

(i) The probability of a symbol error, i.e. that something other than 00 will be chosen.

$$\Rightarrow P_r(2) + P_r(3) + P_r(4) = \boxed{0.1512}$$

(ii) The probability that one bit error will be made.

$$\Rightarrow P_r(2) + P_r(4) = \boxed{0.145}$$

(iii) The probability that two bit errors will be made.

$$\Rightarrow P_r(3) = \boxed{0.0062}$$

(iv) The bit error rate: assuming that a very large number of bits are transmitted using this system, what fraction are expected to be decoded incorrectly?

$$BER = \frac{SER}{\log_2 4} = \left(\frac{1}{2}\right)(0.1512) = \boxed{0.0756}$$