Homework Set 2 EE 1473

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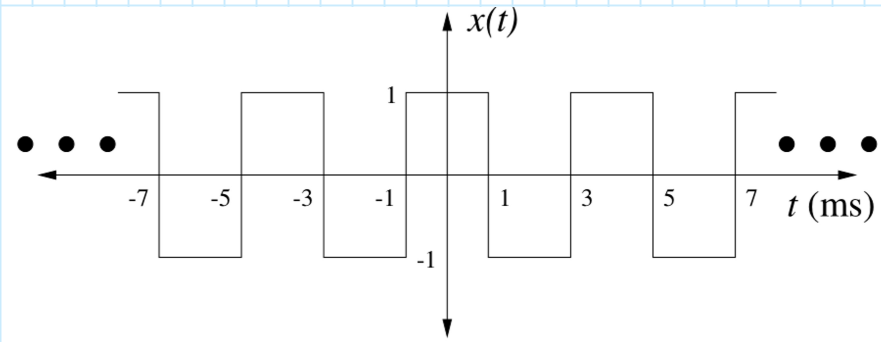
1. Consider the waveform x(t) shown below. Note that the t axis is in units of milliseconds.



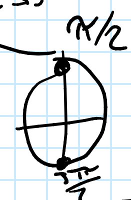
(a) Determine the complex Fourier series coefficients c n for this waveform

$$x(t) = \sum_{\infty} C_n e^{jn\omega_0 t} \qquad C_n = \frac{1}{T_0}\int_a^{a+T_0} x(t) e^{-jn\omega_0 t}\, dt$$

$$\omega_0 = 2\pi f_0 = 2\pi / T_0 \qquad \underline{T_0 = 4ms} \qquad \underline{\omega_0 = 500\pi}$$

$$C_n = \frac{1}{T_0}\int_{-1}^{3} x(t) e^{-jn\omega_0 t} \qquad x(t) = \begin{cases} 1 & -1 \le t \le 1 \\ -1 & 1 \le t \le 3 \end{cases}$$

$$C_n = \frac{1}{T_0}\left[\int_{-1m}^{1m} e^{-jn\omega_0 t} - \int_{1m}^{3m} e^{-jn\omega_0 t}\right]$$



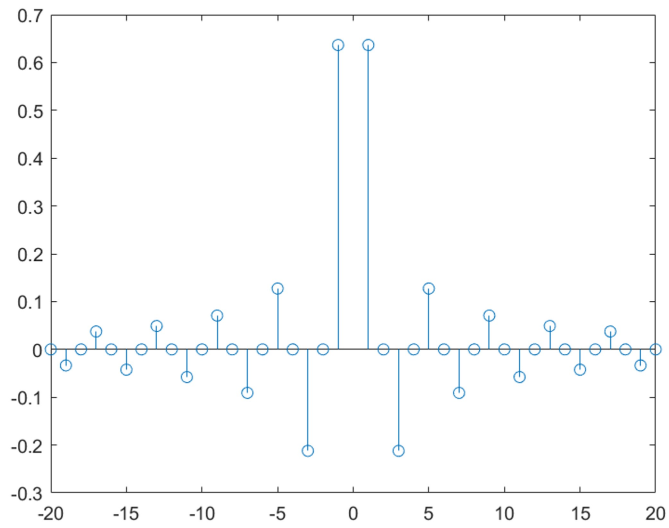
$$\frac{\pm j}{j} \in \mathbb{R}$$

$$\Rightarrow \frac{1}{jn\omega_0 T_0}\left[e^{-jn\omega_0 t}\Big|_{1/1000}^{3/1000} - e^{-jn\omega_0 t}\Big|_{-1/1000}^{1/1000}\right]$$

$$C_n = \frac{e^{-\frac{jn\omega_0 3}{1000}} - 2e^{-\frac{jn\omega_0}{1000}} + e^{\frac{jn\omega_0}{1000}}}{j2\pi n} = \boxed{\frac{e^{-\frac{jn3\pi}{2}} - 2e^{-\frac{jn\pi}{2}} + e^{\frac{jn\pi}{2}}}{j2\pi n}}$$

(b) Use MATLAB to plot the Fourier series coefficients for x(t) that you determined in part (a)

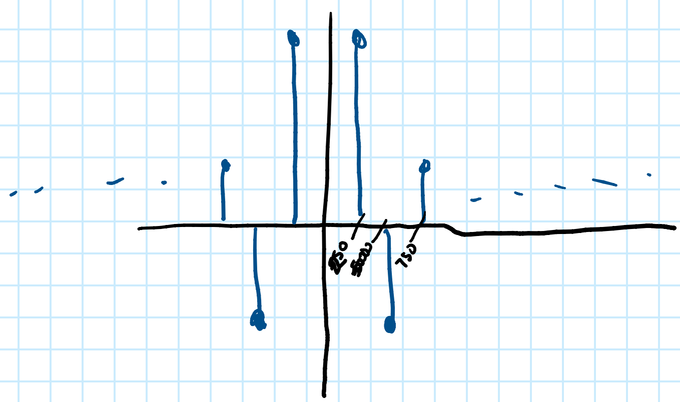
max(imag(Cn)) = 9.8501e-17 for [-20, 20]



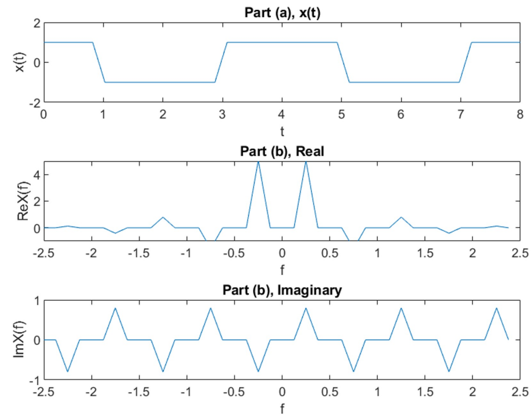
(c) Determine X(f), the Fourier transform of x(t), using the results of part (a). Sketch X(f) versus f by hand.

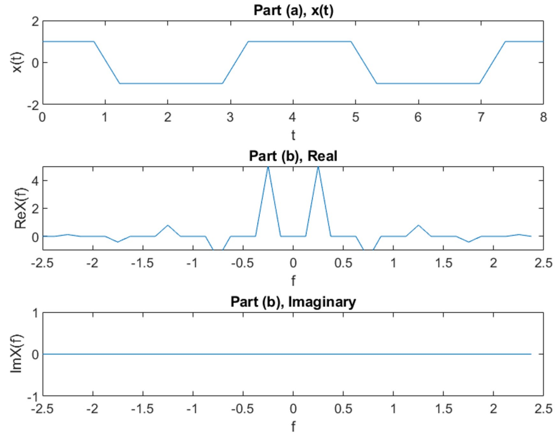
for periodic $x(t)$: $X(f) = \sum_n C_n \delta(f - n f_0)$

i.e. → The plot looks like the plot for C_n with indices of $n \to k f_0$

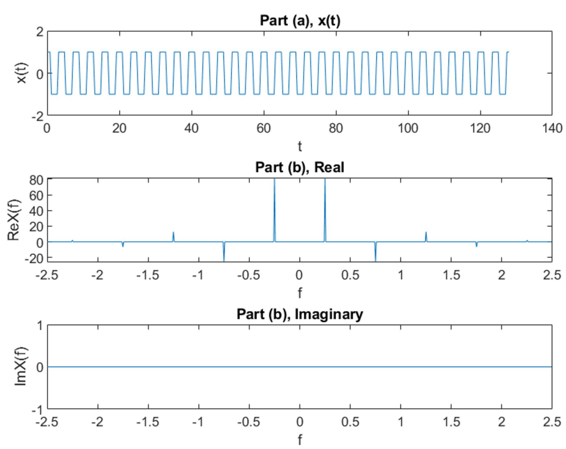


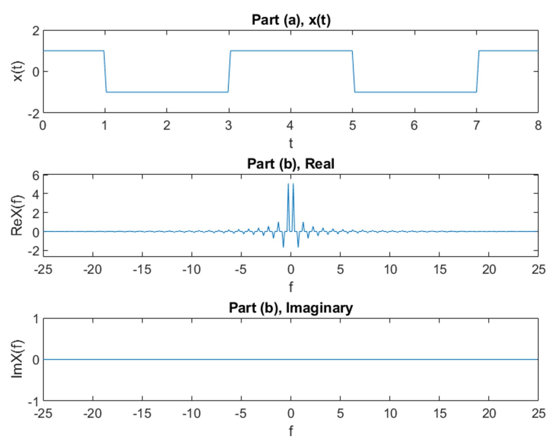
Problem 2





These two sets of plots show the fft of the wave form before and after the symmetry was enforced using the code provided in part (c) . It is clear from these plots that the imaginary portion of the spectrum becomes negligible.





This plot shows the effect of increasing the signal duration on the FFT results. The longer signal duration causes higher precision in isolating the fundamental frequencies of the signal, the triangular peaks at a short duration become pulses at the longer duration.

This plot shows the effect of an increased sample rate. A higher sample rate broadens the spectrum that will be shown in the FFT, i.e. a higher frequency will be included.

3. Let H(f) be a bandlimited lowpass spectrum that decreases smoothly with frequency according to

$$H(f) = \begin{cases} \dfrac{1}{2f_0} & |f| < (1-r)f_0 \\[2ex] \dfrac{1}{4f_0}\left\{1 + \cos\left[\dfrac{\pi\left(|f| - (1-r)f_0\right)}{2rf_0}\right]\right\} & (1-r)f_0 \le |f| \le (1+r)f_0 \\[2ex] 0 & |f| > (1+r)f_0. \end{cases}$$

a) $r = 2f_0 = 1$

i) show $H(f) = \cos^2\left(\frac{\pi f}{2}\right) \cdot \Pi(f/2)$

from conch:

$$\Pi\left(\frac{t}{T}\right) = \begin{cases} 1 & |t| \leq T/2 \\ 0 & |t| > T/2 \end{cases} \rightarrow \Pi\left(\frac{f}{2}\right) = \begin{cases} 1 & |f| \leq 1 \\ 0 & |f| > 1 \end{cases}$$

$\underline{r = 2f_0 = 1}$

$$H(f) = \begin{cases} 1 & |f| < 0 \\ \frac{1}{2}\left(1 + \cos\left(\frac{\pi |f|}{1}\right)\right) & 0 \leq |f| \leq 1 \\ 0 & |f| > 1 \end{cases}$$

$$|f| < 0 = \{\emptyset\}$$

$$\frac{1}{2} + \frac{1}{2}\cos 2\theta = \cos^2\theta$$

$$\Rightarrow \frac{1}{2}(1 + \cos(\pi|f|)) = \cos^2\left(\frac{\pi|f|}{2}\right)$$

cos is even $\rightarrow$ $\cos(|f|) = \cos(f)$

$$\boxed{\therefore H(f) = \cos^2\left(\frac{\pi f}{2}\right)\Pi\left(\frac{f}{2}\right)}$$

ii) Duality $\mathcal{F}^{-1}\left\{\cos\left(\frac{\pi f}{2}\right)\right\} = \frac{1}{2}\left(\delta(t - \tfrac{1}{4}) + \delta(t + \tfrac{1}{4})\right)$

$x(t) \longleftrightarrow X(f)$

$X(t) \longleftrightarrow x(f)$ $\mathcal{F}\{\cos(2\pi f t)\} = \frac{1}{2}\left(\delta\left(f - \frac{f}{2\pi}\right) + \delta\left(f + \frac{f}{2\pi}\right)\right)$

$$\mathcal{F}^{-1}\left\{\cos\left(\frac{\pi f}{2}\right)\right\} = \frac{1}{2}\left(\delta\left(t - \frac{\pi/2}{2\pi}\right) + \delta\left(t + \frac{\pi/2}{2\pi}\right)\right)$$

$$\boxed{\therefore \mathcal{F}\left\{\cos\left(\frac{\pi f}{2}\right)\right\} = \frac{1}{2}\left[\delta(t - \tfrac{1}{4}) + \delta(t + \tfrac{1}{4})\right]}$$

iii) Convolution

$x(t) * y(t) = X(f)Y(f)$

Show $\mathcal{F}^{-1}\{H(f)\} = \frac{1}{2}\left[\frac{\sin(2\pi t - \pi)}{2\pi t - \pi}\right] + \frac{\sin(2\pi t)}{2\pi t} + \frac{1}{2}\left[\frac{\sin(2\pi t + \pi)}{2\pi t + \pi}\right]$

$$\mathcal{F}^{-1}\{H(f)\} = \mathcal{F}^{-1}\left\{\cos\left(\frac{\pi f}{2}\right)\right\} * \mathcal{F}^{-1}\left\{\cos\left(\frac{\pi f}{2}\right)\right\} * \mathcal{F}^{-1}\left\{\Pi\left(\frac{f}{2}\right)\right\}$$

$$\Rightarrow \left[\tfrac{1}{2}\delta(t + \tfrac{1}{4}) + \tfrac{1}{2}\delta(t - \tfrac{1}{4})\right] * \left[\tfrac{1}{2}\delta(t + \tfrac{1}{4}) + \tfrac{1}{2}\delta(t - \tfrac{1}{4})\right] * \left[2\,\frac{\sin(2\pi t)}{2\pi t}\right]$$

$$x(t) * y(t) = \int_{\infty} x(\tau)\, y(t - \tau)\, d\tau \quad ; \quad x(t) * \delta(t - T) = x(t - \tau)$$

$$\frac{1}{T}\int_{\infty}\left[\delta(\tau-1/4)+\delta(\tau+1/4)\right]\cdot\left[\delta(t+1/4-\tau)+\delta(t-1/4-\tau)\right] * 2\,Sa(2\pi t)$$

$$\Rightarrow \frac{1}{T}\left[\delta(t-1/2)+\delta(t)+\delta(t)+\delta(t+1/2)\right] * 2Sa(2\pi t)$$

$$\Rightarrow \boxed{\frac{1}{2}\left[Sa(2\pi(t-1/2))+2\,Sa(2\pi(t))+Sa(2\pi(t+1/2))\right]}$$

$$Sa(x)=\frac{\sin(x)}{x}$$

b) $2f_0 = 1$; $0 \le r \le 1$

i) $f_0 = 1/2$

$$\frac{1}{2f_0}=\frac{1}{1}=1 \qquad (1-r)f_0=(1-r)\frac{1}{2}=\frac{1-r}{2}$$

$$\frac{1}{4f_0}\left\{1+\cos\left(\frac{\pi|f|-(1-r)f_0}{2rf_0}\right)\right\} \qquad \frac{1}{4f_0}=\frac{1}{2}$$

$$\frac{\pi(f-(1-r)f_0)}{2rf_0}=\frac{\pi(f-(1-r)\frac{1}{2})}{r}=\frac{\pi(2f-(1-r))}{2r}$$

$$(1+r)f_0=\frac{(1+r)}{2}$$

$$\therefore \quad H(f)=\begin{cases}1 & |f|<\frac{1-r}{2}\\ \frac{1}{2}\left\{1+\cos\left(\frac{\pi(2|f|-1+r)}{2r}\right)\right\} & \frac{1-r}{2}\le|f|\le\frac{1+r}{2}\\ 0 & |f|>\frac{1+r}{2}\end{cases}$$

for $2f_0=1$ and $0\le r\le 1$

ii) $$H(f)=\Pi(f) * \frac{\pi}{2r}\cos\left(\frac{\pi f}{r}\right)\cdot\Pi\left(\frac{f}{r}\right)$$

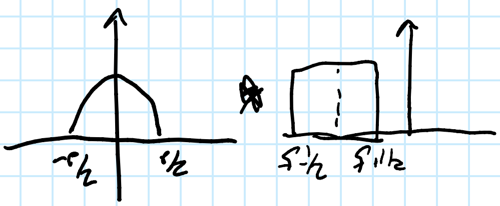
$$\Pi(f)=\begin{cases}1 & |f|<1/2\\ 0 & |f|>1/2\end{cases}$$

$$\Pi(f) = \begin{cases} 1 & |f| < 1/2 \\ 0 & |f| > 1/2 \end{cases}$$

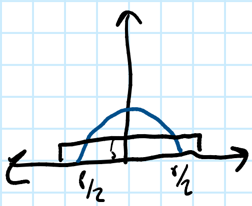
$$\Rightarrow \frac{\pi}{2r}\int_{-\infty}^{\infty} \cos\left(\frac{\pi u}{r}\right)\cdot \Pi\left(\frac{u}{r}\right)\cdot \Pi(f-u)$$

$$\Pi\left(\frac{u}{r}\right) = \begin{cases} 1 & |u| < \frac{r}{2} \\ 0 & \_\_\_ \end{cases}$$

$$\Pi(f-u) = \begin{cases} 1 & |f-u| < 1/2 \\ 0 & \_\_\_ \end{cases} \Rightarrow \begin{array}{l} < f + 1/2 \\ > f - 1/2 \end{array}$$



$$\Rightarrow 0 \begin{cases} f + 1/2 < -r/2 \\ f - 1/2 > r/2 \\ f < \frac{-r-1}{2} \\ f > \frac{r+1}{2} \end{cases} \quad |f| > \frac{1+r}{2}$$



$$\Rightarrow 1 \quad \begin{array}{l} f - 1/2 > -\frac{r}{2} \\ f + 1/2 < \frac{r}{2} \\ f > \frac{1-r}{2} \\ f < \frac{-(1-r)}{2} \end{array} \quad |f| < \frac{1-r}{2}$$

O/W

$$\frac{\pi}{2r}\int_{-r/2}^{f+1/2} \cos\left(\frac{\pi u}{r}\right) du \quad \text{or} \quad \int_{f-1/2}^{r/2} \cos\left(\frac{\pi u}{r}\right) du$$

$$\Rightarrow \cancel{\frac{\pi}{2r}}\left[\cancel{\frac{r}{\pi}} \sin\left(\frac{\pi f}{r}\right)\right]\Bigg|_{-r/2}^{f+1/2} = \frac{1}{2}\left[\sin\left(\frac{\pi(f+1/2)}{r}\right) - \sin\left(\frac{\pi(-r/2)}{r}\right)\right]$$

$$\Rightarrow \frac{1}{\ }\left[\sin\left(2\pi(f+1)\ \right) + 1\right] \quad f < 0$$

$$\Rightarrow \frac{1}{2}\left[\sin\left(\frac{2\pi(f+1)}{2r}\right)+1\right] \quad f<0$$

Positive case

$$\frac{1}{2}\left[\sin\left(\frac{\pi f}{r}\right)\right]\Bigg|_{f-1/2}^{r/2} = \frac{1}{2}\left[1-\sin\left(\frac{\pi(f-1/2)}{r}\right)\right] \quad f>0$$

$$\sin(\theta) = -\sin(-\theta)$$

$$\Pi * \cos\pi \begin{cases} \left[1+\sin\left(\frac{2\pi(|f|-1)}{2r}\right)\right]\frac{1}{2} & \frac{1-r}{2}<|f|<\frac{1+r}{2} \\ 1 & |f|<\frac{1-r}{2} \\ 0 & |f|>\frac{1+r}{2} \end{cases}$$

$$\sin(x) = \cos\left(\frac{\pi}{2}-x\right) \Longleftrightarrow \sin\left(\frac{2\pi(|f|-1)}{2r}\right) = \cos\left(\pi\left(\frac{1}{2}-\frac{2(|f|-1)}{2r}\right)\right)$$

$$\cos\left(\pi\left(\frac{1}{2}-\frac{2(|f|+1)}{2r}\right)\right) = \cos\left(\pi\left(\frac{r-2(|f|+1)}{2r}\right)\right)$$

$$\Rightarrow \cos\left(\pi\left(\frac{2|f|+r-1}{2r}\right)\right) \qquad \cos(\theta) = \cos(-\theta)$$

$$\therefore \Pi(f) * \frac{\pi}{2r}\cos\left(\frac{\pi f}{r}\right)\Pi\left(\frac{f}{r}\right) = \begin{cases} 0 & |f|>\frac{1+r}{2} \\ \frac{1}{2}\left\{1+\cos\left(\frac{2|f|+r-1}{2r}\right)\right\} & \frac{1-r}{2}<|f|<\frac{1+r}{2} \\ 1 & |f|<\frac{1-r}{2} \end{cases}$$

iii) $\mathcal{F}^{-1}\left\{\cos\left(\frac{\pi f}{r}\right)\Pi\left(\frac{f}{r}\right)\right\} = \mathcal{F}\left\{\cos\frac{\pi f}{r}\right\} * \mathcal{F}^{-1}\left\{\Pi\left(\frac{f}{r}\right)\right\}$

$$\Rightarrow \left[\frac{r}{2\pi}\left(\delta\left(t-\frac{1}{2r}\right)+\delta\left(t+\frac{1}{2r}\right)\right)\right] * \left[\frac{\sin(\pi r t)}{\pi t}\right]$$

$$x(t) * \delta(t-T) = x(t-T)$$

$$\Rightarrow \frac{r}{2\pi}\left[\frac{\sin(\pi r(t-1/2r))}{\pi(t-1/2r)} + \frac{\sin(\pi r(t+1/2r))}{\pi(t+1/2r)}\right]$$

$$\Rightarrow \frac{r}{2\pi}\left[\frac{\sin(\pi r t - \pi/2)}{\pi t - \pi/2r} + \frac{\sin(\pi r t + \pi/2)}{\pi t + \pi/2r}\right]$$

$\sin(-\pi/2) = \cos$

$$\Rightarrow \frac{r}{2\pi}\left[\frac{\cos(\pi r t)}{\pi t - \pi/2r} - \frac{\cos(\pi r t)}{\pi t + \pi/2r}\right]$$

$$\Rightarrow \frac{2r}{\pi}\left[\frac{\cos(\pi r t)}{1-(2rt)^2}\right]$$

iv) $H(f) = \Pi(f) * \frac{\pi}{} \cos\left(\frac{\pi f}{r}\right) \frac{}{2r} \Pi\left(\frac{f}{r}\right)$ (from (ii))

$$\Rightarrow H(f) = \mathcal{F}\left\{\mathcal{F}^{-1}\{\Pi(f)\} \cdot \mathcal{F}^{-1}\left\{\frac{\pi}{2r}\cos\left(\frac{\pi f}{r}\right) \cdot \Pi\left(\frac{f}{r}\right)\right\}\right\}$$

$$\therefore \mathcal{F}^{-1}\{H(f)\} = \left[\frac{\sin(\pi t)}{\pi t}\right] \cdot \left[\frac{\cos(\pi r t)}{1-(2rt)^2}\right]$$

v) $\mathcal{F}\{h(t)\} = H(f)$

$\therefore \mathcal{F}^{-1}\{H(f)\} = h(t)$

c) Time scaling: $x(\alpha t) \longleftrightarrow \frac{1}{|\alpha|} X\left(\frac{f}{\alpha}\right)$

from b $\rightarrow$ $h(t) = \left[\frac{\sin(\pi t)}{\pi t}\right]\left[\frac{\cos(\pi r t)}{\ldots}\right]$

From b → $h(t) = \left[\frac{\sin(\pi t)}{\pi t}\right]\left[\frac{\cos(\pi r t)}{1-(2rt)^2}\right]$

to general case time scale $\alpha = 2f_0$

<u>so if</u>

$$\left[\frac{\sin(\pi t)}{\pi t}\right]\left[\frac{\cos(\pi r t)}{1-(2rt)^2}\right] \xleftrightarrow{\mathcal{F}} \begin{cases} 1 & |f| > \frac{1+r}{2} \\ \frac{1}{2}\left\{1+\cos\left(\pi\left(\frac{2|f|-1+r}{2r}\right)\right)\right\} & \frac{1-r}{2} < |f| < \frac{1+r}{2} \\ 0 & |f| < \frac{1+r}{2} \end{cases}$$

<u>then</u>

$$\overbrace{\left[\frac{\sin(2f_0\pi t)}{2\pi f_0 t}\right]\left[\frac{\cos(2\pi f_0 r t)}{1-(4f_0 r t)^2}\right]}^{h(2f_0 t)} \longleftrightarrow \underbrace{\begin{cases} 1/2f_0 & |f| > \left(\frac{1+r}{2}\right)2f_0 \\ \frac{1}{2f_0}\left(\frac{1}{2}\right)\left\{1+\cos\left(\pi\left(\frac{\frac{2|f|}{2f_0}-1+r}{2r}\right)\right)\right. & \frac{(1-r)2f_0}{2} < |f| < \frac{(1+r)2f_0}{2} \\ 0 & |f| < \left(\frac{1-r}{2}\right)2f_0 \end{cases}}_{\frac{1}{2f_0}H\left(\frac{f}{2f_0}\right)}$$

<u>Problem 4</u>

$$H(f) = \frac{f_0^2}{f_0^2 - f^2 + j(f_0/Q)f}$$

a) 3-dB $= f_0 \iff Q = 1/\sqrt{2}$

max $H(f) = 1$ @ $f = 0$ 3dB bandwidth →

$$1/\sqrt{2} = \left|\frac{f_0^2}{f_0^2 - f^2 + j(f_0/Q)f}\right|$$

$$\Rightarrow \sqrt{2}\,f_0^2 = f_0^2 - f^2 + j(f_0/Q)f \qquad (f = f_0)$$

$$\Rightarrow \sqrt{2}\,\cancel{f_0^2} = \left|j\frac{\cancel{f_0^2}}{Q}\right|$$

$$\sqrt{2} = \left|\frac{j}{Q}\right|$$

$$\boxed{\Rightarrow Q = 1/\sqrt{2}}$$

$$|H(f)|^2 = \left|\frac{f_0^2}{\quad}\right|^2$$

$\boxed{\Rightarrow \alpha = \sqrt{2}}$

$$|H(f)|^2 = \left| \frac{f_0^2}{f_0^2 - f^2 + j f_0 f \sqrt{2}} \right|^2$$

$$\Rightarrow \frac{(f_0^2)^2}{\sqrt{(f_0^2-f^2)^2 + (f_0 f \sqrt{2})^2}^{\,2}} = \frac{f_0^4}{f_0^4 + f^4 - \cancel{2f_0^2 f^2} + \cancel{2f_0^2 f^2}}$$

$$= \left(\frac{f_0^4}{f_0^4 + f^4}\right)\frac{(1/f_0^4)}{(1/f_0^4)} = \boxed{\frac{1}{1 + f^4/f_0^4}}$$

b) 40 dB BW

40 dB = 10^4 power ratio

$$\frac{1}{10^4} = \frac{1}{1 + f^4/f_0^4}$$

$$1 + \left(\frac{f}{f_0}\right)^4 = 10000$$

$$\left(\frac{f}{f_0}\right)^4 = 9999 \rightarrow \frac{f}{f_0} = 9.99$$

$$\boxed{40\text{ dB BW} \approx 10 f_0}$$

c) null Bandwidth

$$0 = \frac{1}{1 + (f/f_0)^4} \qquad \lim_{f \to \infty} = 0$$

$$\boxed{\text{null BW} = \infty}$$

d) Absolute Bandwidth $\Leftrightarrow$ null since there's no roots

$$\boxed{\text{Absolute BW} = \infty}$$

e) EQ BW:

$$B_{eq} = \frac{1}{|X(f)|^2} \int_0^\infty |X(f)|^2 \, df$$

$$\Rightarrow \left(1 + \left(\frac{f_0}{f_0}\right)^4\right) \int^\infty \frac{1}{1 + (f/f_0)^4} \, df$$

$$\Rightarrow \left(1+\left(\frac{f_b}{f_0}\right)^4\right) \int_0 \frac{1}{1+(f/f_0)^4}\, df$$

$$\frac{f_0}{4\sqrt{2}} \quad \frac{\ln(f^2+\sqrt{2}f_0f+f_0^2)}{\ln(f^2-\sqrt{2}f_0f+f_0^2)} + 2\operatorname{atan}\left(\frac{\sqrt{2}f}{f_0}+1\right) + 2\operatorname{atan}\left(\frac{\sqrt{2}f}{f_0}-1\right)\Bigg]$$

$\ln(\infty) \to \infty$ $\quad$ $\operatorname{atan}(1) \to \pi/4$

$\operatorname{atan}(\infty) \to \pi/2$ $\quad$ $\operatorname{atan}(-1) = -\pi/4$

$$\Big|_0^\infty \Rightarrow \left[\frac{f_0}{4\sqrt{2}}\left(1+\pi+\pi\right)\right] - \left[\frac{f_0}{4\sqrt{2}}\left(\frac{\cancel{\ln(f_0^2)}}{\cancel{\ln(f_0^2)}} + \pi/2 - \pi/2\right)\right]$$

$$\Rightarrow \left(\frac{f_0}{4\sqrt{2}}\right)2\pi \qquad \frac{1}{|X(f_0)|^2} = 2$$

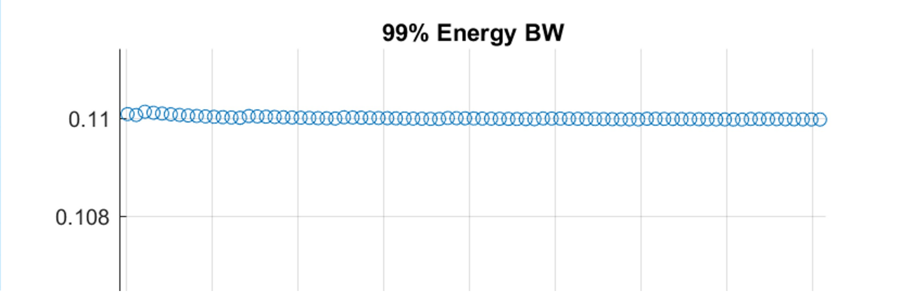
$$B_{eq} = (2)\left(\frac{f_0}{4\sqrt{2}}\right)(2\pi) = \boxed{\frac{\pi f_0}{\sqrt{2}}}$$

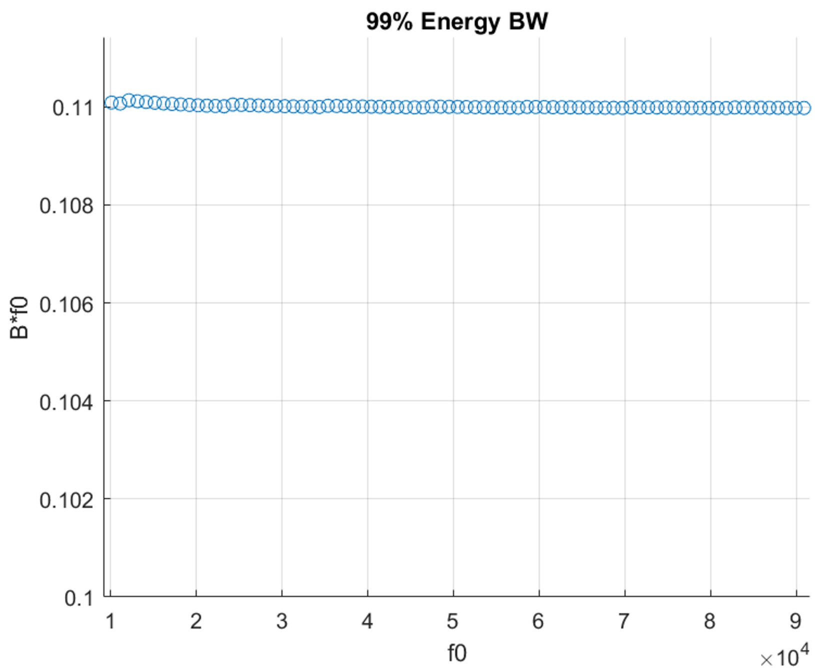
f) 99% BW Energy not Power

$$\int_0^B |H(f)|^2 = 0.99\int_0^\infty |H(f)|^2$$

$$\int_0^\infty \frac{1}{1+(f/f_0)^4} = \left(\frac{B_{eq}}{2}\right)0.99 = \left(\frac{\pi f_0}{2\sqrt{2}}\right)0.99$$

$$\Rightarrow 1.099 f_0 = \int_0^B |X(f)|^2 df$$





$$\boxed{B \approx \frac{f_0}{9}}$$