Homework Set 4 EE 1473

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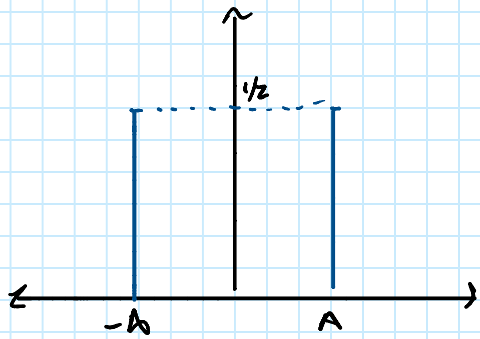
1. In this problem, we consider a simple model of a binary communications system,

r = s + n

where s represents a symbol being transmitted, n represents noise caused by the channel, and r represents the received symbol. Let s be a binary random variable that takes on the values +A and −A with equal probability. Let n be a Gaussian random variable with mean 0 and variance σ^2 . Assume that s and n are independent.

(a) Determine and sketch the PDF $f_s(s)$.

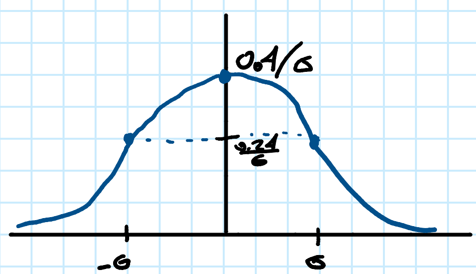
$$f_s(s) = \begin{cases} 1/2 & A \\ 1/2 & -A \\ 0 & \text{—} \end{cases}$$



(b) Determine and sketch the PDF $f_n(n)$.

$$f_n(n) = \frac{1}{\sigma\sqrt{2\pi}} e^{-n^2/2\sigma^2}$$

$$= \sim N(0, \sigma^2)$$



(c) Determine expressions for the conditional PDF's f r (r|s = −A) and f r (r|s = +A). Sketch uboth of these functions the same set of axes. One way to do this is to recognize that, if s = +A, then the random variable r is a simple function of the random variable n,

r = n + A			if s = +A.

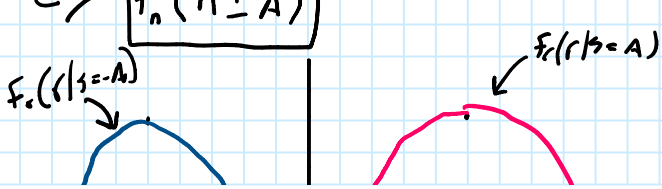
Therefore, one can obtain f_r(r|s = +A) using the standard formula for the functional transformation of a random variable. See the lecture notes or Couch, Section B-8.

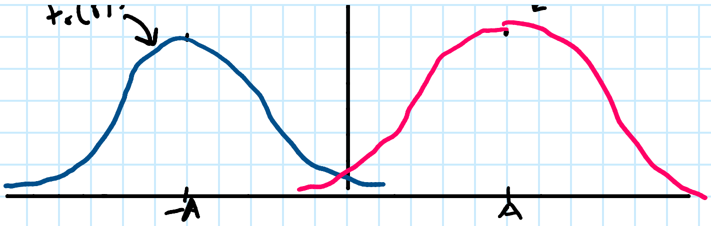
$f_r(r|s=\pm A)$, $r = n \pm A$

$$F_r(r|s=\pm A) = P(r \le A \mid s = \pm A)$$

$$f_r = \frac{d}{da}\left(\frac{P(\{r \le A\} \cap \{s = A\})}{P(s=A)}\right)$$

$$\Rightarrow \boxed{f_n(n \pm A)}$$





(d) Suppose you observe the voltage r, and are asked to estimate whether s = −A or s = +A. Let the estimate be s~, which is determined by the rule

s~ = { −A if r < T }
{ +A if r > T }

where T is a threshold value. Determine an expression for the probability P(s~ != s) in terms of the Q function, the signal amplitude A, the noise variance σ 2 , and the threshold value T. This is the bit error rate for this communications system. The Q function is defined in Couch Section B-7, and tabulated in Section A-9.

$$Q(z) = \frac{1}{\sqrt{2\pi}} \int_z^{\infty} e^{-\lambda^2/2} \, d\lambda$$

$$P(\tilde{s} \neq s) \Leftrightarrow P(s = A \mid r < T) + P(s = -A \mid r > T)$$

$$\Rightarrow \frac{1}{2}\int_{-\infty}^{T} f_n(n-A) + \frac{1}{2}\int_{T}^{\infty} f_n(n+A)$$

$$\Rightarrow \frac{1}{2}\int_{-\infty}^{T} \frac{1}{\sigma\sqrt{2\pi}} e^{-(n-A)^2/2\sigma^2} + \frac{1}{2}\int_{T}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-(n+A)^2/2\sigma^2}$$

$$\lambda = \frac{n \pm A}{\sigma} \Rightarrow \int_{-(T-A)/\sigma}^{\infty} e^{-\lambda^2/2} + \int_{(T+A)/\sigma}^{\infty} e^{-\lambda^2/2}$$

$$\boxed{P_e = \frac{1}{2}\left[Q\left(\frac{A-T}{\sigma}\right) + Q\left(\frac{A+T}{\sigma}\right)\right]}$$

(e) Show that the optimum threshold value is T = 0, in the sense that this value minimizes the bit error rate. Do this by taking the derivative of the bit error rate with respect to T, and setting the derivative equal to zero.

$$\frac{d\left[\int_{a(x)}^{b(x)} f(\lambda, x)\, d\lambda\right]}{dx} = f(b(x), x)\frac{db(x)}{dx} - f(a(x), x)\frac{da(x)}{dx} + \int_{a(x)}^{b(x)} \frac{\partial f(\lambda, x)}{\partial x}\, d\lambda \quad \text{(Leibniz's rule)}$$

$$\Rightarrow \frac{dP_e}{dT} = \frac{1}{2\sigma\sqrt{2\pi}}\left(e^{-(T-A)^2/2\sigma^2} - e^{-(T+A)^2/2\sigma^2}\right)$$

$$\frac{dP_e}{dT} = 0 \Rightarrow \frac{(T-A)^2}{2\sigma^2} = \frac{(T+A)^2}{2\sigma^2}$$

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$$(T-A)^2 = (T+A)^2$$

$$\therefore \boxed{T=0}$$

(f) Evaluate the bit error r
ate for T = 0, A = +5 Volts, and σ^2 = 2500, 100, 25, 4, and 1.5625.

σ^2	2500	100	25	4	1.5625
BER	0.2301	0.1543	0.0793	0.0031	1.5979e-05

(g) Let the signal-to-noise ratio for this system be defined as

S/N = E{s^2}/E{n^2 } .

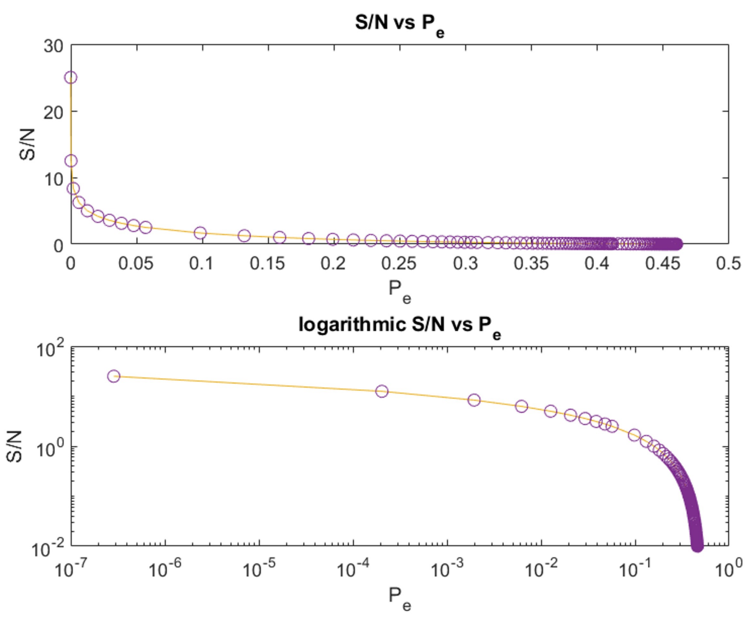
Evaluate the signal-to-noise ratio for A = 5 Volts and σ 2 = 2500, 100, 25, 4, and 1.5625.

$$E\{s^2\} = \int_{-\infty}^{\infty} s^2 f_s(s)\, ds = \tfrac{1}{2}A^2 + \tfrac{1}{2}(-A^2) \Rightarrow \boxed{A^2}$$

$$E\{n^2\}\ \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} n^2 e^{-n^2/2\sigma^2} = \sigma^2$$

σ^2	2500	100	25	4	1.5625
S/N	0.01	0.25	1	6.25	16

(h) Using the data you generated in parts (f) and (g), make a plot of bit error rate versus signal-to-noise ratio.



2. A digital source produces bits at a rate of 768 kpbs. These data are to be transmitted using a baseband signal constructed from RCRO pulses with rolloff

factor r, as described by Couch, equation (3-73). The spectrum for this pulse is provided in equation (3-69).

(a) Suppose that Polar NRZ signaling is used, i.e. a positive pulse is used to represent each binary 1, and a negative pulse is used to represent each binary zero. Determine the parameters f_0 , f_1 , f_Δ and B, such that the baseband signal will satisfy the criterion for zero intersymbol interference, for each of the following rolloff factors:

i. r = 0.2, ii. r = 0.5, iii. r = 0.8.

Zero ISI when $R = 2f_0 = \frac{2B}{1+r}$

$f_0 = \frac{R}{2}$

$B = f_0(1+r)$

$f_\Delta = B - f_0$

$f_1 = B - 2f_\Delta$

R	0.2	0.5	0.8
f_0	384 kHz	384 kHz	384 kHz
B	460.8 kHz	576 kHz	691.2 kHz
f_Δ	76.8 kHz	192 kHz	307.2 kHz
f_1	307.2 kHz	192 kHz	76.8 kHz

(b) Recompute the 6-dB bandwidths f_0 and the absolute bandwidths B from part (a) is Unipolar RZ signaling is used.

Unipolar RZ first null at 2R → $\boxed{f_0 = R}$

$\boxed{f_0 = 768\text{ kHz}}$ $B = 2B$

$\boxed{B = 921.6\text{ kHz},\ 1.152\text{ MHz},\ 1.382\text{ MHz}}$

(c) Next, suppose that Polar NRZ signaling is used, but the data are grouped into two-bit symbols, and each symbol is transmitted using a RCRO pulse with one of four amplitudes. Recompute the bandwidths f_0 and B from part (a) for this system.

$D = R/\ell$ $\ell = 2$ $D = 384\text{ k baud}$

$B = (1+r)\frac{D}{2}$

$\Rightarrow \frac{1}{2}B_a$

$B = 230.4\text{ kHz},\ 288\text{ kHz},\ 345.6\text{ kHz}$

$f_0 = 192\text{ kHz}$

(d) Repeat part (c) if three-bit symbols are used, and transmitted using eight-level signaling.

$D = R/3 = 256\text{ kHz} \Rightarrow \frac{1}{3}B_a$

$f_0 = 128\text{ kHz}$

$B = 153.6\text{ kHz},\ 192\text{ kHz},\ 230.4\text{ kHz}$

3. An analog signal with a bandwidth of 32 MHz is to be transmitted over a PCM system. The peak signal-to-noise ratio due to quantization must be at least 40 dB, and the probability of a bit error during transmission is $P_e = 0$.

(a) Determine the minimum sampling rate required for this system.

$f_s \geq 2B \Rightarrow 64\text{ MHz}$

(b) Determine the number of bits required for each PCM word in order to meet the SNR constraint, and the corresponding number of quantization levels.

6 dB per bit at $P_e = 0 \rightarrow \left(\frac{S}{N}\right)_{pk} = 10\log(3M^2)$

$20\log_{10}(M) + 10\log_{10}(3) = 40$

$6.02n + 4.77 = 40$

$n = 5.85 \Rightarrow$ $\boxed{\text{6 samples per bit, 64 levels}}$

(c) Determine the bit rate of the system if 2 times oversampling is applied to the analog signal.

$R = nf_s$, $f_s = 2f_{s_{min}} = 128\ \text{MHz}$

$\boxed{R = 768\ \text{Mb/s}}$

(d) Determine the null bandwidth of the PCM signal Polar NRZ signaling is used, and rectangular pulses are used to represent the bits.

$B_{PCM} = nf_s = R$

$\boxed{\text{768 MHz w/ the oversampling; 384 MHz minimum}}$

(e) How will the answers to parts (a) through (d) change if the average quantization SNR must be at least 40 dB?

$\frac{S}{N}_{peak} = 3\frac{S}{N}_{avg}$

$6.02n = 40$ $\boxed{\text{7 bits; 126 levels; 896 Mb/s; 896 MHz}}$

4. Consider again the system from Problem 4, in which the peak quantization SNR must be at least 40 dB, but now suppose that the probability of a bit error is not zero.

(a) Suppose that $P_e = 10^{-6}$. Determine the number of bits per sample n and the number of quantization levels M necessary to meet the constraint. Comment on your results.

$\frac{S}{N}_{pk} = \frac{3M^2}{1 + 4(M^2-1)P_e} \geq 10^4$

$3M^2$ [illegible] 4 [illegible] $4M^2 - 4$ [illegible] $\Rightarrow 2.96M^2 \geq 9999.96$

N_{pk} $\quad \overline{1+4(M^2-1)P_e} \geq 10$

$$3M^2 \geq 10^4 + \frac{4M^2-4}{100} \Rightarrow 2.96M^2 \geq 9999.96$$

$M = 58.1 \qquad n = \lceil \log_2 M \rceil = \boxed{\begin{array}{l}6 \text{ bits/sample} \\ 64 \text{ levels}\end{array}}$

$P_e = 10^{-6}$ does not affect the S/N constraint enough to require additional bits, this is the same result as $P_e = 0$

(b) Repeat part (a) for the case when $P_e = 2 \times 10^{-5}$.

$$3M^2 \geq 10^4 + \frac{4M^2-4}{5} \Rightarrow 2.2M^2 \geq 9999.2$$

$n = 7$ bits

$M = 128$ levels

(c) Determine the minimum value for P_e such that n = 9 quantization bits and M = 512 quantization levels are required.

$$10^4 \leq \frac{3(512)^2}{1+4(512^2-1)P_e} \Rightarrow 10^4 + (4(10^4)M^2 - 10^4)P_e \leq 3M^2$$

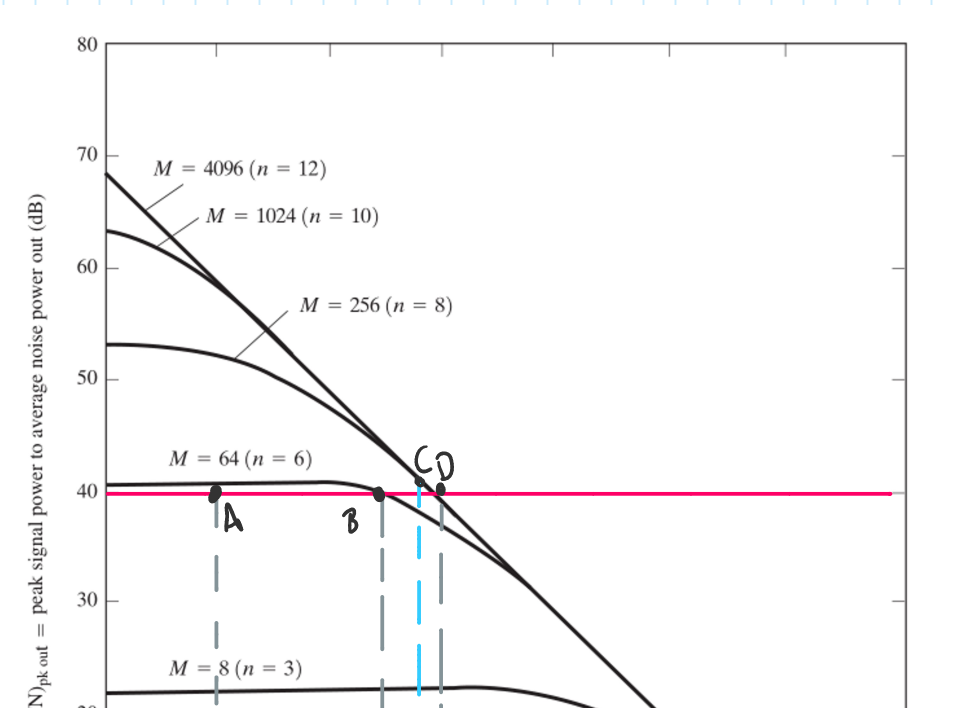
$$\boxed{P_e = 7.1 \times 10^{-5}}$$

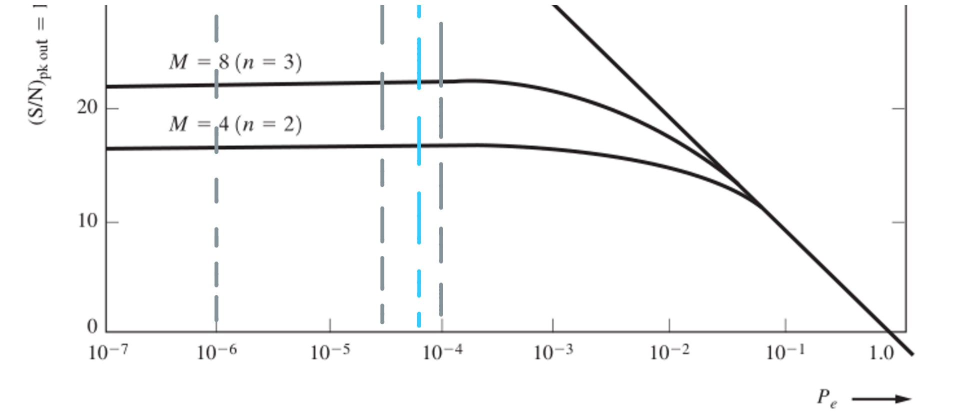
(d) Repeat part (a) for the case when $P_e = 10^{-4}$.

$$3M^2 \geq 10^4 + 4M^2 - 4$$

$-M^2 \geq 9996$ This scenario is impossible

(e) Examine Couch, Figure 7-17. Interpret your answers to parts (a) through (d), making specific reference to this figure.





This figure shows S/N as a function of P_e for increasing values of M and n. points above a curve show values of S/N and P_e that are impossible w/o increasing n. Points above the diagonal for $n \geq 12$ cannot be obtained regardless of increasing n.

5. So far, we have considered a rectangular pulses, sinc pulses and RCRO pulses to represent digital data. In this problem, we consider using a sinc-squared pulse for this purpose, $p(t) = (\sin\omega_0 t/\omega_0 t)^2$.

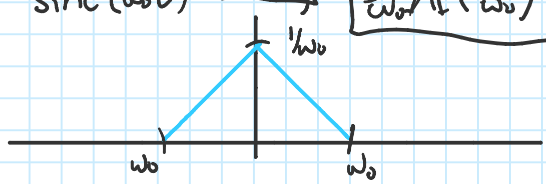
(a) Use Tables 2-1 and 2-2 to determine P(f), the Fourier transform of p(t). Then sketch P(f) vs. f.

Duality $W(t) \xrightarrow{\mathcal{F}} w(-f)$

given $w(t) \xrightarrow{\mathcal{F}} W(f)$

$$T\,\mathrm{sinc}^2(\pi f T) \xleftrightarrow{\mathcal{F}^{-1}} \Lambda\left(\frac{t}{T}\right)$$

$$\therefore\ \mathrm{sinc}^2(\omega_0 t) \xrightarrow{\mathcal{F}} \boxed{\frac{1}{\omega_0}\Lambda\left(\frac{f}{\omega_0}\right)}$$



(b) Use Parseval's relation to determine E_p, the energy in p(t).

$$E_p\ \int_{-\infty}^{\infty} |P(f)|^2\,df = 2\int_0^{\omega_0}\left(1 - \frac{f}{\omega_0}\right)^2 df = \boxed{\frac{2\omega_0}{3}}$$

(c) How should the frequency $f_0 = \omega_0/2\pi$ be chosen, so that a data signal using the pulse p(t) will satisfy the zero-ISI condition? Give your answer in terms of the bit rate R.

$\mathrm{sinc}^2 = 0$ when $\omega_0 t = n\pi$

$\Rightarrow\ 2\pi f_0 = n\pi$

$f_0 = n/2$

n represents the samples $\Rightarrow$ $\boxed{f_0 = R/2}$

Λ Geplesents the samples $\Rightarrow$ $\boxed{f_0 = R/2}$

(d) Determine the absolute and equivalent bandwidths of p(t), assuming that f_0 is chosen as in part (c). Give your answers in terms of the bit rate R.

abs BW is width of the Λ which is ω_0

$\omega_0 = 2\pi f_0$

$B = \frac{2\pi R}{2} = \boxed{\pi R}$

$$B_{eq} = \frac{1}{|X(f_0)|^2} \underbrace{\int_0^\infty |X(f)|^2 \, df}_{2\frac{\omega_0}{3}} \Rightarrow B = \frac{2\omega_0^3}{3}$$

$|X(0)|^2 = \left(\frac{1}{\omega_0}\right)^2$

$$\boxed{\Rightarrow B_{eq} = \frac{2(\pi R)^3}{3}}$$