EE 1473 - Digital Communication Systems
Homework Set 3

Due: Thursday, January 31, 2019

The purpose of the first problem on this assignment is to get everyone started on the Simulation Project. In it, you will write various pieces of MATLAB code that could form part of the baseband signaling block of your simulated system. For your project, you may ultimately end up with a different design than the one considered here, but this assignment will give you a good idea of how some of the blocks could be written.

Although you are permitted to work in pairs on the project, this is a homework assignment, and it is meant to be completed individually. Of course you may discuss your design strategies with other members of the class, but **do not share your MATLAB code with other students.** Making use of code obtained from someone other than yourself, including students who are currently taking the class or those who completed it in the past, is plagiarism and will not be tolerated.

Many of the steps below require that you write MATLAB functions; recall that a function takes certain variables as input and produces other variables as output. This is different from a script, which simply contains a list of commands to be executed. For example, the following function returns the mean-square value of N elements of a vector x:

$$\sigma = \frac{1}{N}\sum_{n=1}^{N} x_n^2.$$

```
function sigma = ms_value(x,N)

if N > length(x)
    disp('Error: the input vector contains fewer than N elements')
    sigma = -1;
    return
end

temp = x(1:N);

sigma = mean(temp.^2);
```

Please remember that you must submit hard copies of all MATLAB figures and other elements as identified below, and you must also submit your MATLAB code electronically to courseweb. An assignment will be created on the Homework page for this purpose.

1. Simulation of Baseband Digital Signaling

    (a) Write a MATLAB function that will produce random binary data values. The inputs to this function should be the number of bits to generate, and the data values a_n to assign to the bits. Call the `rand` function to generate a pseudo-random sequence of numbers in the range $[0, 1]$ and then map them to the correct data values. Your function should be equally likely to produce a binary 0 or a binary 1.

Call your function to produce 100 bits, and assign the value $a_n = -1$ for a binary 0 and $a_n = +1$ for a binary 1. Use the MATLAB `stem` function to plot the data sequence a_n versus n for $1 \leq n \leq 100$, and determine the relative frequency of each data value.

(b) Write a MATLAB function that will produce a rectangular pulse,

$$h(t) = \begin{cases} 1, & |t| < \frac{T_b}{2} \\ 0, & \frac{T_b}{2} < |t| < 5T_b \end{cases}$$

where T_b is the bit period. Note that the above specification has you compute the pulse over 10 bit periods. (The reason for this will become clear in future homework assignments.) Also note that this specification does not tell you what value to assign when $t = -T_b/2$. This is a problem that has annoyed nearly every student who has attempted this assignment, and your choice on this question will affect whether the signal formed in part (c) is correct.

The inputs to your function should be T_b and the number of samples per bit, i.e. the number of instants during each bit period at which to compute the value of the pulse. The outputs of your function should be the pulse samples and a vector of times at which the pulse was computed.

Call your function with $T_b = 0.2$ seconds and with at least 16 samples per bit. Plot the resulting pulse versus time.

(c) Write a MATLAB script that will call your random data function from part (a) with at least $N = 20$ bits and data values $a_n = \pm 1$. Then call the rectangular pulse function from part (a) with $T_b = 0.2$ seconds and at least 16 samples per bit. Form a baseband signal from the data and pulse according to the equation

$$s(t) = \sum_{n=0}^{N-1} a_n h(t - nT_b).$$

Plot the resulting signal versus time. If you have done this correctly your signal should be equal to zero for $t < -\frac{1}{2}T_b$ and $t > \left(N - \frac{1}{2}\right) T_b$. The time range $-\frac{1}{2}T_b < t < \left(N - \frac{1}{2}\right) T_b$ is the part of signal corresponding to the data values.

(d) Repeat part (c) using sinc pulses,

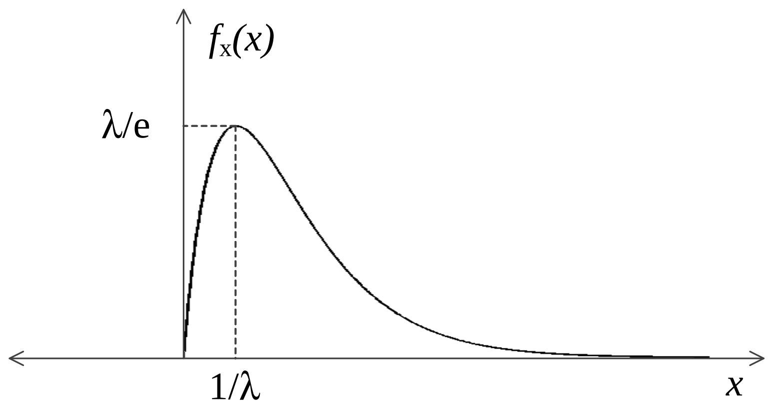
$$h(t) = \frac{\sin \pi Rt}{\pi Rt},$$

where R is the bit rate.

2. An Erlang-2 random variable has pdf

$$f_{\mathrm{x}}(x) = \lambda^2 x e^{-\lambda x} u(x),$$

where $u(\cdot)$ is the unit step function. A plot of this function is shown below. The *rate parameter* λ controls the peak height of the pdf, as well as the rate at which it falls off as $x \to \infty$.



(a) Show that $f_{\mathrm{x}}(x)$ is a valid pdf, i.e. that it is nonnegative and integrates to 1.

(b) Determine the mean.

(c) Determine the second moment.

(d) Determine the variance.

Hint: The integrals you need can be found in Couch, Section A-5.

3. Preamble: In applying probability to the study of digital communication, it is important to become skilled at *computing moments of random variables.* Nearly every problem we wish to solve will ultimately reduce to computing the moments of one or more random variables, and then evaluating some mathematical expressions involving the moments. For a single Gaussian random variable, the solution to every meaningful question we can pose will be expressed as some function of the mean and variance.

- **Example 1:** Let x be a Guassian random variable with mean μ and variance σ^2. The probability that $x \in (x_1, x_2]$ is given by

$$P(x_1 < x \leq x_2) = Q\left(\frac{\mu - x_2}{\sigma}\right) - Q\left(\frac{\mu - x_1}{\sigma}\right).$$

Therefore, if μ and σ^2 are known, and we have some way of evaluating the $Q(\cdot)$ function, we can compute the probability that x lies in any interval.

- **Example 2:** All of the moments for a Gaussian random variable can be calculated from the mean and variance. Let $y \sim N(\mu, \sigma^2)$, and suppose we want to compute $E\{y^n\}$ for some positive integer n. We are already familiar with the expressions for $n = 1$ and $n = 2$,

$$\begin{aligned} E\{y\} &= \mu \\ E\{y^2\} &= \mu^2 + \sigma^2. \end{aligned}$$

For $n \geq 3$, we can express the moment in terms of lower-order moments,

$$E\{y^n\} = \mu E\{y^{n-1}\} + \sigma^2 (n-1) E\{y^{n-2}\}.$$

We can use this expression to recursively compute $E\{y^n\}$ for $n = 3$, 4, etc. (A proof of this relationship is provided in the Appendix to this assignment.)

- **Example 3:** Let x and y be jointly Gaussian random variables. Then x and y are independent if and only if they are uncorrelated. This means that, in order to show that two Gaussian random variables are independent, we need only show

$$E\{xy\} = E\{x\}E\{y\},$$

a computation involving only moments, whereas for general random variables it would be necessary to show that

$$f_{\mathrm{xy}}(x, y) = f_{\mathrm{x}}(x) f_{\mathrm{y}}(y).$$

(A proof of this property for jointly Gaussian random variables is provided in the Appendix to this assignment.)

- **Example 4:** Let x be a random variable, and let $y = h(x)$, where $h(\cdot)$ is some known function. Then the expected value of y can be computed either by computing the integral of y times its pdf, or by computing the integral of $h(x)$ times the pdf for x, i.e.

$$E\{y\} = \int_{-\infty}^{\infty} y\, f_{\mathrm{y}}(y)\, dy = \int_{-\infty}^{\infty} h(x)\, f_{\mathrm{x}}(x)\, dx.$$

In some cases, the second expression is considerably easier to evaluate then the first, because it does not require obtaining the pdf for y.

This result can be extended to multiple random variables. An important special case is when one random variable is expressed as the weighted sum of some set of random variables, for example

$$y = h_1 x_1 + h_2 x_2,$$

where x_1, x_2 are random variables, and h_1, h_2 are known coefficients. We can apply the result given above to find the mean of y,

$$E\{y\} = \int_{-\infty}^{\infty} y\, f_{\mathrm{y}}(y)\, dy$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (h_1 x_1 + h_2 x_2)\, f_{\mathrm{x}_1}(x_1)\, dx_1\, f_{\mathrm{x}_2}(x_2)\, dx_2 \\
&= h_1 \int_{-\infty}^{\infty} x_1 f_{\mathrm{x}_1}(x_1)\, dx_1 \int_{-\infty}^{\infty} f_{\mathrm{x}_2}(x_2)\, dx_2 \\
&\quad + h_2 \int_{-\infty}^{\infty} f_{\mathrm{x}_1}(x_1)\, dx_1 \int_{-\infty}^{\infty} x_2 f_{\mathrm{x}_2}(x_2)\, dx_2 \\
&= h_1 \int_{-\infty}^{\infty} x_1 f_{\mathrm{x}_1}(x_1)\, dx_1 + h_2 \int_{-\infty}^{\infty} x_2 f_{\mathrm{x}_2}(x_2)\, dx_2 \\
&= h_1 E\{x_1\} + h_2 E\{x_2\}.
\end{aligned}
$$

This equation says that we can compute the mean of a weighted sum of random variables, and the result is the same weighted sum applied to the individual means. This can be very useful in a variety of contexts, because it allows us to compute the mean of y without having to work directly with the pdf for *any* of the random variables.

Note that this property holds for any type of random variables, not just Gaussian, but if x_1 and x_2 are Gaussian, then y will also be Gaussian. Furthermore, we can extend this result to the weighted sum of any number of random variables,

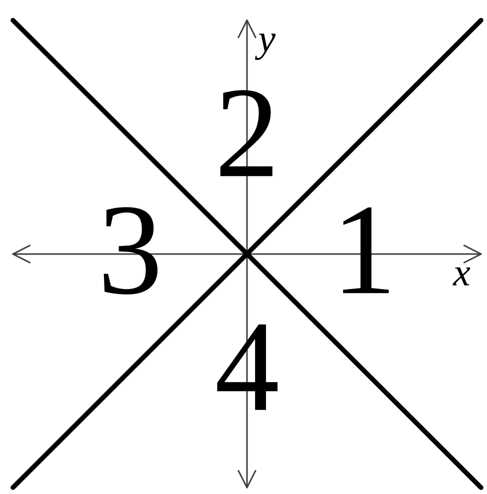
$$
y = \sum_{i=1}^{N} h_i x_i \qquad \Rightarrow \qquad E\{y\} = \sum_{i=1}^{N} h_i E\{x_i\}.
$$

In this problem, you will apply the results from Examples 1, 2, 3 and 4 to a case of jointly Gaussian random variables.

(a) Let $x \sim N(2,1)$ and $y \sim N(0,1)$ be independent random variables, and let $(x,y) \in \mathbb{R}^2$ represent a point in the plane. What is the joint pdf for x and y? Use the `mesh` function in MATLAB to plot $f_{\mathrm{xy}}(x,y)$.

(b) Use the result of Example 1 to determine the probability that the point (x,y) lies in the first quadrant. Repeat for the other quadrants.

(c) The figure below shows the plane divided into four regions that one might describe as rotated quadrants.



Suppose we wish to compute the probability that (x, y) lies in Region 1. Express this probability as an integral, and explain why it is difficult to evaluate this integral without performing numerical integration.

(d) Define two new random variables,

$$s = y + x \qquad \text{and} \qquad d = y - x.$$

Because they are linear combinations of Gaussian random variables, s and d are also Gaussian. Determine the means and variances of s and d.

(e) Use the results of Examples 3 and 4 to show that s and d are statistically independent.

(f) Suppose that (x, y) lies in Region 1, as shown in part (c). What values must s and d take when this is true? Repeat for regions 2, 3, and 4.

(g) Use the results of part (f) to compute the probability that (x, y) lies in Region 1. Repeat for Regions 2, 3, and 4.

(h) We will use the joint pdf for x and y to model a communication system that transmits two-bit symbols. Assume that the transmitted bit-pair is 00, and this is encoded at the transmitter as $(x, y) = (2, 0)$, but noise corrupts the signal during transmission such that other bit pairs may be decoded at the receiver. In this model, the random variables x and y represent samples of noisy output signals from the receiver, and the decoding rule for choosing which bits were transmitted is as follows:

$$\begin{array}{ll} \text{Choose 00} & \text{if } (x,y) \in \text{Region 1} \\ \text{Choose 10} & \text{if } (x,y) \in \text{Region 2} \\ \text{Choose 11} & \text{if } (x,y) \in \text{Region 3} \\ \text{Choose 01} & \text{if } (x,y) \in \text{Region 4.} \end{array}$$

Determine the following performance indices for this system.

(i) The probability of a symbol error, i.e. that something other than 00 will be chosen.

(ii) The probability that one bit error will be made.

(iii) The probability that two bit errors will be made.

(iv) The bit error rate: assuming that a very large number of bits are transmitted using this system, what fraction are expected to be decoded incorrectly?

APPENDIX: Proofs for two properties discussed in the preamble to Problem 3.

I. Proof that two jointly Gaussian random variables are independent if and only if they are uncorrelated.

The joint pdf for a pair of Gaussian random variables is given by

$$f_{\mathrm{xy}}(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \cdot e^{-z^2},$$

where

$$z^2 = \frac{\left(x-\mu_x\right)^2}{2\sigma_x^2\left(1-\rho^2\right)} + \frac{\rho\left(x-\mu_x\right)\left(y-\mu_y\right)}{\sigma_x\sigma_y\left(1-\rho^2\right)} + \frac{\left(y-\mu_y\right)^2}{2\sigma_y^2\left(1-\rho^2\right)},$$

μ_x and σ_x^2 are the mean and variance of x, and μ_y and σ_y^2 are the mean and variance of y. The parameter ρ is the correlation coefficient, defined as

$$\rho = \frac{E\left\{(x-\mu_x)(y-\mu_y)\right\}}{\sigma_x\sigma_y}.$$

We can express the correlation coefficient as

$$\rho = \frac{E\left\{xy\right\} - \mu_x E\left\{y\right\} - \mu_y E\left\{x\right\} + \mu_x\mu_y}{\sigma_x\sigma_y} = \frac{E\left\{xy\right\} - \mu_x\mu_y}{\sigma_x\sigma_y}.$$

Assuming that σ_x and σ_y are both positive and finite[1], we see that

$$\rho = 0 \qquad \text{if and only if} \qquad E\left\{xy\right\} = \mu_x\mu_y,$$

which by definition means that x and y are uncorrelated.

The individual pdfs for x and y are

$$f_{\mathrm{x}}(x) = \frac{1}{\sigma_x\sqrt{2\pi}} \cdot e^{-(x-\mu_x)^2/2\sigma_x^2} \qquad \text{and} \qquad f_{\mathrm{y}}(y) = \frac{1}{\sigma_y\sqrt{2\pi}} \cdot e^{-(y-\mu_y)^2/2\sigma_y^2},$$

and are known as the *marginal densities*. By definition, x and y will be independent if and only if the joint density can be expressed as the product of the marginal densities,

$$\begin{aligned}
f_{\mathrm{xy}}(x,y) &= f_{\mathrm{x}}(x) f_{\mathrm{y}}(y) \\
\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\, e^{-z^2} &= \frac{1}{\sigma_x\sqrt{2\pi}}\, e^{-(x-\mu_x)^2/2\sigma_x^2} \cdot \frac{1}{\sigma_y\sqrt{2\pi}}\, e^{-(y-\mu_y)^2/2\sigma_y^2} \\
\frac{e^{-z^2}}{\sqrt{1-\rho^2}} &= e^{-(x-\mu_x)^2/2\sigma_x^2} \cdot e^{-(y-\mu_y)^2/2\sigma_y^2}.
\end{aligned}$$

If we take the natural logarithm of this equation, and multiply by -1, we have

$$z^2 + \frac{1}{2}\ln\left(1-\rho^2\right) = \frac{\left(x-\mu_x\right)^2}{2\sigma_x^2} + \frac{\left(y-\mu_y\right)^2}{2\sigma_y^2}.$$

Returning to out expression for z^2, we note that it includes the term

$$\frac{\rho xy}{\sigma_x\sigma_y\left(1-\rho^2\right)},$$

[1]This is not a restrictive assumption. Surely, there is no meaning to a negative value for the variance or standard deviation of a random variable, and a GRV with variance zero collapses to a deterministic variable that is equal to the mean with probability 1. An infinite variance would mean that the random variable is equally likely to take any real value, an impractical case that we will ignore here.

while no term of the form xy appears in the product of the marginal densities. Therefore, if $\rho \neq 0$, the joint density will not satisfy the requirement for independence.

On the other hand, if $\rho = 0$, then the expression for z^2 reduces to

$$\begin{aligned} z^2 &= \frac{(x-\mu_x)^2}{2\sigma_x^2(1-0^2)} + \frac{0\,(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y(1-0^2)} + \frac{(y-\mu_y)^2}{2\sigma_y^2(1-0^2)} \\ &= \frac{(x-\mu_x)^2}{2\sigma_x^2} + \frac{(y-\mu_y)^2}{2\sigma_y^2}, \end{aligned}$$

and the equation that must be satisfied for independence simplifies to

$$z^2 + \frac{1}{2}\ln\left(1-0^2\right) = \frac{(x-\mu_x)^2}{2\sigma_x^2} + \frac{(y-\mu_y)^2}{2\sigma_y^2},$$

which is clearly true in this case.

In summary, we have shown that

- the random variables x and y will be uncorrelated if and only if $\rho = 0$, and
- $\rho = 0$ if and only if the random variables x and y are independent.

Therefore, we may conclude that x and y will be independent if and only if they are uncorrelated.

II. Recursion relation for moments of a Gaussian random variable.

Let $y \sim N(\mu, \sigma^2)$ be a Guassian random variable with mean μ and variance σ^2. We will establish formulas for computing all of the moments of y, i.e.

$$E\{y^n\}$$

for all positive integers n. By definition, the first moment is the mean,

$$E\left\{y^1\right\} = \mu.$$

The second moment can be computed through Couch, equation (B-29),

$$\text{Var}\{y\} = E\left\{y^2\right\} - E^2\{y\} \qquad \Rightarrow \qquad E\left\{y^2\right\} = \sigma^2 + \mu^2.$$

How can we find moments of order three and higher? To do so, let us define the *characteristic function* for the random variable x, which is the Fourier transform of the pdf,

$$\Phi_{\mathrm{y}}(\omega) = \int_{-\infty}^{\infty} f_{\mathrm{y}}(y)\, e^{j\omega y}\, dy = E\left\{e^{j\omega y}\right\},$$

which, we recognize, is equal to the Fourier transform of the probability density function, except for a change of sign in the exponent.

At this point, you may be wondering "Why on earth would I ever want to compute the Fourier transform of a pdf?" The answer is that the characteristic function allows us to compute the moments of y. Note that the n^{th} derivative of the characteristic function is

$$\begin{aligned}
\frac{d^n}{d\omega^n}\Phi_{\mathrm{y}}(\omega) &= \int_{-\infty}^{\infty} f_{\mathrm{y}}(y)\,(jy)^n\,e^{j\omega y}\,dy \\
\Rightarrow\ \frac{1}{j^n}\frac{d^n}{d\omega^n}\Phi_{\mathrm{y}}(\omega)\bigg|_{\omega=0} &= \int_{-\infty}^{\infty} f_{\mathrm{y}}(y)\,y^n\,dy \\
&= E\left\{\mathrm{y}^n\right\}.
\end{aligned}$$

Therefore, we can compute all of the moments of y, by taking derivatives of the characteristic function, setting $\omega = 0$, and dividing by j^n.

Let $x \sim N(0,1)$ be a Gaussian random variable with mean 0 and variance 1. The characteristic function is

$$\begin{aligned}
\Phi_{\mathrm{x}}(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2}e^{j\omega x}\,dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2}\cos(\omega x)\,dx + \frac{j}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2}\sin(\omega x)\,dx.
\end{aligned}$$

In the second term above, the integrand is the product of an even function and an odd function, and is therefore and odd function, so the integral over the entire real line is zero. Thus we are left with

$$\Phi_{\mathrm{x}}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2}\cos(\omega x)\,dx.$$

From Couch, Section (A-5) we obtain the following definite integral.

$$\int_0^{\infty} e^{-a^2x^2}\cos(bx)\,dx = \frac{\sqrt{\pi}e^{-b^2/4a^2}}{2a}.$$

Therefore, with $a = 1/\sqrt{2}$ and $b = \omega$, the characteristic function for x is

$$\Phi_{\mathrm{x}}(\omega) = 2\cdot\frac{1}{\sqrt{2\pi}}\cdot\frac{\sqrt{\pi}e^{-\omega^2/2}}{\sqrt{2}} = e^{-\omega^2/2}.$$

Now let $y = \mu + \sigma x$, so that the mean and variance of y are μ and σ^2, respectively,

$$\begin{aligned}
E\{y\} &= E\{\mu + \sigma x\} \\
&= \mu + \sigma E\{x\} \\
&= \mu \\
E\left\{y^2\right\} &= E\left\{\mu^2 + 2\mu\sigma x + \sigma^2 x^2\right\} \\
&= \mu^2 + 2\mu\sigma E\{x\} + \sigma^2 E\left\{x^2\right\} \\
&= \mu^2 + \sigma^2 \\
\mathrm{Var}\{y\} &= E\left\{y^2\right\} - E^2\{y\} \\
&= \mu^2 + \sigma^2 - \mu^2 \\
&= \sigma^2.
\end{aligned}$$

The characteristic function for y can be expressed in terms of the characteristic function for x,

$$\Phi_{\mathrm{y}}(\omega) = E\left\{e^{j\omega y}\right\} = E\left\{e^{j\omega(\mu+\sigma x)}\right\} = e^{j\mu\omega}\, E\left\{e^{j\omega\sigma x}\right\} = e^{j\mu\omega}\,\Phi_{\mathrm{x}}(\omega\sigma).$$

Therefore,

$$\Phi_{\mathrm{y}}(\omega) = e^{j\mu\omega - \sigma^2\omega^2/2}.$$

We can express this function as

$$\Phi_{\mathrm{y}}(\omega) = e^{g(\omega)} \qquad \text{where} \qquad g(\omega) = j\mu\omega - \frac{\sigma^2\omega^2}{2}.$$

With this result, we can generate the moments of y, beginning with the mean. The first derivative of the characteristic function is

$$\begin{aligned}
\frac{d\Phi_{\mathrm{y}}}{d\omega} &= e^{g(\omega)} \cdot \frac{dg}{d\omega} \\
&= e^{g(\omega)} \cdot \left(j\mu - \sigma^2\omega\right) \\
\Rightarrow E\{y\} &= \frac{1}{j}\, e^{g(0)} \cdot (j\mu - 0) \\
g(0) &= 1 \\
\Rightarrow E\{y\} &= \mu.
\end{aligned}$$

Before generating the second moment, notice that the derivative of the characteristic function can be expressed as

$$\frac{d\Phi_{\mathrm{y}}}{d\omega} = \Phi_{\mathrm{y}}(\omega) \cdot \frac{dg}{d\omega}. \tag{1}$$

This property holds for any single Gaussian random variable. If we take the derivative equation (1),

$$\frac{d^2\Phi_{\mathrm{y}}}{d\omega^2} = \frac{d\Phi_{\mathrm{y}}}{d\omega} \cdot \frac{dg}{d\omega} + \Phi_{\mathrm{y}}(\omega) \cdot \frac{d^2 g}{d\omega^2}. \tag{2}$$

This expression is useful, because it expresses the second derivative in terms of the first derivative and the function itself. We could proceed to substitute the expressions for $\Phi_{\mathrm{y}}(\omega)$ and its derivative on the right side, but that would only serve to complicate the notation. Instead, we can directly evaluate the second moment as follows,

$$\begin{aligned}
\left.\frac{d^2\Phi_{\mathrm{y}}}{d\omega^2}\right|_{\omega=0} &= \left.\frac{d\Phi_{\mathrm{y}}}{d\omega}\right|_{\omega=0} \cdot \left.\frac{dg}{d\omega}\right|_{\omega=0} + \Phi_{\mathrm{y}}(0) \cdot \left.\frac{d^2 g}{d\omega^2}\right|_{\omega=0} \\
\Phi_{\mathrm{y}}(0) &= 1 \\
\left.\frac{d\Phi_{\mathrm{y}}}{d\omega}\right|_{\omega=0} &= \left.\frac{dg}{d\omega}\right|_{\omega=0} = j\mu \\
\frac{d^2 g}{d\omega^2} &= -\sigma^2
\end{aligned}$$

$$\Rightarrow \left.\frac{d^2\Phi_{\mathrm{y}}}{d\omega^2}\right|_{\omega=0} = j\mu \cdot j\mu + 1\left(-\sigma^2\right) = -\mu^2 - \sigma^2$$

$$E\left\{y^2\right\} = \frac{1}{j^2} \cdot \left.\frac{d^2\Phi_{\mathrm{y}}}{d\omega^2}\right|_{\omega=0} = \mu^2 + \sigma^2.$$

This means that the variance is

$$\mathrm{Var}\left\{y\right\} = E\left\{y^2\right\} - E^2\left\{y\right\} = \sigma^2.$$

To this point, we have only verified what we already know: that the mean of y is μ and the variance is σ^1, but we can extend these results to all moments. Return to equation (2), and take the derivative, to obtain,

$$\frac{d^3\Phi_{\mathrm{y}}}{d\omega^3} = \frac{d^2\Phi_{\mathrm{y}}}{d\omega^2} \cdot \frac{dg}{d\omega} + 2 \cdot \frac{d\Phi_{\mathrm{y}}}{d\omega} \cdot \frac{d^2g}{d\omega^2} + \Phi_{\mathrm{y}}(\omega) \cdot \frac{d^3g}{d\omega^3},$$

but

$$\frac{d^3g}{d\omega^3} = 0,$$

therefore

$$\frac{d^3\Phi_{\mathrm{y}}}{d\omega^3} = \frac{d^2\Phi_{\mathrm{y}}}{d\omega^2} \cdot \frac{dg}{d\omega} + 2 \cdot \frac{d\Phi_{\mathrm{y}}}{d\omega} \cdot \frac{d^2g}{d\omega^2}. \tag{3}$$

We can use this result to find the third moment of y,

$$\begin{aligned}
\left.\frac{d^3\Phi_{\mathrm{y}}}{d\omega^3}\right|_{\omega=0} &= \left(-\mu^2 - \sigma^2\right)(j\mu) + 2\,(j\mu)\left(-\sigma^2\right) \\
&= -j\mu^3 - j3\mu\sigma^2 \\
E\left\{y^3\right\} &= \frac{1}{j^3} \cdot \left.\frac{d^3\Phi_{\mathrm{y}}}{d\omega^3}\right|_{\omega=0} \\
&= \mu^3 + 3\mu\sigma^2.
\end{aligned}$$

We could proceed to take further derivatives of the characteristic function and generate as many moments as we like, but can we determine a formula for the general case? To do so, consider rewriting equation (3) as a formula for the n^{th} derivative of $\Phi_{\mathrm{y}}(\omega)$, where in this case $n = 3$,

$$\frac{d^n\Phi_{\mathrm{y}}}{d\omega^n} = \frac{d^{n-1}\Phi_{\mathrm{y}}}{d\omega^{n-1}} \cdot \frac{dg}{d\omega} + (n-1) \cdot \frac{d^{n-2}\Phi_{\mathrm{y}}}{d\omega^{n-2}} \cdot \frac{d^2g}{d\omega^2}. \tag{4}$$

Is this formula valid for $n > 3$? To find out, take the derivative of equation (4),

$$\frac{d^{n+1}\Phi_{\mathrm{y}}}{d\omega^{n+1}} = \frac{d^n\Phi_{\mathrm{y}}}{d\omega^n} \cdot \frac{dg}{d\omega} + \frac{d^{n-1}\Phi_{\mathrm{y}}}{d\omega^{n-1}} \cdot \frac{d^2g}{d\omega^2}$$

$$
\begin{aligned}
& +(n-1)\cdot\frac{d^{n-1}\Phi_{\mathrm{y}}}{d\omega^{n-1}}\cdot\frac{d^2g}{d\omega^2}+(n-1)\cdot\frac{d^{n-1}\Phi_{\mathrm{y}}}{d\omega^{n-1}}\cdot\frac{d^3g}{d\omega^3} \\
\frac{d^3g}{d\omega^3} &= 0 \\
\Rightarrow \frac{d^{n+1}\Phi_{\mathrm{y}}}{d\omega^{n+1}} &= \frac{d^{n}\Phi_{\mathrm{y}}}{d\omega^{n}}\cdot\frac{dg}{d\omega}+n\cdot\frac{d^{n-1}\Phi_{\mathrm{y}}}{d\omega^{n-1}}\cdot\frac{d^2g}{d\omega^2}.
\end{aligned}
$$

What we have obtained is a version of equation (4), except that the indices on the derivatives of the characteristic function have all increased by 1. In other words, we have shown that if equation (4) is true for order n, then it is also true for order $n+1$. And we have already demonstrated that equation (4) is valid for $n=3$, so we may now conclude that it is true for $n=4,, 5, \ldots$. This is a *proof by induction*, that equation (4) is a general *recursion relation* for generating derivatives.

Next, we can use equation (4) to establish a recursion relation for moments

$$
\begin{aligned}
E\left\{y^n\right\} &= \frac{1}{j^n}\cdot\left.\frac{d^n\Phi_{\mathrm{y}}}{d\omega^n}\right|_{\omega=0} \\
&= \frac{1}{j^n}\cdot j^{n-1}E\left\{y^{n-1}\right\}(j\mu)+\frac{1}{j^n}\cdot(n-1)j^{n-2}E\left\{y^{n-2}\right\}\left(-\sigma^2\right) \\
E\left\{y^n\right\} &= \mu E\left\{y^{n-1}\right\}+\sigma^2(n-1)E\left\{y^{n-2}\right\}. \qquad (5)
\end{aligned}
$$

Equation (5) applies for all $n \geq 3$,

$$
\begin{aligned}
E\left\{y^3\right\} &= \mu E\left\{y^2\right\}+2\sigma^2E\left\{y\right\} \\
&= \mu\left(\mu^2+\sigma^2\right)+2\mu\sigma^2 \\
&= \mu^3+3\mu\sigma^2
\end{aligned}
$$

$$
\begin{aligned}
E\left\{y^4\right\} &= \mu E\left\{y^3\right\}+3\sigma^2E\left\{y^2\right\} \\
&= \mu\left(\mu^3+3\mu\sigma^2\right)+3\sigma^2\left(\mu^2+\sigma^2\right) \\
&= \mu^4+6\mu^2\sigma^2+3\sigma^4
\end{aligned}
$$

$$
\begin{aligned}
E\left\{y^5\right\} &= \mu E\left\{y^4\right\}+4\sigma^2E\left\{y^3\right\} \\
&= \mu\left(\mu^4+6\mu^2\sigma^2+3\sigma^4\right)+4\sigma^2\left(\mu^3+3\mu\sigma^2\right) \\
&= \mu^5+10\mu^3\sigma^2+15\mu\sigma^4,
\end{aligned}
$$

etc.