Homework Set 5 EE 1473

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2. Let w and x be two independent random variables, with means and variances

$\mu_w = E\{w\}$ $\mu_x = E\{x\}$		$\sigma^2_w = Var\{w\}$ $\sigma^2_x = Var\{x\}$

respectively. Determine the mean of each of the following random variables.

(a) The random variables $y = x^2$ and $z = w^2$.

$$\sigma_x^2 = E[x^2] - E[x]^2$$

$$\Rightarrow \sigma_x^2 = \mu_y - \mu_x^2$$

$$\Rightarrow \boxed{\mu_y = \sigma_x^2 + \mu_x^2}$$

likewise $\rightarrow$ $\boxed{\mu_z = \sigma_w^2 + \mu_w^2}$

(b) The random variables s = w + x and d = w − x.

$$\boxed{\begin{aligned} \mu_s &= \mu_w + \mu_x \\ \mu_d &= \mu_w - \mu_x \end{aligned}}$$

(c) The random variables $a = s^2$, $b = s*d$ and $c = d^2$.

$$E[a] = E[s^2] = E\{(w+x)^2\}$$

$$\Rightarrow E(w^2) + E(x^2) + E\{2wx\}$$

$$\Rightarrow \mu_y + \mu_z + 2\mu_w\mu_x = \mu_a$$

$$\boxed{\mu_a = \sigma_x^2 + \mu_x^2 + \sigma_w^2 + \mu_w + 2\mu_w\mu_x}$$

$$E\{b\} = E\{sd\} = \mu_s\mu_d$$

$$\Rightarrow (\mu_w + \mu_x)(\mu_w - \mu_x)$$

$$\Rightarrow \boxed{\mu_w^2 - \mu_x^2 = \mu_b}$$

$$E\{c\} = E\{d^2\} = E\{(w-x)^2\}.$$

$$\Rightarrow E\{w^2\} - 2E\{wx\} + E\{x^2\}$$

$$\Rightarrow E\{w^2\} - 2E\{wx\} + E\{x^2\}$$

$$\boxed{\mu_c = \sigma_x^2 + \mu_x^2 + \sigma_w^2 + \mu_w^2 - 2\mu_w\mu_x}$$

(d) The random variable r = 2a + 3b + 2c – y + z.

$$E\{r\} = 2E\{a\} + 3E\{b\} + 2E\{c\} - E\{y\} + E\{z\}$$

$$\Rightarrow 2\mu_a + 3\mu_b + 2\mu_c - \mu_y + \mu_z$$

$$\Rightarrow 4(\sigma_x^2 + \mu_x^2 + \sigma_w^2 + \mu_w^2) + 3(\mu_w^2 - \mu_x^2) - (\sigma_x^2 + \mu_x^2) + (\sigma_w^2 + \mu_w^2)$$

$$\boxed{3\sigma_x^2 + 5\sigma_w^2 + 8\mu_w^2}$$

3. Let the random process x(t) is defined as

x(t) = s(t) + w(t),

where s(t) is a rectangular pulse,

***s(t) = { A 0<= t <=T, 0* elsewhere},**

and w(t) is a WGN process with mean E{w(t)} = 0 and autocorrelation function

$R_w(t, t+\tau) = E\{w(t)w(t+\tau)\} = \sigma^2\delta(\tau)$.

(a) Determine the mean of x(t).

$$E\{x(t)\} = E\{s(t) + w(t)\} = E\{s(t)\} + E\{w(t)\}$$

$\mu_w = 0 \qquad \mu_s = s(t)$ (deterministic)

$$\boxed{E\{x(t)\} = s(t)}$$

(b) Determine the autocorrelation function for x(t),

$$R_x(\tau) = E\{x(t)x(t+\tau)\}$$

$$\Rightarrow E\{[s(t) + w(t)][s(t+\tau) + w(t+\tau)]\}$$

$$\Rightarrow E\{s(t)s(t+\tau)\} + E\{s(t)w(t+\tau)\} + E\{w(t)s(t+\tau)\} + E\{w(t)w(t+\tau)\}$$

$$E\{s(t)s(t+\tau)\} + s(t)\underbrace{E\{w(t+\tau)\}}_{0} + s(t+\tau)\underbrace{E\{w(t)\}}_{0} + R_w(\tau)$$

$$\Rightarrow \int_{-\infty}^{\infty} s(t)s(t+\tau)\,dt + \sigma^2\delta(\tau)$$

$$\boxed{R_x(\tau) = \begin{cases} A(T-\tau) + \sigma^2\delta(\tau) & -T < \tau < T \\ 0 & \text{---} \end{cases}}$$

x(t) is WSS, $R_x(\tau)$ is only dependent on τ, and the mean is constant for $[0 < t < T]$

$X(t)$ is WSS, $R_x(\tau)$ is only dependent on τ, and the mean is constant for $\boxed{0<t<T}$

(c) Suppose that x(t) is processed by integrating it over 0 ≤ t ≤ T to produce the random variable x_T, Compute d^2. Is it possible to choose T as small as possible, while simultaneously maximizing d^2?

$$X_T = \int_0^T x(t)\,dt \quad d^2 = \frac{E\{X_T\}^2}{Var\{X_T\}} \quad Var\{X_T\} = E\{X_T\}^2 - E\{X_T^2\}$$

$$X_T = \int_0^T s(t) + \int_0^T w(t) = AT + \int_0^T w(t)\,dt$$

$$X_T^2 = \left(\int_0^T s(t)\right)^2 + 2\left(\int_0^T s(t)\right)\left(\int_0^T w(t)\right) + \left(\int_0^T w(t)\right)^2$$

$$E\{X_T\} = E\{AT\} + \int_0^T E\{w(t)\}dt = AT$$

$$E\{X_T^2\} = E\{AT^2\} + 2[0] + R_w(0)$$

$$\Rightarrow d^2 = \frac{E\{X_T\}^2}{Var\{X_T\}} = \frac{(AT)^2}{AT^2 - AT^2 - \sigma^2}$$

$$\boxed{d^2 = \frac{(AT)^2}{\sigma^2}}$$

d^2 is maximized for small T by increasing Amplitude A of the pulse and by reducing channel noise variance

4. Let the random process y(t) be a linear function of t, observed in AWGN,

y(t) = mt + w(t).

where m is the slope of the linear term. The noise w(t) is a Gaussian random process with mean

$\mu_w(t) = E\{w(t)\} = 0,$

and autocorrelation function,

$R_w(t, u) = E\{w(t)w(u)\} = \sigma^2\delta(t - u).$

(a) Determine the mean of y(t), $\mu_y(t) = E\{y(t)\}$.

$$E\{y(t)\} = E\{mt\} + E\{w(t)\}$$

$$\Rightarrow \boxed{mt}$$

$\Rightarrow \boxed{mt}$

(b) Determine the autocorrelation function for y(t), Ry(t, u) = E{y(t)y(u)}. Is y(t) wide-sense stationary? Why or why not?

$R_y(t,u) = E\{y(t)y(u)\} = E\{m^2tu + mtw(u) + muw(t) + w(t)w(u)\}$

$\Rightarrow \boxed{m^2tu}$ • This is not WSS

→ The function is not even $R_x(\tau) \neq R_y(-\tau)$

→ There are values $\tau > 0$ where $R_y(\tau) > R_y(0)$

(c) Now suppose that we wish to estimate the value of m, by observing y(t) for 0 ≤ t ≤ T. Let m^ be the estimate, which is computed as follows,

$$\hat{m} = \frac{2}{T^2}\int_0^T y(t)\,dt$$

Show that the mean of the estimate is
E {m^ } = m.
Is this a good property for the estimate to have? Why or why not?

$$E\{\hat{m}\} = E\left\{\frac{2}{T^2}\int_0^T y(t)\,dt\right\} = \frac{2}{T^2}\int_0^T E\{y(t)\}\,dt$$

$$\Rightarrow \frac{2}{T^2}\int_0^T m_y(t)\,dt = \frac{2}{T^2}\,\frac{mt^2}{2}\Big|_0^T = \frac{\cancel{2}}{\cancel{T^2}}\,\frac{m\cancel{T^2}}{\cancel{2}}$$

$\boxed{m}$ The goal was to estimate m, so it is a good property that the average is m, and is independent of t and T

(d) Determine a constraint on T such that Var {m^ } < kσ².

$$Var\{\hat{m}\} = E\{\hat{m}\}^2 - E\{\hat{m}^2\}$$

$$E\{\hat{m}^2\} = E\left\{\left(\frac{2}{T^2}\int_0^T y(t)\,dt\right)^2\right\}$$

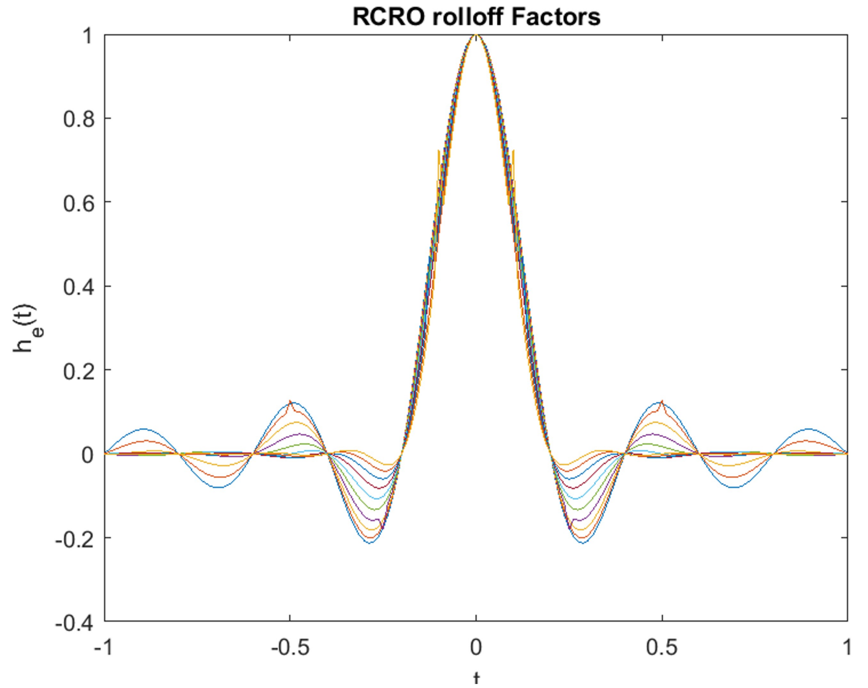
$$\Rightarrow \frac{4}{T^4}E\left\{\left(\int_0^T mt\right)^2 + 2\int_0^T mt(w(t)) + \left(\int_0^T w(t)\right)^2\right.$$

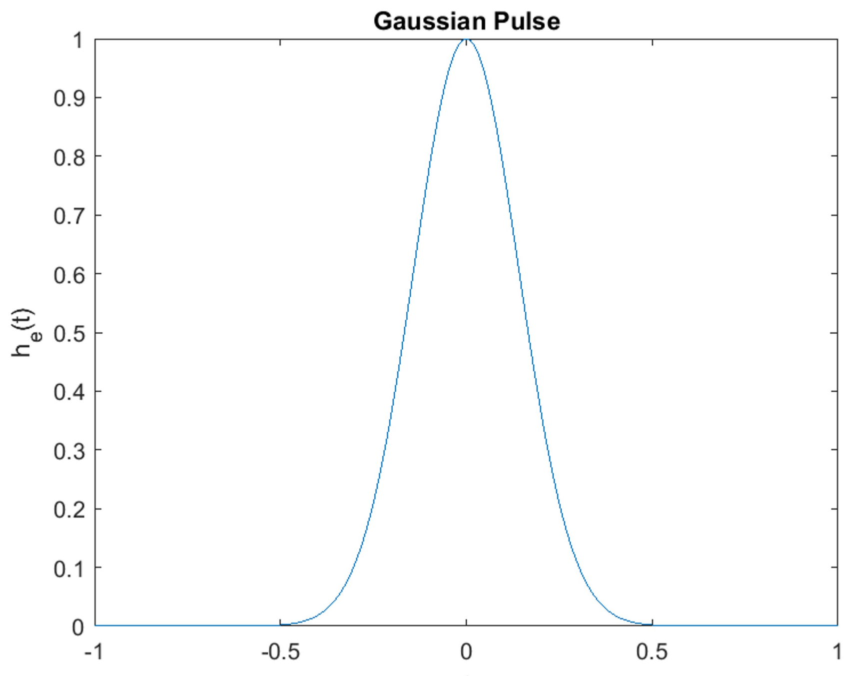
$$\Rightarrow \left(\frac{4}{T^4}\right)\left[\left(\frac{mT^2}{2}\right)^2 + \sigma^2\right] = m^2 + \frac{4\sigma^2}{T^4}$$

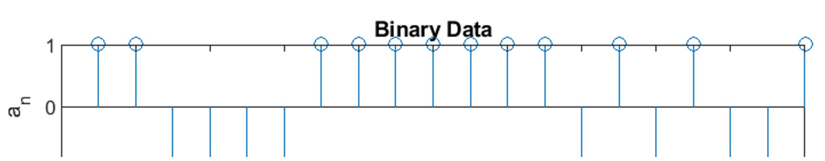
$$Var\{\hat{m}\} = \frac{4\sigma^2}{T^4} < k\sigma^2$$

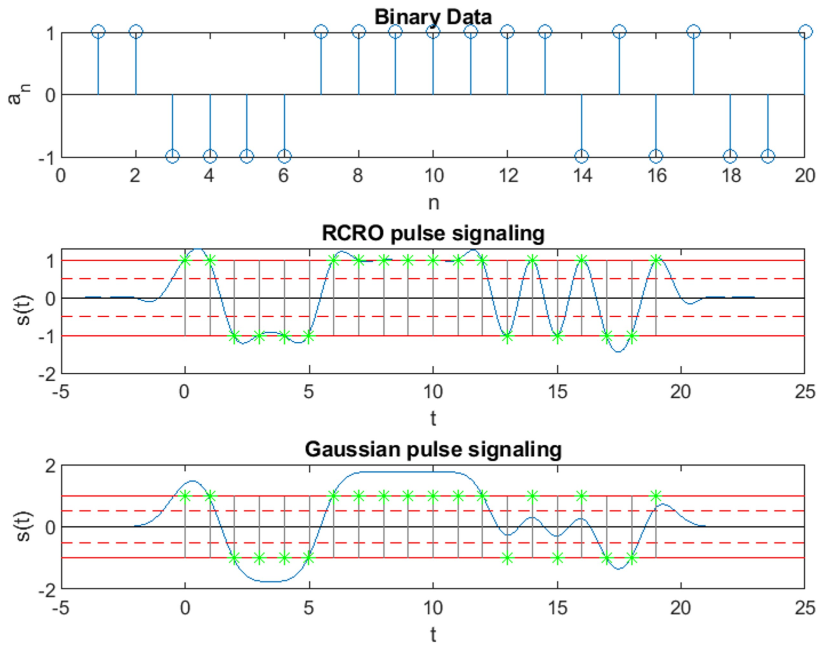
$$4 < kT^4 \qquad \boxed{T > \left(\frac{k}{4}\right)^{1/4}}$$

MATLAB Plots problem 1









It is clear in the figures that the RCRO containg zero ISI. There is ISI in the gaussian pulse when there are three or more alternating bits, when neither adjacent bit is the same value.