Johns Hopkins Engineering

Power Electronics 525.725

Module 6 Lecture 6

Small Signal Modeling

7.1. Introduction

Objective: maintain $v(t)$ equal to an accurate, constant value V.

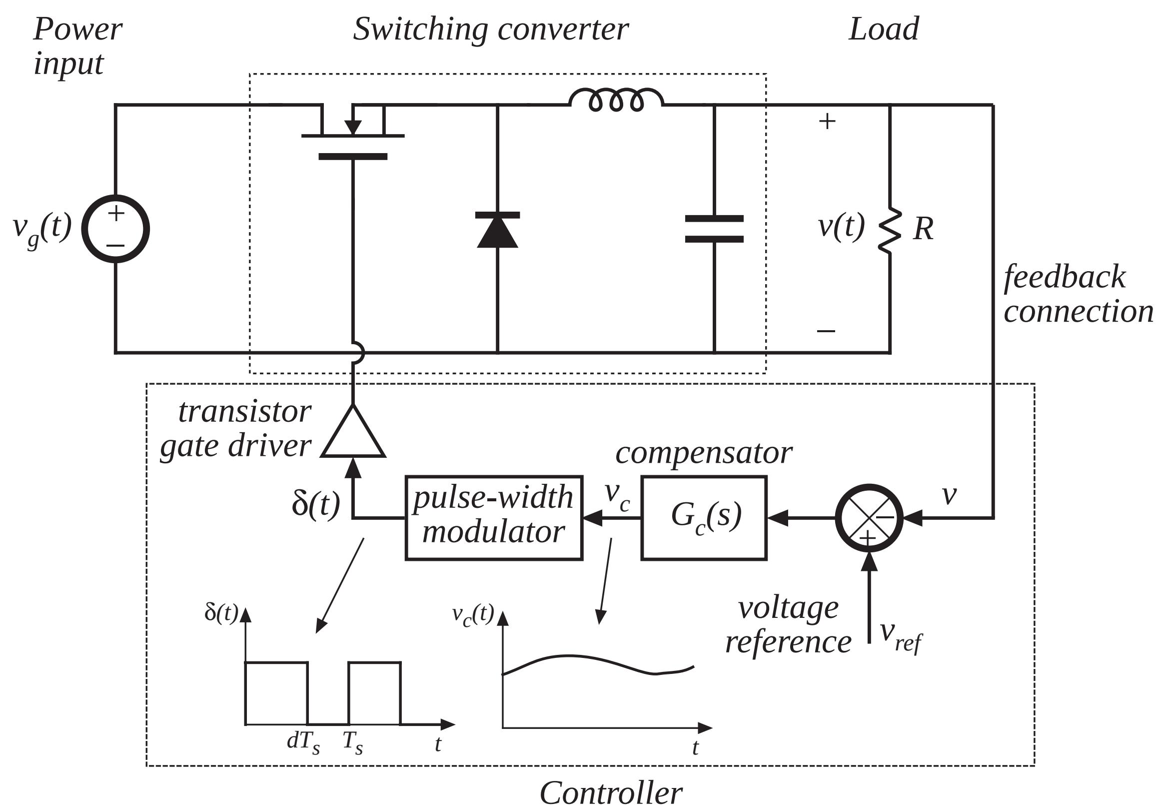
There are disturbances:

- in $v_g(t)$
- in R

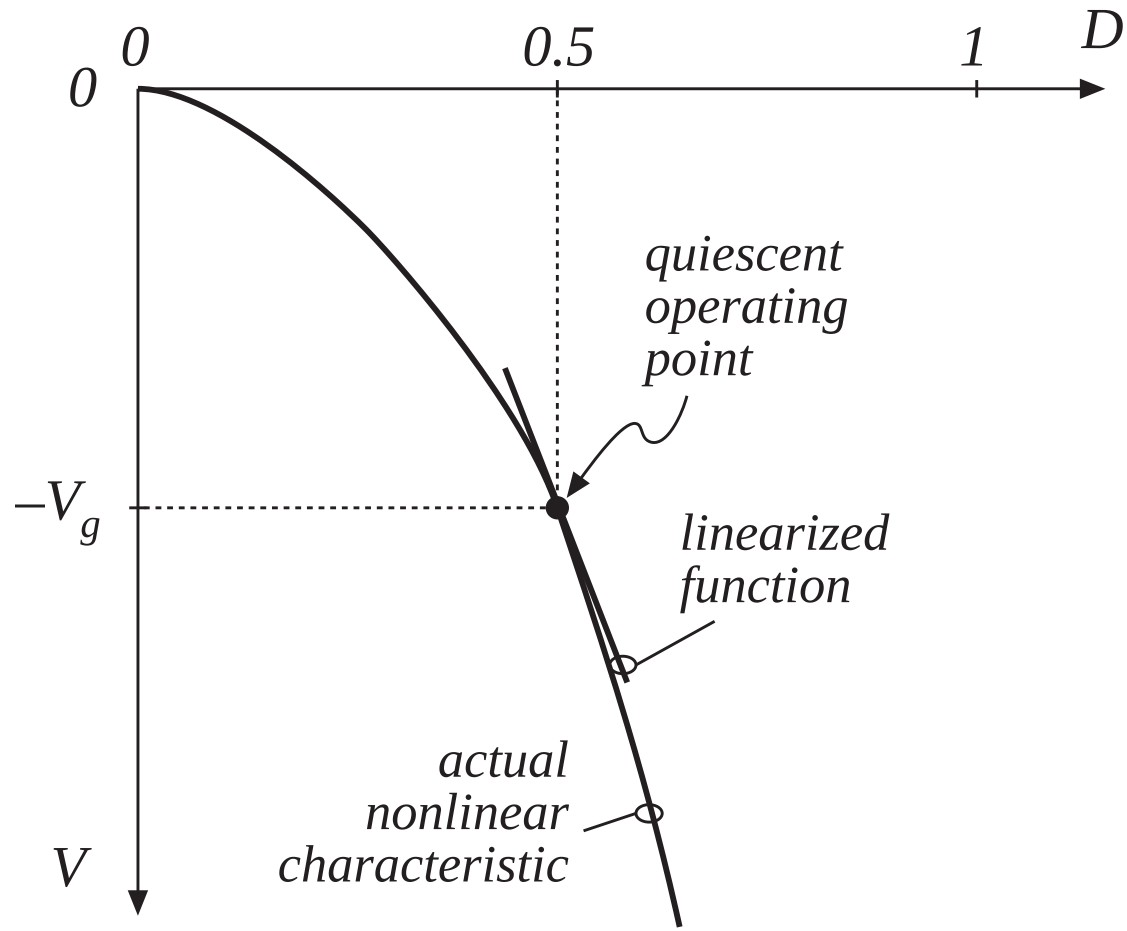
There are uncertainties:

- in element values
- in V_g
- in R

A simple dc-dc regulator system, employing a buck converter



Buck-boost converter: nonlinear static control-to-output characteristic

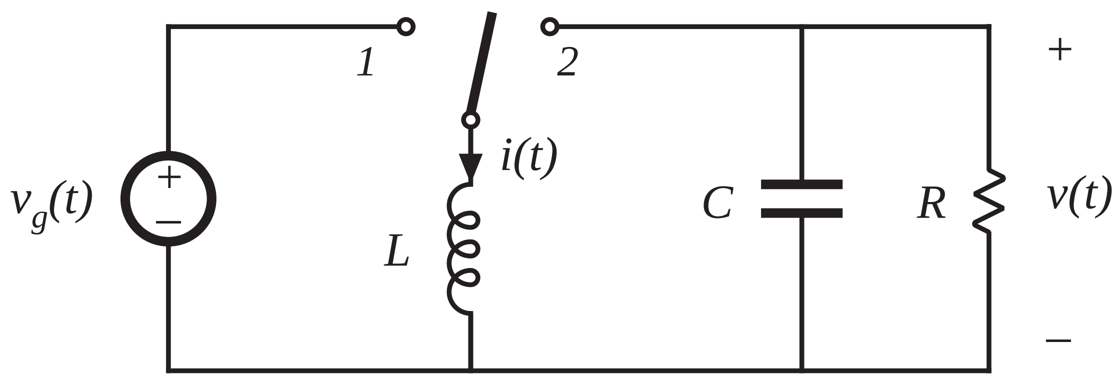


Example: linearization at the quiescent operating point

$$D = 0.5$$

7.2. The basic ac modeling approach

Buck-boost converter example



Buck Boost Large Signal Model Equations

Converter averaged equations:

$$L\,\frac{d\left\langle i(t)\right\rangle_{T_s}}{dt} = d(t)\,\left\langle v_g(t)\right\rangle_{T_s} + d'(t)\,\left\langle v(t)\right\rangle_{T_s}$$

$$C\,\frac{d\left\langle v(t)\right\rangle_{T_s}}{dt} = -\,d'(t)\,\left\langle i(t)\right\rangle_{T_s} - \frac{\left\langle v(t)\right\rangle_{T_s}}{R}$$

$$\left\langle i_g(t)\right\rangle_{T_s} = d(t)\,\left\langle i(t)\right\rangle_{T_s}$$

—nonlinear because of multiplication of the time-varying quantity $d(t)$ with other time-varying quantities such as $i(t)$ and $v(t)$.

Construct small-signal model: Linearize about quiescent operating point

If the converter is driven with some steady-state, or quiescent, inputs

$$d(t) = D$$
$$\left\langle v_g(t) \right\rangle_{T_s} = V_g$$

then, from the analysis of Chapter 2, after transients have subsided the inductor current, capacitor voltage, and input current

$$\left\langle i(t) \right\rangle_{T_s}, \left\langle v(t) \right\rangle_{T_s}, \left\langle i_g(t) \right\rangle_{T_s}$$

reach the quiescent values I, V, and I_g, given by the steady-state analysis as

$$V = -\frac{D}{D'} V_g$$
$$I = -\frac{V}{D' R}$$
$$I_g = D\, I$$

Perturbation

So let us assume that the input voltage and duty cycle are equal to some given (dc) quiescent values, plus superimposed small ac variations:

$$\left\langle v_g(t) \right\rangle_{T_s} = V_g + \hat{v}_g(t)$$
$$d(t) = D + \hat{d}(t)$$

In response, and after any transients have subsided, the converter dependent voltages and currents will be equal to the corresponding quiescent values, plus small ac variations:

$$\left\langle i(t) \right\rangle_{T_s} = I + \hat{i}(t)$$
$$\left\langle v(t) \right\rangle_{T_s} = V + \hat{v}(t)$$
$$\left\langle i_g(t) \right\rangle_{T_s} = I_g + \hat{i}_g(t)$$

The small-signal assumption

If the ac variations are much smaller in magnitude than the respective quiescent values,

$$\left| \hat{v}_g(t) \right| << \left| V_g \right|$$
$$\left| \hat{d}(t) \right| << \left| D \right|$$
$$\left| \hat{i}(t) \right| << \left| I \right|$$
$$\left| \hat{v}(t) \right| << \left| V \right|$$
$$\left| \hat{i}_g(t) \right| << \left| I_g \right|$$

then the nonlinear converter equations can be linearized.

Perturbation of inductor equation

Insert the perturbed expressions into the inductor differential equation:

$$L\,\frac{d\left(I+\hat{i}(t)\right)}{dt}=\left(D+\hat{d}(t)\right)\left(V_g+\hat{v}_g(t)\right)+\left(D'-\hat{d}(t)\right)\left(V+\hat{v}(t)\right)$$

note that $d'(t)$ is given by

$$d'(t)=\left(1-d(t)\right)=1-\left(D+\hat{d}(t)\right)=D'-\hat{d}(t) \qquad \text{with } D'=1-D$$

Multiply out and collect terms:

$$L\left(\cancelto{0}{\frac{dI}{dt}}+\frac{d\,\hat{i}(t)}{dt}\right)=\underbrace{\left(DV_g+D'V\right)}_{\textit{Dc terms}}+\underbrace{\left(D\hat{v}_g(t)+D'\hat{v}(t)+\left(V_g-V\right)\hat{d}(t)\right)}_{1^{st}\textit{ order ac terms (linear)}}+\underbrace{\hat{d}(t)\left(\hat{v}_g(t)-\hat{v}(t)\right)}_{2^{nd}\textit{ order ac terms (nonlinear)}}$$

The perturbed inductor equation

$$L\left(\cancelto{0}{\frac{dI}{dt}} + \frac{d\hat{i}(t)}{dt}\right) = \underbrace{\left(DV_g + D'V\right)}_{\text{Dc terms}} + \underbrace{\left(D\hat{v}_g(t) + D'\hat{v}(t) + \left(V_g - V\right)\hat{d}(t)\right)}_{\substack{1^{st}\text{ order ac terms}\\(\text{linear})}} + \underbrace{\hat{d}(t)\left(\hat{v}_g(t) - \hat{v}(t)\right)}_{\substack{2^{nd}\text{ order ac terms}\\(\text{nonlinear})}}$$

Since I is a constant (dc) term, its derivative is zero

The right-hand side contains three types of terms:

- Dc terms, containing only dc quantities
- First-order ac terms, containing a single ac quantity, usually multiplied by a constant coefficient such as a dc term. These are linear functions of the ac variations
- Second-order ac terms, containing products of ac quantities. These are nonlinear, because they involve multiplication of ac quantities

Neglect of second-order terms

$$L\left(\cancelto{0}{\frac{dI}{dt}} + \frac{d\,\hat{i}(t)}{dt}\right) = \underbrace{\left(DV_g + D'V\right)}_{\textit{Dc terms}} + \underbrace{\left(D\hat{v}_g(t) + D'\hat{v}(t) + \left(V_g - V\right)\hat{d}(t)\right)}_{\begin{array}{c}1^{st}\textit{ order ac terms}\\(\textit{linear})\end{array}} + \underbrace{\hat{d}(t)\left(\hat{v}_g(t) - \hat{v}(t)\right)}_{\begin{array}{c}2^{nd}\textit{ order ac terms}\\(\textit{nonlinear})\end{array}}$$

Provided

$$\left|\hat{v}_g(t)\right| << \left|V_g\right|$$
$$\left|\hat{d}(t)\right| << \left|D\right|$$
$$\left|\hat{i}(t)\right| << \left|I\right|$$
$$\left|\hat{v}(t)\right| << \left|V\right|$$
$$\left|\hat{i}_g(t)\right| << \left|I_g\right|$$

then the second-order ac terms are much smaller than the first-order terms. For example,

$$\left|\hat{d}(t)\,\hat{v}_g(t)\right| << \left|D\,\hat{v}_g(t)\right| \quad \text{when} \quad \left|\hat{d}(t)\right| << D$$

So neglect second-order terms.
Also, dc terms on each side of equation are equal.

Linearized inductor equation

Upon discarding second-order terms, and removing dc terms (which add to zero), we are left with

$$L\,\frac{d\hat{i}(t)}{dt} = D\hat{v}_g(t) + D'\hat{v}(t) + \left(V_g - V\right)\hat{d}(t)$$

This is the desired result: a linearized equation which describes small-signal ac variations.

Note that the quiescent values D, D', V, V_g, are treated as given constants in the equation.

Capacitor equation

Perturbation leads to

$$C\,\frac{d\left(V+\hat{v}(t)\right)}{dt} = -\left(D' - \hat{d}(t)\right)\left(I + \hat{i}(t)\right) - \frac{\left(V+\hat{v}(t)\right)}{R}$$

Collect terms:

$$C\left(\cancel{\frac{dV}{dt}}^{\,0} + \frac{d\hat{v}(t)}{dt}\right) = \underbrace{\left(-D'I - \frac{V}{R}\right)}_{\textit{Dc terms}} + \underbrace{\left(-D'\hat{i}(t) - \frac{\hat{v}(t)}{R} + I\hat{d}(t)\right)}_{\substack{1^{st}\ \textit{order ac terms} \\ (\textit{linear})}} + \underbrace{\hat{d}(t)\hat{i}(t)}_{\substack{2^{nd}\ \textit{order ac term} \\ (\textit{nonlinear})}}$$

Neglect second-order terms. Dc terms on both sides of equation are equal. The following terms remain:

$$C\,\frac{d\hat{v}(t)}{dt} = -D'\hat{i}(t) - \frac{\hat{v}(t)}{R} + I\hat{d}(t)$$

This is the desired small-signal linearized capacitor equation.

Average input current

Perturbation leads to

$$I_g + \hat{i}_g(t) = \left(D + \hat{d}(t)\right)\left(I + \hat{i}(t)\right)$$

Collect terms:

$$\underbrace{I_g}_{\textit{Dc term}} + \underbrace{\hat{i}_g(t)}_{1^{st}\textit{ order ac term}} = \underbrace{\left(DI\right)}_{\textit{Dc term}} + \underbrace{\left(D\hat{i}(t) + I\hat{d}(t)\right)}_{\substack{1^{st}\textit{ order ac terms} \\ (\textit{linear})}} + \underbrace{\hat{d}(t)\hat{i}(t)}_{\substack{2^{nd}\textit{ order ac term} \\ (\textit{nonlinear})}}$$

Neglect second-order terms. Dc terms on both sides of equation are equal. The following first-order terms remain:

$$\hat{i}_g(t) = D\hat{i}(t) + I\hat{d}(t)$$

This is the linearized small-signal equation which described the converter input port.

7.2.6. Construction of small-signal equivalent circuit model

The linearized small-signal converter equations:

$$L \frac{d\hat{i}(t)}{dt} = D\hat{v}_g(t) + D'\hat{v}(t) + \left(V_g - V\right)\hat{d}(t)$$

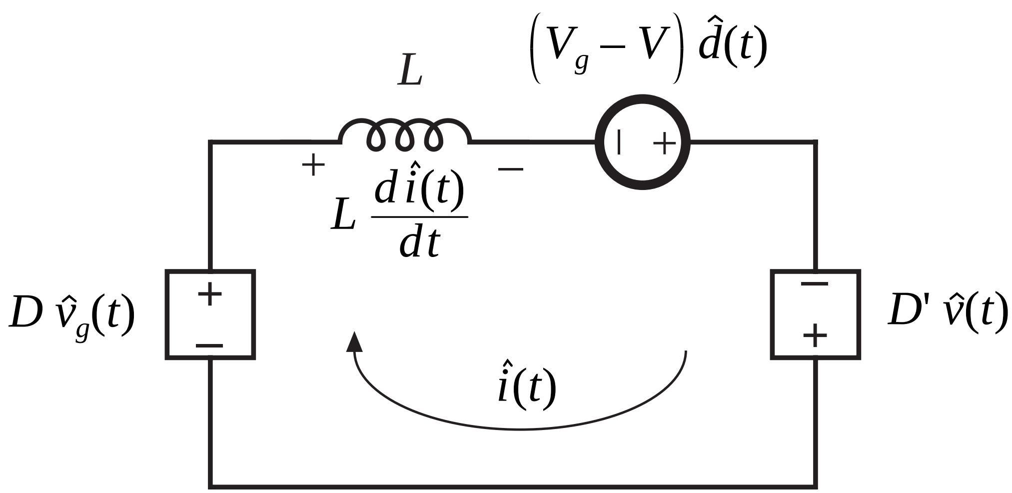
$$C \frac{d\hat{v}(t)}{dt} = -D'\hat{i}(t) - \frac{\hat{v}(t)}{R} + I\hat{d}(t)$$

$$\hat{i}_g(t) = D\hat{i}(t) + I\hat{d}(t)$$

Reconstruct equivalent circuit corresponding to these equations, in manner similar to the process used in Chapter 3.

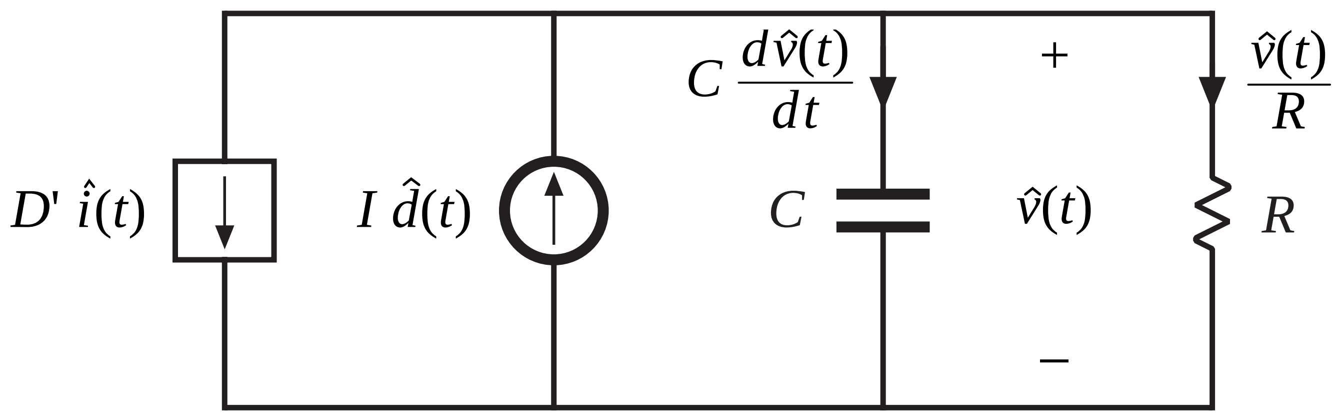
Inductor loop equation

$$L \frac{d\hat{i}(t)}{dt} = D\hat{v}_g(t) + D'\hat{v}(t) + \left(V_g - V\right)\hat{d}(t)$$



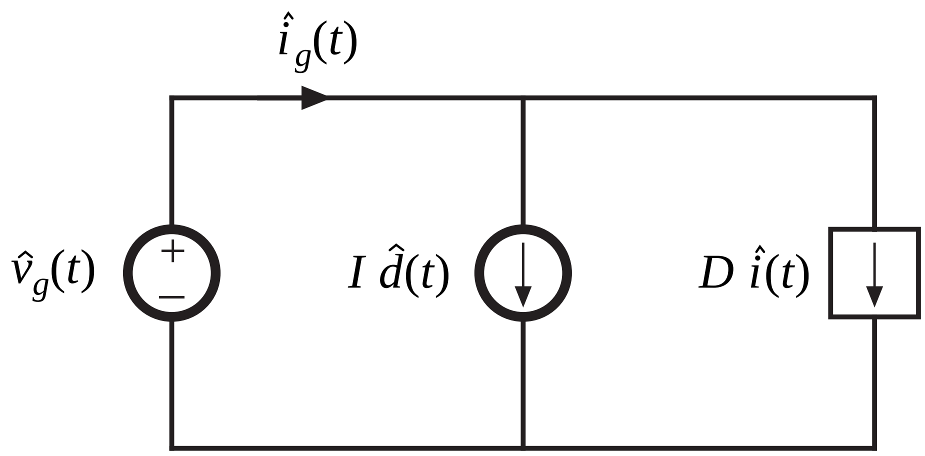
Capacitor node equation

$$C\,\frac{d\hat{v}(t)}{dt} = -\,D'\,\hat{i}(t) - \frac{\hat{v}(t)}{R} + I\hat{d}(t)$$



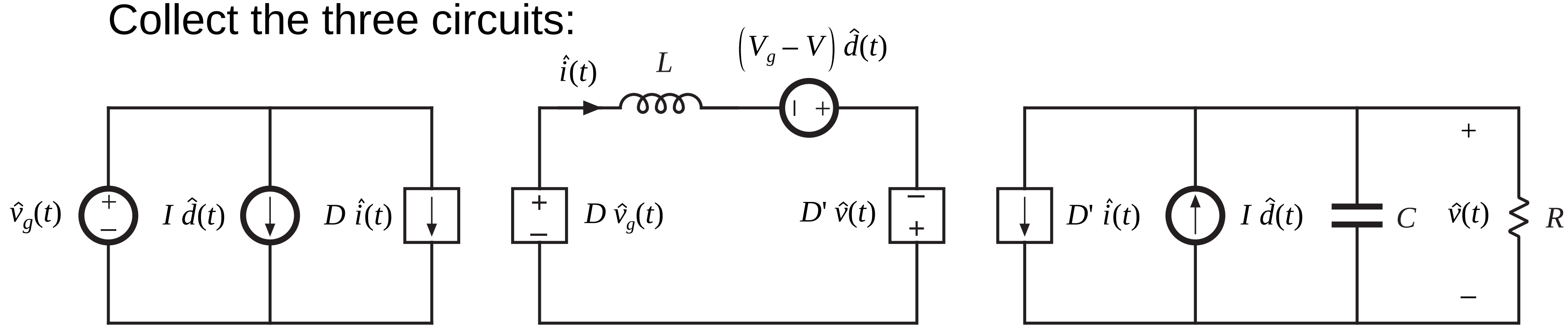
Input port node equation

$$\hat{i}_g(t) = D\hat{i}(t) + I\hat{d}(t)$$



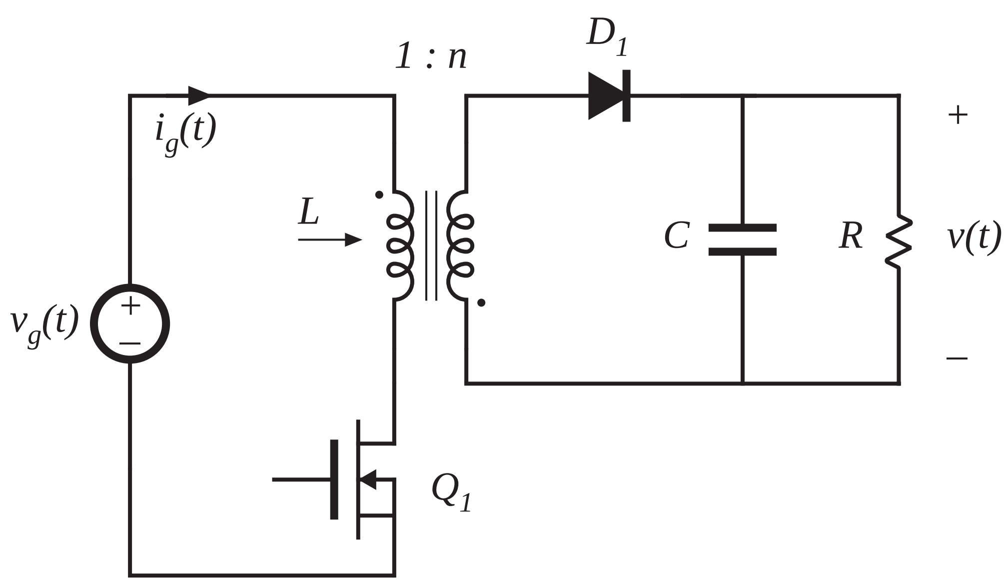
Complete Small Signal equivalent circuit

Collect the three circuits:



7.3. Example: a nonideal flyback converter

Flyback converter example



- MOSFET has on-resistance R_{on}
- Flyback transformer has magnetizing inductance L, referred to primary

The averaged converter equations

$$L\frac{d\left\langle i(t)\right\rangle_{T_s}}{dt} = d(t)\left\langle v_g(t)\right\rangle_{T_s} - d(t)\left\langle i(t)\right\rangle_{T_s} R_{on} - d'(t)\frac{\left\langle v(t)\right\rangle_{T_s}}{n}$$

$$C\frac{d\left\langle v(t)\right\rangle_{T_s}}{dt} = d'(t)\frac{\left\langle i(t)\right\rangle_{T_s}}{n} - \frac{\left\langle v(t)\right\rangle_{T_s}}{R}$$

$$\left\langle i_g(t)\right\rangle_{T_s} = d(t)\left\langle i(t)\right\rangle_{T_s}$$

LARGE SIGNAL MODEL

— a system of nonlinear differential equations

Next step: perturbation and linearization. Let

$$\left\langle v_g(t)\right\rangle_{T_s} = V_g + \hat{v}_g(t)$$

$$d(t) = D + \hat{d}(t)$$

$$\left\langle i(t)\right\rangle_{T_s} = I + \hat{i}(t)$$

$$\left\langle v(t)\right\rangle_{T_s} = V + \hat{v}(t)$$

$$\left\langle i_g(t)\right\rangle_{T_s} = I_g + \hat{i}_g(t)$$

Perturbation of the averaged inductor equation

$$L \frac{d\left\langle i(t) \right\rangle_{T_s}}{dt} = d(t) \left\langle v_g(t) \right\rangle_{T_s} - d(t) \left\langle i(t) \right\rangle_{T_s} R_{on} - d'(t) \frac{\left\langle v(t) \right\rangle_{T_s}}{n}$$

$$L \frac{d\left(I + \hat{i}(t)\right)}{dt} = \left(D + \hat{d}(t)\right) \left(V_g + \hat{v}_g(t)\right) - \left(D' - \hat{d}(t)\right) \frac{\left(V + \hat{v}(t)\right)}{n} - \left(D + \hat{d}(t)\right) \left(I + \hat{i}(t)\right) R_{on}$$

$$L \left(\cancelto{0}{\frac{dI}{dt}} + \frac{d\hat{i}(t)}{dt} \right) = \underbrace{\left(DV_g - D'\frac{V}{n} - DR_{on}I \right)}_{Dc\ terms} + \underbrace{\left(D\hat{v}_g(t) - D'\frac{\hat{v}(t)}{n} + \left(V_g + \frac{V}{n} - IR_{on}\right)\hat{d}(t) - DR_{on}\hat{i}(t) \right)}_{1^{st}\ order\ ac\ terms\ (linear)}$$

$$+ \underbrace{\left(\hat{d}(t)\hat{v}_g(t) + \hat{d}(t)\frac{\hat{v}(t)}{n} - \hat{d}(t)\hat{i}(t)R_{on} \right)}_{2^{nd}\ order\ ac\ terms\ (nonlinear)}$$

Linearization of averaged inductor equation

Dc terms:

$$0 = DV_g - D'\frac{V}{n} - DR_{on}I$$

Second-order terms are small when the small-signal assumption is satisfied. The remaining first-order terms are:

$$L\frac{d\hat{i}(t)}{dt} = D\hat{v}_g(t) - D'\frac{\hat{v}(t)}{n} + \left(V_g + \frac{V}{n} - IR_{on}\right)\hat{d}(t) - DR_{on}\hat{i}(t)$$

This is the desired linearized inductor equation.

Perturbation of averaged capacitor equation

Original averaged equation:

$$C\,\frac{d\left\langle v(t)\right\rangle_{T_s}}{dt} = d'(t)\,\frac{\left\langle i(t)\right\rangle_{T_s}}{n} - \frac{\left\langle v(t)\right\rangle_{T_s}}{R}$$

Perturb about quiescent operating point:

$$C\,\frac{d\left(V + \hat{v}(t)\right)}{dt} = \left(D' - \hat{d}(t)\right)\frac{\left(I + \hat{i}(t)\right)}{n} - \frac{\left(V + \hat{v}(t)\right)}{R}$$

Collect terms:

$$C\left(\cancelto{0}{\frac{dV}{dt}} + \frac{d\hat{v}(t)}{dt}\right) = \underbrace{\left(\frac{D'I}{n} - \frac{V}{R}\right)}_{\textit{Dc terms}} + \underbrace{\left(\frac{D'\hat{i}(t)}{n} - \frac{\hat{v}(t)}{R} - \frac{I\hat{d}(t)}{n}\right)}_{\substack{1^{st}\ \textit{order ac terms} \\ (\textit{linear})}} - \underbrace{\frac{\hat{d}(t)\hat{i}(t)}{n}}_{\substack{2^{nd}\ \textit{order ac term} \\ (\textit{nonlinear})}}$$

Linearization of averaged capacitor equation

Dc terms:

$$0 = \left(\frac{D'I}{n} - \frac{V}{R} \right)$$

Second-order terms are small when the small-signal assumption is satisfied. The remaining first-order terms are:

$$C \frac{d\hat{v}(t)}{dt} = \frac{D'\hat{i}(t)}{n} - \frac{\hat{v}(t)}{R} - \frac{I\hat{d}(t)}{n}$$

This is the desired linearized capacitor equation.

Perturbation of averaged input current equation

Original averaged equation:

$$\left\langle i_g(t) \right\rangle_{T_s} = d(t) \left\langle i(t) \right\rangle_{T_s}$$

Perturb about quiescent operating point:

$$I_g + \hat{i}_g(t) = \left(D + \hat{d}(t) \right) \left(I + \hat{i}(t) \right)$$

Collect terms:

$$\underbrace{I_g}_{Dc\ term} + \underbrace{\hat{i}_g(t)}_{1^{st}\ order\ ac\ term} = \underbrace{\left(DI \right)}_{Dc\ term} + \underbrace{\left(D\hat{i}(t) + I\hat{d}(t) \right)}_{\substack{1^{st}\ order\ ac\ terms \\ (linear)}} + \underbrace{\hat{d}(t)\hat{i}(t)}_{\substack{2^{nd}\ order\ ac\ term \\ (nonlinear)}}$$

Linearization of averaged input current equation

Dc terms:

$$I_g = DI$$

Second-order terms are small when the small-signal assumption is satisfied. The remaining first-order terms are:

$$\hat{i}_g(t) = D\hat{i}(t) + I\hat{d}(t)$$

This is the desired linearized input current equation.

Summary: dc and small-signal ac converter equations

Dc equations:

$$0 = DV_g - D'\frac{V}{n} - DR_{on}I$$

$$0 = \left(\frac{D'I}{n} - \frac{V}{R}\right)$$

$$I_g = DI$$

Small-signal ac equations:

$$L\,\frac{d\hat{i}(t)}{dt} = D\hat{v}_g(t) - D'\frac{\hat{v}(t)}{n} + \left(V_g + \frac{V}{n} - IR_{on}\right)\hat{d}(t) - DR_{on}\hat{i}(t)$$

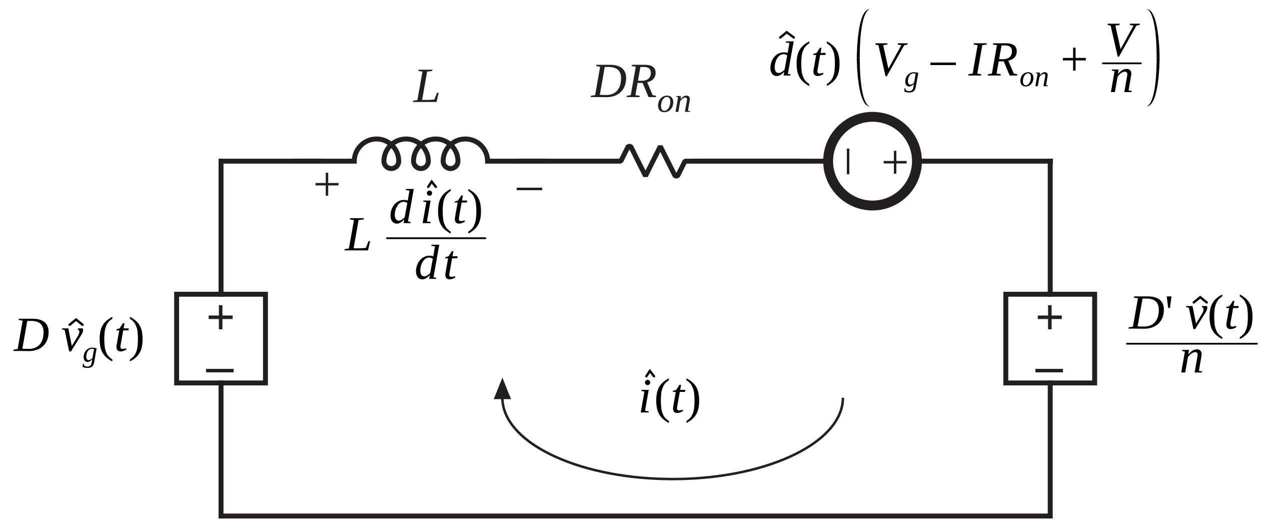
$$C\,\frac{d\hat{v}(t)}{dt} = \frac{D'\hat{i}(t)}{n} - \frac{\hat{v}(t)}{R} - \frac{I\hat{d}(t)}{n}$$

$$\hat{i}_g(t) = D\hat{i}(t) + I\hat{d}(t)$$

Next step: construct equivalent circuit models.

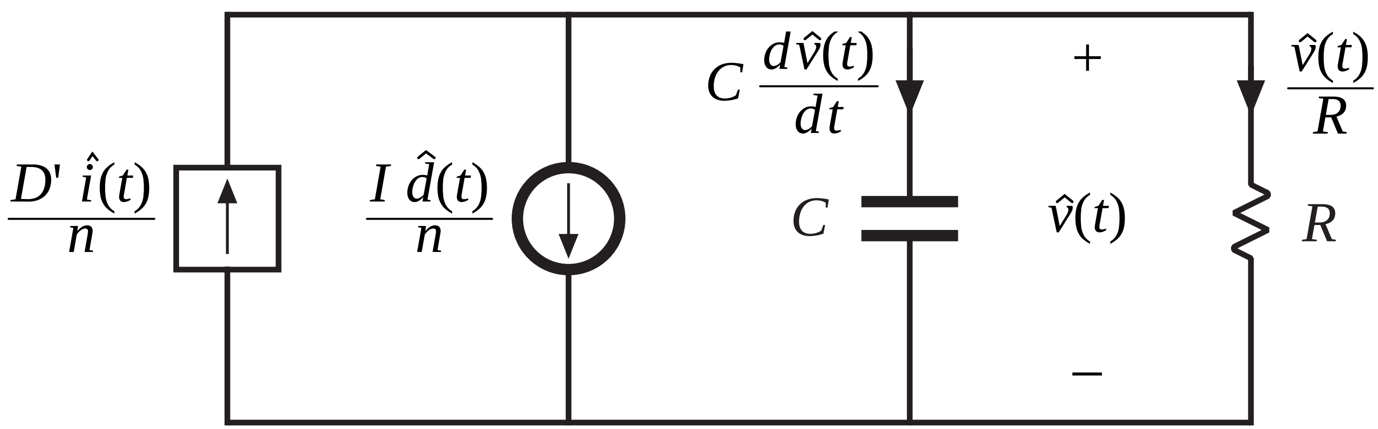
Small-signal ac equivalent circuit: inductor loop

$$L\frac{d\hat{i}(t)}{dt} = D\hat{v}_g(t) - D'\frac{\hat{v}(t)}{n} + \left(V_g + \frac{V}{n} - IR_{on}\right)\hat{d}(t) - DR_{on}\hat{i}(t)$$



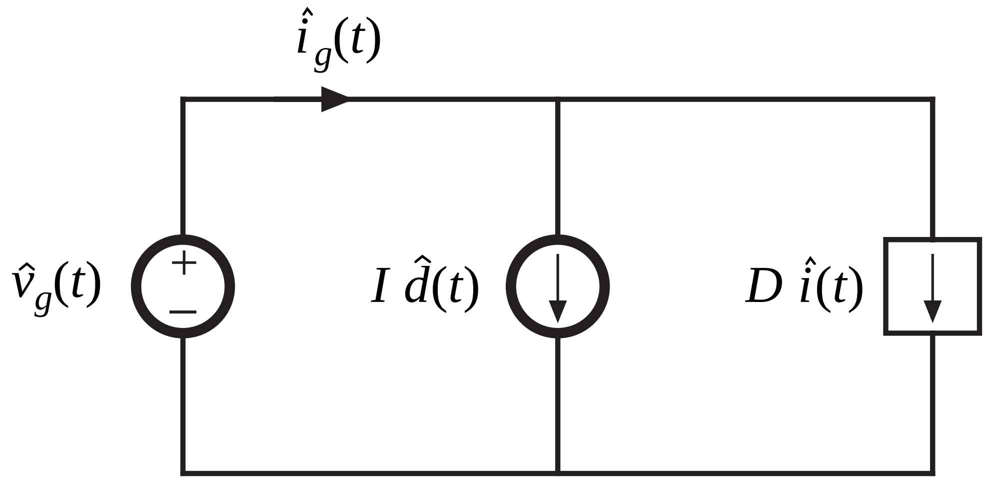
Small-signal ac equivalent circuit: capacitor node

$$C\,\frac{d\hat{v}(t)}{dt} = \frac{D'\hat{i}(t)}{n} - \frac{\hat{v}(t)}{R} - \frac{I\hat{d}(t)}{n}$$



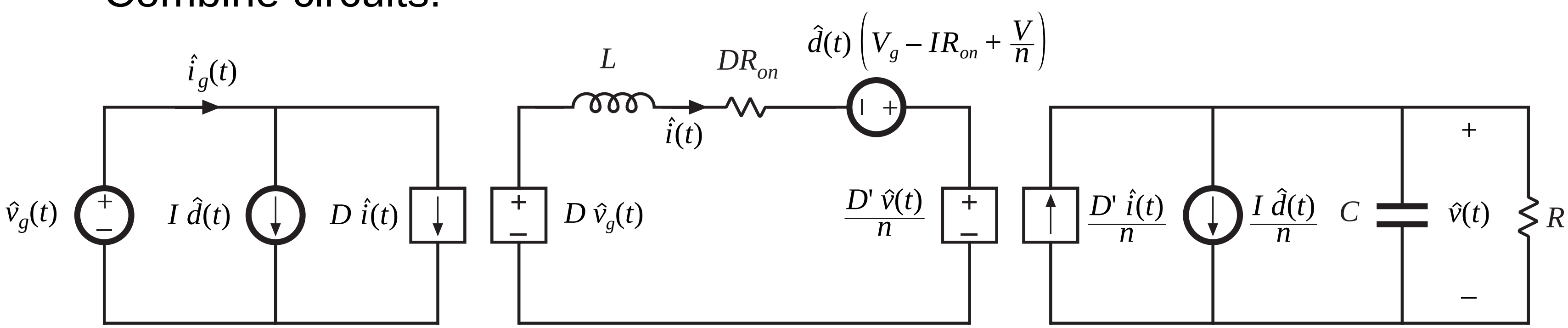
Small-signal ac equivalent circuit: converter input node

$\hat{i}_g(t) = D\hat{i}(t) + I\hat{d}(t)$



Small-signal ac model, nonideal flyback converter example

Combine circuits:



SIMULINK SIMULATION FLYBACK CONVERTER SMALL SIGNAL MODEL

$$L\frac{d\hat{i}(t)}{dt} = D\hat{v}_g(t) - D'\frac{\hat{v}(t)}{n} + \left(V_g + \frac{V}{n} - IR_{on}\right)\hat{d}(t) - DR_{on}\hat{i}(t)$$

$$C\frac{d\hat{v}(t)}{dt} = \frac{D'\hat{i}(t)}{n} - \frac{\hat{v}(t)}{R} - \frac{I\hat{d}(t)}{n}$$

$$\hat{i}_g(t) = D\hat{i}(t) + I\hat{d}(t)$$

$V_g = 48$ V

V = 12 V

P_{out} = 150 W

$f_{sw} = 100$ kHz

$L_m = 250$ uH

$R_{on} = 25$ mΩ

$C_{out} = 100$ uF

n = 0.5 V