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"I pledge my honor that I have abided by the Stevens Honor System."

Problem 1 X and Y are independent, identically distributed, binomial discrete random variables. Find the distribution of Z = X + Y.

Problem 1

$\left.\begin{matrix} X \\ Y \end{matrix}\right\}$ independent, identically distr., binomial descrete r.v.

Find distribution of $Z = X + Y$

pmF: $P\{X = x\} = \binom{n}{x} p^x (1-p)^{n-x}$

cdF: $P\{X \le x\} = \sum_{y=0}^{x} \binom{n}{y} p^y (1-p)^{n-y}$

$E(x) = np$

$Var(x) = np(1-p)$

let
$X \sim Bin(n, p)$
$Y \sim Bin(m, p)$

$0 \le k \le n+m$

$$\mathbb{P}(X+Y=k) = \sum_{i=0}^{k} \mathbb{P}(X=i, Y=k-i)$$

$$= \sum_{i=0}^{k} \mathbb{P}(X=i)\mathbb{P}(Y=k-i) \quad \text{independence}$$

$$= \sum_{i=0}^{k} \binom{n}{i} p^i (1-p)^{n-i} \binom{m}{k-i} p^{k-i} (1-p)^{m-k+i}$$

$$= p^k (1-p)^{n+m-k} \sum_{i=0}^{k} \binom{n}{i}\binom{m}{k-i}$$

$$\boxed{= \binom{n+m}{k} p^k (1-p)^{n+m-k}}$$

Problem 2 X and Y are independent, identically and geometrically distributed discrete random variables. Find the distribution of Z = X + Y.

Problem 2

X, Y $\left\{ \begin{array}{l} \text{independent} \\ \text{identically + geomerically distr.} \\ \text{discrete r.v.} \end{array} \right.$

find distr. $Z = X + Y$

$$P(X = x) = p(1-p)^{x-1} \quad x\ 1, 2, 3 \ldots$$

$$P(Z = x + y) = \sum_{x=1}^{x=z-1} P(X = x, Y = z - x)$$

$$= \sum_{x=1}^{x=z-1} p(1-p)^{x-1} p(1-p)^{z-x-1}$$

$$= p^2 \sum_{x=1}^{x=z-1} (1-p)^z (1-p)^{-2}$$

$$\boxed{= (z-1)p^2(1-p)^{z-2}}$$

Problem 3 Given two independent random variables, X and Y, with probability densities

$$f_X(x) = \begin{cases} \frac{1}{2}e^{-\frac{x}{2}}, & x \geq 0 \\ 0, & x < 0 \end{cases} \qquad f_Y(y) = \begin{cases} \frac{1}{2}e^{-\frac{y}{2}}, & y \geq 0 \\ 0, & y < 0 \end{cases}$$

Find the probability density of the random variable Z = X + Y.

Problem 3

X

$f_X(x) = \begin{cases} \frac{1}{2}e^{-x/2}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

Y

$f_Y(y) = \begin{cases} \frac{1}{2}e^{-y/2}, & y \geq 0 \\ 0, & y < 0 \end{cases}$

Find

Probability density of $Z = X + Y$

exponential distribution pdfs

$f_X(x) = \lambda e^{-\lambda x}$ $\qquad f_Y(y) = \lambda e^{-\lambda y}$

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y)\, f_Y(y)\, dy =$$

$$= \int_0^z \lambda e^{-\lambda(z-y)} (\lambda e^{-\lambda y} dy)$$

$$= \lambda^2 e^{\lambda z} \int_0^z 1\, dy = \lambda^2 e^{-\lambda z} [y]_0^z$$

$$= \lambda^2 z e^{-\lambda z}$$

where ~~λ~~

$$\lambda = 1/2$$

$$f_Z(z) = \left(\frac{1}{2}\right)^2 z e^{-0.5z}$$

$$\boxed{= 0.25 z e^{-0.5z}}$$

Problem 4 Random variables X and Y have the joint pdf $f_{XY}(x,y) = x + y$, for $0 < x < 1$ and $0 < y < 1$.

a) Determine marginal pdf's of X and Y.
b) Are X and Y statistically independent?

Problem 4

r.v.'s X and Y

joint pdf..

$$F_{XY}(x,y) = x+y \quad \begin{cases} 0<x<1 \\ 0<y<1 \end{cases}$$

a) Marginal pdfs of X and Y

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 (x+y)\,dy$$

$$= \left(xy + y^2/2\right)\Big|_0^1$$

$$\boxed{= x + 1/2, \text{ for } 0<x<1}$$ ← Marginal PDF of X

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^1 (x+y)\,dx$$

$$= \left(x^2/2 + xy\right)\Big|_0^1$$

$$\boxed{= y + 1/2, \text{ for } 0<y<1}$$ ← Marginal PDF of Y

b) X and Y are NOT statistically independent

$$f(x,y) \neq f(x)f(y)$$

$$x+y \neq (x+1/2)(y+1/2)$$

$$xy + 1/2y + 1/2x + xy$$

$$x+y \neq 2xy + 1/2y + 1/2x$$

Problem 5 X_1 and X_2 are independent random variables with identical distribution

$$p_X = \begin{bmatrix} 0 & 1 & 2 \\ \frac{1}{8} & \frac{3}{8} & \frac{1}{2} \end{bmatrix}$$

Find the distribution of their sum, $X_1 + X_2$

Problem 5

X_1, X_2 independent r.v.

identical distribution

$$P_X = \begin{bmatrix} 0 & 1 & 2 \\ 1/8 & 3/8 & 1/2 \end{bmatrix}$$

Find dist. of $X_1 + X_2$

let $y = x_1 + x_2$

let $x_1 = k$ $z = 2$

$x_2 = z - k$

$$P(z=2) = \sum_{k=-\infty}^{\infty} P(x_1 = k) \cdot P(x_2 = 2-k)$$

0 → $P(Z = x_1 + x_2 = 0)$

$= P(x_1 = 0) \cdot P(x_2 = 0)$

$\cancel{P()} = 1/8 \cdot 1/8 = 1/64$

→ $P(x_1 + x_2 = 1)$

$= P(x_1 = 0) \cdot P(x_2 = 1) + P(x_1 = 1) \cdot P(x_2 = 0)$

$= \left(\frac{1}{8} \cdot \frac{3}{8}\right) + \left(\frac{3}{8} \cdot \frac{1}{8}\right) = \frac{3}{64} + \frac{3}{64} = \cancel{3} \; 6/64$

$P(Z = x_1 + x_2 = 2) = P(x_1 = 0) \cdot P(x_2 = 2) + P(x_1 = 1)$

$P(x_1 = 1) + P(x_1 = 2) \cdot P(x_2 = 0)$

$= \left(\frac{1}{8} \cdot \frac{1}{2}\right) + \left(\frac{3}{8} \cdot \frac{3}{8}\right) + \left(\frac{1}{2} \cdot 1/8\right)$

$= \frac{4}{64} + \frac{9}{64} + \frac{4}{64} \boxed{= \frac{17}{64}}$