# Problem Set 7 11/4/19

*Give high-level descriptions of TMs and any PDAs in solutions*

## Problem 1

The class of TM-decidable languages is closed under...

**Union:**

Let $L_1$ and $L_2$ be decidable languages
TMs $M_1$ and $M_2$ decide them
let TM $M$ decide the language $L : L_1 \cup L_2$

On input $\omega$:
1. Run $M_1$ on $\omega$ → if accept then ACCEPT
2. Run $M_2$ on $\omega$ → if accept then ACCEPT
otherwise → REJECT

TM M ACCEPTS $\omega$ if
→ $M_1$ or $M_2$ ACCEPT

if $\omega \in L_1$ or $L_2$
M will Accept $\omega$
$L_1 \cup L_2$ is Decidable

TM M REJECTS $\omega$ if
→ $M_1$ and $M_2$ both REJECT
M decides $L_1 \cup L_2$

**Concatenation:**

Let $L_1$ and $L_2$ be decidable language
concatenation → $L_1 L_2 = \{xy \mid x \in L_1 \text{ and } y \in L_2\}$
let TMs $M_1$ and $M_2$ decide $L_1$ and $L_2$
let TM M decide $L_1 L_2$

On input $\omega$:
1. partition $\omega$ into 2 substrings $xy$
2. Run $M_1$ on $x$
3. Run $M_2$ on $y$

If both accept → M ACCEPTs $\omega$

(Partition must be adjusted + rerun through M, if none accept after trying all → REJECT)

TM → accepts when both $M_1$ and $M_2$ accept.

$L_1 L_2$ is decidable

Star:

for language $L$,
$L^* = \{x \in L \cup LL \cup LLL \cup \ldots\}$ let $M_1$ decide $L$
let $M$ decide $L^*$

On input $\omega$:

① For each possible partition of $\omega$, $\omega_1\omega_2\omega_3\omega_4\ldots\omega_n$

② Run $M_1$ on $\omega_1 \ldots \omega_n$ (each subsection)

③ if $M_1$ accepts each $\omega_i \rightarrow$ ACCEPT
otherwise $\rightarrow$ reject

(similar to last, must partition each possible way to check)

$L^*$ is decidable

Intersection:

let $L_1$ and $L_2$ be decidable languages
TMs $M_1$ and $M_2$ decide them
let $M$ decide $L_1 \cap L_2$

on input $\omega$:

① Run $M_1$ on $\omega \rightarrow$ if rejects, REJECT

② If doesnt reject, run $M_2$ on $\omega$
if $M_2$ accepts $\rightarrow$ ACCEPT
otherwise $\rightarrow$ REJECT

$L_1 \cap L_2$ is decidable

Complement:

let language $L$ be decidable
TM $M_1$ decides $L$
TM $M$ decides $\overline{L}$

on input $\omega$:

① Run $M_1$ on $\omega \rightarrow$ if rejects, ACCEPT
otherwise $\rightarrow$ REJECT

$\overline{L}$ is decidable

(if one TM of $L$ rejects it means $\omega$ is $\overline{L}$.)

## Problem 2

The class of TM-recognizable languages is closed under...

**Union:**

let $L_1$ and $L_2$ be recognizable languages
TMs $M_1$ and $M_2$ recognize them
TM $M$ recognizes $L_1 \cup L_2$

On input $w$:

① Alternate running $M_1$ and $M_2$ on $w$ each step
② if $M_1$ or $M_2$ accept $\longrightarrow$ ACCEPT
if $M_1$ and $M_2$ reject/halt $\rightarrow$ REJECT
(looping) $\rightarrow$ M loops

M accepts when it reaches Accept state after all the steps. $\rightarrow$ $L_1 \cup L_2$ is Recognizable
M rejects when both $M_1$ and $M_2$ reject

**Concatenation:**

let $L_1$ and $L_2$ be recognizable languages
TMs $M_1$ and $M_2$ recognize them
TM $M$ recognizes $L_1 L_2$

On input $w$:

① Partition $w$ into 2 parts $w_1$ and $w_2$
② Run $M_1$ on $w_1$ $\longrightarrow$ if halt/reject, REJECT
③ Run $M_2$ on $w_2$ $\longrightarrow$ if accept, ACCEPT
otherwise $\longrightarrow$ REJECT

$L_1 L_2$ is recognizable

**Star:**

let $L$ be a recognizable language
TM $M_1$ recognizes $L$
TM $M$ recognizes $L^*$

on input $w$:

① Partition $w$ into parts $w_1 w_2 \dots w_n$
② Run $M_1$ on $w_i$ (all parts)
if $M_1$ accepts ALL $w_i \rightarrow$ ACCEPT
if $M_1$ halt/rejects any $w_i \rightarrow$ REJECT
(must try different partitions)

if $w_i \in L$ then there exists a TM M that recognizes $L^*$

Intersection:

let $L_1$ and $L_2$ be recognizable languages
TMs $M_1$ and $M_2$ recognize them
let TM $M$ recognize $L_1 \cap L_2$

On input $w$:

① Run $M_1$ on $w$. → if halt/reject, REJECT
② Then Run $M_2$ on $w$ → if halt/reject, REJECT
otherwise → ACCEPT

if $M_1$ and $M_2$ accept $w$ → $M$ accepts $w$
$w \in L_1 \cap L_2$

$L_1 \cap L_2$ is recognizable

Complement:

if $L$ and $\bar{L}$ are TM-recognizable
the $L$ is decidable
from theorem, ~~L and~~ $\bar{L}$
A lang is Decidable
iff both it and its complement
are recognizable
TM would NOT Halt for strings not
in the language

Problem 3

Proposition: A language is decidable if and only if there is an enumerator which prints out a strings in lexicographic order (increasing length)

Proof:

We have a TM, M, that decides language L
We can construct an enumerator E.
- ↳ that generates strings in standard order.
- ↳ tests each string ∈ L? (by using M)
- ↳ prints string if ∈ L

Consider...

① if L is finite: {E may loop w/ no more outputs; show L is decidable w/o E to avoid loop}
→ then it is decidable → (all finite lang Decidable)

② if L is Infinite:
→ Then TM M
on input w:
① E enumerates all strings in L (in standard order)
*Until a string ordered after w is generated
② If w is printed by enumerator E
then → ACCEPT
otherwise → REJECT

Problem 4

Proposition: Every infinite TM-recognizable language has an infinite Decidable subset.

Proof:

let L be an infinite Recognizable language and has an infinite decidable subset.

There exists an enumerator E for all strings in L

let $L' = \{w_1, w_2, w_3, \dots w_n\}$ $i > 1$

$w_i$ is next string in order after $w_{i-1}$

Consider:

① L' is Infinite and $L' \subseteq L$

let L' be finite

→ E generates all strings < $w_i$, the largest in L'

→ because E generates finite # strings < $w_i$

L is finite → CONTRADICTION!

Also: M only ACCEPTS if in Stnd Order

→* L' is NOT finite → L' is Infinite

Because E printed all strings ∈ L',

→* $L' \subseteq L$

② L' is decidable

Can construct E'

on no input:

① prints out strings accepted by M

→ Meaning all in Standard Order

Because L is infinite, E' is infinitely enumerating

E' is decidable because $\boxed{\text{prints Stnd order}}$

→ Therefore, the language L', of E' is decidable

→ L' is an infinite decidable subset of L, an infitie Recognizable language