How to write a proof using the pumping lemma to show that a language is not regular.

Problem 1. Prove that $L = \{ww^R : w \in \{0,1\}^*\}$ is not regular.

Proof. The comments in red address some common mistakes, not part of the proof.

For sake of contradiction, we assume that L is regular, in which case the pumping lemma for regular language applies. Do NOT use the pumping lemma for context free languages here!

Let p be the pumping length the pumping lemma guarantees.

Consider the string $s = 0^p 1^{2p} 0^p \in L$. Remember that the counterexample string we choose MUST be in the language, and its length must be p or greater.

Let $s = xyz$, where $|xy| \leq p$ and $|y| > 0$. The pumping lemma says that every string s of length at least p, can be divided into three pieces so that these two conditions (and a third one) must be satisfied. Our goal is to argue that no matter how s is divided all three conditions cannot be simultaneously satisfied. All that remains is to argue that the third condition can never be satisfied for s when these two conditions are satisfied. Since I must argue for all choices of x, y that satisfy these two conditions I cannot make any assumptions about what $x \text{ and } y$ are.

Since $|xy| \leq p$ and the first p symbols of s are all 0, it follows that $y = 0^k$, for some $k, 1 \leq k \leq p$.

Now consider the string $xyyz = 0^p 0^k 1^{2p} 0^p$. This is not in the language L. But this violates the pumping lemma which states that it must be in L. From this contradiction we conclude that L is not regular.

Problem 2. Prove that $L = \{ww : w \in \{0,1\}^*\}$ is not context-free.

Proof. For sake of contradiction, we assume that L is context-free, in which case the pumping lemma for context-free language applies. Do NOT use the pumping lemma for regular languages here!

Let p be the pumping length the pumping lemma guarantees.

Consider the string $s = 0^p 1^p 0^p 1^p \in L$. Remember that the counterexample string we choose MUST be in the language, and its length must be p or greater.

Let $s = uvxyz$, where $|vxy| \leq p$ and $|vy| > 0$. The pumping lemma for CFLs says that every string s of length at least p, can be divided into five pieces so that these two conditions (and a third one) must be satisfied. Our goal is to argue that no matter how s is divided all three conditions cannot be simultaneously satisfied. All that remains is to argue that the third condition can never be satisfied for s when these two conditions are satisfied. Since I must argue for all choices of v, x, y that satisfy these two conditions I cannot make any assumptions about what they are, nor can I assume anything about $u\ and\ z$.

Since $|vxy| \leq p$ the possible cases are:

Case 1. vxy is contained within a single contiguous block of 0s (or of 1s). When the string is pumped up, the number of 0s (or 1s) within that block will be greater than p while the remaining blocks remain at size p. The pumped string is not in the language.

Case 2. Since $|vxy| \leq p$, vxy can contain symbols from at most two contiguous blocks. For example, it can contain symbols from the 1st and 2nd blocks, or from the 2nd and 3rd blocks, or from the 3rd and 4th blocks. In all these cases, when we pump the string the numbers of 0s in blocks 1 and 3, or the numbers of 1s in blocks 2 and 4 will be different, so that the resulting string is not in the language.

Thus, in every case, pumping the string s results in a string not in the language. This contradicts the pumping lemma for context-free languages; therefore, L is not context-free.