CS 135 Spring 2018: Problem Set 5.

Problem 1. (10 points) In lecture we proved that the Principle of Induction follows from the Well-ordering principle. In this problem we will prove the converse; namely, that $PI \rightarrow WOP$.

Nothing could be simpler, thought Lem E. Hackett. He had proved tougher theorems previously. Here's the hypothesis that Lem came up with:

$$P(k)\text{: } \textit{Every set containing } k \geq 1 \textit{ elements has a least element.}$$

Lem quickly established $P(1)$, the base case. He then established $P(k) \rightarrow P(k+1)$ as follows: If you remove one element from any set with $k+1$ elements, then you are left with a set with k elements. By the inductive hypothesis, this has a least element. We can compare this with the element that was removed; the smaller one is the least in the original set of $k+1$ elements.

Done! Or so, Lem figured. Alas, his professor told him that although there was no flaw in his argument, the problem was that Lem had not established the well-ordering principle.

(a) Why did Lem's proof not establish $PI \rightarrow WOP$?
(b) Thinking it over, Lem figured out a different approach. If the well-ordering principle is false, then there must be a non-empty set A that does not contain a least element. So Lem changed his hypothesis to: $P(k)\text{: } k \notin A$. Help complete the proof for Lem.

Problem 2. (10 points) The Fibonacci numbers $F(n)$ for $n \in \boldsymbol{N}$ are defined as follows:

$$F(n) = \begin{cases} 0, & n = 0 \\ 1, & n = 1 \\ F(n-1) + F(n-2), & n > 1 \end{cases}$$

Using strong induction[1], prove that

$$F(n) = \frac{1}{\sqrt{5}}\,(p^n - q^n\,),$$

where $p = \frac{1+\sqrt{5}}{2}$ and $q = \frac{1-\sqrt{5}}{2}$.

Hint: p and q are the two roots of the equation $x^2 - x - 1 = 0$.

[1] In strong Induction the inductive hypothesis is $\forall i, 0 \leq i \leq k\; P(i)$. In other words, in the inductive step you can show that $P(k+1)$ follows from $P(0) \wedge P(1) \wedge \cdots \wedge P(k)$. It is called strong induction because the inductive hypothesis seems stronger. Although induction and strong induction are equivalent, some proofs are simpler using strong induction.