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Beveridge Sequences

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1 Introduction to the sequence

Our Catalan group is defined as a sequence of $a_1, \ldots, a_n + 1$ of non-negative integers such that $\sum_{i=1}^{n+1} (2^{-a_i}) = 1$ and for all $2 \leq j \leq n+1$ the number $2^{a_j} \sum_{i=1}^{j-1} (2^{-a_i})$ is an integer. We will call any sequence that fits these rules a beveridge sequence.

2 Examples

2.1 Enumerations for $0 \leq n \leq 4$

We enumerate all possible sequences for $0 \leq n \leq 4$.

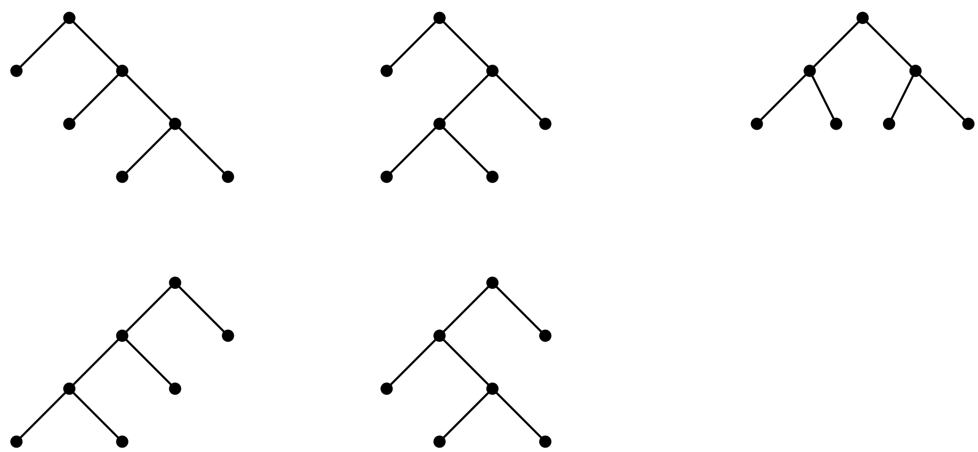
n	**Beveridge Sequences of length $n+1$**	**Number of Sequences**
0	0	1
1	1,1	1
2	1,2,2 2,1,1	2
3	1,2,3,3 1,3,3,2 2,2,2,2 2,3,3,1 3,3,2,1	5
4	3,3,2,2,2 4,4,3,2,1 3,4,4,2,1 2,4,4,3,1 2,3,4,4,1 1,4,4,3,2 1,3,4,4,2 1,2,4,4,3 1,2,3,4,4 3,3,3,3,1 1,3,3,3,3 2,3,3,2,2 2,2,3,3,2 2,2,2,3,3	14

2.2 Visualization of the Sequence

In the course of studying this sequence, we have noticed that the values in a sequence correspond to the depth of the leaves in a plane binary tree with $2n+1$

vertices. A *binary tree* is a tree data structure in the field of computer science in which each vertex has at most two children (referred to as the left child and right child. A *plane binary tree* (or full binary tree) is a binary tree where every vertex has either *zero* of *two* children. We noted that the *Beveridge* sequence offers a way to represent plane binary trees with $2n+1$ vertices.

Below is a visualization of the sequence as plane binary trees when $n = 3$, in which case there are 5 possible sequences.



Each binary tree above corresponds to one of the Beveridge sequence with $n = 3$, from left to right: $(1,2,3,3), (1,3,3,2), (2,2,2,2), (3,3,2,1), (2,3,3,1)$.

3 Proof

3.1 Groundwork

Let us start by laying down a few theorems and definitions that will be used throughout the rest of the paper. Some theorems will be left unproven as their proof will be left for a real analysis student to complete.

Definition 3.1.1. Beveridge A beveridge sequence is any sequence, S, of non-negative integers for which both the following two properties hold.

1. $\sum_{i=1}^{|S|} \left(2^{-S_i}\right) = 1$

2. For all indices $j \in \{2, 3, \ldots, |S|\}$, $2^{S_j} \sum_{i=1}^{j-1} \left(2^{-S_i}\right)$ is an integer.

Theorem 3.1.1. *For any sequence S of real numbers*

$$\sum_{n=1}^{k} \left(2^{-S_n}\right) > \sum_{n=1}^{k-1} \left(2^{-S_n}\right)$$

for any integer $k > 1$.

Theorem 3.1.2. *If a sequence S is beveridge and $|S| > 1$ then all elements in S are positive.*

Proof. Let S be a beveridge sequence. Thus all elements in S must be non-negative integers. Assume that S contains some index i for which S_i is not positive. Zero is the only non-negative, non-positive integer, thus $S_i = 0$. We know that $2^{-0} = 1$, thus S must only contain 0 as otherwise

$$\sum_{n=1}^{|S|} \left(2^{-S_n}\right) > 2^{-S_i} = 1$$

which is impossible as S is beveridge. Thus if S contains a non-positive element, then that may be it's only element, making $|S| = 1$. If $|S| > 1$ then S may only contain non-negative and non-zero integers, which is the same as only containing positive integers. $\square$

3.2 Midpoint

Theorem 3.2.1. *If a sequence S is beveridge and $|S| > 1$ then there exists exactly one index $p(S)$ such that*

$$\sum_{i=1}^{p(S)} \left(2^{-S_i}\right) = \frac{1}{2}$$

Proof. Let S be a beveridge sequence with $|S| > 1$. Assume for the sake of contradiction that there does not exist an index p such that $\sum_{i=1}^{p} \left(2^{-S_i}\right) = \frac{1}{2}$. Because of theorem 3.1.1, we know that there is some index, j such that

$$\sum_{i=1}^{j} \left(2^{-S_i}\right) > \frac{1}{2} > \sum_{i=1}^{j-1} \left(2^{-S_i}\right)$$

Because 2^{S_j} must be positive regardless of j or S, we may multiply it through and our inequality will hold, giving us

$$2^{S_j}\sum_{i=1}^{j}\left(2^{-S_i}\right) = 1+2^{S_j}\sum_{i=1}^{j-1}\left(2^{-S_i}\right) > 2^{S_j}\frac{1}{2} > 2^{S_j}\sum_{i=1}^{j-1}\left(2^{-S_i}\right)$$

At this point we must observe that $j > 1$. To prove this we will assume that $j = 1$ (note that $j < 1$ makes no sense as it would be a sum of 0 things, thus resulting in a result of 0 which is less then $\frac{1}{2}$), and we find

$$\sum_{i=1}^{j}\left(2^{-S_i}\right) = \sum_{i=1}^{1}\left(2^{-S_i}\right) = 2^{-S_1} > \frac{1}{2}$$

and the only non-negative integer value for S_1 that satisfies this is $S_1 = 0$, which by theorem 3.1.2 means that $|S| = 1$, which is a contradiction as we defined S to have $|S| > 1$. We thus know that $j > 1$, and because of that we know that $2^{S_j}\sum_{i=1}^{j-1}\left(2^{-S_i}\right)$ is an integer. Now to finally finish this off we know that for any integer k there are no integers in the interval $(k, k+1)$, thus we know that $2^{S_j}\frac{1}{2} = 2^{S_j-1}$ is not an integer. As S_j must be an integer this means that $S_j - 1 < 0$ or alternatively $S_j < 1$ which would mean that S_j is not positive, however by theorem 3.1.2 we know that S_j is positive and thus we have our contradiction.

Finally we notice that there is only one possible point for any beveridge sequence p via theorem 3.1.1, any index before p must yield a sum less than $\frac{1}{2}$ and any index after p must yield a sum greater than $\frac{1}{2}$.[1]

Now we have shown that for all beveridge sequences S of length greater than one, there exists a unique index k, such that $\sum_{i=1}^{k}\left(2^{-S_i}\right) = \frac{1}{2}$. Thus we may define a function $p(S) = k$ where

$$\sum_{i=1}^{k}\left(2^{-S_i}\right) = \frac{1}{2}$$

$\square$

Theorem 3.2.2. *For any beveridge sequence S with $|S| > 1$, $p(S) < |S|$.*

1. When I say sum I really mean a sum in the form $\sum_{i=0}^{k}\left(2^{-S_i}\right)$, where k is some index in S.

Proof. S is beveridge thus, $\sum_{i=1}^{|S|}\left(2^{-S_i}\right) = 1 > \frac{1}{2}$, and by theorem 3.1.1 we know that $p < |S|$. $\square$

3.3 Decomposition

In this section we will show that using the index $p(S)$ we may break any beveridge sequence S into two smaller beveridge sequences, in exactly one way, as long as $|S| \neq 1$.

Theorem 3.3.1. *For any beveridge sequence S, of length greater than 1, there exists two sequences*

$$L(S) = (S_1 - 1, S_2 - 1, \ldots, S_{p(S)} - 1)$$
$$R(S) = (S_{p(S)+1} - 1, S_{p(S)+2} - 1, \ldots, S_{|S|} - 1)$$

both of which are beveridge.

Proof. Let S be a beveridge sequence with $|S| > 1$. First notice that $|L(S)| \geq 1$ and that $|R(S)| \geq 1$. $L(S)$ always contains $S_{p(S)} - 1$ as an element, thus must contain at least one element. By theorem 3.2.2 we know that $p(S) + 1 < |S|$, thus $S_{p(s)+1} - 1$ is always in $R(S)$, so $R(S)$ may also not be empty. Second we would also like to notice that $L(S)$ and $R(S)$ contain only non-negative integers, as S contains only positive integers (see theorem 3.1.2).

Now that we have shown that $L(S)$ and $R(S)$ are non-empty sequences of non-negative integers, we can start to show that they fit the requirements to be beveridge, this is simply a fair amount of fairly simple manipulation.

First we want to show that $\sum_{i=1}^{|L(S)|}\left(2^{-L(S)_i}\right) = 1$.

$$\begin{aligned}
\sum_{i=1}^{|L(S)|}\left(2^{-L(S)_i}\right) &= \sum_{i=1}^{p}\left(2^{-(S_i-1)}\right) \\
&= \sum_{i=1}^{p}\left(2^{-S_i+1}\right) \\
&= 2\sum_{i=1}^{p}\left(2^{-S_i}\right) = 2 \cdot \frac{1}{2} = 1
\end{aligned}$$

Next we wish to show that $\sum_{i=1}^{|R(S)|} \left(2^{-R(S)_i}\right) = 1$.

$$\begin{aligned}
\sum_{i=1}^{|R(S)|} \left(2^{-R(S)_i}\right) &= \sum_{i=p+1}^{|S|} \left(2^{-(S_i-1)}\right) \\
&= \sum_{i=p+1}^{|S|} \left(2^{-S_i+1}\right) \\
&= 2\sum_{i=p+1}^{|S|} \left(2^{-S_i}\right) \\
&= 2\left[\sum_{i=1}^{|S|} \left(2^{-S_i}\right) - \sum_{i=1}^{p} \left(2^{-S_i}\right)\right] = 2\left[1-\frac{1}{2}\right] = 1
\end{aligned}$$

Next we will show that for any $j \in \{2, 3, \ldots, |L(S)|\}$, $\sum_{i=1}^{j-1} \left(2^{L(S)_j - L(S)_i}\right)$ is an integer. Let us start by choosing $j \in \{2, 3, \ldots, |L(S)|\}$.

$$\begin{aligned}
\sum_{i=1}^{j-1} \left(2^{L(S)_j - L(S)_i}\right) &= \sum_{i=1}^{j-1} \left(2^{(S_j-1)-(S_i-1)}\right) \\
&= \sum_{i=1}^{j-1} \left(2^{S_j - S_i}\right) \in \mathbb{Z}
\end{aligned}$$

We know this is an integer as S is beveridge, thus we have shown that $L(S)$ is beveridge.

Finally we will show that for any $j \in \{2, 3, \ldots, |R(S)|\}$, $\sum_{i=1}^{j-1} \left(2^{R(S)_j - R(S)_i}\right)$ is an integer.

$$\begin{aligned}
\sum_{i=1}^{j-1} \left(2^{R(S)_j - R(S)_i}\right) &= \sum_{i=p+1}^{j-1+p} \left(2^{(S_j-1)-(S_i-1)}\right) \\
&= \sum_{i=p+1}^{j-1+p} \left(2^{S_j - S_i}\right) \\
&= \sum_{i=1}^{j-1+p} \left(2^{S_j - S_i}\right) - \sum_{i=1}^{p} \left(2^{S_j - S_i}\right) \\
&= \sum_{i=1}^{j-1+p} \left(2^{S_j - S_i}\right) - 2^{S_j} \cdot \frac{1}{2}
\end{aligned}$$

Now we notice that $\sum_{i=1}^{j-1+p} \left(2^{S_j - S_i}\right)$ is an integer as S is beveridge and $j \neq 1$. Additionally we know that S_j is positive via theorem 3.1.2 and thus $S_j - 1 \geq 0$. Also notice that $2^{S_j} \cdot \frac{1}{2} = 2^{S_j - 1}$, and as $S_j - 1$ is non-negative, 2^{S_j-1} must be an integer. Finally the integers are closed on addition so we know $\sum_{i=1}^{j-1+p} \left(2^{S_j - S_i}\right) - 2^{S_j} \cdot \frac{1}{2}$ is an integer. Now we have shown that $R(S)$ is beveridge. □

3.4 Composition

In the last section we showed that one could decompose any beveridge sequence S of length greater than 1 into two smaller beveridge sequences. We now want to show that if we take two beveridge sequences L and R we can compose them to construct a larger beveridge sequence. Let us start by defining a composition function

$$C(L, R) = (L_1 + 1, L_2 + 1, \ldots, L_{|L|} + 1, \quad R_1 + 1, R_2 + 1, \ldots, R_{|R|} + 1)$$

Theorem 3.4.1. *If L and R and beveridge sequences then $C(L, R)$ is also beveridge.*

Proof. Let L and R be beveridge sequences. First let us notice that $C(L, R)$ is a sequence of positive integers, as L and R are both sequences of non-negative integers.

Next we would like to show that $\sum_{i=1}^{|C(L,R)|}\left(2^{-C(L,R)_i}\right) = 1$.

$$\begin{aligned}\sum_{i=1}^{|C(L,R)|}\left(2^{-C(L,R)_i}\right) &= \sum_{i=1}^{|L|}\left(2^{-(L_i+1)}\right) + \sum_{i=1}^{|R|}\left(2^{-(R_i+1)}\right) \\ &= \frac{1}{2}\left[\sum_{i=1}^{|L|}\left(2^{-L_i}\right) + \sum_{i=1}^{|R|}\left(2^{-R_i}\right)\right] \\ &= \frac{1}{2}\left[1+1\right] = 1\end{aligned}$$

Finally we would like to show that for any $j \in \{2, 3, \ldots, |C(L,R)|\}$, $\sum_{i=1}^{j-1}\left(2^{C(L,R)_j - C(L,R)_i}\right)$ is an integer. To do this we must do a proof by cases. Let us consider the three possible cases:

1. $j < |L| + 1$
2. $j = |L| + 1$
3. $j > |L| + 1$

First if $j < |L| + 1$, this is the same as saying $j \in \{2, 3, \ldots, |L|\}$. First we should notice that

$$\sum_{i=1}^{j-1}\left(2^{C(L,R)_j - C(L,R)_i}\right) = \sum_{i=1}^{j-1}\left(2^{(L_j+1)-(L_i+1)}\right)$$

as we are never referencing any index in $C(L,R)$ after the j^{th} index which we know is at most $|L|$. Now we simplify and find

$$\sum_{i=1}^{j-1}\left(2^{(L_j+1)-(L_i+1)}\right) = \sum_{i=1}^{j-1}\left(2^{L_j - L_i}\right) \in \mathbb{Z}$$

as L is beveridge.

Second if $j = |L| + 1$, this means that we want to show that $\sum_{i=1}^{|L|}\left(2^{C(L,R)_{|L|} - C(L,R)_i}\right)$ is an integer. Let us start by rewriting this as $2^{C(L,R)_{|L|+1}}\sum_{i=1}^{|L|}\left(2^{-C(L,R)_i}\right)$. Now we notice that $\sum_{i=1}^{|L|}\left(2^{-C(L,R)_i}\right) = \sum_{i=1}^{|L|}\left(2^{-(L_i+1)}\right)$. Well we now show that

$$\sum_{i=1}^{|L|}\left(2^{-(L_i+1)}\right) = \frac{1}{2}\sum_{i=1}^{|L|}\left(2^{-L_i}\right) = \frac{1}{2}\cdot 1 = \frac{1}{2}$$

We now look at the $2^{C(L,R)_{|L|+1}}$ part. Notice that $C(L,R)_{|L|+1} = R_1 + 1$, thus $2^{C(L,R)_{|L|+1}} = 2^{R_1+1}$. Now we may put the parts together and we find that

$$\sum_{i=1}^{|L|} \left(2^{C(L,R)_{|L|}-C(L,R)_i}\right) = 2^{R_1+1} \cdot \frac{1}{2} - 2^{R_1}$$

which is an integer as R is composed of non-negative integers.

Finally if $j > |L| + 1$ we may do the following manipulation.

$$\begin{aligned}
\sum_{i=1}^{j-1} \left(2^{C(L,R)_j - C(L,R)_i}\right) &= 2^{C(L,R)_j} \sum_{i=1}^{j-1} \left(2^{-C(L,R)_i}\right) \\
&= 2^{R_{j-|L|}+1} \left[\sum_{i=1}^{|L|} \left(2^{-(L_i+1)}\right) + \sum_{i=1}^{j-1-|L|} \left(2^{-(R_i+1)}\right)\right] \\
&= 2^{R_{j-|L|}+1} \cdot \frac{1}{2} \left[1 + \sum_{i=1}^{j-1-|L|} \left(2^{-R_i}\right)\right] \\
&= 2^{R_{j-|L|}} + 2^{R_{j-|L|}} \sum_{i=1}^{j-|L|-1} \left(2^{-R_i}\right)
\end{aligned}$$

Now $j - |L|$ must be a member of $\{2, 3, \ldots, |R|\}$ and so we know that $2^{R_{j-|L|}} \sum_{i=1}^{j-|L|-1} \left(2^{-R_i}\right)$ is an integer as R is beveridge, thus we have the sum of two integers, and that must be an integer.

Now we have demonstrated that for any pair of beveridge sequences L and R, $C(L,R)$ is beveridge. $\square$

We now must demonstrate that if we have two separate pairs of beveridge sequences we can't possibly compose both pairs respectively and get the same sequence

Theorem 3.4.2. *If A, B, $\bar{A}$ and $\bar{B}$ are beveridge sequences such that $A \neq \bar{A}$ or $B \neq \bar{B}$ then $C(A,B) \neq C(\bar{A},\bar{B})$.*

Proof. This will be a proof by contra-positive. First we state

$$\sum_{i=1}^{|L|} \left(2^{-C(L,R)_i}\right) = \sum_{i=1}^{|L|} \left(2^{-L_i-1}\right) = \sum_{i=1}^{|L|} \left(2^{-L_i}\right) \cdot \frac{1}{2} = \frac{1}{2}$$

and we find that the sum across the first $|L|$ terms of $2^{-C(L,R)}$ is $\frac{1}{2}$, thus $L = L(C(L,R))$, and thus there is exactly one way to construct L from $C(L,R)$ and further there will be exactly one way to construct R. Thus we know that compositions are unique. □

4 Complete induction

Here we will finally prove that there are C_n beveridge sequences of length $n+1$, where C_n is the n^{th} Catalan number defined as

$$C_0 = 1$$
$$C_n = \sum_{i=0}^{n-1} C_i \cdot C_{n-i-1}$$

Proof. First we wish to show that there is exactly one beveridge sequence of length 1. If a sequence is beveridge and of length one then we know that $\sum_{i=1}^{1} \left(2^{-A_i}\right) = 1$, where A is our beveridge sequence. We may rewrite this without the sigma as $2^{-A_1} = 1$. There is only one possible non-negative integer value for A_1, and this is 0. Thus the sequence (0) is the only sequence of length 1 that is beveridge.[2] Now we have shown a base case

Now choose $k \in \mathbb{N}$, let us assume that for all non-negative integers $n < k$, the number of beveridge sequences of length $n+1$ is C_n. Now we wish to show that there are exactly C_k beveridge sequences of length $k+1$. First by theorem 3.3.1 we know that all beveridge sequences of length $k+1$ can be decomposed into two shorter beveridge sequences. Next by theorem 3.4.1 and 3.4.2 we know that each composition of two beveridge sequences produces a unique beveridge sequence with length $l+r$ where l and r are the lengths of the two sequences being composed respectively. Thus for every beveridge sequence of length $t < k+1$ (of which there are C_{t-1}) we may compose it with a beveridge sequence of length $k+1-t$ (of which there are C_{k-t}) to get a unique beveridge sequence of length $k+1$. Thus we find that there are $\sum_{t=1}^{k} \left(C_{t-1} \cdot C_{k-t}\right) = \sum_{i=0}^{k-1} \left(C_i \cdot C_{k-i-1}\right)$ beveridge sequences of length $k+1$ that can be composed of smaller beveridge sequences, this is of course all of them as we have already stated.

2. Technically I haven't shown that (0) is beveridge, however it is rather trivial.

Now by the principle of complete induction we know there are C_n beveridge sequences of length $n+1$. $\square$