Topological Graph Theory

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Contents

1 Introduction

Topological graph theory is an entire field within topology and as such this paper is by no means meant to cover all of topological graph theory in any depth. This paper instead will first cover a rather shallow overview of the field, followed by a more in depth study of graphs and their genus. The overview will mainly be focused on giving a thorough understanding of what topological graph theory is as well as to briefly cover the history of the field. In giving an overview of the field we will cover some of the basic concepts and definitions needed for the more rigorous part of the paper; as this may be confusing, all definitions will be provided in a glossary at the end. After the overview we will dive into Kuratowski's Theorem, We will go through and attempt to have an intuitive understanding of a Kuratowski's Theorem and it's proof. After proving Kuratowski's Theorem, we will continue onto talking about generalizations of the theorem and map colorings, however their coverage will be rather shallow and lacking proofs.

2 Overview

2.1 Graphs

Before we talk about topological graph theory with any level of understanding we must first understand what a graph is.

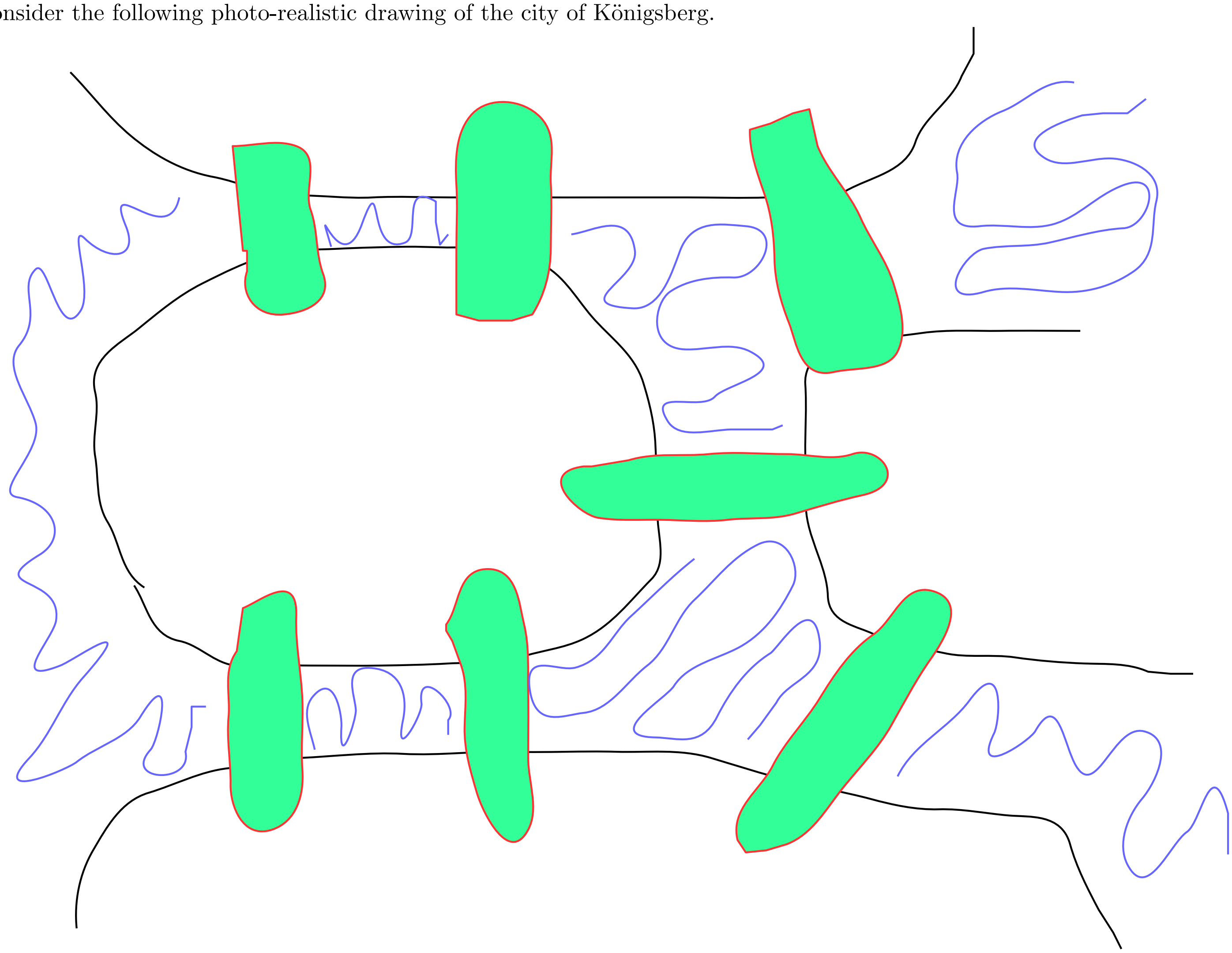
A graph is generally defined as a set of verticies combined with a set of edges between verticies, however here it may be more useful to think about them visually with a simple representation.

Consider first a set of points, this may be thought of as just drawing dots on a sheet of paper. Each of these points will be called a vertex. Now we may start drawing lines between verticies. Lines may cross over each other and need not be straight. There is no requirement that all verticies have a line going to it. Each of these lines are called an edge. We will simply insist that no edge connects two verticies and that we do not have multiple edges between the same pair of verticies.

Once we have drawn this we have a representation of a graph. If we were to move the verticies around on the paper but leave them having the same edges (the same verticies are connected to the point as they were before). we would be left with the same graph. That is to say, it doesn't mater where we put a vertex on our sheet, the graph exists independently of the representation we draw.

2.2 Königsberg, and it's seven bridges

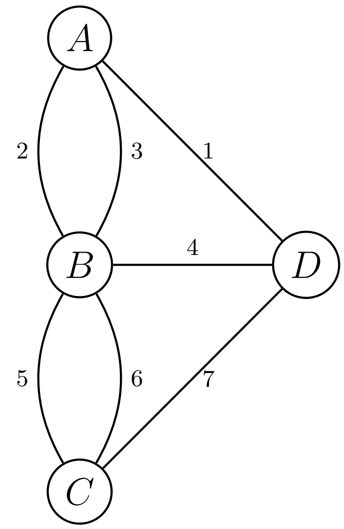
Consider the following photo-realistic drawing of the city of Königsberg.



Now the question is, if we get to choose where we start, can we go for a stroll and cross every bridge exactly once?

I first came across this question in the 8^{th} grade, and it was presented to us during geometry class. While undoubtedly an interesting problem, it is quite misleading to try and think of this as a geometry problem. Instead we will try and reduce it to a graph problem.

Let us start by thinking of every island as a vertex and every bridge as an edge. We find the following graph.



It is worth noting two things about the above diagram. First that the labels on the verticies and edges are unrelated to the problem, but have been added simply to make referring to parts of the graph much easier. Second that whatever the above image depicts, does not fit our definition of a graph.

Notice that edge 2 and edge 3 both connect vertex A to B, as well edges 5 and 6 do for B and C. This is a strict violation of our definition for a graph. The issue of course then comes to what would one call such a beast as this where, presumably, one is able to have as many connections between any pair of verticies and could even have connections from a vertex to itself.

I am particularly glad that you're paying enough attention to notice that the diagram does not depict a graph. This is what we will refer to as a multigraph. It is worth noting that graphs are a type of multigraph, so anything we show to be true for all multigraphs, is also true for all graphs.

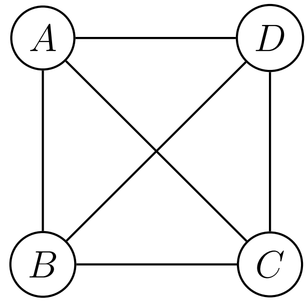
Now to solve this problem we need to make one simple observation about how we walk. If we are to go to island (or vertex) we must also leave that island, unless it is the last island we arrive on. This means, that with the exception of the island we start on and end on, each island must have an even number of bridges connected to it. On the multigraph we would say that we need all verticies but a start and end vertex to have an even number of edges. If we look at the multigraph above, we have four verticies that have an odd number of edges connected to them.

2.3 History

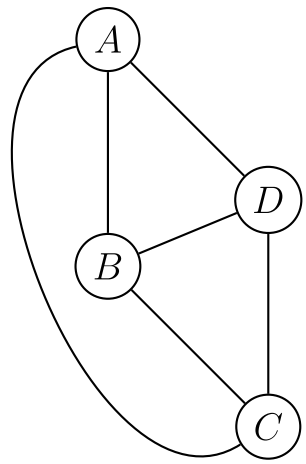
The seven bridges of Königsberg problem was solved by Euler in 1736. In mathematics this problem is of great historical significance as it is considered to be the beginning of graph theory as well as a sort of precursor to topology.

3 Planar Graphs

Consider the following graph.



This graph is called a complete graph as every vertex is connected to every other vertex by an edge; in fact this graph in particular is called K_4 as it is the complete four vertex graph. We would like to find out if we can draw this above graph without having any lines crossing. We can in fact draw this graph without any intersections and for any skeptics who may being reading this, the below is K_4 without any edges intersecting.



So if this graph can be drawn without intersection, can any graph be drawn without intersections? If some can and some can't how do we tell which can be drawn and which can not?

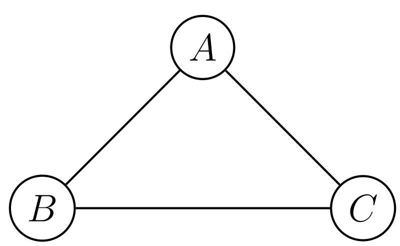
4 Kuratowski's Theorem

Kuratowski's theorem tell us exactly what graphs can be drawn on a plane or sphere. We say either a plane or sphere as it turns out to be the same question. Now before we tackle the theorem let's lay down some terminology.

4.1 Terminology

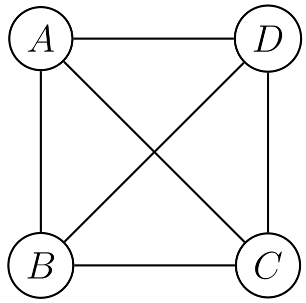
4.1.1 Terminology: face

Consider the following graph.

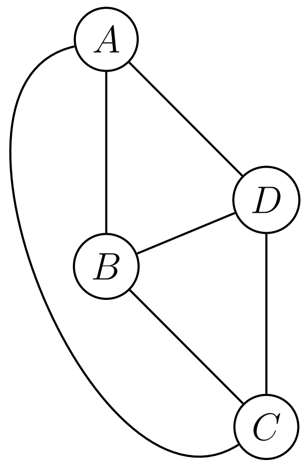


We say this graph has two faces. One face is easy to see and contained within triangle ABC. The other face is a bit harder to see, however it is all the space outside of the triangle ABC. We may call this second face the outer face for this drawing of the graph, however it is also possible to draw it such that the inner face may be the outer face. You can think of this as kind of turning the triangle inside out.

A face only makes sense to talk about for a graph that can be embedded in a certain space. For example the graph K_4 may look like it has 5 faces in the following drawing.



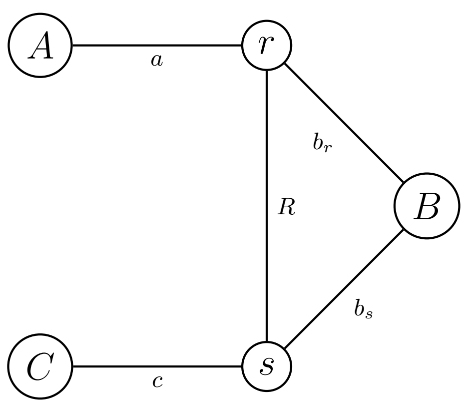
This would be wrong as a face assumes a drawing without edges intersecting. This can be rectified if the graph can be drawn without intersection and thus we find this drawing.



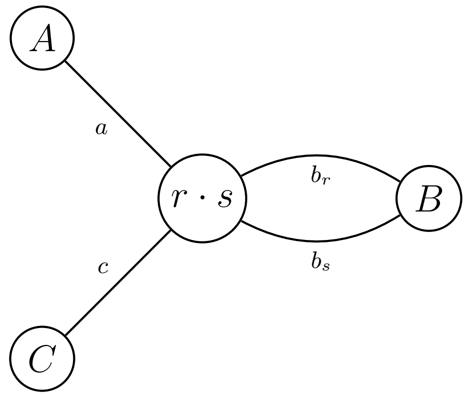
This means there are actually 4 faces in K_4.

4.1.2 Terminology: contraction

Let us consider the following graph.



We wish to preform a contraction on edge R. To be clear all graph contractions are on edges. So we will make a new vertex, $r \cdot s$ which has all the edges of r and all the edges of s except the edge we are contracting across. In this case the edge we are contracting across is R and thus we get the following multigraph.



This can then be reduced into a graph again by simply treating b_r and b_s as the same edge. This operation is particularly important as we will prove that if a graph is planar, then so is any graph or multigraph obtained by contracting an edge. The proof for this will be provided.

4.1.3 Terminology: subgraph

We can say that a graph A is a subgraph of another graph B, if all verticies in A are also in B and each edges in A is also in B.

4.2 Groundwork

Here we would like to prove some lemmas that we would like to use while proving Kuratowski's theorem.

Now for some extensions of Kuratowski's theorem

Theorem 4.1 (Spherical and Planar graphs are the same)**.** *If and only if a graph may be drawn on a plane with no edges intersection then it may be drawn on a sphere without self intersection.*

Now this we won't actually prove as it's quite difficult to do so, however notice that the plane is homeomorphic to the punctured sphere. Now it seems intuitive that if you can draw a finite graph, or even multi-graph on a sphere then it would not have to cover every single point on the sphere. This means that we could puncture the sphere at one point and thus we have the plane.

This is not needed for our proof of Kuratowski's theorem, however it does help to think about drawing graphs on planes sometimes as well as helping to think about extensions beyond Kuratowski's theorem.

Glossary

complete graph A graph with all possible edges included, the notation K_n is used to denote the complete with n verticies. 3

contraction A graph operation where one removes an edge by fusing two verticies together. 5

edge A connection or line between verticies in a graph. 1, 2, 3, 5, 6

face An section of the space we are embedding our graph in that is separated from the rest of the space by edges. 4, 5

graph A set of verticies and edges. 1, 2, 3, 4, 5, 6

multigraph Like a graph, but multiple edges may connect the same vertex pair, and an edge may connect a vertex to itself. 3, 5, 6

set A collection of objects. 1

subgraph A is a subgraph of some graph $\mathcal{G}$ iff you could add verticies and edges to A and somehow get $\mathcal{G}$. 6

vertex A point or node in a graph. 1, 2, 3, 5, 6

References