

Algebra Homework

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1 Chapter 1

1.1 Section 1

1.1.1 Question 1

Choose $a, b \in S$. We find

$$a = a * b = b * a = b$$

, and thus all elements in S must be the same element, so there is most one element of S .

1.1.2 Question 2

Let us choose $a, b, c \in S$.

(a) We have

$$a * b = a - b = -(b - a) = -(b * a)$$

, thus iff $0 = a * b = a - b$ we have $a * b = b * a$ as $0 = -0$, however for any other value of $a * b$, $a * b \neq b * a$. We also may notice that iff $a = b$, then $a * b = a - b = 0$. Thus for all $a \neq b$, $a * b \neq b * a$.

(b) We have

$$\begin{aligned} a * (b * c) &= a - (b - c) \\ &= a + (c - b) \\ &= a + c - b \\ &= a - b + c \\ &= a - b - (-c) \\ &= (a - b) - (-c) \\ &= (a * b) * -c \end{aligned}$$

so $a * (b * c) = (a * b) * c$ iff $c = -c$ which is only true if $c = 0$.

(c) We have $a * 0 = a - 0 = a$.

(d) We have $a * a = a - a = 0$.

1.2 Section 2

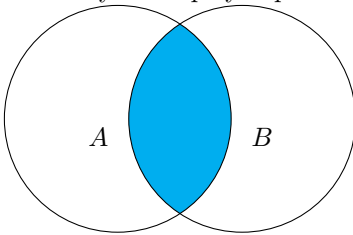
1.2.1 Question 8

Let $x \in (A - B) \cup (B - A)$ then either $x \in A - B$ or $x \in B - A$. If $x \in A - B$ then we get that $x \in A$ and $x \notin B$, thus $x \in A \cup B$ and $x \notin A \cap B$, which would mean $x \in (A \cup B) - (A \cap B)$. If $x \in B - A$ then we get that $x \in B$ and $x \notin A$, thus $x \in A \cup B$ and $x \notin A \cap B$, which would mean $x \in (A \cup B) - (A \cap B)$. It has now been demonstrated that $(A - B) \cup (B - A) \subset (A \cup B) - (A \cap B)$.

Now let $x \in (A \cup B) - (A \cap B)$. We have that $x \in A \cup B$ and $x \notin A \cap B$. It follows that either $x \in A$ or $x \in B$, however, x is not in both A and B . This may be written as: $x \in A$ and $x \notin B$, or $x \in B$ and $x \notin A$. This then translates to $x \in A - B$ or $x \in B - A$, therefore, $x \in (A - B) \cup (B - A)$. It has now been demonstrated that $(A \cup B) - (A \cap B) \subset (A - B) \cup (B - A)$.

Now it has been shown that both sets are subsets of each-other, thus $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$.

This may be displayed pictorially as follows:



1.2.2 Question 9

Let $x \in A \cap (B \cup C)$, thus $x \in A$ and $x \in B \cup C$. We then have that $x \in B$ or $x \in C$. Now as we already know that $x \in A$ then we get that either $x \in B \cap A$ or $x \in C \cap A$ and therefore $x \in (A \cap B) \cup (A \cap C)$. Thus it has been shown that $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$.

Let $x \in (A \cap B) \cup (A \cap C)$, thus $x \in (A \cap B)$ or $x \in (A \cap C)$. We then get that either $x \in A$ and $x \in B$ or that $x \in A$ and $x \in C$, either way $x \in A$, thus we may write that $x \in A$ and either $x \in B$ or $x \in C$. This would be the same as $x \in A$ and $x \in B \cup C$, which then translates to $x \in A \cap (B \cup C)$. Thus it has been shown that $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$.

We have now shown that both sets are subsets of each-other, thus $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

1.2.3 Question 10

Let $x \in A \cup (B \cap C)$, assume then for the sake of contradiction that $x \notin (A \cup B) \cap (A \cup C)$. Because $x \in A \cup (B \cap C)$ we have that $x \in A$ or $x \in B \cap C$. Because $x \notin (A \cup B) \cap (A \cup C)$ we have that $x \notin A \cup B$ or $x \notin A \cup C$. We then get that either $x \notin A$ and $x \notin B$ or $x \notin A$ and $x \notin C$, either way $x \notin A$, so we have $x \in B \cap C$. We know that $x \notin B$ or $x \notin C$, however we also have that $x \in B$ and $x \in C$ due to $x \in B \cap C$, thus we have a contradiction. Thus $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$.

Let $x \in (A \cup B) \cap (A \cup C)$ and assume for the sake of contradiction that $x \notin A \cup (B \cap C)$. We then get that $x \notin A$ and $x \notin B \cap C$. We also have that $x \in A \cup B$ and $x \in A \cup C$, so if $x \notin A$ then we get $x \in B$ and $x \in C$. This is then translated to $x \in B \cap C$ which is a direct contradiction with $x \notin B \cap C$ and again we have a contradiction. Thus $(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$.

We have now shown that both sets are subsets of each other, thus $A \cap (B \cup C) = (A \cup B) \cap (A \cup C)$.

1.2.4 Question 12

(a)

$$\begin{aligned}(A \cup B)' &= \{x \in S \mid x \notin A \cup B\} \\ &= \{x \in S \mid x \notin A \text{ and } x \notin B\} \\ &= \{x \in S \mid x \in A' \text{ and } x \in B'\} \\ &= A' \cap B'\end{aligned}$$

(b)

$$\begin{aligned}(A \cap B)' &= \{x \in S \mid x \notin A \cap B\} \\ &= \{x \in S \mid x \notin A \text{ or } x \notin B\} \\ &= \{x \in S \mid x \in A' \text{ or } x \in B'\} \\ &= A' \cup B'\end{aligned}$$

1.2.5 Question 13

(a)

$$\begin{aligned}A + B &= (A - B) \cup (B - A) \\ &= (B - A) \cup (A - B) \\ &= B + A\end{aligned}$$

(b) First notice that for any set X , $X - \emptyset = X$ and that $\emptyset - X = \emptyset$.

$$\begin{aligned}A + \emptyset &= (A - \emptyset) \cup (\emptyset - A) \\ &= A \cup \emptyset \\ &= A\end{aligned}$$

(c)

$$\begin{aligned}A \cdot A &= A \cap A \\ &= A\end{aligned}$$

(d)

$$\begin{aligned}A + A &= (A - A) \cup (A - A) \\ &= \emptyset \cup \emptyset \\ &= \emptyset\end{aligned}$$

(e) To simplify this question let me introduce the logical operation, $a \oplus b$ which is defined as either a or b but not both, and we will show that $a \oplus (b \oplus c) = (a \oplus b) \oplus c$ using truth tables.

a	b	c	$a \oplus b$	$b \oplus c$	$a \oplus (b \oplus c)$	$(a \oplus b) \oplus c$
False	False	False	False	False	False	False
False	False	True	False	True	True	True
False	True	False	True	True	True	True
False	True	True	True	False	False	False
True	False	False	True	False	True	True
True	False	True	True	True	False	False
True	True	False	False	True	False	False
True	True	True	False	False	True	True

Now we wish to show that $A + B = \{x \in S \mid x \in A \oplus x \in B\}$. To do this we will first show that $a \oplus b = (a \wedge \neg b) \vee (b \wedge \neg a)$, where \neg is a logical not, \wedge is a logical and, and \vee is a logical or. We again show this by the following truth table:

a	b	$\neg b$	$a \wedge \neg b$	$\neg a$	$b \wedge \neg a$	$(a \wedge \neg b) \vee (b \wedge \neg a)$	$a \oplus b$
False	False	True	False	True	False	False	False
False	True	False	False	True	True	True	True
True	False	True	True	False	False	True	True
True	True	False	False	False	False	False	False

Now we find

$$\begin{aligned}
A + B &= \{x \in S \mid x \in A + B\} \\
&= \{x \in S \mid x \in (A - B) \cup (B - A)\} \\
&= \{x \in S \mid x \in (A - B) \vee x \in (B - A)\} \\
&= \{x \in S \mid (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)\} \\
&= \{x \in S \mid x \in A \oplus x \in B\}
\end{aligned}$$

so we then have

$$\begin{aligned}
A + (B + C) &= \{x \in S \mid x \in A \oplus x \in B + C\} \\
&= \{x \in S \mid x \in A \oplus (x \in B \oplus x \in C)\} \\
&= \{x \in S \mid (x \in A \oplus x \in B) \oplus x \in C\} \\
&= \{x \in S \mid x \in A + B \oplus x \in C\} \\
&= (A + B) + C
\end{aligned}$$

(f) Suppose $B \neq C$. Because $B \neq C$ there exists some $x \in S$ such that either $x \in B$ and $x \notin C$ or $x \in C$ and $x \notin B$, we will assume without loss of generality that $x \in B$ and $x \notin C$. Now if $x \in A$ then we would find $x \notin A + B$ and $x \in A + C$. If $x \notin A$ we would find that $x \in A + B$ and $x \notin A + C$. We now have shown that $B \neq C \implies A + B \neq A + C$, thus by contrapositive we have $A + B = A + C \implies B = C$.

(g) First we will want to show logical equivalence between the statement $a \wedge (b \oplus c)$ and $(a \wedge b) \oplus (a \wedge c)$.

a	b	c	$b \oplus c$	$a \wedge b$	$a \wedge c$	$a \wedge (b \oplus c)$	$(a \wedge b) \oplus (a \wedge c)$
False	False	False	False	False	False	False	False
False	False	True	True	False	False	False	False
False	True	False	True	False	False	False	False
False	True	True	False	False	False	False	False
True	False	False	False	False	False	False	False
True	False	True	True	False	True	True	True
True	True	False	True	True	False	True	True
True	True	True	False	True	True	False	False

now we may show

$$\begin{aligned}
A \cdot (B + C) &= A \cap (B + C) \\
&= \{x \in S \mid x \in A \cap (B + C)\} \\
&= \{x \in S \mid x \in A \wedge x \in (B + C)\} \\
&= \{x \in S \mid x \in A \wedge (x \in B \oplus x \in C)\} \\
&= \{x \in S \mid (x \in A \wedge x \in B) \oplus (x \in A \wedge x \in C)\} \\
&= \{x \in S \mid x \in A \cap B \oplus x \in A \cap C\} \\
&= \{x \in S \mid x \in (A \cap B) + (A \cap C)\} \\
&= (A \cap B) + (A \cap C) \\
&= (A \cdot B) + (A \cdot C)
\end{aligned}$$

1.2.6 Question 14

First notice that if A and B are disjoint then $m(A \cup B) = m(A) + m(B)$. So now we get the three disjoint sets $A - B$, $A \cap B$, and $B - A$, notice that $A = (A - B) \cup (A \cap B)$, that $B = (B - A) \cup (A \cap B)$, and $A \cup B = (A - B) \cup (A \cap B) \cup (B - A)$. Now we get $m(A) = m(A - B) + m(A \cap B)$, $m(B) = m(B - A) + m(A \cap B)$, and $m(A \cup B) = m(A - B) + m(A \cap B) + m(B - A)$. We then get

$$\begin{aligned}
m(A) + m(B) &= m(A - B) + m(A \cap B) + m(B - A) + m(A \cap B) \\
&= m(A \cup B) + m(A \cap B) \\
m(A) + m(B) - m(A \cap B) &= m(A \cup B)
\end{aligned}$$

1.2.7 Question 22

- (a) To construct a subset of any set we go through each element and choose to include it or not to, this gives us two possibilities per element. For a set of size n then there are n independent choices to be made in constructing a subset, thus 2^n subsets.
- (b) There are exactly $\binom{n}{m} = \frac{n!}{m!(n-m)!}$ subsets of a set with n elements that have m elements.

Proof. Let us start by defining $\binom{n}{m}$ as the number of ways to choose a subset with m elements from a set with n elements. Now we must recognize that $k!$ is the number of ways to order a set with k elements. Then we get that $\binom{n}{m} m!(n-m)! = n!$ as we may order our set with n elements by choosing the first m elements in our order ($\binom{n}{m}$ possible ways), then ordering those elements ($m!$ ways), and finally ordering the rest of the elements ($(n-m)!$ ways). This gives us $\binom{n}{m} m!(n-m)! = n!$ and from there we divide and get $\binom{n}{m} = \frac{n!}{m!(n-m)!}$. \square

1.3 Section 3

1.3.1 Question 7

Let $g : S \rightarrow T$, $h : S \rightarrow T$ and $f : T \rightarrow U$ be functions such that f is 1-1 and $f \circ g = f \circ h$. Assume for the sake of contradiction that $g \neq h$, then there exists some $s \in S$ such that $g(s) \neq h(s)$. We know that $f \circ g(s) = f \circ h(s)$, thus $f(g(s)) = f(h(s))$ so $g(s) = h(s)$ by f being 1-1. Thus we have a contradiction and we know that $g = h$.

1.3.2 Question 8

- (a) Yes, as all integers are either even or odd and none are both even and odd.
- (b) Let us break this into cases:
- If s_1 and s_2 are even, then there exists $k_1 \in \mathbb{Z}$ and $k_2 \in \mathbb{Z}$ such that $2k_1 = s_1$ and $2k_2 = s_2$. Thus $s_1 + s_2 = 2k_1 + 2k_2 = 2(k_1 + k_2)$, thus $f(s_1 + s_2) = 1$. We also find that $f(s_1) \cdot f(s_2) = 1 \cdot 1 = 1$.

- If s_1 is even and s_2 is odd, then there exists $k_1 \in \mathbb{Z}$ and $k_2 \in \mathbb{Z}$ such that $s_1 = 2k_1$ and $s_2 = 2k_2 + 1$. Thus $s_1 + s_2 = 2k_1 + 2k_2 + 1 = 2(k_1 + k_2) + 1$ so $f(s_1 + s_2) = -1$. We also find that $f(s_1)f(s_2) = 1 \cdot -1 = -1$.
- If s_1 is odd and s_2 is even we may write that $f(s_1 + s_2) = f(s_2 + s_1)$ and that $f(s_1)f(s_2) = f(s_2)f(s_1)$ because both addition and multiplication are commutative. Now we see that we have reproduced our previous case and thus in this case the equality holds.
- If s_1 and s_2 are odd, then there exists $k_1 \in \mathbb{Z}$ and $k_2 \in \mathbb{Z}$ such that $2k_1 + 1 = s_1$ and $2k_2 + 1 = s_2$, thus $s_1 + s_2 = 2k_1 + 1 + 2k_2 + 1 = 2(k_1 + k_2 + 1)$ so $f(s_1 + s_2) = 1$. We also find that $f(s_1)f(s_2) = -1 \cdot -1 = 1$.

Thus for all possible integers s_1 and s_2 , we have $f(s_1 + s_2) = f(s_1)f(s_2)$.

This tells us that even integers are closed under addition. that odd integers added together always are even, and finally that an odd added to an even is odd.

- (c) No, as $f(1 \cdot 2) = f(2) = 1$ and $f(1)f(2) = -1 \cdot 1 = -1$.

1.3.3 Question 12

- (a) No f is not a function as $2/3 = 4/6$ and $f(2/3) = 2^2 3^3 \neq 2^4 3^6 = f(4/6)$.
(b) We may define $f(m/n) = 2^m 3^n$ iff m and n are coprime.

1.3.4 Question 19

Let $f(x) = x^2 + ax + b$, thus $f'(x) = 2x + a$. $f'(x)$ is linear so there exists only one $x \in \mathbb{R}$ for which $f'(x) = 0$, and thus this x is a global extrema for f , so f can not be surjective. Now consider $x_1 = -\frac{a}{2} - 1$ and $x_2 = -\frac{a}{2} + 1$, thus

$$\begin{aligned} f(x_1) &= \left(-\frac{a}{2} - 1\right)^2 + a\left(-\frac{a}{2} - 1\right) + b \\ &= \frac{a^2}{4} + 2\frac{a}{2} + 1 - \frac{a^2}{2} - a + b \\ &= \frac{a^2}{4} + 1 + b \\ f(x_2) &= \left(-\frac{a}{2} + 1\right)^2 + a\left(-\frac{a}{2} + 1\right) + b \\ &= \frac{a^2}{4} - 2\frac{a}{2} + 1 - \frac{a^2}{2} + a + b \\ &= \frac{a^2}{4} + 1 + b \end{aligned}$$

so f must be 1-1.

1.3.5 Question 23

Ugly proof:

First let us show that there exists some bijection from \mathbb{N} to $\mathbb{Z}_{\geq 0}^2$. Consider the 1 norm on $\mathbb{Z}_{\geq 0}^2$, defined as $\|(a, b)\|_1 = a + b$. Then we may partition $\mathbb{Z}_{\geq 0}^2$ into subsets $P_n = \{x \in \mathbb{Z}_{\geq 0}^2 \mid \|x\|_1 = n\}$, for any $n \in \mathbb{Z}_{\geq 0}$. Notice that for $(a, b) \in P_n$ then $a \leq n$ and $b \leq n$, thus forcing P_n to be finite. Now we can construct a function mapping from \mathbb{N} to $\mathbb{Z}_{\geq 0}^2$ by giving each element of P_1 a number from 1 to $|P_1|$ (inclusive), then the next $|P_1|$ will be given to elements of P_2 and so on infinitely. Notice that by construction $x \neq y \implies f(x) \neq f(y)$, so we get this being 1-1, additionally for any $(a, b) \in \mathbb{Z}_{\geq 0}^2$, $(a, b) \in P_{a+b}$ and thus receives a number greater than $\sum_{n=0}^{a+b-1} |P_n|$ and less than or equal to $\sum_{n=0}^{a+b} |P_n|$. This means that we can label each element of $\mathbb{Z}_{\geq 0}^2$ with a single natural number and thus have a bijection.

Now we can also construct a trivial bijection, $h : \mathbb{Z}_{\geq 0}^2 \rightarrow S$ as $h(a, b) = 2^a 3^b$. Now we may compose the bijections to get a 1-1 correspondence $\mathbb{N} = S$ onto T .

Nice proof:

First notice that $T \subset S$ so there exists the trivial injective function from T to S . Second notice that $f : S \rightarrow T$ defined as $f(s) = 2^s$ is both well defined as injective. By the Schröder-Bernstein theorem there must be some bijection from S to T .

1.3.6 Question 28

Let S be a finite set, with $f : S \rightarrow S$. Now let $f(x) = f(y)$, for some $x \neq y$, then there remain $|S| - 2$ elements in $S - \{x, y\}$ and $|S| - 1$ elements in $S - \{f(x)\}$. This means that for any definition of f on $S - \{x, y\}$ it can not possibly be onto $S - \{f(x)\}$. We have now shown f not being 1-1 implies f not being onto, by contrapositive f being onto implies f is 1-1.

1.3.7 Question 29

Let S be a finite set, with $f : S \rightarrow S$ injective. Now as f is 1-1 each $s \in S$ has a unique $f(s) \in S$, so $f(S)$ must have exactly $|S|$ unique elements, thus $f(S) \subset S$ with exactly $|S|$ elements.¹ Because S is finite, this implies $f(S) = S$.

1.4 Section 4

1.4.1 Question 5

- (a) First identity:

$$\begin{aligned} f^2 g^2 &= f f g g \\ &= f(fg)g \\ &= f(gf)f \\ &= (fg)^2 \end{aligned}$$

- (b) Second Identity: Let i be the identity function.

$$\begin{aligned} f^{-1} g^{-1} g f &= i \\ f^{-1} g^{-1} g f (g f)^{-1} &= i (g f)^{-1} \\ &= f^{-1} g^{-1} i = i (f g)^{-1} \\ &= f^{-1} g^{-1} = (f g)^{-1} \end{aligned}$$

1.4.2 Question 9

1.4.3 Question 10

1.4.4 Question 14

1.4.5 Question 21

1.4.6 Question 27

1.4.7 Question 30

1.4.8 Question 32

¹ $f(A)$ is defined as $\{y \in \mathbf{Rng}(f) \mid \exists x \in \mathbf{Dom}(f) f(x) = y\}$ when $A \subset \mathbf{Dom}(f)$ and $A \not\subset \mathbf{Dom}(f)$.