Topology Midterm

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17.5 In order to show that, for any order topology, $\overline{(a,b)} \subset [a,b]$ we first notice that $[a,b] \supset (a,b)$ and that $[a,b]$ is closed. By definition we know $\overline{(a,b)} = \bigcap$ all closed supersets of (a,b). We now notice that $[a,b]$ is one such closed superset of (a,b), thus $\overline{(a,b)} \subset [a,b]$.

We now will look to see when $\overline{(a,b)} = [a,b]$. We already know that $\overline{(a,b)} \subset [a,b]$, and to have equality we only need $[a,b] \subset \overline{(a,b)}$. Let us start by noticing that $[a,b]$ is the union of disjoint sets (a,b) and $\{a,b\}$. Now if $[a,b]$ is to be a subset of $\overline{(a,b)}$ then that would be the same as saying $(a,b) \cup \{a,b\} \subset \overline{(a,b)}$ thus both (a,b) and $\{a,b\}$ must be subsets of $\overline{(a,b)}$. We know that $(a,b) \subset \overline{(a,b)}$ as $\overline{(a,b)} = (a,b) \cup (a,b)'$, and because we know that $\{a,b\}$ is disjoint from (a,b) we can then say $[a,b] \subset \overline{(a,b)} \implies \{a,b\} \subset (a,b)'$. We also can say

$$\begin{aligned} \{a,b\} \subset (a,b)' &\implies \{a,b\} \subset \overline{(a,b)} \\ &\implies \{a,b\} \cup (a,b) \subset \overline{(a,b)} \\ &\implies [a,b] \subset \overline{(a,b)} \end{aligned}$$

and thus, iff a and b are limit points for the interval (a,b), then our equality $([a,b] = \overline{(a,b)})$ holds.

17.17 Consider the lower limit topology on $\mathbb{R}$, and the topology given by the basis $\mathcal{C}$ of Exercise 8 §13. Determine the closures of the intervals $A = \big(0, \sqrt{2}\big)$ and $B = \big(\sqrt{2}, 3\big)$ in these two topologies.

Basis $\mathcal{C}$ of Exercise 8 §13:

$$\mathcal{C} = \{[a,b) \mid a < b \textit{ and } a,b \in \mathbb{Q}\}$$

First we will consider our topology to be $\mathbb{R}_\ell$:

Let C be an interval in the form (a,b), where $a,b \in \mathbb{R}$. By definition we know that $\overline{C}$ is the intersection of all closed sets that contain C. We know that $(-\infty, a) \cup [b, \infty) \in \mathbb{R}_\ell$ and thus $[a,b)$ is closed in $\mathbb{R}_\ell$, thus $\overline{C} \subset [a,b)$.

Now if we can show that $[a,b) \subset \overline{C}$ then we will know that $[a,b) = \overline{C}$.

First we note that by theorem 17.6 $\overline{C} = C \cup C'$, now because we know $C \subset \overline{C}$ then we can say if $[a,b) \setminus C \subset \overline{C} \setminus C$ then $[a,b) = \overline{C}$. We also know that $\overline{C} \setminus C \subset C'$, thus we can say that if $[a,b) \setminus C \subset C'$ then $[a,b) = \overline{C}$. Next we find that $[a,b) \setminus C = \{a\}$ so if $a \in C'$ then $[a,b) = \overline{C}$. We will show $a \in C'$ by contradiction.

Let us assume $a \notin C'$ then there is an interval $[x,y)$, where $x,y \in \mathbb{R}$, that contains a but no elements in C. By definition $[x,y) = \{k \mid x \le k < y\}$, so if $a \in [x,y)$ then $x \le a < y$. Now we can construct an interval $(a,y) \subset [x,y)$ which is not empty as $y > a$ and thus it will contain some elements of C. We now have a contradiction, thus $a \in C'$, thus

$$[a,b) = \overline{C}$$

Now if we let $a = 0$ and $b = \sqrt{2}$ then we know $\overline{\big(0,\sqrt{2}\big)} = \overline{A} = \big[0, \sqrt{2}\big)$.

Now if we let $a = \sqrt{2}$ and $b = 3$ then we know $\overline{\big(\sqrt{2},3\big)} = \overline{B} = \big[\sqrt{2}, 3\big)$.

Now we will to continue on to the topology $\mathscr{C}$, which is given by basis $\mathcal{C}$.

Let us first attempt to find $\overline{\left(0,\sqrt{2}\right)}$. We will consider the set $\left[0,\sqrt{2}\right]$, and attempt to show that it is closed by showing it's compliment is open.

$$\begin{aligned}\left[0,\sqrt{2}\right]^{\mathsf{c}} &= (-\infty,0)\cup\left(\sqrt{2},\infty\right)\\ &= \left(\bigcup_{a<b<0 \text{ and } a,b\in\mathbb{Q}}[a,b)\cup\bigcup_{\sqrt{2}<a<b \text{ and } a,b\in\mathbb{Q}}[a,b)\right)\in\mathscr{C}\end{aligned}$$

Thus $\left[0,\sqrt{2}\right]$ is closed, and thus $\overline{\left(0,\sqrt{2}\right)}\subset\left[0,\sqrt{2}\right]=\left(0,\sqrt{2}\right)\cup\left\{0,\sqrt{2}\right\}$. Now to find $\overline{\left(0,\sqrt{2}\right)}$ we simply must determine if 0 is a limit point of $\left(0,\sqrt{2}\right)$ and if $\sqrt{2}$ is a limit point of $\left(0,\sqrt{2}\right)$.

Any open interval around 0 must have an upper bound greater then 0, and thus 0 is a limit point for $\left(0,\sqrt{2}\right)$.

Any open interval around $\sqrt{2}$ must have a lower bound that is a rational number, $\sqrt{2}$ is not rational, thus there must be a rational number less then $\sqrt{2}$ in the interval, and thus $\sqrt{2}$ is a limit point for $\left(0,\sqrt{2}\right)$.

Thus we have shown

$$\left[0,\sqrt{2}\right]=\overline{\left(0,\sqrt{2}\right)}$$

Next we will find $\overline{\left(\sqrt{2},3\right)}$. We first will check if $\sqrt{2}$ and 3 are limit points of $\left(\sqrt{2},3\right)$.

For an open interval in $\mathscr{C}$ to include $\sqrt{2}$ there must a rational number greater then $\sqrt{2}$ as the upper bound, thus there is some value between $\sqrt{2}$ and that upper bound that is in $\left(\sqrt{2},3\right)$. Thus $\overline{\left(\sqrt{2},3\right)}\supset\left[\sqrt{2},3\right)$.

The interval $[3,4)$ includes 3 and is in $\mathcal{C}$, thus it is a nood[1] of 3 which contains no values in $\left(\sqrt{2},3\right)$. Thus we know $3\notin\overline{\left(\sqrt{2},3\right)}$.

Now we wish to show that $\left[\sqrt{2},3\right)$ is closed, if we can do that then we know that $\left[\sqrt{2},3\right)\subset\overline{\left(\sqrt{2},3\right)}\subset\left[\sqrt{2},3\right)$ or $\left[\sqrt{2},3\right)=\overline{\left(\sqrt{2},3\right)}$.

$$\begin{aligned}\left[\sqrt{2},3\right)^{\mathsf{c}} &= \left(-\infty,\sqrt{2}\right)\cup[3,\infty)\\ &= \left(\bigcup_{a<b<\sqrt{2} \text{ and } a,b\in\mathbb{Q}}[a,b)\cup\bigcup_{3\le a<b \text{ and } a,b\in\mathbb{Q}}[a,b)\right)\in\mathscr{C}\end{aligned}$$

Thus we have shown $\left[\sqrt{2},3\right)$ is closed, and thus have shown

$$\left[\sqrt{2},3\right)=\overline{\left(\sqrt{2},3\right)}$$

18.5 Consider the linear function $f:\mathbb{R}\to\mathbb{R}$ defined as $f(x)=\frac{x-a}{b-a}$. We know all linear functions are continuous, we have the homeomorphisms

$$\begin{aligned}f((a,b)) &= \left(\frac{a-a}{b-a},\frac{b-a}{b-a}\right)\\ &= (0,1)\end{aligned}$$

$$\begin{aligned}f([a,b]) &= \left[\frac{a-a}{b-a},\frac{b-a}{b-a}\right]\\ &= [0,1]\end{aligned}$$

thus we have shown homeomorphism between (a,b) and $(0,1)$, and between $[a,b]$ and $[0,1]$ for any $a<b\in\mathbb{R}$.

18.8(a) Consider first $\{x\in X\mid f(x)\le g(x)\}^{\mathsf{c}}=\{x\in X\mid f(x)>g(g)\}=S$. Let us now choose $x\in S$.

[1] nood = neighborhood

By theorem 17.11 we know that Y is Hausdorff, thus there exists disjoint open sets $\mathfrak{A}$ and $\mathfrak{B}$ such that $f(x) \in \mathfrak{A}$ and $g(x) \in \mathfrak{B}$. Let $A \subset \mathfrak{A}$ be a basis element in Y's order topology such that $f(x) \in A$ and let $B \subset \mathfrak{B}$ be a basis element in Y's order topology such that $g(x) \in B$. Now we have $\forall_{a\in A}\forall_{b\in B}[a > b]$ due to the definition of the order topology.

Next we notice that $g^{-1}(B)$ is open by continuity of g and $f^{-1}(A)$ is open by continuity of f. Finite intersections are open so $g^{-1}(B)\cap f^{-1}(A)$ must be open. Now for any $\hat{x} \in g^{-1}(B)\cap f^{-1}(A)$ we have $g(\hat{x}) \in B$ and $f(\hat{x}) \in A$ thus $g(\hat{x}) < f(\hat{x})$. We now have shown that for every $x \in S$ there exists a nood of x that is completely contained in S. We may now take the union of a nood for each x and we get that S is the union of open sets, thus S is open. We now conclude that $\{x \in X \mid f(x) \leq g(x)\}$ is closed.

(b) Let $A = \{x \in X \mid f(x) \leq g(x)\}$ and $B = \{x \in X \mid g(x) \leq f(x)\}$, thus $\forall_{a\in A} f(a) = h(a)$ and $\forall_{b\in B} g(b) = h(b)$; additionally $X = A \cup B$. Notice that if $x \in A \cap B$ then $f(x) \leq g(x)$ and $g(x) \leq f(x)$, thus $g(x) = f(x)$. Now the final thing we must show is that A and B are both closed, however we just showed that in part a of this problem, so we may use the pasting lemma and we know that $h(x)$ is continuous.

19.7 **Box topology:** Let $x \in \mathbb{R}^\omega \setminus \mathbb{R}^\infty$, therefore $x = (x_\alpha)_{\alpha\in J}$ where $x_a \neq 0$ for infinitely many $\alpha \in J$. For each $\alpha \in J$ such that $x_\alpha \neq 0$ let A_α be a nood of x_α that does not include 0, for all other $\alpha \in J$ let A_α be a nood of 0. We find that $\prod\limits_{\alpha\in J} A_\alpha$ is open in $\mathbb{R}^\omega$ as it is a basis element. We have now shown that for any $x \in \mathbb{R}^\omega \setminus \mathbb{R}^\infty$ there is an open set $A(x)$ such that $x \in A(x) \subset \mathbb{R}^\omega \setminus \mathbb{R}^\infty$, thus we may take the union and find $\bigcup\limits_{x\in\mathbb{R}^\omega\setminus\mathbb{R}^\infty} A(x) = \mathbb{R}^\omega \setminus \mathbb{R}^\infty$, thus $\mathbb{R}^\omega \setminus \mathbb{R}^\infty$ is open, and thus $\mathbb{R}^\infty$ is closed so $\mathbb{R}^\infty = \overline{\mathbb{R}^\infty}$.

Product topology: Let $x \in \mathbb{R}^\omega$ and let N be a nood of x, thus there exists some basis element of $\mathbb{R}^\omega$, A such that $x \in A \subset N$. There must then exists $(x_\alpha)_{\alpha\in J} = x$ and $\prod\limits_{\alpha\in J} A_\alpha = A$ with $x_\alpha \in \mathbb{R}$ for all $\alpha \in J$ and A_α open in $\mathbb{R}$ for all $\alpha \in J$. We also know that for only finitely many $\alpha \in J$, $A_\alpha \neq \mathbb{R}$. We will now let $y_\alpha = 0$ for all $\alpha \in J$ where $A_\alpha = \mathbb{R}$, and let $y_\alpha = x_\alpha$ for all $\alpha \in J$ where $A_\alpha \neq \mathbb{R}$, thus $y = (y_\alpha)_{\alpha\in J} \in A$. We also notice that there are at most finitely many $\alpha \in J$ such that $y_\alpha \neq 0$ thus $y \in \mathbb{R}^\infty$. This means that for any $x \in \mathbb{R}^\omega$ and any nood of x, there is some point $y \in \mathbb{R}^\infty$ such that y is in the chosen nood of x, thus all points in $\mathbb{R}^\omega$ are limit points of $\mathbb{R}^\infty$. Finally we conclude that $\overline{\mathbb{R}^\infty} = \mathbb{R}^\omega$.

20.4

		box topology	uniform topology	product topology
(a)	f	not continuous	not continuous	continuous
	g	not continuous	continuous	continuous
	h	not continuous	continuous	continuous
(b)	$\mathbf{w}$	does not converge	does not converge	converges
	$\mathbf{x}$	does not converge	converges	converges
	$\mathbf{y}$	does not converge	converges	converges
	$\mathbf{z}$	converges	converges	converges