Topology Homework 5

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18.2. Suppose that $f : X \to Y$ is continuous. If x is a limit point of the subset A of X, is it necessarily true that $f(x)$ is a limit point of $f(A)$?

No, it is not true. Suppose $X = Y = \mathbb{R}$ and we give X and Y the standard topology on $\mathbb{R}$. Then define f to be the zero function and let $A = (8, 9)$. Then we can choose x to be a limit point of A, for example $x = 9$, and then $f(A) = \{0\}$ and $f(x) = 0$ and 0 is not a limit point of $\{0\}$.

6. Find a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at precisely one point.

We assume the standard topology on $\mathbb{R}$. Now consider

$$f(x) = \begin{cases} 0 & x \notin \mathbb{Q} \\ x & x \in \mathbb{Q} \end{cases}$$

Let us first show that f is continuous at 0. $f(0) = 0$, so let V be an open set containing 0. Now because V is an open set in the standard topology on $\mathbb{R}$ containing 0 there will some $a > 0$ such that the interval $(-a, a) \subset V$. Now let $U = (-a, a)$, then find that $f(U) \subset V$ as for any $x \in U$, $f(x) = 0 \in V$ or $f(x) = x \in U \subset V$. Thus by our definition of point continuity f is continuous at the point 0.

Let us now consider a point $x \neq 0$ in $\mathbb{R}$. We will show that there exists some nhood V of $f(x)$, such that for any nhood U of x, we have $U \not\subset V$. Now if $x \in \mathbb{Q}$ then we will let V be an nhood of x not containing 0 and because the irrational's are dense on $\mathbb{R}$ there exists some irrational number in any nhood U of x, thus $0 \in f(U)$. If $x \notin \mathbb{Q}$ then $f(x) = 0$ so we may let $V = (-|x|, |x|)$. Now we will find that any nhood, U, of x contains some rational, q, such that $|q| > |x|$ and thus $f(q) = q \notin V$ and thus $f(U) \not\subset V$.

7.(a) First before we embark upon this little adventure let us give a rigorous definition for the limit from above, as I could not find any in this book. This definition is not complete as it does not deal with function of subsets of $\mathbb{R}$ and it does not deal with limits that tend to ∞ or $-\infty$, however it will be all we need.

Definition 0.1 (Limit from above)**.** Let $f : \mathbb{R} \to \mathbb{R}$, then

$$\lim_{x \to c^+} f(x) = a \in \mathbb{R}$$

if for every $\epsilon > 0$ there exists some $\delta > 0$ such that for any $x \in (c, c + \delta)$, $f(x) \in (a - \epsilon, a + \epsilon)$.

Now let $f : \mathbb{R} \to \mathbb{R}$ be continuous from above. Now choose $r \in \mathbb{R}$, and let V be a nhood of $f(r)$. Because V is open in $\mathbb{R}$ and contains $f(r)$, there must be some $\epsilon > 0$ such that $(f(r) - \epsilon, f(r) + \epsilon) \subset V$. Now by definition of continuity from above we know that $\lim_{x \to r^+} f(x) = f(r)$, thus we have some $\delta > 0$ such that for any $x \in (r, r + \delta)$, $f(x) \in (f(r) - \epsilon, f(r) + \epsilon)$; this may also be written as $f((r, r + \delta)) \subset (f(r) - \epsilon, f(r) + \epsilon)$. It is also trivial that $f(r) \in (f(r) - \epsilon, f(r) + \epsilon)$ so we may say that $f([r, r + \delta)) \subset (f(r) - \epsilon, f(r) + \epsilon) \subset V$. If we consider f then as a map from $\mathbb{R}_\ell$ to $\mathbb{R}$ we get that for any point $r \in \mathbb{R}$ and any nhood of V of $f(r)$, there exists a nhood $U = [r, r + \delta)$ of r such that $f(U) \subset f(V)$. Now by the powers vested in me by theorem 18.1 part 4 in the book I hereby declare f continuous when thought of as a map from $\mathbb{R}_\ell$ to $\mathbb{R}$.

(b) Notice how this section merely asks for conjectures, not proofs nor even for the conjectures to be correct.

Now I state this without proof as I really do not know how to prove it, but from what I can tell it seems to be right. First there are no continuous function from $\mathbb{R}$ to $\mathbb{R}_\ell$, in some sense there seems to be 'to many' open sets in $\mathbb{R}_\ell$ for each one to have an inverse that is open in $\mathbb{R}$. My next statement I have less of a feel for, but I feel that continuous functions from $\mathbb{R}$ to $\mathbb{R}$ are the same as continuous function from $\mathbb{R}_\ell$ to $\mathbb{R}_\ell$. I would like to talk to you more about this and see if we can't come up with a proof or a disproof of either of these statements.

19.2. Let A_α be a subspace of X_α, for each $\alpha \in J$. Show that that $\prod A_\alpha$ is a subspace of $\prod X_\alpha$ if both products are given the box topology, or if both products are given the product topology.

Box topology:

Let B be a basis element of $\prod A_\alpha$ as a box topology, then for each $\alpha \in J$ there exists B_α open in A_α such that $\prod B_\alpha = B$. For any given $\alpha \in J$ there must be some open C_α in X_α such that $B_\alpha = C_\alpha \cap A_\alpha$ because B_α is open in A_α which is a subspace of C_α. Now we find that $B = \prod B_\alpha = \prod [C_\alpha \cap A_\alpha] = \prod C_\alpha \cap \prod A_\alpha$ which is a basis element in $\prod A_\alpha$ as a subspace of $\prod X_\alpha$.

Let B now be a basis element of $\prod A_\alpha$ as a subspace of $\prod X_\alpha$, then there exists an open $C \in \prod X_\alpha$ such that $\prod A_\alpha \cap C = B$. By virtue of C being open in $\prod X_\alpha$ there exists some open $C_\alpha \subset X_\alpha$ for each $\alpha \in J$ such that $\prod C_\alpha = C$ Thus we have $B = \prod A_\alpha \cap C = \prod A_\alpha \cap \prod C_\alpha = \prod [A_\alpha \cap C_\alpha]$, which is a basis element of $\prod A_\alpha$ as the box topology of all A_α.

Product topology:

Let B be an open set in $\prod A_\alpha$ as a product topology, then (by theorem 19.1) for each $\alpha \in J$ there exists some B_α that is open in A_α, such that $\prod B_\alpha = B$. We also know (by theorem 19.1) that for only finitely many $\alpha \in J$, $B_\alpha \neq A_\alpha$. Now for any $\alpha \in J$ such that $B_\alpha = A_\alpha$ we let $C_\alpha = X_\alpha$ and for all other $\alpha \in J$ there must be some open set, C_α in X_α such that $A_\alpha \cap C_\alpha = B_\alpha$. Now for all but finitely many $\alpha \in J$, $C_\alpha = X_\alpha$, thus $\prod C_\alpha$ is an open set in $\prod X_\alpha$. Finally we say $B = \prod B_\alpha = \prod [C_\alpha \cap X_\alpha] = \prod C_\alpha \cap \prod X_\alpha$, thus B would be open in $\prod A_\alpha$ as a subspace topology.

Now let B be an open set in $\prod A_\alpha$ as a subspace of $\prod X_\alpha$, then there exists some open $C \subset \prod X_\alpha$ such that $C \cap \prod A_\alpha = \prod B_\alpha$. Because C is open in $\prod X_\alpha$ then for each $\alpha \in J$ there exists C_α open in X_α such that $\prod C_\alpha = C$ and for all but finitely many $\alpha \in J$, $C_\alpha = X_\alpha$. Now for each $\alpha \in J$ we let $B_\alpha = C_\alpha \cap A_\alpha$, thus B_α will be open in A_α and there may be only finitely many $B_\alpha \neq A_\alpha$. Finally we say that $B = C \cap \prod A_\alpha = \prod C_\alpha \cap \prod A_\alpha = \prod [C_\alpha \cap A_\alpha] = \prod B_\alpha$ which must be open in $\prod A_\alpha$ with a product topology.

3. Show that if for all $\alpha \in J$, X_α is Hausdorff then $\prod X_\alpha$ is Hausdorff given either a box or product topology.

First we will state and prove a theorem to help us in this proof.

Theorem 0.1. *Let a space X has two topologies $\mathcal{T}$ and $\mathcal{S}$ with $\mathcal{S}$ finer than $\mathcal{T}$. If $\mathcal{T}$ is Hausdorff then so must $\mathcal{S}$ be.*

Proof. Let us choose $x, y \in X$ such that $x \neq y$. As $\mathcal{T}$ is Hausdorff there exists a nhood U of x and nhood V of y in $\mathcal{T}$ such that U and V are disjoint. $\mathcal{S}$ is finer than $\mathcal{T}$ so U and V must also be in $\mathcal{S}$, thus for arbitrary point $x, y \in X$ there is are disjoint nhoods of them in $\mathcal{S}$. □

First we notice that the box topology is finer then the product topology, thus because of theorem 0.1 if we show that the product topology on $\prod X_\alpha$ is Hausdorff then so is the box topology on $\prod X_\alpha$.

Consider $\prod X_\alpha$ given the product topology, with X_α Hausdorff for all $\alpha \in J$. Let $x, y \in \prod X_\alpha$ be given with $x \neq y$. For each $\alpha \in J$ there exists some $x_\alpha, y_\alpha \in X_\alpha$ such that $\prod x_\alpha = x$ and $\prod y_\alpha = y$. There then must be some $\beta \in J$ for which $x_\beta \neq y_\beta$ as if there was not then we would have $x = y$. We have that X_β is Hausdorff, thus we may choose an nhoods of U_β of x_β and V_β of y_β such that $U_\beta \cap V_\beta = \varnothing$. Now for all $\alpha \in J \setminus \{\beta\}$ let us define $U_\alpha = V_\alpha = X_\alpha$. Now we have $\prod U_\alpha$ and $\prod V_\alpha$ open in $\prod X_\alpha$ with $\prod U_\alpha$ disjoint from $\prod V_\alpha$, and $x \in \prod U_\alpha$ and $y \in \prod V_\alpha$. Thus $\prod X_\alpha$ is Hausdorff.

Now by theorem 0.1 we have $\prod X_\alpha$ with the box topology as Hausdorff.

5. If f is continuous then all $\alpha \in J$, f_α is continuous.

Proof. Let there be some $\beta \in J$ such that f_β is not continuous. Let us also define $B_\alpha = X_\alpha$ for all $\alpha \neq \beta$, and let B_β an open set in X_β such that $f_\beta^{-1}(B_\beta)$ is not open in A. Now for each $\alpha \neq \beta$ we have $f_\alpha^{-1}(B_\alpha) = A$

by definition of a function (it must be defined at all points on A). Now we have

$$\begin{aligned}
f^{-1}\left(\prod B_\alpha\right) &= \left\{a \in A \mid f(a) \in \prod B_\alpha\right\} \\
&= \left\{a \in A \mid \prod f_\alpha(a) \in \prod B_\alpha\right\} \\
&= \{a \in A \mid \forall_{\alpha \in J}\, [f_\alpha(a) \in B_\alpha]\} \\
&= \bigcap_{\alpha \in J} [\{a \in A \mid f_\alpha(a) \in B_\alpha\}] \\
&= \bigcap_{\alpha \in J} \left[f_\alpha^{-1}(B_\alpha)\right] \\
&= \bigcap_{\alpha \in J \setminus \{\beta\}} \left[f_\alpha^{-1}(B_\alpha)\right] \cap f_\beta^{-1}(B_\beta) \\
&= A \cap f_\beta^{-1}(B_\beta) \\
&= f_\beta^{-1}(B_\beta)
\end{aligned}$$

$\square$