Combinatorics E xam 2

Benji Altman

November 18, 2017

1

1.a What is the OGF for integer partitions where each part occurs an even number of times?

Any integer partition, p on n may be written as a sequence $(a_1, a_2, a_3, \dots, a_k)$. If p has the property that every part in it appears an even number of times then we may order it's sequence representation such that $a_i = a_{i+k/2}$ for all $i \in [k/2]$. Now look at the partition composed of $(a_1, a_2, a_3, \dots, a_{k/2})$ and we find that this is a partition on $\frac{n}{2}$. Additionally if $(a_1, a_2, a_3, \dots, a_k)$ is a partition on $\frac{n}{2}$ then $(a_1, a_2, a_3, \dots, a_k, a_1, a_2, a_3, \dots, a_k)$ will be a partition with an even number of each part on n. Now we have constructed a bijection from partitions on n to partitions on $2n$ with an even number of each part. We know that the number of partitions on n has an OGF of

$$f(x) = \prod_{i \in \mathbb{N}} \left(\frac{1}{1 - x^i} \right)$$

thus the number of partitions on $2n$ with only even parts would have the same OGF and the number of partitions on n with only even parts would have an OGF of

$$A(x) = f(x^2) = \prod_{i \in \mathbb{N}} \left(\frac{1}{1 - x^{2i}} \right)$$

1.b What is the OGF for integer partitions where each part is even?

We know the OGF for integer partitions on n using only parts of size k is $\frac{1}{1-x^k}$ as it is only doable one way and only if n is a multiple of k. Thus to get the number of partitions on n using only parts that are even in size we get

$$B(x) = \frac{1}{1 - x^2} + \frac{1}{1 - x^4} + \frac{1}{1 - x^6} \cdots = \prod_{i \in \mathbb{N}} \left(\frac{1}{1 - x^{2i}} \right)$$

1.c What conclusions may be drawn?

Notice that $A(x) = B(x)$ where $A(x)$ is the number of partitions on n where each part occurs an even number of times and $B(x)$ is the number of partitions on n where each part is even. This means that there are the same number of partitions on n that are composed solely of even parts as there are if each part occurs an even number of times.

2

Definition 2.1. Words are formed using the letters A, B, C and digits $1, 2, 3, 4$. Set $a_0 = 0$ and for $n \geq 1$, let a_n be the number of words made up of n of these symbols in which there are not two letters in succession. (Two numbers in succession are okay.)

2.a find a_1 and a_2

For a word of length 1 we simply may choose any valid symbol to be the first character and there are 7 of them and then we are done, thus $a_1 = 7$.

For a word of length 2 we simply find all words of length two where we allow for any string of symbols to be valid and the remove the bad ones. There are 7^2 strings on our 7 symbols and 3^2 strings containing two letters. Thus we have $a_2 = 7^2 - 3^2 = 40$.

2.b Show that $a_n = 4a_{n-1} + 12a_{n-2}$ for $n \geq 3$

For any word with length $n > 2$ we may either end with a number or letter.

If it ends in a number then it may be composed by taking a word of length $n-1$ and appending a number. There are 4 numbers that may be appended a_{n-1} words of length $n-1$ thus we have $4a_{n-1}$ words of length n that end in a number.

If the word ends with a letter then the character before it must be a number giving use a 4 numbers followed by 3 letters that are possible and then before the number we may have any word of length $n-2$, which there are a_{n-2} of. Thus we have $3 \cdot 4 \cdot a_{n-2}$ words of length n that end with a letter.

Now we may put this together and we find $a_n = 4a_{n-1} + 12a_{n-2}$.

2.c Show that the OGF, $A(x) = \frac{7x+12x^2}{1-4x-12x^2}$

$$
\begin{aligned}
a_0 &= 0 \\
a_1 &= 7 \\
a_2 &= 40 \\
A(x) &= \sum_{n=0}^{\infty}(a_n x^n) \\
&= \sum_{n=3}^{\infty}(a_n x^n) + a_2x^2 + a_1x + a_0 \\
&= \sum_{n=3}^{\infty}((4a_{n-1} + 12a_{n-2})x^n) + a_2x^2 + a_1x + a_0 \\
&= \sum_{n=3}^{\infty}(4a_{n-1}x^n + 12a_{n-2}x^n) + a_2x^2 + a_1x + a_0 \\
&= \sum_{n=3}^{\infty}(4a_{n-1}x^n) + \sum_{n=3}^{\infty}(12a_{n-2}x^n) + a_2x^2 + a_1x + a_0 \\
&= 4x\sum_{n=2}^{\infty}(a_n x^n) + 12x^2\sum_{n=1}^{\infty}(a_n x^n) + a_2x^2 + a_1x + a_0 \\
&= 4x(A(x) - a_1x - a_0) + 12x^2(A(x) - a_0) + a_2x^2 + a_1x + a_0 \\
&= 4xA(x) - 4a_1x^2 - 4a_0x + 12x^2A(x) - 12a_0x^2 + a_2x^2 + a_1x + a_0 \\
&= A(x)(4x + 12x^2) + (-4a_1 - 12a_0 + a_2)x^2 + (-4a_0 + a_1)x + a_0 \\
&= A(x)(4x + 12x^2) + (-4(7) - 12(0) + 40)x^2 + (-4(0) + 7)x + 0 \\
&= A(x)(4x + 12x^2) + 12x^2 + 7x \\
A(x) - A(x)(4x + 12x^2) &= 12x^2 + 7x \\
A(x)(-4x - 12x^2 + 1) &= 12x^2 + 7x \\
A(x) &= \frac{12x^2 + 7x}{-4x - 12x^2 + 1} = \frac{7x + 12x^2}{1 - 4x - 12x^2}
\end{aligned}
$$

2.d Determine a simple cosed formula for a_n

$$\begin{aligned} A(x) + 1 &= \frac{7x + 12x^2 + 1 - 4x - 12x^2}{1 - 4x - 12x^2} \\ &= \frac{1 + 3x}{(-6x + 1)(2x + 1)} \\ &= \frac{a}{-6x + 1} + \frac{b}{2x + 1} \\ &= \frac{a(2x + 1) + b(-6x + 1)}{(-6x + 1)(2x + 1)} \end{aligned}$$

Thus $2a - 6b = 3$ and $a + b = 1$. Mathematica then is nice enough to tell us that $a = \frac{9}{8}$ and $b = -\frac{1}{8}$. Now armed with that we find

$$\begin{aligned} A(x) + 1 &= \frac{9/8}{-6x + 1} + \frac{-1/8}{2x + 1} \\ &= \frac{9}{8}\sum_{n=0}^{\infty}((6x)^n) - \frac{1}{8}\sum_{n=0}^{\infty}((-2x)^n) \\ &= \sum_{n=0}^{\infty}\left(\left(\frac{9}{8}(6)^n - \frac{1}{8}(-2)^n\right)x^n\right) \end{aligned}$$

We only need find a_n for $n \geq 1$ thus we don't care that we added 1 to $A(x)$ as that would only effect the x^0 term. Thus

$$a_n = \frac{9}{8}(6)^n - \frac{1}{8}(-2)^n$$

for all $n \geq 1$.

3 Vandermonde's formula

Definition 3.1. Vandermonde's formula

$$\sum_{j=0}^{k}\left(\binom{m}{j}\binom{n}{k-j}\right) = \binom{m+n}{k}$$

3.a Show that both sides of Vandermonde's formula are 0 if $k > m + n$

Let $k > m + n$.

First $\binom{m+n}{k} = 0$ as $k > m + n$.

Now for all $j \in [0, m]$ we find that $\binom{n}{k-j} = 0$ as $m \geq j$ thus $k - j \geq k - m > n$. For all $j > m$ we find that $\binom{m}{j} = 0$ and thus for all $j \in [0, k]$, $\binom{m}{j}\binom{n}{k-j} = 0$ and by extension $\sum_{n=0}^{j}\left(\binom{m}{j}\binom{n}{k-j}\right) = 0$.

3.b Give a combinatorial proof for Vandermode's formula

You have n blue marbles, numbered 1 through n, and m red marbles, numbered 1 through m, all in a bin. You want to take out k marbles. Well you have a set of $n + m$ unique objects and you wish to choose k of them, then this must simply be $\binom{n+m}{k}$.

You also however could break this up into cases. You could pick j of the red marbles, and thus would have picked $k - j$ of the blue marbles, where $0 \leq j \leq k$ (as you can not pick less than 0 red marbles or more than k red marbles and result in k marbles at the end). Thus we would find that you have $\binom{m}{j}$ ways to pick red marbles and $\binom{n}{k-j}$ ways to pick blue marbles for a fixed j. We thus must take the sum for all possible j and we find you have $\sum_{n=0}^{k}\left(\binom{m}{j}\binom{n}{k-j}\right)$ ways to do this.

Thus Vandermonde's formula must be true as we found that the right hand side and left hand side are both counting the same thing.

3.c Prove using OGF

Let our task $\mathcal{H}$ be choosing some number of elements from a set of cardinality $m+n$. We obviously have $h_k = \binom{m+n}{k}$ ways of doing this, and thus a generating function

$$H(x) = \sum_{k=0}^{\infty}\left(h_k x^k\right) = \sum_{k=0}^{\infty}\left(\binom{m+n}{k}x^k\right)$$

We may also split $\mathcal{H}$ into two tasks by first doing task $\mathcal{F}$ (choosing some number of elements from a set of size m) and $\mathcal{G}$ (choosing some number of elements from a set of size n). There are $f_k = \binom{n}{k}$ ways to do task $\mathcal{F}$, where k is the number of elements being chosen. There are $g_k = \binom{m}{k}$ ways to do task $\mathcal{G}$, where k is the number of elements being chosen. Thus $\mathcal{F}$ has OGF

$$\sum_{k=0}^{\infty}\left(\binom{n}{k}x^k\right)$$

and $\mathcal{G}$ has OGF

$$\sum_{k=0}^{\infty}\left(\binom{m}{k}x^k\right)$$

By the product formula for OGF we know that

$$H(x) = F(x)G(x)$$

$$\sum_{k=0}^{\infty}\left(\binom{m+n}{k}x^k\right) = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\left(\binom{m}{k}\binom{n}{k-j}\right)x^k\right)$$

Thus Vandermonde's theorem must be true.

4 Find the exponential generating function for the number of partitions of $[n]$ into an even number of odd-sized blocks. (You can only use odd-sized blocks. So the number of even-sized blocks is zero.

First let us have a zero-indexed sequence $a = (0,1,0,1,0,1,\ldots)$, which has EGF $A(x) = \sinh x$. This sequence can be thought of as 'accepting' any set with odd cardinality and 'rejecting' a set of even cardinality.

Second let us have another zero-indexed sequence $b = (1,0,1,0,1,0,\ldots)$, which has EGF $B(x) = \cosh x$. This sequence can be thought of as 'accepting' any set with even cardinality and 'rejecting' any other finite set.

Now we can use the composition rule for EGF and find $H(x) = B(A(x)) = \cosh(\sinh x)$ and it would represent splitting a set into any number of blocks, if a block is even in size then the configuration is 'rejected' and otherwise do nothing to it. Then on the set of blocks reject it if it is not an even sized set and accept if it is even sized.

$$\cosh(\sinh x)$$

5 Let r_n be the number of ways to partition $[n]$ without any singleton blocks

5.a Give combinatorial proof that $r_n = \sum_{k=0}^{n-2} \left(\binom{n-1}{k} r_k\right)$

The element n will be in a block with some number of other elements. Let k be the number of elements not in the block with n. Thus $k \in [0, n-2]$ as n must be joined by at least 1 other element. Now choose the k elements out of $[n] \setminus \{n\} = [n-1]$ to not be in a block with n (n must be in it's own block, and thus is excluded from this), and partition them such that there are no singleton blocks. There are $\binom{n-1}{k}$ ways to make the choice and r_k ways to partition them, thus we have $\binom{n-1}{k} r_k$ ways to partition when there are exactly k elements not in the block with n.

Now sum across all possible values for k and find $r_n = \sum_{k=0}^{n-2} \left(\binom{n-1}{k} r_k\right)$

5.b Prove that r_{n+1} counts the number of partitions on $[n]$ with at least 1 singleton block.

Let $t = n+1$. Now to partition $[n]$ such that it has some number of singleton sets we find that there must be some number of elements, $k \in [0, n-1]$, not in singleton blocks. We simply choose those k elements and then partition them into a set without any singleton blocks and find that we $\binom{n}{k} r_k$ ways to do this for a fixed k. We have only 1 way of distributing the remaining elements as they all go into singletons. Thus we simply sum across all possible values of k and find that we have $\sum_{k=0}^{n-1} \left(\binom{n}{k} r_k\right)$ ways to partition $[n]$ such that it has at least one singleton block. If we change over to using $t = n+1$ we find $\sum_{k=0}^{t-2} \left(\binom{t-1}{k} r_k\right)$ and that is the equation for $r_t = r_{n+1}$ thus r_{n+1} is the number of ways to partition $[n]$ such that there is at least one singleton block.

5.c Show that $B(n) = r_n + r_{n+1}$.

$B(n)$ is the number of partitions on $[n]$. Now either there is or there is not a singleton block in a partition. If there is at least one singleton block we have r_{n+1} ways to partition as shown above, and if there are no singleton sets then we have r_n ways to do this by definition of r_n. Thus $B(n) = r_n + r_{n+1}$.

5.d Find a closed formula for the EGF for r_n.

Consider the zero-indexed sequence $a = (1, 0, 1, 1, 1, \ldots)$ for task $\mathcal{A}$, which may be thought of 'accepting' any finite set whose cardinality is non-empty and it has EGF of $A(x) = e^x - x$. We now use theorem 3.30 from the book and set $a_0 = 0$ thus changing our EGF to $\hat{A}(x) = e^x - x - 1$. Then we would find that $\exp(e^x - x - 1)$ would be the EGF for splitting a set $[n]$ into as many blocks as we like and then doing task $\mathcal{A}$ to it. This thus must be the EGF for r_n it would split a set into any number of blocks and reject any singleton blocks.

$$R(x) = \exp\left(e^x - x - 1\right)$$

5.e Use the EGF to show that $r_n = \sum_{k=0}^{n} \left(\binom{n}{k}(-1)^{n-k} B(k)\right)$

Let us denote the n^{th} bell number as $B(n)$ and let $\mathscr{B}(x)$ be the EGF for $B(n)$. We find that $\mathscr{B}(x)$ by using the exponential rule for EGF and we end with

$$\mathscr{B}(x) = \exp(e^x - 1)$$

We also will denote

Now we do the following work

$$\begin{aligned}R(x) &= \exp(e^x - x - 1)\\ &= \exp(e^x - 1)e^{-x}\\ &= \mathscr{B}(x)e^{-x}\\ &= \sum_{n=0}^{\infty}\left(B(n)\frac{x^n}{n!}\right)\sum_{n=0}^{\infty}\left((-1)^n\frac{x^n}{n!}\right)\end{aligned}$$

By proposition 3.18 from the book we find that $R(x) = \sum_{n\geq 0}\left(\sum_{k=0}^{n}\left(\binom{n}{k}B(k)(-1)^{n-k}\right)\frac{x^n}{n!}\right)$ and thus we find

$$r_n = \sum_{k=0}^{n}\left(\binom{n}{k}B(k)(-1)^{n-k}\right)$$