Homework 3

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February 19, 2018

Question 1

Part a

So first notice that $L_k = I - E_k$ for some $n \times n$ matrix E_k. Notice that

$$E_k = \begin{bmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & \ell_{k+1,k} & 0 & & \\ & & \vdots & & \ddots & \\ & & \ell_{n,k} & & & 0 \end{bmatrix}$$

Notice now that $\ell_k e_k^T = E_k$ as $f(\vec{v}) = e_k^T \vec{v}$ will take a column vector $\vec{v}$ and put it into the k^{th} column of a matrix. Additionally we would like to notice that ${E_k}^2 = 0$ as E_k has zeros on the diagonal and all indicies above it, thus for any index i, j in ${E_k}^2$ the dot product that gives it would contain a zero in either the first or second vector in every index.

Now let A be the matrix that we would like to show is T_k^{-1}

$$\begin{aligned} L_k &= I - E_k \\ A &= I + E_k \\ L_k A &= (I - E_k)(I + E_k) \\ &= I^2 - (E_k)^2 \\ &= I \end{aligned}$$

And now by definition A is the inverse of T_k.

Part b

Now notice that

$$\begin{aligned} \prod_{k=1}^{n} T_k^{-1} &= \prod_{k=1}^{n} (I + E_k) \\ &= (I + E_1) \cdot (I + E_2) \cdots (I + E_n) \\ &= (I^2 + E_1 + E_1 + E_2 + E_1 \cdot E_2)(I + E_3) \cdots (I + E_n) \\ &= (I + E_1 + E_2)(I + E_3) \cdots (I + E_n) \\ &= \ldots \\ &= I + E_1 + E_2 + \ldots + E_n \end{aligned}$$

Again notice that $E_a E_b = 0_{n \times n}$ as E_a and E_b have zeros on the diagonal and everywhere above it.

Question 2

Part a

$$A = \begin{bmatrix} 4 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 4 & -1 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 4 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 4 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 4 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 4 \end{bmatrix}$$

and

$$b = \begin{bmatrix} 5 \\ 15 \\ 0 \\ 10 \\ 0 \\ 10 \\ 20 \\ 30 \end{bmatrix}$$

Part b

```
#Takes in matrix, A, and does a L-U decomposition
#Complete, tested
LUDecompose = function(A)
{
  if(nrow(A) != ncol(A))
    return(NULL)
  out = new.env()
  out$U = A
  out$L = diag(nrow(A))

  #now lets start getting to it
  i = 1
  while(i < nrow(A))
  {
    #make ith column 0
    row = i + 1
    while(row <= nrow(A))
    {
      #set row
      op = out$U[row, i] / out$U[i, i]
      out$L[row, i] = op
      out$U[row,] = out$U[row,] - out$U[i,] * op
      row = row + 1
    }
    i = i + 1
  }
  return(out)
```

```r
}
#Now to type in A
A = matrix(
  c(4,-1,-1,0,0,0,0,0,
    -1,4,0,-1,0,0,0,0,
    -1,0,4,-1,-1,0,0,0,
    0,-1,-1,4,0,-1,0,0,
    0,0,-1,0,4,-1,-1,0,
    0,0,0,-1,-1,4,0,-1,
    0,0,0,0,-1,0,4,-1,
    0,0,0,0,0,-1,-1,4),
  nrow = 8, ncol = 8, byrow = TRUE)


factorization = LUDecompose(A)
print(factorization$U)
```

```
##      [,1]  [,2]      [,3]      [,4]       [,5]      [,6]       [,7]
## [1,]    4 -1.00 -1.000000  0.000000  0.0000000  0.000000  0.0000000
## [2,]    0  3.75 -0.250000 -1.000000  0.0000000  0.000000  0.0000000
## [3,]    0  0.00  3.733333 -1.066667 -1.0000000  0.000000  0.0000000
## [4,]    0  0.00  0.000000  3.428571 -0.2857143 -1.000000  0.0000000
## [5,]    0  0.00  0.000000  0.000000  3.7083333 -1.083333 -1.0000000
## [6,]    0  0.00  0.000000  0.000000  0.0000000  3.391854 -0.2921348
## [7,]    0  0.00  0.000000  0.000000  0.0000000  0.000000  3.7051760
## [8,]    0  0.00  0.000000  0.000000  0.0000000  0.000000  0.0000000
##           [,8]
## [1,]  0.000000
## [2,]  0.000000
## [3,]  0.000000
## [4,]  0.000000
## [5,]  0.000000
## [6,] -1.000000
## [7,] -1.086128
## [8,]  3.386790
```

```r
print(factorization$L)
```

```
##       [,1]        [,2]       [,3]        [,4]       [,5]        [,6]
## [1,]  1.00  0.00000000  0.0000000  0.00000000  0.0000000  0.00000000
## [2,] -0.25  1.00000000  0.0000000  0.00000000  0.0000000  0.00000000
## [3,] -0.25 -0.06666667  1.0000000  0.00000000  0.0000000  0.00000000
## [4,]  0.00 -0.26666667 -0.2857143  1.00000000  0.0000000  0.00000000
## [5,]  0.00  0.00000000 -0.2678571 -0.08333333  1.0000000  0.00000000
## [6,]  0.00  0.00000000  0.0000000 -0.29166667 -0.2921348  1.00000000
## [7,]  0.00  0.00000000  0.0000000  0.00000000 -0.2696629 -0.08612836
## [8,]  0.00  0.00000000  0.0000000  0.00000000  0.0000000 -0.29482402
##            [,7] [,8]
## [1,]  0.0000000    0
## [2,]  0.0000000    0
## [3,]  0.0000000    0
## [4,]  0.0000000    0
## [5,]  0.0000000    0
## [6,]  0.0000000    0
```

```
## [7,]  1.0000000    0
## [8,] -0.2931381    1
```

Part c

```r
mySolve(factorization$L, factorization$U, c(5,15,0,10,0,10,20,30))
```

```
## [1]  3.956938  6.588517  4.239234  7.397129  5.602871  8.760766  9.411483
## [8] 12.043062
```

Part d

```r
mySolve(factorization$L, factorization$U, c(5,15,0,10,0,10,30,40))
```

```
## [1]  4.138756  6.770335  4.784689  7.942584  7.057416 10.215311 13.229665
## [8] 15.861244
```

Part e

```r
solve(A)
```

```
##             [,1]        [,2]       [,3]       [,4]       [,5]       [,6]
## [1,] 0.295264808 0.086553374 0.09450586 0.05094869 0.03180993 0.02273552
## [2,] 0.086553374 0.295264808 0.05094869 0.09450586 0.02273552 0.03180993
## [3,] 0.094505857 0.050948688 0.32707474 0.10928890 0.10450421 0.05913216
## [4,] 0.050948688 0.094505857 0.10928890 0.32707474 0.05913216 0.10450421
## [5,] 0.031809932 0.022735522 0.10450421 0.05913216 0.32707474 0.10928890
## [6,] 0.022735522 0.031809932 0.05913216 0.10450421 0.10928890 0.32707474
## [7,] 0.009998350 0.008183468 0.03180993 0.02273552 0.09450586 0.05094869
## [8,] 0.008183468 0.009998350 0.02273552 0.03180993 0.05094869 0.09450586
##             [,7]        [,8]
## [1,] 0.009998350 0.008183468
## [2,] 0.008183468 0.009998350
## [3,] 0.031809932 0.022735522
## [4,] 0.022735522 0.031809932
## [5,] 0.094505857 0.050948688
## [6,] 0.050948688 0.094505857
## [7,] 0.295264808 0.086553374
## [8,] 0.086553374 0.295264808
```

Question 6

Take a matrix A, one may define x_i such that x_i maximizes the i^{th} value in Ax_i. One does this by letting $x_{ij} = \begin{cases} 1 & A_{i,j} \geq 0 \\ -1 & A_{i,j} < 0 \end{cases}$. Now we may maximize $||Ax_i||_\infty$, by letting i be the row in A's sum of absolute values is greatest. This means that $||A||_\infty$ is the maximum sum of absolute values of rows

Question 7

let

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$$

Now we will do this in steps

$$P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 3 & 1 & 1 \\ -\frac{1}{3} & \frac{1}{3} & \frac{4}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 3 & 1 & 1 \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$$

Thus we have

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 3 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ -\frac{1}{3} & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 0 & -\frac{2}{3} & \frac{1}{3} \\ 0 & 0 & \frac{3}{2} \end{bmatrix}$$