Finding the image of a function

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1. **Definition**

   Let $f : A \to B$. The *image* of f is the set
   $$\operatorname{Im} f = \{f(a) : a \in A\}.$$

2. **Remarks**

   It follows that $b \in \operatorname{Im} f$, if and only if $\exists a \in A$ such that $f(a) = b$. Therefore, to check that a given element $b \in B$ is in $\operatorname{Im} f$ it suffices to show that the equation: $f(x) = b$ has a solution $x \in A$. The image of f is just the set of $b \in B$ such that $f(x) = b$ has a solution. Finding this set may be easy or very hard depending on the nature of the function f. We illustrate with some simple examples.

3. **Example**

   Let $f : \{x \in \mathbb{R} : x \geq 0\} \to \mathbb{R}$ be given by
   $$f(x) = \sqrt{x}.$$
   Find $\operatorname{Im} f$.

   Solution The equation $f(x) = b$ is equivalent to $\sqrt{x} = b, \quad x \geq 0$. We are required to find the set of all elements b in the codomain $\mathbb{R}$ for which this equation has a solution. If $b < 0$ there is no solution since, by definition, the principal square root is non negative. If $b \geq 0$ then $\sqrt{b^2} = |b| = b$ and so $x = b^2$ is a solution of the equation $f(x) = b$. Thus
   $$\operatorname{Im} f = \{x \in \mathbb{R} : x \geq 0\}$$

4. **Example**

   Let $f : \{x \in \mathbb{R} : x \geq 0\} \to \mathbb{R}$ be given by $f(x) = \sqrt{x} + 7$. Find $\operatorname{Im} f$.

   Solution The equation: $f(x) = b$ is equivalent to $\sqrt{x} + 7 = b, \quad x \geq 0$. We must find the set of all elements b in the codomain $\mathbb{R}$ for which this equation has a solution. If $b \geq 7$, then $b - 7 \geq 0$. In this case:
   $$\begin{aligned} \sqrt{(b-7)^2} + 7 &= |b-7| + 7 \\ &= b - 7 + 7 \\ &= b \end{aligned}$$
   Therefore $x = (b-7)^2$ is a solution $f(x) = b$. On the other hand if $b < 7$, the equation $f(x) = b$ has no solution since $\sqrt{(b-7)^2}$ is non negative by definition. We conclude that
   $$\operatorname{Im} f = \{b \in \mathbb{R} : b \geq 7\}.$$

5. **Example**

   Let $f : \mathbb{R} \to \mathbb{R}$ be given by
   $$f(x) = \sqrt{x^2 + 1}.$$
   Find $\operatorname{Im} f$.

   Solution The equation $f(x) = b$ is equivalent to $\sqrt{x^2+1} = b, \quad x \in \mathbb{R}$. If $b < 0$ this clearly has no solutions. On the other hand if $b \geq 0$
   $$\begin{aligned} & \sqrt{x^2+1} &= b \\ \Rightarrow & \quad x^2 + 1 &= b^2 \\ \Rightarrow & \quad x^2 &= b^2 - 1 \end{aligned}$$
   Clearly if $b^2 < 1$ there is no solution. If $b^2 \geq 1$, substitution reveals that both $x = \sqrt{b^2-1}$ and $x = -\sqrt{b^2-1}$ are solutions. Thus $f(x) = b$ has solutions if and only if $b \geq 0$ and $b^2 \geq 1$. But $b \geq 0$ and $b^2 \geq 1$ both hold if and only if $b \geq 1$. Hence
   $$\operatorname{Im} f = \{b \in \mathbb{R} : b \geq 1\}$$

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6. **Example**

Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = -3 - |2x|.$$

Find $\operatorname{Im} f$.

Solution

$$\begin{array}{ll} & f(x) = b \\ \Leftrightarrow & -3 - |2x| = b, \quad x \in \mathbb{R} \\ \Leftrightarrow & |2x| = -(b+3) \end{array}$$

Clearly if $b > -3$ this can have no solutions. On the other hand, if $b \leq -3$, then $x = \dfrac{b+3}{2}$ is a solution since

$$\begin{aligned} f(x) &= -3 - \left| 2\frac{b+3}{2} \right| \\ &= -3 - |b+3| \\ &= (-3) + (b+3) \\ &= b \end{aligned}$$

and so

$$\operatorname{Im} f = \{b \in \mathbb{R} : b \leq -3\}.$$