Summary of Definitions and Theorems from Exercise Sheets

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Disclaimer: I do *not* attend the lecture and therefore cannot guarantee that the content presented here aligns with the official material of the course. These notes are based solely on my interpretation and understanding of the assignments. This can **not replace the professor's lecture notes**, and it is **not binding** for the exam in any form. Use at your own discretion.

Contents

Sheet 1 Set Operations in Probability

Set Operations

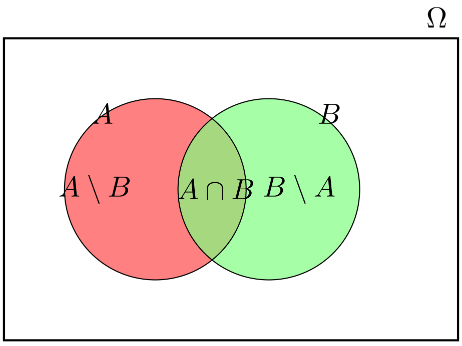
Definition 1.1 (Sample Space)**.** *The* sample space Ω *is the set of all possible outcomes of a random experiment.*

Definition 1.2 (Event)**.** *An* event *is a subset* $A \subset \Omega$.

Example 1.1 (Die Roll)**.** *Let* $\Omega = \{1, 2, 3, 4, 5, 6\}$. *The event "a die shows an odd face" is represented by the subset* $A = \{1, 3, 5\}$.

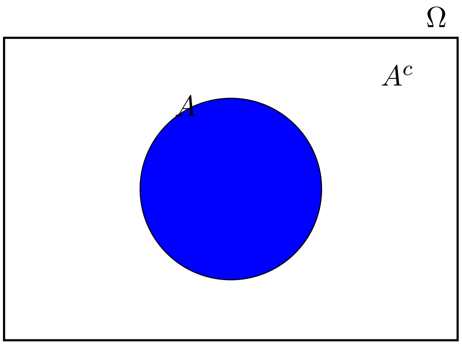
Basic Set Operations (for $A, B \subset \Omega$)

- **Union:** $A \cup B$ – all elements in A **or** in B
- **Intersection:** $A \cap B$ – all elements in both A **and** B
- **Complement:** $A^c = \Omega \setminus A$ – all elements **not** in A
- **Set Difference:** $A \setminus B = A \cap B^c$ – elements in A but not in B



Properties of Set Operations

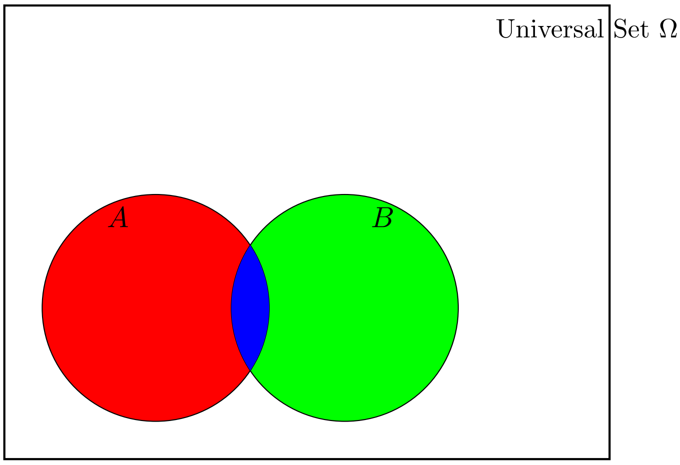
- $(A^c)^c = A$



Obviously is the complement of A^c, the white area, again blue, i.e. A.

- $(A \cup B)^c = A^c \cap B^c$

  In order to find $(A \cup B)^c$, the white area, just take all elements which are not in A, hence A^c. But this includes the green area, so just subtract B from A^c to get the white area. This implies $(A \cup B)^c = A^c \cap B^c$.



- $(A \cap B)^c = A^c \cup B^c$ (De Morgan's laws)

  We want to catch everything which is not in blue, i.e white + green + red. So take all elements in A^c (which is white + green) and add all elements in B^c (which is white + red) in order to achieve that $\implies (A \cap B)^c = A^c \cup B^c$.

- Distributivity:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Basic Observations

- $A \cap \Omega = A$
- $A \cap \emptyset = \emptyset$
- If $A, B \subset \Omega$, then:

$$A \cap B \subset A, \quad A \subset A \cup B$$

Remark 1.1. *By taking unions or intersections of events (subsets of Ω), we generate new events.*

Topic: Infinite Unions and Intersections

Definition 1.3 (Infinite Intersection). *Let $\{A_i\}_{i\in\mathbb{N}} \subseteq \Omega$ be a sequence of events. Then:*

$$x \in \bigcap_{i=1}^{\infty} A_i \quad \Leftrightarrow \quad x \in A_i \textit{ for all } i \in \mathbb{N}$$

This means x lies in every single set A_i.

Observation: The intersection is always a subset of each of the individual sets:

$$\bigcap_{i=1}^{\infty} A_i \subseteq A_j \quad \textit{for all } j \in \mathbb{N}$$

Definition 1.4 (Infinite Union). *Let $\{A_i\}_{i\in\mathbb{N}} \subseteq \Omega$. Then:*

$$x \in \bigcup_{i=1}^{\infty} A_i \quad \Leftrightarrow \quad \textit{there exists } j \in \mathbb{N} \textit{ such that } x \in A_j$$

This means x lies in at least one of the sets A_i, but it is not specified which.

Remark 1.2. *The union is always at least as large as any single set:*

$$A_j \subseteq \bigcup_{i=1}^{\infty} A_i \quad \textit{for all } j \in \mathbb{N}$$

Remark 1.3 (Distributive Laws with Infinite Collections). *The distributive properties also hold in the infinite case:*

$$B \cup \bigcap_{i=1}^{\infty} A_i = \bigcap_{i=1}^{\infty} (B \cup A_i)$$
$$B \cap \bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} (B \cap A_i)$$