Statistics – assignment no. 6

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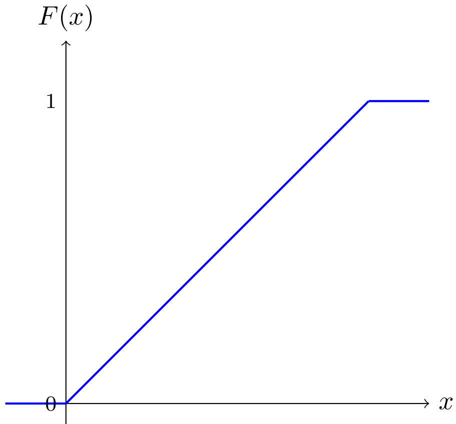
June 4, 2025

solution 30. Since X is continuous, its distribution function F is continuous. We know (or see the summary) what that means. This is how a continuous distribution function may look.



Continuous distribution function

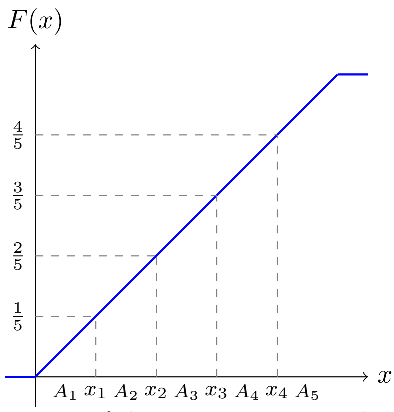
We see that it attains **ALL** values in the open interval $(0, 1)$ (and not only those discrete points).

Let $n \in \mathbb{N}$ with $n \geq 3$. By the intermediate value theorem, for each $i = 1, \ldots, n-1$ there exists a real number x_i such that

$$F(x_i) = \frac{i}{n}.$$

We may assume that $x_1 < x_2 < \cdots < x_{n-1}$. These values divide the real line into n disjoint intervals:

$$A_1 := (-\infty, x_1], \quad A_i := (x_{i-1}, x_i] \text{ for } i = 2, \ldots, n-1, \quad A_n := (x_{n-1}, \infty).$$



Partition of the support via quantiles

Each of these intervals has the same probability under X:

$$\mathbb{P}(X \in A_i) = \frac{1}{n}, \quad \text{for all } i = 1, \ldots, n.$$

We now bound the probability $\mathbb{P}(X = Y)$. Since X and Y are independent, we have

$$\mathbb{P}(X = Y) = \sum_{i=1}^{n} \mathbb{P}(X = Y,\, X \in A_i) \leq \sum_{i=1}^{n} \mathbb{P}(X \in A_i,\, Y \in A_i).$$

Using independence of X and Y:

$$\sum_{i=1}^{n} \mathbb{P}(X \in A_i) \cdot \mathbb{P}(Y \in A_i) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{P}(Y \in A_i).$$

But since the A_i form a partition of $\mathbb{R}$, we have:

$$\sum_{i=1}^{n} \mathbb{P}(Y \in A_i) = 1.$$

Hence:

$$\mathbb{P}(X = Y) \leq \frac{1}{n}.$$

This holds for *every* $n \geq 3$. Therefore, choosing it arbitrarily large:

$$\mathbb{P}(X = Y) = 0.$$

Conclusion: If X is continuous and independent of Y, then $\mathbb{P}(X = Y) = 0$.