Statistics – assignment no. 2

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solution 10. As our sample space Ω, we use $\Omega := \{1, 2, \dots, 1000\}$, and we define our probability measure P as usual by $P(A) = \frac{|A|}{|\Omega|} = \frac{|A|}{1000}$ for each event $A \subset \Omega$. Consider the events

$$A_k := \{\omega \in \Omega : \omega \text{ is divisible by } k\}, \quad k = 2, 3, \dots$$

We want to compute $P(A_3 \cup A_5 \cup A_7)$. But attention! The events A_3, A_5, and A_7 are not pairwise disjoint. E.g., obviously the natural number $15 \in A_3 \cap A_5$. In this case, we can use the inclusion-exclusion formula:

Inclusion-Exclusion Principle

For two events A and B (easy, we already know that), we have:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

This also works for three events A, B, and C, where we must subtract the pairwise intersections, but then the intersection of all three is subtracted as well (which we do not want because it is clearly included) and must be added back(draw a Venn-diagram to verify the 3 steps (all three - subtract intersections - but then observe that $A \cap B \cap C$ is removed, hence add it back):

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) \tag{1}$$
$$- P(A \cap B) - P(A \cap C) - P(B \cap C) \tag{2}$$
$$+ P(A \cap B \cap C). \tag{3}$$

Btw: The same procedure can be applied to unions of 4 (where then a fourth line is added above with intersec. of all four sets), 5, 6, and so on...

Hence, in our case:

$$\begin{aligned} P(A_3 \cup A_5 \cup A_7) = {} & P(A_3) + P(A_5) + P(A_7) \\ & - P(A_3 \cap A_5) - P(A_3 \cap A_7) - P(A_5 \cap A_7) \\ & + P(A_3 \cap A_5 \cap A_7). \end{aligned}$$

Therefore, we just have to calculate the probabilities of all the sets above. Note that a natural number m which is divisible by k can be written in the form kn=m, where n is another natural number (example: 10= 5*2). So we write our events A_k in the form

$$A_k = \{\omega \in \Omega : \omega \text{ is divisible by } k\} = \{kn : n \in \mathbb{N}, \quad \underbrace{n \le \left\lfloor \frac{1000}{k} \right\rfloor}_{\text{to ensure that } kn \le 1000} \quad \},$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x (example: $\lfloor 333.33 \rfloor = 333$. We see that

$$P(A_k) = \frac{|A_k|}{|\Omega|} = \frac{1}{1000}|A_k| = \frac{1}{1000}\left\lfloor \frac{1000}{k} \right\rfloor.$$

In particular:

$$\begin{aligned} P(A_3) &= \frac{\lfloor 1000/3 \rfloor}{1000} = \frac{333}{1000}, \\ P(A_5) &= \frac{\lfloor 1000/5 \rfloor}{1000} = \frac{200}{1000}, \\ P(A_7) &= \frac{\lfloor 1000/7 \rfloor}{1000} = \frac{142}{1000}. \end{aligned}$$

An integer is divisible by 3 and 5 if and only if it is divisible by $3 \cdot 5 = 15$. Thus:

$$\begin{aligned}
P(A_3 \cap A_5) = P(A_{3*5}) = P(A_{15}) &= \frac{\lfloor 1000/15 \rfloor}{1000} = \frac{66}{1000}, \\
P(A_3 \cap A_7) = P(A_{3*7}) = P(A_{21}) &= \frac{\lfloor 1000/21 \rfloor}{1000} = \frac{47}{1000}, \\
P(A_5 \cap A_7) = P(A_{5*7}) = P(A_{35}) &= \frac{\lfloor 1000/35 \rfloor}{1000} = \frac{28}{1000}, \\
P(A_3 \cap A_5 \cap A_7) = P(A_{3*5*7}) = P(A_{105}) &= \frac{\lfloor 1000/105 \rfloor}{1000} = \frac{9}{1000}.
\end{aligned}$$

Altogether, we have:

$$\begin{aligned}
P(A_3 \cup A_5 \cup A_7) &= \frac{333 + 200 + 142 - 66 - 47 - 28 + 9}{1000} \\
&= \frac{543}{1000}.
\end{aligned}$$

Answer: $\boxed{54.3\%}$