Prof. Dr. L. Imhof Bonn, Summer semester 2025

Statistics Tutorial
Exercise Sheet 2

Exercise 6. Suppose that A and B are disjoint events, that C is another event, and that

$$P(B \setminus C) = \frac{1}{4}, \qquad P((A \cup B)^c \setminus C) = \frac{1}{5}.$$

Compute $P(A \cup C)$.

Solution. The events $B \setminus C$ and $(A \cup B)^c \setminus C$ are disjoint (the first event is a subset of B and the second event is a subset of B^c). Their union is

$$\begin{aligned}(B \setminus C) \cup [(A \cup B)^c \setminus C] &= [B \cup (A \cup B)^c] \cap C^c \\ &= [B \cup (A^c \cap B^c)] \cap C^c \\ &= [(B \cup A^c) \cap (B \cup B^c)] \cap C^c \\ &= (B \cup A^c) \cap C^c \\ &= A^c \cap C^c.\end{aligned}$$

In the last equation we used that fact $B \subset A^c$, because A and B are disjoint. Using the two given probabilities, we find

$$P(A^c \cap C^c) = \frac{1}{4} + \frac{1}{5} = \frac{9}{20},$$

and the desired probability is

$$P(A \cup C) = 1 - P((A \cup C)^c) = 1 - P(A^c \cap C^c) = \frac{11}{20}.$$

Exercise 7. How many passwords satisfy the following requirements?

- The password consists of exactly 10 characters.
- Exactly one of the 10 characters is a digit $(0, 1, \ldots, 9)$.
- Exactly one of the 10 characters is one of the 11 special characters ! $? % # § & ( ) [ ]
- The 8 remaining characters are lowercase or uppercase letters (a, b, ..., z, A, B, ..., Z).

Suppose a password that satisfies these conditions is generated at random with all passwords of this type being equally likely to be generated. What is the probability that the password generated consists of 10 different characters?

Solution. Each password of the given type can be created as follows.

1. Choose the position for the digit	10 possibilities
2. Choose the digit	10 possibilities
3. Then choose the position for the special character	9 possibilities
4. Choose the special character	11 possibilities
5. Choose letters for the remaining 8 positions	52^8 possibilities

According to the multiplication rule, there are

$$10 \cdot 10 \cdot 9 \cdot 11 \cdot 52^8 \approx 5.293 \cdot 10^{17}$$

passwords that satisfy the given requirements.
To compute the desired probability, we determine the number of passwords of the given type that consist of 10 different characters. This number can be calculated as above; the only difference is that in the 5th step, we first choose one out of 52 letters, then one out of the remaining 51 letters, then one out of the remaining 50 letters, and so on. Instead of 52^8 possibilities, there are now $52 \cdot 51 \cdots 45$ possibilities. The desired probability is, after cancellation of common factors,

$$\frac{51 \cdot 50 \cdots 45}{52^7} \approx 0.568.$$

Exercise 8. A six-sided die is rolled twice. For every $k \in \mathbb{N}$, find the probability that the sum of the face values is k.

Solution. We use the sample space $\Omega = \{(i,j) : i,j \in \{1,\dots,6\}\}$ and the probability measure P given by $P(A) = |A|/|\Omega| = |A|/36$, $A \subset \Omega$. For every $k \in \mathbb{N}$, let A_k be the event that the sum of the face values is k. Thus, $A_k = \{(i,j) \in \Omega : i + j = k\}$. For $k = 1$ and for every $k \geq 13$, $A_k = \emptyset$, and so $P(A_k) = 0$.
For $k = 2, 3, \dots, 7$,

$$A_k = \{(i, k-i) : 1 \leq i \leq k-1\},$$

so that $|A_k| = k - 1$, and

$$P(A_k) = \frac{k-1}{36}.$$

For $k = 8, 9, \dots, 12$,

$$A_k = \{(i, k-i) : k-6 \leq i \leq 6\},$$

so that $|A_k| = 6 - (k-6) + 1 = 13 - k$, and

$$P(A_k) = \frac{13-k}{36}.$$

Exercise 9. A six-sided die is rolled three times. Which probability is larger: the probability that the sum of the face values is 9 or the probability that the sum of the face values is 10?

Solution. We use the sample space $\Omega = \{(i,j,k) : i,j,k \in \{1,\dots,6\}\}$ and the probability measure P given by $P(A) = |A|/|\Omega|$, $A \subset \Omega$. For $\ell \in \{9, 10\}$, let B_ℓ denote the event that the sum of the three face values is ℓ.

To find the number of elements of B_9, we partition the event according to the number that turned up on the first roll:

$$B_9 = \bigcup_{n=1}^{6} \{(i,j,k) \in \Omega : i = n, (j,k) \in A_{9-n}\},$$

where $A_m := \{(j,k) \in \{1,\dots,6\}^2 : j + k = m\}$. The number of elements of A_m was determined in the previous exercise. We have

$$|\{(i,j,k) \in \Omega : i = n, (j,k) \in A_{9-n}\}| = |A_{9-n}| = \begin{cases} 5, & n = 1, \\ 8 - n, & n = 2, \dots, 6. \end{cases}$$

Hence,

$$|B_9| = \sum_{n=1}^{6} |A_{9-n}| = 5 + 6 + 5 + 4 + 3 + 2 = 25.$$

Similarly,

$$B_{10} = \bigcup_{n=1}^{6} \{(i,j,k) \in \Omega : i = n, (j,k) \in A_{10-n}\},$$

$$\begin{aligned} |\{(i,j,k) \in \Omega : i = n, (j,k) \in A_{10-n}\}| &= |A_{10-n}| \\ &= \begin{cases} 13 - (10 - n) = 3 + n, & n = 1, 2, \\ 9 - n, & n = 3, \dots, 6, \end{cases} \end{aligned}$$

and

$$|B_{10}| = \sum_{n=1}^{6} |A_{10-n}| = 4 + 5 + 6 + 5 + 4 + 3 = 27.$$

Thus, $|B_{10}| > |B_9|$. The probability that the sum of the three face values is 10 is larger than the probability that the sum is 9.

Exercise 10. A natural number is chosen at random from the set $\{1, 2, \dots, 1000\}$ and each element has the same probability of being chosen. What is the probability that the number chosen can be divided by at least one of the numbers 3, 5, 7?

Solution. We use the sample space $\Omega = \{1, 2, \dots, 1000\}$ and the probability measure P defined by $P(A) = |A|/|\Omega|$ for each $A \subset \Omega$. Consider the events

$$A_k := \{\omega \in \Omega : \omega \text{ is divisible by } k\}, \qquad k = 2, 3, \dots$$

We want to compute $P(A_3 \cup A_5 \cup A_7)$. The events A_3, A_5, A_7 are not pairwise disjoint. E.g., $15 \in A_3 \cap A_5$. In this case, we can use equation (1.7) from the lectures, which gives

$$\begin{aligned} P(A_3 \cup A_5 \cup A_7) =& P(A_3) + P(A_5) + P(A_7) \\ & - P(A_3 \cap A_5) - P(A_3 \cap A_7) - P(A_5 \cap A_7) \\ & + P(A_3 \cap A_5 \cap A_7). \end{aligned}$$

Writing A_k in the form $A_k = \{kn : n \in \mathbb{N}, n \leq \frac{1000}{k}\}$, we see that

$$P(A_k) = \frac{|A_k|}{|\Omega|} = \frac{1}{1000}\left\lfloor \frac{1000}{k} \right\rfloor,$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. In particular,

$$P(A_3) = \frac{\lfloor 1000/3 \rfloor}{1000} = \frac{333}{1000}, \quad P(A_5) = \frac{\lfloor 1000/5 \rfloor}{1000} = \frac{200}{1000}, \quad P(A_7) = \frac{\lfloor 1000/7 \rfloor}{1000} = \frac{142}{1000}.$$

An integer is divisible by 3 and by 5 if and only if the integer is divisible by $3 \cdot 5 = 15$. Thus, $A_3 \cap A_5 = A_{15}$, and

$$P(A_3 \cap A_5) = P(A_{15}) = \frac{\lfloor 1000/15 \rfloor}{1000} = \frac{66}{1000}.$$

Similarly,

$$P(A_3 \cap A_7) = P(A_{21}) = \frac{\lfloor 1000/21 \rfloor}{1000} = \frac{47}{1000},$$

$$P(A_5 \cap A_7) = P(A_{35}) = \frac{\lfloor 1000/35 \rfloor}{1000} = \frac{28}{1000},$$

$$P(A_3 \cap A_5 \cap A_7) = P(A_{3\cdot 5\cdot 7}) = P(A_{105}) = \frac{\lfloor 1000/105 \rfloor}{1000} = \frac{9}{1000}.$$

Altogther, we have

$$P(A_3 \cup A_5 \cup A_7) = \frac{333 + 200 + 142 - 66 - 47 - 28 + 9}{1000} = \frac{543}{1000}.$$