exercise2 (Score: 14.0 / 14.0)

1. Written response (Score: 3.0 / 3.0)
2. Test cell (Score: 1.0 / 1.0)
3. Test cell (Score: 1.0 / 1.0)
4. Test cell (Score: 2.0 / 2.0)
5. Test cell (Score: 3.0 / 3.0)
6. Test cell (Score: 1.0 / 1.0)
7. Test cell (Score: 3.0 / 3.0)

Lab 2

1. 提交作業之前，建議可以先點選上方工具列的**Kernel**，再選擇**Restart & Run All**，檢查一下是否程式跑起來都沒有問題，最後記得儲存。
2. 請先填上下方的姓名(name)及學號(stduent_id)再開始作答，例如：

```
name = "我的名字"
student_id= "B06201000"
```

3. 四個求根演算法的實作可以參考lab-2 (https://yuanyuyuan.github.io/itcm/lab-2.html)，裡面有教學影片也有範例程式可以套用。
4. **Deadline: 10/9(Wed.)**

In [1]:

```
name = "李澤諺"
student_id = "B05902023"
```

Exercise 2

Kepler's equation

In celestial mechanics, *Kepler's equation*

$$M = E - e\sin(E)$$

relates the mean anomaly M to the eccentric anomaly E of an elliptical orbit of eccentricity e, where $0 < e < 1$, see *Wiki website* (https://en.wikipedia.org/wiki/Kepler's_laws_of_planetary_motion) for the details.

1. Prove that fixed-point iteration using the iteration function

$$g(E) = M + e\sin(E)$$

is convergent locally.

[Hint: You may use Ostrowski's Theorem mentioned in the lecture note.]

proof.

(Top)

令 $h(E) = g(E) - E = M + esinE - E$，$\forall E \in \mathrm{R}$
取 $a \in \mathrm{R}$ 且 $a > M + e$，則有 $h(a) = M + esina - a < M + esina - (M + e) = e(sina - 1) \leq 0$
取 $b \in \mathrm{R}$ 且 $b < M - e$，則有 $h(b) = M + esinb - b > M + esinb - (M - e) = e(sinb + 1) \geq 0$
因為 $h(E) = M + esinE - E$ 在 $[a, b]$ 上連續，且 $h(a) < 0 < h(b)$
所以由Intermediate Value Theorem，可得 $\exists\, c \in (a, b)$，使得 $h(c) = g(c) - c = 0$
即 $g(c) = c$，由此可知 g 存在fixed point
接著，令 α 為 g 的一個fixed point
因為 $g \in C^1(\mathrm{R})$，且 $\forall E \in \mathrm{R}$，皆有 $|g'(E)| = |\frac{d}{dE}(M + esinE)| = |ecosE| \leq e < 1$
因此由Ostrowski's Theorem，可知 $\exists\, \delta > 0$，使得 $\forall \{x_n\} \subset \mathrm{R}$，若 $|x_0 - \alpha| < \delta$，則 $\{x_n\}$ 會converge到 α
因此可得 g 會locally converge

---

2. Use the fixed-point iteration scheme in "Q.1" to solve Kepler's equation for the eccentric anomaly E corresponding to a mean anomaly $M = \frac{2\pi}{3}$ and an eccentricity $e = 0.5$.

Part 0. Import libraries

In [2]:

```
import matplotlib.pyplot as plt
import numpy as np
```

Part 1. Define the fixed point function

In [3]:

(Top)

```python
def fixed_point(
    func,
    x_0,
    tolerance=1e-7,
    max_iterations=5,
    report_history=False
):
    '''Approximate solution of f(x)=0 on interval [a,b] by the secant method.

    Parameters
    ----------
    func : function
        The target function.
    x_0 : float
        Initial guess point for a solution f(x)=0.
    tolerance: float
        One of the termination conditions. Error tolerance.
    max_iterations : (positive) integer
        One of the termination conditions. The amount of iterations allowed.
    report_history: bool
        Whether to return history.

    Returns
    -------
    solution : float
        Approximation of the root.
    history: dict
        Return history of the solving process if report_history is True.
    '''

    x = x_0
    iteration = 0
    history = {'estimation' : [] , 'error' : []}
    while True:
        error = abs(func(x) - x)

        history['estimation'].append(x)
        history['error'].append(error)

        iteration += 1
        if (iteration >= max_iterations or error < tolerance):
            return (x , history) if report_history else x

        x = func(x)
```

Test your implementaion with the assertion below.

In [4]:

```python
          test_fixed_method                                                    (Top)

root = fixed_point(lambda x: x - (x**2 - 4*x + 3.5), 2, tolerance=1e-7, max_iterations=100, report_histor
y=False)

error = np.inf
for solution in np.roots([1, -4, 3.5]):
    if abs(root - solution) < error:
        exact_solution = solution
        error = abs(root - solution)

assert error < 1e-7
```

Part 2. Assign values to variables anomaly mean "*M*" and eccentricity "*e*".

$$M = \frac{2\pi}{3} \quad \text{and} \quad e = 0.5$$

In [5]:

(Top)

```
M = 2 * np.pi / 3
e = 0.5
```

In [6]:

M_and_e (Top)

```
print('M =', M)
print('e =', e)

### BEGIN HIDDEN TESTS
assert M == 2*np.pi/3, 'M is wrong!'
assert e == 0.5, 'e is wrong!'
### END HIDDEN TESTS
```

```
M = 2.0943951023931953
e = 0.5
```

Part 3. Define the function of Kepler's equation

Recall Kepler's equation **:**

$$M = E - e\sin(E).$$

So we let the function $f(E) = E - e\sin(E) - M$, then

$$g(E) = E - f(E) = M + e\sin(E)$$

For the instance:

If we want to implement "$\sin(x)$", we will call `np.sin(x)` with numpy in python.

In [7]:

(Top)

```
def f(E):
    return E - e * np.sin(E) - M

def g(E):
    return M + e * np.sin(E)
```

In [8]:

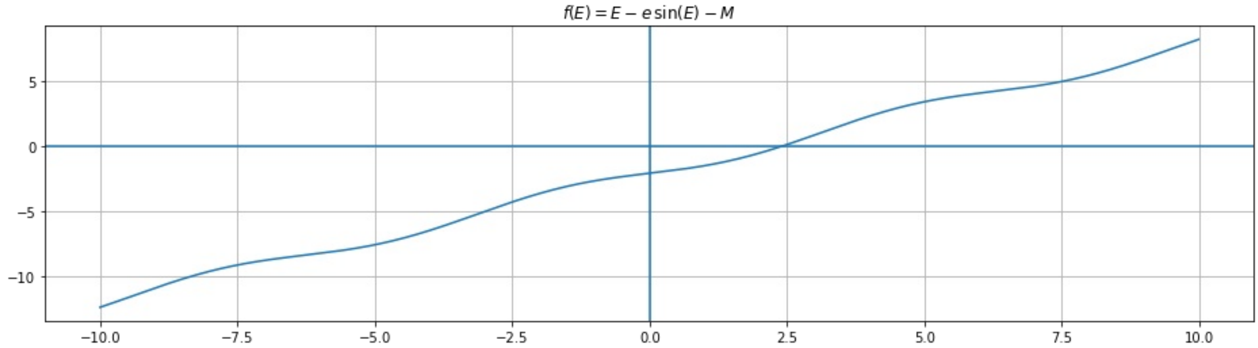
```
        test_f_and_g                                                    (Top)

print('M =', M)

# f(0) = -M, g(0) = M
print('f(0) =', f(0))
print('g(0) =', g(0))

### BEGIN HIDDEN TESTS
from random import random
rd_number = random()
assert f(rd_number) == rd_number - 0.5*np.sin(rd_number) - 2*np.pi/3, 'f is wrong!'
assert g(rd_number) == 2*np.pi/3 + 0.5*np.sin(rd_number), 'g is wrong!'
### END HIDDEN TESTS
```

```
M = 2.0943951023931953
f(0) = -2.0943951023931953
g(0) = 2.0943951023931953
```

Part 4. Plot the function $f(E)$ and $g(E)$

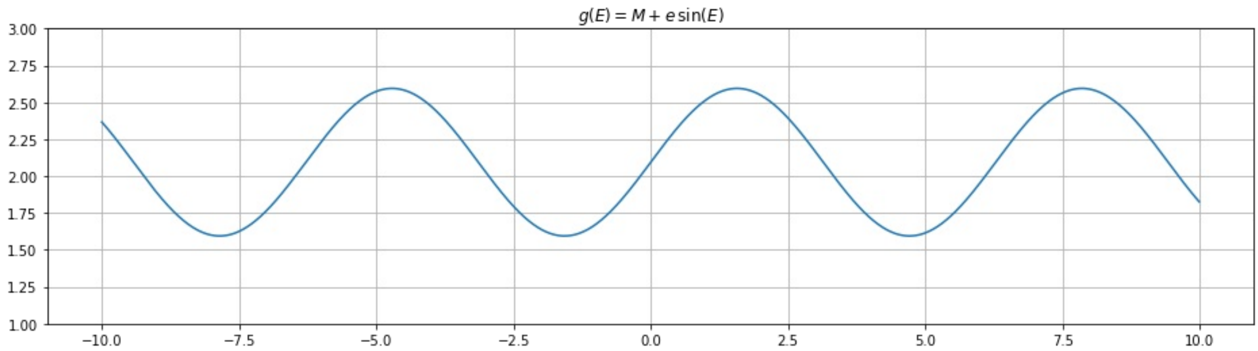
In [9]:

```
fig, ax = plt.subplots(figsize=(16, 4))
search_range = np.arange(-10, 10, 0.01)
ax.plot(search_range, f(search_range))
ax.set_title(r'$f(E) = E - e\,\sin(E) - M$')
ax.grid(True)
ax.axhline(y=0)
ax.axvline(x=0)
plt.show()
```



In [10]:

```
fig, ax = plt.subplots(figsize=(16, 4))
search_range = np.arange(-10, 10, 0.01)
ax.plot(search_range, g(search_range))
ax.set_title(r'$g(E) = M + e\,\sin(E)$')
ax.grid(True)
ax.axhline(y=0)
plt.ylim(1,3)
plt.show()
```



Part 5. Find the solution of "E"

In [11]:

(Top)

```
root = fixed_point(func = g , x_0 = 2 , tolerance = 1e-7 , max_iterations = 100 , report_history = False)
```

In [12]:

the_root_of_E (Top)

```
print('My estimation of root:', root)

### BEGIN HIDDEN TESTS
assert abs(root - 2.425) < 0.002, 'root is wrong!'
### END HIDDEN TESTS
```

```
My estimation of root: 2.423405402184932
```

---

3. An " *exact* " formula for E is known:

$$E = M + 2\sum_{m=1}^{\infty} \frac{1}{m} J_m(me)\sin(mM);$$

where $J_m(x)$ is the Bessel function of the first kind of order m.

Use this formula to compute E. How many terms are needed to produce the value obtained in "Q.2" until convergence?

Part 0. Import package

In [13]:

```
from scipy.special import jn  # Bessel function
```

Part 1. Define the function

For the convenience, we define the function $h(m)$ as

$$h(m) \triangleq \frac{2}{m} J_m(me)\sin(mM)$$

If we want to implement " ***Bessel function*** " $J_m(x)$, we can call `jn(m,x)` in Python.

In [14]:

(Top)

```
def h(m):
    return (2 / m) * jn(m , m * e) * np.sin(m * M)
```

In [15]:

```
h                                                    (Top)
```

```
# test the function of h
print('h(1) =', h(1))
assert round(h(1), 5) == 0.41962

### BEGIN HIDDEN TESTS
from random import random
rd_number = random()
assert h(rd_number) == 2*jn(rd_number, rd_number*0.5)*np.sin(rd_number*(2*np.pi/3))/rd_number, 'h is wron
g!'
### END HIDDEN TESTS
```

```
h(1) = 0.41962127776423175

---------------------------------------------------------------------------
AssertionError                            Traceback (most recent call last)
<ipython-input-15-69cf0965b94d> in <module>
      6 from random import random
      7 rd_number = random()
----> 8 assert h(rd_number) == 2*jn(rd_number, rd_number*0.5)*np.sin(rd_number*(2*np.pi/3))/r
d_number, 'h is wrong!'
      9 ### END HIDDEN TESTS

AssertionError: h is wrong!
```

Part 2. Find how many terms we need to achieve the result obtained Q.2 in a tolerance 10^{-7}.

That is to find _num*terms* such that

$$\left| \text{root} - \left(M + \sum_{k=1}^{\text{num\_terms}} h(k) \right) \right| < 10^{-7}$$

For example, the following cell shows the implmentation with only 1 term.

In [16]:

```
LHS = root
RHS = M + h(1)
error = abs(LHS-RHS)
print('Left hand side is the estimation of root by the fixed-point method:', LHS)
print('Right hand side is the approximation by the formula in only 1 term:', RHS)
print('The error between LHS and RHS:', error)
```

```
Left hand side is the estimation of root by the fixed-point method: 2.423405402184932
Right hand side is the approximation by the formula in only 1 term: 2.514016380157427
The error between LHS and RHS: 0.09061097797249529
```

In [17]:

```
                                                     (Top)
```

```
LHS = root
RHS = M + h(1)
num_terms = 1
tolerance = 1e-7

while (abs(LHS - RHS) >= tolerance):
    num_terms += 1
    RHS += h(num_terms)
```

In [18]:

number_of_term (Top)

```
print('Number of terms to approximate:', num_terms)

### BEGIN HIDDEN TESTS
assert num_terms > 20 , '%d is too few!' % num_terms
### END HIDDEN TESTS
```

```
Number of terms to approximate: 23
```

In [ ]: