exercise1 (Score: 14.0 / 14.0)

1. Test cell (Score: 2.0 / 2.0)
2. Test cell (Score: 1.0 / 1.0)
3. Test cell (Score: 1.0 / 1.0)
4. Test cell (Score: 1.0 / 1.0)
5. Test cell (Score: 1.0 / 1.0)
6. Test cell (Score: 3.0 / 3.0)
7. Test cell (Score: 2.0 / 2.0)
8. Task (Score: 3.0 / 3.0)

Lab 5

1. 提交作業之前，建議可以先點選上方工具列的**Kernel**，再選擇**Restart & Run All**，檢查一下是否程式跑起來都沒有問題，最後記得儲存。
2. 請先填上下方的姓名(name)及學號(stduent_id)再開始作答，例如：

```
name = "我的名字"
student_id= "B06201000"
```

3. 演算法的實作可以參考lab-5 (https://yuanyuyuan.github.io/itcm/lab-5.html), 有任何問題歡迎找助教詢問。
4. **Deadline: 12/11(Wed.)**

In [1]:

```
name = "李澤諺"
student_id = "B05902023"
```

Exercise 1

An $m \times m$ ***Hilbert matrix*** H_m has entries $h_{ij} = 1/(i+j-1)$ for $1 \le i,j \le m$, and so it has the form

$$\left [

$$\begin{matrix} 1 & 1/2 & 1/3 & \cdots \\ 1/2 & 1/3 & 1/4 & \cdots \\ 1/3 & 1/4 & 1/5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

\right].$$

In [2]:

```
import numpy as np
from numpy import linalg as LA
import matplotlib.pyplot as plt
```

Part 1

Generate the Hilbert matrix of order m, for $m = 2, 3, \dots, 12$.

For each m, compute the condition number of H_m, *ie* , in p-norm for $p = 1$ and 2, and make a plot of the results.

Part 1.1

Define the function of Hilbert matrix

In [3]:

(Top)

```
def hilbert_matrix(m):
    '''
    Return:
        2D np.array, the Hildert Matrix of order m
    '''
    H = np.zeros((m , m))
    for i in range(m):
        for j in range(m):
            H[i][j] = 1 / (i + j + 1)
    return H
```

Test your function.

In [4]:

hilbert_matrix (Top)

```
print('H_2:\n', hilbert_matrix(2))
### BEGIN HIDDEN TESTS
assert np.mean(np.array(hilbert_matrix(3)) - np.array([[1, 1/2, 1/3], [1/2, 1/3, 1/4], [1/3, 1/4, 1/5]]))
< 1e-7
### END HIDDEN TESTS
```

```
H_2:
 [[1.         0.5       ]
 [0.5        0.33333333]]
```

Part 1.2

Collect all Hilbert matrices into the list `H_m` for m = 2, 3, ..., 12.

In [5]:

(Top)

```
H_m = []
for m in range(2 , 13):
    H_m.append(hilbert_matrix(m))
```

Check your Hilbert matrix list.

In [6]:

hilbert_matrices (Top)

```
for i in range(len(H_m)):
    print('H_%d:' % (i+2))
    print(H_m[i])
    print()
### BEGIN HIDDEN TESTS
error = 0
for m in range(2, 13):
    error += LA.norm(hilbert_matrix(m) - np.array([[1/(i + j + 1) for j in range(m)] for i in range(m)]))
assert error < 1e-16
### END HIDDEN TESTS
```

```
H_2:
[[1.         0.5       ]
 [0.5        0.33333333]]

H_3:
[[1.         0.5        0.33333333]
 [0.5        0.33333333 0.25      ]
 [0.33333333 0.25       0.2       ]]

H_4:
[[1.         0.5        0.33333333 0.25      ]
 [0.5        0.33333333 0.25       0.2       ]
 [0.33333333 0.25       0.2        0.16666667]
 [0.25       0.2        0.16666667 0.14285714]]

H_5:
[[1.         0.5        0.33333333 0.25       0.2       ]
 [0.5        0.33333333 0.25       0.2        0.16666667]
 [0.33333333 0.25       0.2        0.16666667 0.14285714]
 [0.25       0.2        0.16666667 0.14285714 0.125     ]
 [0.2        0.16666667 0.14285714 0.125      0.11111111]]

H_6:
[[1.         0.5        0.33333333 0.25       0.2        0.16666667]
 [0.5        0.33333333 0.25       0.2        0.16666667 0.14285714]
 [0.33333333 0.25       0.2        0.16666667 0.14285714 0.125     ]
 [0.25       0.2        0.16666667 0.14285714 0.125      0.11111111]
 [0.2        0.16666667 0.14285714 0.125      0.11111111 0.1       ]
 [0.16666667 0.14285714 0.125      0.11111111 0.1        0.09090909]]

H_7:
[[1.         0.5        0.33333333 0.25       0.2        0.16666667
  0.14285714]
 [0.5        0.33333333 0.25       0.2        0.16666667 0.14285714
  0.125     ]
 [0.33333333 0.25       0.2        0.16666667 0.14285714 0.125
  0.11111111]
 [0.25       0.2        0.16666667 0.14285714 0.125      0.11111111
  0.1       ]
 [0.2        0.16666667 0.14285714 0.125      0.11111111 0.1
  0.09090909]
 [0.16666667 0.14285714 0.125      0.11111111 0.1        0.09090909
  0.08333333]
 [0.14285714 0.125      0.11111111 0.1        0.09090909 0.08333333
  0.07692308]]

H_8:
[[1.         0.5        0.33333333 0.25       0.2        0.16666667
  0.14285714 0.125     ]
 [0.5        0.33333333 0.25       0.2        0.16666667 0.14285714
  0.125      0.11111111]
 [0.33333333 0.25       0.2        0.16666667 0.14285714 0.125
  0.11111111 0.1       ]
 [0.25       0.2        0.16666667 0.14285714 0.125      0.11111111
  0.1        0.09090909]
 [0.2        0.16666667 0.14285714 0.125      0.11111111 0.1
  0.09090909 0.08333333]
 [0.16666667 0.14285714 0.125      0.11111111 0.1        0.09090909
  0.08333333 0.07692308]
 [0.14285714 0.125      0.11111111 0.1        0.09090909 0.08333333
  0.07692308 0.07142857]
 [0.125      0.11111111 0.1        0.09090909 0.08333333 0.07692308
  0.07142857 0.06666667]]

H_9:
[[1.         0.5        0.33333333 0.25       0.2        0.16666667
  0.14285714 0.125      0.11111111]
 [0.5        0.33333333 0.25       0.2        0.16666667 0.14285714
  0.125      0.11111111 0.1       ]
 [0.33333333 0.25       0.2        0.16666667 0.14285714 0.125
  0.11111111 0.1        0.09090909]
 [0.25       0.2        0.16666667 0.14285714 0.125      0.11111111
  0.1        0.09090909 0.08333333]
 [0.2        0.16666667 0.14285714 0.125      0.11111111 0.1
  0.09090909 0.08333333 0.07692308]
 [0.16666667 0.14285714 0.125      0.11111111 0.1        0.09090909
  0.08333333 0.07692308 0.07142857]
 [0.14285714 0.125      0.11111111 0.1        0.09090909 0.08333333
  0.07692308 0.07142857 0.06666667]
 [0.125      0.11111111 0.1        0.09090909 0.08333333 0.07692308
```

```
  0.07142857 0.06666667 0.0625    ]
 [0.11111111 0.1        0.09090909 0.08333333 0.07692308 0.07142857
  0.06666667 0.0625     0.05882353]]

H_10:
[[1.         0.5        0.33333333 0.25       0.2        0.16666667
  0.14285714 0.125      0.11111111 0.1       ]
 [0.5        0.33333333 0.25       0.2        0.16666667 0.14285714
  0.125      0.11111111 0.1        0.09090909]
 [0.33333333 0.25       0.2        0.16666667 0.14285714 0.125
  0.11111111 0.1        0.09090909 0.08333333]
 [0.25       0.2        0.16666667 0.14285714 0.125      0.11111111
  0.1        0.09090909 0.08333333 0.07692308]
 [0.2        0.16666667 0.14285714 0.125      0.11111111 0.1
  0.09090909 0.08333333 0.07692308 0.07142857]
 [0.16666667 0.14285714 0.125      0.11111111 0.1        0.09090909
  0.08333333 0.07692308 0.07142857 0.06666667]
 [0.14285714 0.125      0.11111111 0.1        0.09090909 0.08333333
  0.07692308 0.07142857 0.06666667 0.0625    ]
 [0.125      0.11111111 0.1        0.09090909 0.08333333 0.07692308
  0.07142857 0.06666667 0.0625     0.05882353]
 [0.11111111 0.1        0.09090909 0.08333333 0.07692308 0.07142857
  0.06666667 0.0625     0.05882353 0.05555556]
 [0.1        0.09090909 0.08333333 0.07692308 0.07142857 0.06666667
  0.0625     0.05882353 0.05555556 0.05263158]]

H_11:
[[1.         0.5        0.33333333 0.25       0.2        0.16666667
  0.14285714 0.125      0.11111111 0.1        0.09090909]
 [0.5        0.33333333 0.25       0.2        0.16666667 0.14285714
  0.125      0.11111111 0.1        0.09090909 0.08333333]
 [0.33333333 0.25       0.2        0.16666667 0.14285714 0.125
  0.11111111 0.1        0.09090909 0.08333333 0.07692308]
 [0.25       0.2        0.16666667 0.14285714 0.125      0.11111111
  0.1        0.09090909 0.08333333 0.07692308 0.07142857]
 [0.2        0.16666667 0.14285714 0.125      0.11111111 0.1
  0.09090909 0.08333333 0.07692308 0.07142857 0.06666667]
 [0.16666667 0.14285714 0.125      0.11111111 0.1        0.09090909
  0.08333333 0.07692308 0.07142857 0.06666667 0.0625    ]
 [0.14285714 0.125      0.11111111 0.1        0.09090909 0.08333333
  0.07692308 0.07142857 0.06666667 0.0625     0.05882353]
 [0.125      0.11111111 0.1        0.09090909 0.08333333 0.07692308
  0.07142857 0.06666667 0.0625     0.05882353 0.05555556]
 [0.11111111 0.1        0.09090909 0.08333333 0.07692308 0.07142857
  0.06666667 0.0625     0.05882353 0.05555556 0.05263158]
 [0.1        0.09090909 0.08333333 0.07692308 0.07142857 0.06666667
  0.0625     0.05882353 0.05555556 0.05263158 0.05      ]
 [0.09090909 0.08333333 0.07692308 0.07142857 0.06666667 0.0625
  0.05882353 0.05555556 0.05263158 0.05       0.04761905]]

H_12:
[[1.         0.5        0.33333333 0.25       0.2        0.16666667
  0.14285714 0.125      0.11111111 0.1        0.09090909 0.08333333]
 [0.5        0.33333333 0.25       0.2        0.16666667 0.14285714
  0.125      0.11111111 0.1        0.09090909 0.08333333 0.07692308]
 [0.33333333 0.25       0.2        0.16666667 0.14285714 0.125
  0.11111111 0.1        0.09090909 0.08333333 0.07692308 0.07142857]
 [0.25       0.2        0.16666667 0.14285714 0.125      0.11111111
  0.1        0.09090909 0.08333333 0.07692308 0.07142857 0.06666667]
 [0.2        0.16666667 0.14285714 0.125      0.11111111 0.1
  0.09090909 0.08333333 0.07692308 0.07142857 0.06666667 0.0625    ]
 [0.16666667 0.14285714 0.125      0.11111111 0.1        0.09090909
  0.08333333 0.07692308 0.07142857 0.06666667 0.0625     0.05882353]
 [0.14285714 0.125      0.11111111 0.1        0.09090909 0.08333333
  0.07692308 0.07142857 0.06666667 0.0625     0.05882353 0.05555556]
 [0.125      0.11111111 0.1        0.09090909 0.08333333 0.07692308
  0.07142857 0.06666667 0.0625     0.05882353 0.05555556 0.05263158]
 [0.11111111 0.1        0.09090909 0.08333333 0.07692308 0.07142857
  0.06666667 0.0625     0.05882353 0.05555556 0.05263158 0.05      ]
 [0.1        0.09090909 0.08333333 0.07692308 0.07142857 0.06666667
  0.0625     0.05882353 0.05555556 0.05263158 0.05       0.04761905]
 [0.09090909 0.08333333 0.07692308 0.07142857 0.06666667 0.0625
  0.05882353 0.05555556 0.05263158 0.05       0.04761905 0.04545455]
 [0.08333333 0.07692308 0.07142857 0.06666667 0.0625     0.05882353
  0.05555556 0.05263158 0.05       0.04761905 0.04545455 0.04347826]]
```

Part 1.3

Plot the condition number of H_m for $m = 2, 3, \ldots, 12$

Collect all condition numbers in 1-norm of `H_m` into a list `one_norm`

In [7]:

(Top)

```
one_norm = []
for i in range(11):
    one_norm.append(LA.norm(H_m[i] , 1) / LA.norm(LA.inv(H_m[i]) , 1))
```

In [8]:

kappa_one_norm

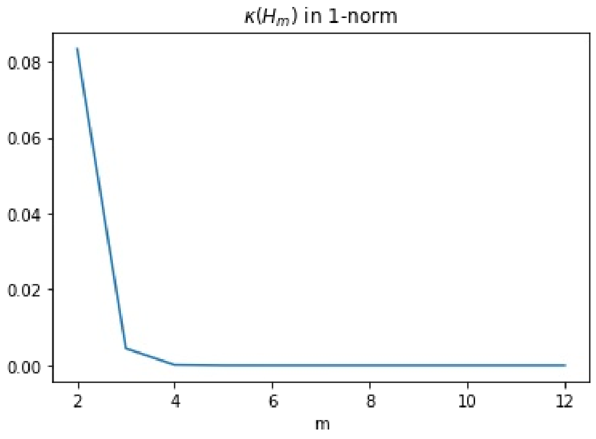
(Top)

```
print('one_norm:\n', one_norm)
### BEGIN HIDDEN TESTS
assert len(one_norm) == 11
### END HIDDEN TESTS
```

```
one_norm:
 [0.08333333333333333, 0.004493464052287565, 0.00015296133137541839, 5.524906439503468e-06, 2.
0648236637913913e-07, 6.8239373086425736e-09, 2.180731769950491e-10, 7.277812828044664e-12, 2
.426363372806177e-13, 7.3871176888843537e-15, 2.2627167250355433e-16]
```

In [9]:

```
plt.plot(range(2,13), one_norm)
plt.xlabel('m')
plt.title(r'$\kappa(H_m)$ in 1-norm')
plt.show()
```



Collect all condition numbers in 2-norm of `H_m` into a list `two_norm`

In [10]:

(Top)

```
two_norm = []
for i in range(11):
    two_norm.append(LA.norm(H_m[i] , 2) / LA.norm(LA.inv(H_m[i]) , 2))
```

In [11]:

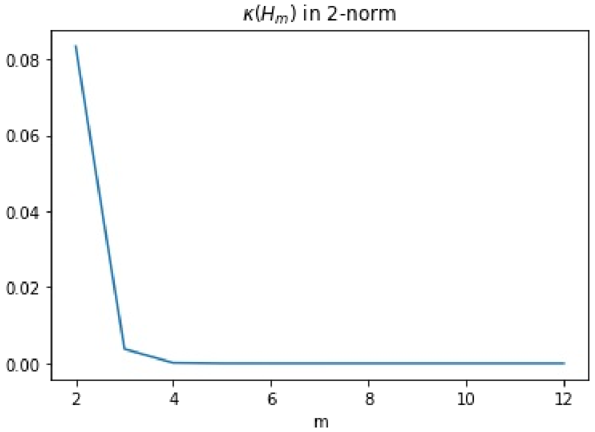
kappa_two_norm (Top)

```
print('two_norm:\n', two_norm)
### BEGIN HIDDEN TESTS
assert len(two_norm) == 11
### END HIDDEN TESTS
```

```
two_norm:
 [0.08333333333333331, 0.0037846322866590615, 0.00014507417740930578, 5.152351054679042e-06,
1.7529439325786222e-07, 5.802964952326552e-09, 1.8851022079708264e-10, 6.040044796749552e-12,
1.9151496323761425e-13, 6.018505882132021e-15, 1.819482930617947e-16]
```

In [12]:

```
plt.plot(range(2,13), two_norm)
plt.xlabel('m')
plt.title(r'$\kappa(H_m)$ in 2-norm')
plt.show()
```



Part 2

Now generate the m-vector $b_m = H_m x$ also, where x is the m-vector with all of its components equal to 1.

Use Gaussian elimination to solve the resulting linear system $H_m x = b_m$ with H_m and b given above, obtaining an approximate solution $\tilde{x}$.

Part 2.1

Construct the m-vector b_m for $m = 2, 3, \ldots, 12$. Store all 1D `np.array` b_m into the list `b_m` .

In [13]:

(Top)

```
b_m = []
for i in range(11):
    b_m.append(np.dot(H_m[i] , np.ones(H_m[i].shape[0])))
```

Print `b_m`

In [14]:

```
        b_m
```

```
for i in range(len(b_m)):
    print('b_%d:' % (i+2))
    print(b_m[i])
    print()
### BEGIN HIDDEN TESTS
error = 0
for m in range(2,13):
    error += LA.norm(b_m[m-2] - np.array([[1/(i + j + 1) for j in range(m)] for i in range(m)])@np.ones(m
))
assert error < 1e-16
### END HIDDEN TESTS
```

```
b_2:
[1.5        0.83333333]

b_3:
[1.83333333 1.08333333 0.78333333]

b_4:
[2.08333333 1.28333333 0.95       0.75952381]

b_5:
[2.28333333 1.45       1.09285714 0.88452381 0.74563492]

b_6:
[2.45       1.59285714 1.21785714 0.99563492 0.84563492 0.73654401]

b_7:
[2.59285714 1.71785714 1.32896825 1.09563492 0.93654401 0.81987734
 0.73013376]

b_8:
[2.71785714 1.82896825 1.42896825 1.18654401 1.01987734 0.89680042
 0.80156233 0.72537185]

b_9:
[2.82896825 1.92896825 1.51987734 1.26987734 1.09680042 0.96822899
 0.86822899 0.78787185 0.72169538]

b_10:
[2.92896825 2.01987734 1.60321068 1.34680042 1.16822899 1.03489566
 0.93072899 0.84669538 0.77725094 0.7187714 ]

b_11:
[3.01987734 2.10321068 1.68013376 1.41822899 1.23489566 1.09739566
 0.98955252 0.90225094 0.82988251 0.7687714  0.71639045]

b_12:
[3.10321068 2.18013376 1.75156233 1.48489566 1.29739566 1.15621919
 1.04510808 0.95488251 0.87988251 0.81639045 0.761845   0.71441417]
```

Part 2.2

Implement the function of **Gaussian elimination**.

(Note that you need to implement it by hand, simply using some package functions is not allowed.)

In [15]:

(Top)

```
def gaussian_elimination(
    A,
    b
):
    '''
    Arguments:
        A : 2D np.array
        b : 1D np.array

    Return:
        x : 1D np.array, solution to Ax=b
    '''

    m = A.shape[0]
    U = np.copy(A)
    x = np.zeros(m)
    y = np.copy(b)
    for i in range(m - 1):
        L = np.identity(m)
        for j in range(i + 1 , m):
            L[j][i] = -(U[j][i] / U[i][i])
            U[j , i : ] += (L[j][i] * U[i , i : ])
            y[j] += (L[j][i] * y[i])

    x[-1] = y[-1] / U[-1][-1]
    for i in range(1 , m):
        x[-1 - i] = (y[-1 - i] - np.dot(U[-1 - i , -i : ] , x[-i : ])) / U[-1 - i][-1 - i]

    return x
```

Store all approximate solutions $\tilde{x}$ of H_m into a list `x_m` for $m = 2, 3, \ldots, 12$

In [16]:

```
x_m = []
for i in range(len(H_m)):
    x = gaussian_elimination(H_m[i], b_m[i])
    x_m.append(x)
```

Part 3

Investigate the error behavior of the computed solution $\tilde{x}$.

(i) Compute the ∞-norm of the residual $r = b - H_m\tilde{x}$.

(ii) Compute the error $\delta x = \tilde{x} - x$, where x is the vector of all ones.

(iii) How large can you take m before there is no significant digits in the solution ?

Part 3.1

Compute the ∞-norm of the residual $r_m = b_m - H_m\tilde{x}$ for $m = 2, 3, \ldots, 12$. And store the values into the list `r_m` .

In [17]:

(Top)

```
r_m = []
for i in range(11):
    r_m.append(LA.norm(b_m[i] - np.dot(H_m[i] , x_m[i]) , np.inf))
```

In [18]:

```
            infty_norm
```

(Top)

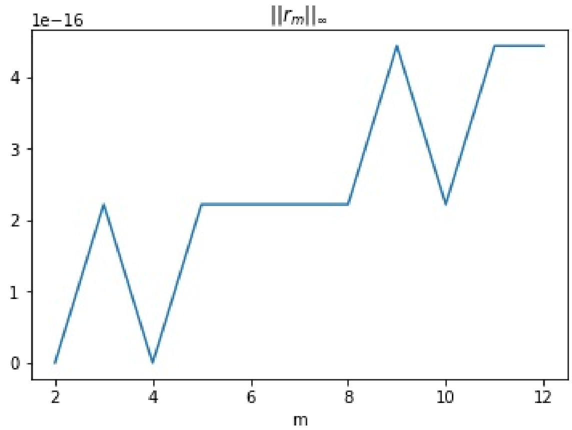
```
print('r_m:\n', r_m)
### BEGIN HIDDEN TESTS
assert np.sum(r_m) < 1e-12
### END HIDDEN TESTS
```

```
r_m:
 [0.0, 2.220446049250313e-16, 0.0, 2.220446049250313e-16, 2.220446049250313e-16, 2.2204460492
50313e-16, 2.220446049250313e-16, 4.440892098500626e-16, 2.220446049250313e-16, 4.44089209850
0626e-16, 4.440892098500626e-16]
```

Plot the figure of the ∞-norm of the residual for $m = 2, 3, \ldots, 12$

In [19]:

```
plt.plot(range(2,13), r_m)
plt.xlabel('m')
plt.title(r'$||r_m||_\infty$')
plt.show()
```



Part 3.2

Compute the error $\delta x = \tilde{x} - x$, where x is the vector of all ones. And store the values into the list `delta_x` .

In [20]:

(Top)

```
delta_x = []
for i in range(11):
    delta_x.append(x_m[i] - np.ones(x_m[i].shape[0]))
```

Collect all errors δx in 2-norm into the list `delta_x_two_norm` for $m = 2, 3, \ldots, 12$

In [21]:

(Top)

```
delta_x_two_norm = []
for i in range(11):
    delta_x_two_norm.append(LA.norm(delta_x[i] , 2))
```

In [22]:

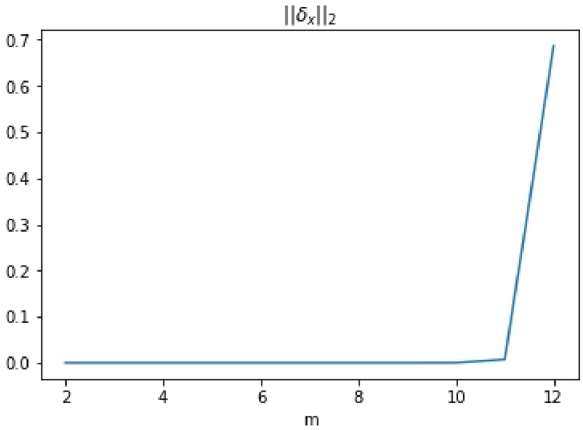
delta_x_two_norm (Top)

```
print('delta_x_two_norm =', delta_x_two_norm)
### BEGIN HIDDEN TESTS
assert (len(delta_x_two_norm) == 11) and (np.mean(delta_x_two_norm) <= 0.1)
### END HIDDEN TESTS
```

```
delta_x_two_norm = [8.005932084973442e-16, 1.762179615616027e-14, 1.531093756529614e-13, 3.47
26989976779822e-12, 3.380198688368993e-10, 9.687978241052163e-09, 4.244691764807721e-07, 1.73
93602379023114e-07, 0.0001452735113877698, 0.007757231295806999, 0.6867691718060317]
```

In [23]:

```
plt.plot(range(2,13), delta_x_two_norm)
plt.xlabel('m')
plt.title(r'$||\delta_x||_2$')
plt.show()
```



(Top)

Part 3.3

How large can you take m before there is no significant digits in the solution ?

由Part 3.2的圖可以看出，當m大於11時，誤差 $||\delta x||_2$ 會突然增加，因此m可以取到的最大值應為11。

In [ ]: