Phase Resetting in the Yamada Model of a Q-Switching Laser

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Abstract: We investigate the phase resetting of a periodic orbit of a self-pulsing laser, described by the Yamada model. We show how the return to the periodic orbit is affected by a brief perturbation during the associated oscillation.

Phase resetting is a technique used to investigate how the phase of an oscillating, or periodic, system is altered or reset due to an external perturbation. While it is often employed in a neuroscience setting, is has found significant use in general dynamical systems. For an oscillating system, phase resetting is characterised by the adjustment of the phase of the system in response to a brief perturbation. This perturbation results in the phase of the periodic orbit – when the pulse arrives – being advanced or delayed, thereby altering its timing relative to the unperturbed state. The resulting phase shift is often portrayed in the form of a *phase-transition curve* (PTC), showing the relationship between the point at which the perturbation is applied, θ_{old}, and the shifted phase, θ_{new}. Each phase specifies a point on the periodic orbit. The attracting periodic orbit has a *basin of attraction*, and each point in the basin of attraction converges back to regular oscillations in synchrony with a specific point on the periodic orbit.

In this work we investigate the phase resetting of a self-pulsing laser, also known as a *Q-switching* laser, with a saturable absorber [1, 2], consisting of a gain and an absorber section. We study a specific model, due to Yamada [3, 4], in the form of three coupled ordinary differential equations:

$$\begin{aligned} \dot{G} &= \gamma(A - G - GI), \\ \dot{Q} &= \gamma(B - Q - aQI), \\ \dot{I} &= (G - Q - 1)I, \end{aligned}$$

where G is the gain, Q the absorption, and I the intensity of the laser; here the dot indicates the derivative with respect to time. In a typical experiment, the dimensionless parameters B and a are fixed, as well as the decay time γ of gain and absorption. The pump current A, however, can be varied experimentally.

The question we address here is how the regular self-pulsating behaviour of this Q-switched device is affected by external perturbations, given as impulses in the intensity I that are into the laser cavity. This behaviour is closely related to the concept of excitability in response to an external input, where the system is just below the self-pulsating regime [5]. To investigate the phase resetting we take a dynamical systems approach and employ recently developed methods [6, 7] based on numerical continuation of boundary value problems in the MATLAB based software COCO [8].

Intensity perturbations are straightforward experimentally, so we set the perturbation in the positive I-direction, given by $\vec{d} = (0, 0, 1)$, with amplitude $A_{\text{p}} > 0$. The associated response of the Yamada model is illustrated in Fig. 1. Panel 1a depicts the time-series of the unperturbed attracting periodic orbit Γ, along with an example of a perturbed orbit. The perturbation is applied at $t = 0$, initially advancing the oscillations compared to the unperturbed orbit. The perturbed orbit is effectively back to the original oscillations at $t \sim 300$, but with a slight delay of $t \approx 4.5$.

Figure 1b shows the phase portrait representation of the periodic orbit Γ and the perturbed orbit which converges back to Γ. The unperturbed attracting periodic orbit Γ is located near the saddle equilibrium point p with non-zero intensity. Its one-dimensional stable manifold $W^s(p)$ "goes through the centre" and forms a boundary of the basin of attraction of Γ (since all points on $W^s(p)$ converge to p rather than to Γ). Note that the perturbed orbit for $A_{\text{p}} = 7.5$ is still within the basin of attraction and, hence, it relaxes back towards Γ.

Figure 1c shows PTCs for four different perturbation amplitudes. The perturbation is applied at each point along the original periodic orbit at phase θ_{old}. The phase at which the perturbed orbit returns, or "resets", is recorded as θ_{new}. If no perturbation is applied, these two phases are the same and thus the PTC is the identity (dashed line in Fig. 1c). For weaker perturbations, such as $A_{\text{p}} = 0.05$ and 0.1, the initial point of the perturbation lies within the basin of attraction and close to the original periodic orbit. We then see qualitatively similar behaviour in the PTC compared to the unperturbed case: all the perturbed orbits converge back to Γ, the new phase θ_{new} also changes over 2π (as does θ_{old}). For much larger perturbation amplitudes, $A_{\text{p}} = 25$ and 30 in Fig. 1c, we see completely

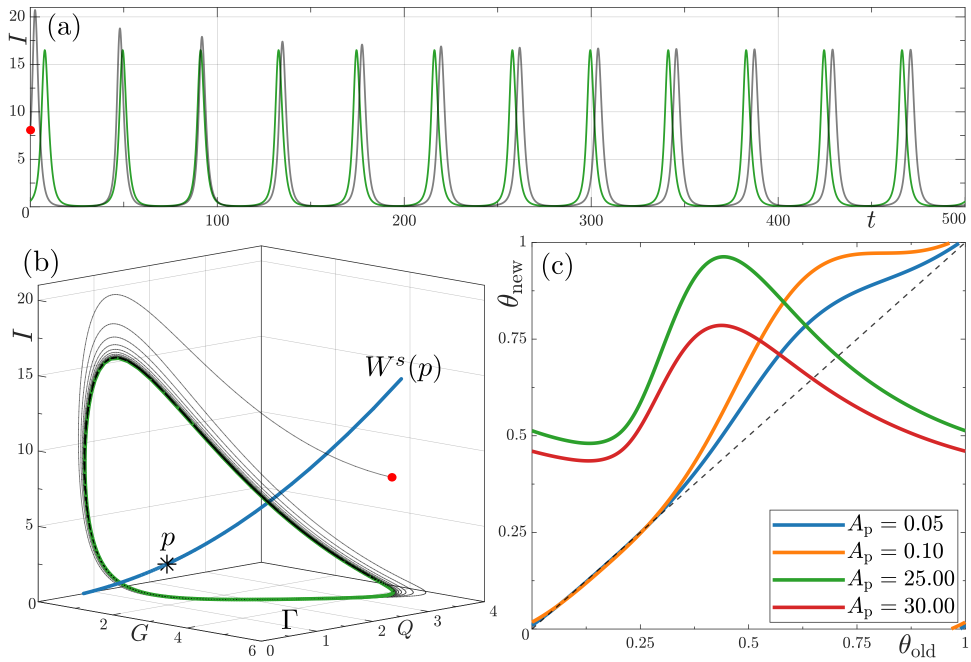

Fig. 1. (a) Time series of the unperturbed pulsations (green). The perturbed response (black) returns back to the original pulsations at $t \sim 300$, with a delay. The perturbation is applied in the positive I-direction, $\vec{d} = (0,0,1)$, with amplitude $A_p = 7.5$ at $t = 0$, highlighted in red. (b) Phase-space representation of the same orbits in the (G,Q,I)-space; also shown is the saddle-point p $(*)$ with its one-dimensional stable manifold $W^s(p)$ (blue). (c) PTCs for four different perturbation amplitudes as shown. The PTCs for the two weaker perturbations are close to the identity (black, dashed), while those for the two larger perturbations, are qualitatively different. Parameters are $\gamma = 0.0354, A = 7.3757, B = 5.8$ and $a = 1.8$.

different behaviour. As the perturbation is applied along the original periodic orbit, there is no longer an increase of 2π in θ_{new}. In fact, there is an exact amplitude, A_p^*, at which this change in that nature of the PTC occurs: when there is a point on Γ that intersects the stable manifold $W^u(p)$. Then the system will evolve towards the steady state p, and *not* back to Γ. Moving past this special value of A_p results in the observed qualitative change of the PTC [6].

PTCs could be measured experimentally by, for example, applying many perturbations to the laser cavity and recording the resulting temporal traces, as in Ref. [5]. From these traces, one could build a PTC for different perturbation amplitudes. The dynamical systems approach taken to investigate this phase resetting phenomena allows us to identify quantitative and qualitative changes in the synchronisation dynamics that could be experimentally observed. This type of information could be useful for the implementation and applications of self-pulsating lasers.

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