

Multi-Mode Filter Equations

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1 Hamiltonian and Master Equation

The Hamiltonian is

$$H = \hbar \frac{\Omega}{2} (\sigma_+ + \sigma_-) + \hbar \sum_{j=-N}^N \omega_j a_j^\dagger a_j + \frac{i\hbar}{2} \sum_{j=-N}^N \left(\mathcal{E}_j^* a_j \sigma_+ - \mathcal{E}_j a_j^\dagger \sigma_- \right), \quad (1)$$

where Ω is the Rabi frequency, σ_+ and σ_- are the atomic raising and lowering operators, a^\dagger and a are the cavity photon creation and annihilation operators, N is the number of modes either side of the central mode ($2N + 1$ total modes),

$$\omega_j = \omega_0 + j\delta\omega, \quad (2)$$

is the resonance frequency of the j^{th} mode, with central frequency ω_0 and mode frequency spacing $\delta\omega$, and

$$\mathcal{E}_j = \sqrt{\frac{\epsilon\gamma\kappa}{2N+1}} e^{im\varphi_j}, \quad (3)$$

is the cascaded systems coupling of the j^{th} mode, where γ is the atomic decay rate, κ is the cavity decay rate, ϵ is the percentage of fluorescence sent to the filter, and

$$\varphi_j = \frac{j\pi}{N}, \quad (4)$$

sets the size of the frequency dependent time delay, with integer m .

The master equation is

$$\begin{aligned} \frac{d\rho}{dt} = & \frac{1}{i\hbar} [H, \rho] + \frac{\gamma}{2} (1 - \epsilon) (2\sigma_- \rho \sigma_+ - \sigma_+ \sigma_- \rho - \rho \sigma_+ \sigma_-) \\ & + \frac{\kappa}{2} \sum_{j=-N}^N \left(2a_j \rho a_j^\dagger - a_j^\dagger a_j \rho - \rho a_j^\dagger a_j \right) \\ & + \frac{1}{2} \sum_{j=-N}^N \left(2C_j \rho C_j^\dagger - C_j^\dagger C_j \rho - \rho C_j^\dagger C_j \right), \end{aligned} \quad (5)$$

where

$$C_j = \sqrt{\frac{\epsilon\gamma}{2N+1}} \sigma_- + \sqrt{\kappa} e^{im\varphi_j} a_j, \quad (6)$$

is the cascaded systems decay operator. Expanding the master equation out into neat terms, we arrive at:

$$\begin{aligned} \frac{d\rho}{dt} = & -i\frac{\Omega}{2}(\sigma_+\rho - \rho\sigma_+) - i\frac{\Omega}{2}(\sigma_-\rho - \rho\sigma_-) + \frac{\gamma}{2}(2\sigma_-\rho\sigma_+ - \sigma_+\sigma_-\rho - \rho\sigma_+\sigma_-) \\ & - i\sum_{j=-N}^N \omega_j (a_j^\dagger a_j \rho - \rho a_j^\dagger a_j) + \kappa \sum_{j=-N}^N (2a_j \rho a_j^\dagger - a_j^\dagger a_j \rho - \rho a_j^\dagger a_j) \\ & - \sum_{j=-N}^N \mathcal{E}_j (a_j^\dagger \sigma_- \rho - \sigma_- \rho a_j^\dagger) - \sum_{j=-N}^N \mathcal{E}_j^* (\rho \sigma_+ a_j - a_j \rho \sigma_+). \end{aligned} \quad (7)$$

2 Operator Averages

2.1 First-Order: Atomic Equations

We may write the Bloch equations in matrix form:

$$\frac{d}{dt}\langle\boldsymbol{\sigma}\rangle = \mathbf{M}\langle\boldsymbol{\sigma}\rangle + \mathbf{M}, \quad (8)$$

where

$$\langle\boldsymbol{\sigma}\rangle = \begin{pmatrix} \langle\sigma_- \rangle \\ \langle\sigma_+ \rangle \\ \langle\sigma_z \rangle \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ -\gamma \end{pmatrix}, \quad (9)$$

and

$$\mathbf{M} = \begin{pmatrix} -\frac{\gamma}{2} & 0 & i\frac{\Omega}{2} \\ 0 & -\frac{\gamma}{2} & -i\frac{\Omega}{2} \\ i\Omega & -i\Omega & -\gamma \end{pmatrix}. \quad (10)$$

This differential equation has the solution

$$\langle\boldsymbol{\sigma}(t)\rangle = e^{\mathbf{M}t} (\langle\boldsymbol{\sigma}\rangle - \langle\boldsymbol{\sigma}\rangle_{ss}) + \langle\boldsymbol{\sigma}\rangle_{ss}, \quad (11)$$

where

$$\langle\boldsymbol{\sigma}\rangle_{ss} = -\mathbf{M}^{-1}\mathbf{B} = \frac{1}{2\Omega^2 + \gamma^2} \begin{pmatrix} -i\gamma\Omega \\ i\gamma\Omega \\ -\gamma^2 \end{pmatrix}. \quad (12)$$

2.2 First-Order: Cavity Equations

$$\frac{d}{dt}\langle a_j \rangle = -(\kappa + i\omega_j)\langle a_j \rangle - \mathcal{E}_j\langle\sigma_- \rangle, \quad (13a)$$

$$\frac{d}{dt}\langle a_j^\dagger \rangle = -(\kappa - i\omega_j)\langle a_j^\dagger \rangle - \mathcal{E}_j^*\langle\sigma_+ \rangle. \quad (13b)$$

2.3 Second-Order: Cavity-Atom Coupled Equations

Using the vector notation, we have moment equations

$$\frac{d}{dt}\langle a_j \boldsymbol{\sigma} \rangle = \mathbf{M}^{(j)}\langle a_j \boldsymbol{\sigma} \rangle + \begin{pmatrix} 0 \\ -\frac{1}{2}\mathcal{E}_j(\langle\sigma_z \rangle + 1) \\ -\gamma\langle a_j \rangle + \mathcal{E}_j\langle\sigma_- \rangle \end{pmatrix}, \quad (14a)$$

and

$$\frac{d}{dt}\langle a_j^\dagger \sigma \rangle = \mathbf{M}^{(j^*)}\langle a_j^\dagger \sigma \rangle + \begin{pmatrix} -\frac{1}{2}\mathcal{E}_j^* (\langle \sigma_z \rangle + 1) \\ 0 \\ -\gamma\langle a_j^\dagger \rangle + \mathcal{E}_j^* \langle \sigma_+ \rangle \end{pmatrix}. \quad (14b)$$

where

$$\mathbf{M}^{(j)} = \mathbf{M} - (\kappa + i\omega_j)\mathbb{1}, \quad \mathbf{M}^{(j^*)} = \mathbf{M} - (\kappa - i\omega_j)\mathbb{1}. \quad (15)$$

2.4 Second-Order: Cavity Equations

$$\frac{d}{dt}\langle a_j a_k \rangle = -(2\kappa + i(\omega_j + \omega_k))\langle a_j a_k \rangle - \mathcal{E}_j\langle a_k \sigma_- \rangle - \mathcal{E}_k\langle a_j \sigma_- \rangle, \quad (16a)$$

$$\frac{d}{dt}\langle a_j^\dagger a_k^\dagger \rangle = -(2\kappa - i(\omega_j + \omega_k))\langle a_j^\dagger a_k^\dagger \rangle - \mathcal{E}_j^*\langle a_k^\dagger \sigma_+ \rangle - \mathcal{E}_k^*\langle a_j^\dagger \sigma_+ \rangle, \quad (16b)$$

$$\frac{d}{dt}\langle a_j^\dagger a_k \rangle = -(2\kappa - i(\omega_j - \omega_k))\langle a_j^\dagger a_k \rangle - \mathcal{E}_j^*\langle a_k \sigma_+ \rangle - \mathcal{E}_k\langle a_j^\dagger \sigma_- \rangle. \quad (16c)$$

2.5 Third-Order: Cavity-Atom Coupled Equations

Using the vector notation, we have moment equations

$$\frac{d}{dt}\langle a_j a_k \sigma \rangle = \mathbf{M}^{(j,k)}\langle a_j a_k \sigma \rangle + \begin{pmatrix} 0 \\ -\frac{1}{2}\mathcal{E}_j (\langle a_k \sigma_z \rangle + \langle a_k \rangle) - \frac{1}{2}\mathcal{E}_k (\langle a_j \sigma_z \rangle + \langle a_j \rangle) \\ -\gamma\langle a_j a_k \rangle + \mathcal{E}_j\langle a_k \sigma_- \rangle + \mathcal{E}_k\langle a_j \sigma_- \rangle \end{pmatrix}, \quad (17a)$$

and

$$\frac{d}{dt}\langle a_j^\dagger a_k^\dagger \sigma \rangle = \mathbf{M}^{(j^*,k^*)}\langle a_j^\dagger a_k^\dagger \sigma \rangle + \begin{pmatrix} -\frac{1}{2}\mathcal{E}_j^* (\langle a_k^\dagger \sigma_z \rangle + \langle a_k^\dagger \rangle) - \frac{1}{2}\mathcal{E}_k^* (\langle a_j^\dagger \sigma_z \rangle + \langle a_j^\dagger \rangle) \\ 0 \\ -\gamma\langle a_j^\dagger a_k^\dagger \rangle + \mathcal{E}_j^*\langle a_k^\dagger \sigma_+ \rangle + \mathcal{E}_k^*\langle a_j^\dagger \sigma_+ \rangle \end{pmatrix}, \quad (17b)$$

and

$$\frac{d}{dt}\langle a_j^\dagger a_k \sigma \rangle = \mathbf{M}^{(j^*,k)}\langle a_j^\dagger a_k \sigma \rangle + \begin{pmatrix} -\frac{1}{2}\mathcal{E}_j^* (\langle a_k \sigma_z \rangle + \langle a_k \rangle) \\ -\frac{1}{2}\mathcal{E}_k (\langle a_j^\dagger \sigma_z \rangle + \langle a_j^\dagger \rangle) \\ -\gamma\langle a_j^\dagger a_k \rangle + \mathcal{E}_j^*\langle a_k \sigma_+ \rangle + \mathcal{E}_k\langle a_j^\dagger \sigma_- \rangle \end{pmatrix}, \quad (17c)$$

where

$$\mathbf{M}^{(j,k)} = \mathbf{M} - (2\kappa + i(\omega_j + \omega_k))\mathbb{1}, \quad (18a)$$

$$\mathbf{M}^{(j^*,k^*)} = \mathbf{M} - (2\kappa - i(\omega_j + \omega_k))\mathbb{1}, \quad (18b)$$

$$\mathbf{M}^{(j^*,k)} = \mathbf{M} - (2\kappa - i(\omega_j - \omega_k))\mathbb{1}. \quad (18c)$$

2.6 Third-Order: Cavity Equations

$$\begin{aligned} \frac{d}{dt}\langle a_j^\dagger a_k a_l \rangle &= -(3\kappa - i\omega_j + i(\omega_k + \omega_l))\langle a_j^\dagger a_k a_l \rangle \\ &\quad - \mathcal{E}_j^*\langle a_k a_l \sigma_+ \rangle - \mathcal{E}_k\langle a_j^\dagger a_l \sigma_- \rangle - \mathcal{E}_l\langle a_j^\dagger a_k \sigma_- \rangle, \end{aligned} \quad (19a)$$

$$\begin{aligned} \frac{d}{dt}\langle a_j^\dagger a_k^\dagger a_l \rangle &= -(3\kappa - i(\omega_j + \omega_k) + i\omega_l)\langle a_j^\dagger a_k^\dagger a_l \rangle \\ &\quad - \mathcal{E}_j^*\langle a_k^\dagger a_l \sigma_+ \rangle - \mathcal{E}_k^*\langle a_j^\dagger a_l \sigma_+ \rangle - \mathcal{E}_l\langle a_j^\dagger a_k^\dagger \sigma_- \rangle. \end{aligned} \quad (19b)$$

2.7 Fourth-Order: Cavity-Atom Coupled Equations

Using the vector notation, we have moment equations

$$\frac{d}{dt}\langle a_j^\dagger a_k a_l \sigma \rangle = \mathbf{M}^{(j^*, k, l)} \langle a_j^\dagger a_k a_l \rangle + \begin{pmatrix} -\frac{1}{2}\mathcal{E}_j^* (\langle a_k a_l \sigma_z \rangle + \langle a_k a_l \rangle) \\ -\frac{1}{2}\mathcal{E}_k (\langle a_j^\dagger a_l \sigma_z \rangle + \langle a_j^\dagger a_l \rangle) - \frac{1}{2}\mathcal{E}_l (\langle a_j^\dagger a_k \sigma_z \rangle + \langle a_j^\dagger a_k \rangle) \\ -\gamma \langle a_j^\dagger a_k a_l \rangle + \mathcal{E}_j^* \langle a_k a_l \sigma_+ \rangle + \mathcal{E}_k \langle a_j^\dagger a_l \sigma_- \rangle + \mathcal{E}_l \langle a_j^\dagger a_k \sigma_- \rangle \end{pmatrix}, \quad (20a)$$

and

$$\frac{d}{dt}\langle a_j^\dagger a_k^\dagger a_l \sigma \rangle = \mathbf{M}^{(j^*, k^*, l)} \langle a_j^\dagger a_k^\dagger a_l \rangle + \begin{pmatrix} -\frac{1}{2}\mathcal{E}_j^* (\langle a_k^\dagger a_l \sigma_z \rangle + \langle a_k^\dagger a_l \rangle) - \frac{1}{2}\mathcal{E}_k^* (\langle a_j^\dagger a_l \sigma_z \rangle + \langle a_j^\dagger a_l \rangle) \\ -\frac{1}{2}\mathcal{E}_l (\langle a_j^\dagger a_k^\dagger \sigma_z \rangle + \langle a_j^\dagger a_k^\dagger \rangle) \\ -\gamma \langle a_j^\dagger a_k^\dagger a_l \rangle + \mathcal{E}_j^* \langle a_k^\dagger a_l \sigma_+ \rangle + \mathcal{E}_k^* \langle a_j^\dagger a_l \sigma_+ \rangle + \mathcal{E}_l \langle a_j^\dagger a_k^\dagger \sigma_- \rangle \end{pmatrix}, \quad (20b)$$

where

$$\mathbf{M}^{(j^*, k, l)} = \mathbf{M} - (3\kappa - i\omega_j + i(\omega_k + \omega_l)), \quad (21a)$$

$$\mathbf{M}^{(j^*, k^*, l)} = \mathbf{M} - (3\kappa - i(\omega_j + \omega_k) - i\omega_l). \quad (21b)$$

2.8 Fourth-Order: Cavity Equation

Finally, we have

$$\begin{aligned} \frac{d}{dt}\langle a_j^\dagger a_k^\dagger a_l a_m \rangle &= -(4\kappa - i(\omega_j + \omega_k) + i(\omega_l + \omega_m)) \langle a_j^\dagger a_k^\dagger a_l a_m \rangle \\ &\quad - \mathcal{E}_j^* \langle a_k^\dagger a_l a_m \sigma_+ \rangle - \mathcal{E}_k^* \langle a_j^\dagger a_l a_m \sigma_+ \rangle \\ &\quad - \mathcal{E}_l \langle a_j^\dagger a_k^\dagger a_m \sigma_- \rangle - \mathcal{E}_m \langle a_j^\dagger a_k^\dagger a_l \sigma_- \rangle. \end{aligned} \quad (22)$$

3 First-Order Correlation Function

The first-order correlation function for the filtered output field is given by

$$G^{(1)}(t, \tau) = \langle A^\dagger(t + \tau) A(\tau) \rangle = \sum_{j=-N}^N \langle a_j^\dagger(t + \tau) A(t) \rangle. \quad (23)$$

We use the Quantum Regression Theorem to solve for this with the moment equations

$$\frac{d}{dt}\langle \sigma(t + \tau) A(t) \rangle = \mathbf{M} \langle \sigma(t + \tau) A(t) \rangle + \mathbf{B}, \quad (24a)$$

$$\frac{d}{dt}\langle a_j^\dagger(\tau) A(t) \rangle = -(\kappa - i\omega_j) \langle a_j^\dagger(t + \tau) A(t) \rangle - \mathcal{E}_j^* \langle \sigma_+(t + \tau) A(t) \rangle, \quad (24b)$$

where

$$\langle \sigma(t + \tau) A(t) \rangle = \begin{pmatrix} \langle \sigma_-(t + \tau) A(t) \rangle \\ \langle \sigma_+(t + \tau) A(t) \rangle \\ \langle \sigma_z(t + \tau) A(t) \rangle \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ -\gamma \langle A(t) \rangle \end{pmatrix}. \quad (25)$$

4 Second-Order Correlation Function

The second-order correlation function for the filtered output field is given by

$$G^{(2)}(t, \tau) = \langle A^\dagger(t) A^\dagger A(t + \tau) A(t) \rangle = \sum_{k, l=-N}^N \langle A^\dagger(0) a_k^\dagger a_l(\tau) A(0) \rangle, \quad (26)$$

with the normalised second-order correlation function given by

$$g^{(2)}(\tau) = \frac{G^{(2)}(\tau)}{\langle A^\dagger A \rangle_{ss}^2}. \quad (27)$$

Using the Quantum Regression Theorem, we can solve for the second-order correlation function by solving the following moment equations

$$\frac{d}{dt} \langle A^\dagger(0) \boldsymbol{\sigma} A(0) \rangle = \mathbf{M} \langle A^\dagger(0) \boldsymbol{\sigma} A(0) \rangle + \mathbf{B}, \quad (28a)$$

$$\frac{d}{dt} \langle A^\dagger(0) a_j^\dagger(\tau) A(0) \rangle = -(\kappa - i\omega_j) \langle A^\dagger(0) a_j^\dagger(\tau) A(0) \rangle - \mathcal{E}_j^* \langle A^\dagger(0) \sigma_+(\tau) A(0) \rangle, \quad (28b)$$

$$\frac{d}{dt} \langle A^\dagger(0) a_j(\tau) A(0) \rangle = -(\kappa + i\omega_j) \langle A^\dagger(0) a_j(\tau) A(0) \rangle - \mathcal{E}_j \langle A^\dagger(0) \sigma_-(\tau) A(0) \rangle, \quad (28c)$$

with

$$\langle \boldsymbol{\sigma} \rangle = \begin{pmatrix} \langle A^\dagger(0) \sigma_-(\tau) A(0) \rangle \\ \langle A^\dagger(0) \sigma_+(\tau) A(0) \rangle \\ \langle A^\dagger(0) \sigma_z(\tau) A(0) \rangle \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ -\gamma \langle A^\dagger A \rangle_{ss} \end{pmatrix}. \quad (29)$$

We also need to solve

$$\frac{d}{dt} \langle A^\dagger(0) a_j \boldsymbol{\sigma}(\tau) A(0) \rangle = \mathbf{M}^{(j)} \langle A^\dagger(0) a_j \boldsymbol{\sigma}(\tau) A(0) \rangle + \begin{pmatrix} 0 \\ -\frac{1}{2} \mathcal{E}_j (\langle A^\dagger(0) \sigma_z(\tau) A(0) \rangle + \langle A^\dagger A \rangle_{ss}) \\ -\gamma \langle A^\dagger(0) a_j(\tau) A(0) \rangle + \mathcal{E}_j \langle A^\dagger(0) \sigma_-(\tau) A(0) \rangle \end{pmatrix}, \quad (30a)$$

and

$$\frac{d}{dt} \langle A^\dagger(0) a_j^\dagger \boldsymbol{\sigma}(\tau) A(0) \rangle = \mathbf{M}^{(j^*)} \langle A^\dagger(0) a_j^\dagger \boldsymbol{\sigma}(\tau) A(0) \rangle + \begin{pmatrix} -\frac{1}{2} \mathcal{E}_j^* (\langle A^\dagger(0) \sigma_z(\tau) A(0) \rangle + \langle A^\dagger A \rangle_{ss}) \\ 0 \\ -\gamma \langle A^\dagger(0) a_j^\dagger(\tau) A(0) \rangle + \mathcal{E}_j^* \langle A^\dagger(0) \sigma_+(\tau) A(0) \rangle \end{pmatrix}. \quad (30b)$$

where

$$\mathbf{M}^{(j)} = \mathbf{M} - (\kappa + i\omega_j) \mathbb{1}, \quad \mathbf{M}^{(j^*)} = \mathbf{M} - (\kappa - i\omega_j) \mathbb{1}. \quad (31)$$

Finally, we will also need to solve

$$\begin{aligned} \frac{d}{dt} \langle A^\dagger(0) a_k^\dagger a_l(\tau) A(0) \rangle &= -(2\kappa - i(\omega_k - \omega_l)) \langle A^\dagger(0) a_k^\dagger a_l(\tau) A(0) \rangle \\ &\quad - \mathcal{E}_k^* \langle A^\dagger(0) a_l \sigma_+(\tau) A(0) \rangle - \mathcal{E}_l \langle A^\dagger(0) a_k^\dagger \sigma_-(\tau) A(0) \rangle. \end{aligned} \quad (32)$$