

# Two-Level Atom: Two Multi-Mode Filters Moment Equations

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## 1 Hamiltonian and Master Equation

The Hamiltonian is

$$H = \hbar \frac{\Omega}{2} (\sigma_+ + \sigma_-) + \hbar \sum_{j=-N}^N \left( \omega_j^{(a)} a_j^\dagger a_j + \omega_j^{(b)} b_j^\dagger b_j \right) + \frac{i\hbar}{2} \sum_{j=-N}^N \left( \mathcal{E}_j^{(a)*} a_j \sigma_+ - \mathcal{E}_j^{(a)} \sigma_- a_j^\dagger \right) + \frac{i\hbar}{2} \sum_{j=-N}^N \left( \mathcal{E}_j^{(b)*} b_j \sigma_+ - \mathcal{E}_j^{(b)} \sigma_- b_j^\dagger \right) \quad (1)$$

where  $\Omega$  is the Rabi frequency,  $\sigma_+$  and  $\sigma_-$  are the atomic raising and lowering operators,  $a^\dagger$  ( $b^\dagger$ ) and  $a$  ( $b$ ) are the cavity photon creation and annihilation operators for filter A (B),  $N$  is the number of modes either side of the central mode ( $2N + 1$  total modes),

$$\omega_j^{(a)} = \omega_0^{(a)} + j\delta\omega^{(a)}, \quad \omega_j^{(b)} = \omega_0^{(b)} + j\delta\omega^{(b)} \quad (2)$$

is the resonance frequency of the  $j^{\text{th}}$  mode, with central frequency  $\omega_0^{(a/b)}$  and mode frequency spacing  $\delta\omega^{(a/b)}$ , and

$$\mathcal{E}_j^{(a)} = \sqrt{\frac{\epsilon\gamma\kappa_a}{2(2N+1)}} e^{im\varphi_j}, \quad \mathcal{E}_j^{(b)} = \sqrt{\frac{\epsilon\gamma\kappa_b}{2(2N+1)}} e^{im\varphi_j}, \quad (3)$$

is the cascaded systems coupling of the  $j^{\text{th}}$  mode for filter A (B), where  $\gamma$  is the atomic decay rate,  $\kappa_a$  ( $\kappa_b$ ) is the cavity decay rate for filter A (B),  $\epsilon$  is the percentage of fluorescence sent to the filter, and

$$\varphi_j = \frac{j\pi}{N}, \quad (4)$$

sets the size of the frequency dependent time delay, with integer  $m$ .

The master equation is

$$\begin{aligned} \frac{d\rho}{dt} = & \frac{1}{i\hbar} [H, \rho] + \frac{\gamma}{2} (1 - \epsilon) (2\sigma_- \rho \sigma_+ - \sigma_+ \sigma_- \rho - \rho \sigma_+ \sigma_-) \\ & + \frac{\kappa_a}{2} \sum_{j=-N}^N \left( 2a_j \rho a_j^\dagger - a_j^\dagger a_j \rho - \rho a_j^\dagger a_j \right) \end{aligned} \quad (5)$$

$$\begin{aligned} & + \frac{1}{2} \sum_{j=-N}^N \left( 2C_j^{(a)} \rho C_j^{(a)\dagger} - C_j^{(a)\dagger} C_j^{(a)} \rho - \rho C_j^{(a)\dagger} C_j^{(a)} \right) \\ & + \frac{\kappa_b}{2} \sum_{j=-N}^N \left( 2b_j \rho b_j^\dagger - b_j^\dagger b_j \rho - \rho b_j^\dagger b_j \right) \end{aligned} \quad (6)$$

$$+ \frac{1}{2} \sum_{j=-N}^N \left( 2C_j^{(b)} \rho C_j^{(b)\dagger} - C_j^{(b)\dagger} C_j^{(b)} \rho - \rho C_j^{(b)\dagger} C_j^{(b)} \right),$$

where

$$C_j = \sqrt{\frac{\epsilon\gamma}{2N+1}} \sigma_- + \sqrt{\kappa} e^{im\varphi_j} a_j, \quad (7)$$

is the cascaded systems decay operator. Expanding the master equation out into neat terms, we arrive at:

$$\begin{aligned} \frac{d\rho}{dt} = & -i\frac{\Omega}{2} (\sigma_+ \rho - \rho \sigma_+) - i\frac{\Omega}{2} (\sigma_- \rho - \rho \sigma_-) + \frac{\gamma}{2} (2\sigma_- \rho \sigma_+ - \sigma_+ \sigma_- \rho - \rho \sigma_+ \sigma_-) \\ & - i \sum_{j=-N}^N \omega_j^{(a)} \left( a_j^\dagger a_j \rho - \rho a_j^\dagger a_j \right) + \kappa_a \sum_{j=-N}^N \left( 2a_j \rho a_j^\dagger - a_j^\dagger a_j \rho - \rho a_j^\dagger a_j \right) \\ & - \sum_{j=-N}^N \mathcal{E}_j^{(a)} \left( a_j^\dagger \sigma_- \rho - \sigma_- \rho a_j^\dagger \right) - \sum_{j=-N}^N \mathcal{E}_j^{(a)*} (\rho \sigma_+ a_j - a_j \rho \sigma_+) \\ & - i \sum_{j=-N}^N \omega_j^{(b)} \left( b_j^\dagger b_j \rho - \rho b_j^\dagger b_j \right) + \kappa_b \sum_{j=-N}^N \left( 2b_j \rho b_j^\dagger - b_j^\dagger b_j \rho - \rho b_j^\dagger b_j \right) \\ & - \sum_{j=-N}^N \mathcal{E}_j^{(b)} \left( b_j^\dagger \sigma_- \rho - \sigma_- \rho b_j^\dagger \right) - \sum_{j=-N}^N \mathcal{E}_j^{(b)*} (\rho \sigma_+ b_j - b_j \rho \sigma_+). \end{aligned} \quad (8)$$

## 2 Operator Averages

### 2.1 First-Order: Atomic Equations

We may write the Bloch equations in matrix form:

$$\frac{d}{dt} \langle \boldsymbol{\sigma} \rangle = \mathbf{M} \langle \boldsymbol{\sigma} \rangle + \mathbf{M}, \quad (9)$$

where

$$\langle \boldsymbol{\sigma} \rangle = \begin{pmatrix} \langle \sigma_- \rangle \\ \langle \sigma_+ \rangle \\ \langle \sigma_z \rangle \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ -\gamma \end{pmatrix}, \quad (10)$$

and

$$\mathbf{M} = \begin{pmatrix} -\frac{\gamma}{2} & 0 & i\frac{\Omega}{2} \\ 0 & -\frac{\gamma}{2} & -i\frac{\Omega}{2} \\ i\Omega & -i\Omega & -\gamma \end{pmatrix}. \quad (11)$$

This differential equation has the solution

$$\langle \boldsymbol{\sigma}(t) \rangle = e^{\mathbf{M}t} (\langle \boldsymbol{\sigma}(0) \rangle - \langle \boldsymbol{\sigma} \rangle_{ss}) + \langle \boldsymbol{\sigma} \rangle_{ss}, \quad (12)$$

where

$$\langle \boldsymbol{\sigma} \rangle_{ss} = -\mathbf{M}^{-1} \mathbf{B} = \frac{1}{2\Omega^2 + \gamma^2} \begin{pmatrix} -i\gamma\Omega \\ i\gamma\Omega \\ -\gamma^2 \end{pmatrix}. \quad (13)$$

## 2.2 First-Order: Cavity Equations

$$\frac{d}{dt}\langle a_j \rangle = -\left(\kappa_a + i\omega_j^{(a)}\right)\langle a_j \rangle - \mathcal{E}_j^{(a)}\langle \sigma_- \rangle, \quad (14a)$$

$$\frac{d}{dt}\langle a_j^\dagger \rangle = -\left(\kappa_a - i\omega_j^{(a)}\right)\langle a_j^\dagger \rangle - \mathcal{E}_j^{(a)*}\langle \sigma_+ \rangle, \quad (14b)$$

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$$\frac{d}{dt}\langle b_j \rangle = -\left(\kappa_b + i\omega_j^{(b)}\right)\langle b_j \rangle - \mathcal{E}_j^{(b)}\langle \sigma_- \rangle, \quad (14c)$$

$$\frac{d}{dt}\langle b_j^\dagger \rangle = -\left(\kappa_b - i\omega_j^{(b)}\right)\langle b_j^\dagger \rangle - \mathcal{E}_j^{(b)*}\langle \sigma_+ \rangle. \quad (14d)$$

## 2.3 Second-Order: Cavity-Atom Coupled Equations

Using the vector notation, we have moment equations

$$\frac{d}{dt}\langle a_j \boldsymbol{\sigma} \rangle = \mathbf{M}_j^{(a)}\langle a_j \boldsymbol{\sigma} \rangle + \begin{pmatrix} 0 \\ -\frac{1}{2}\mathcal{E}_j^{(a)}(\langle \sigma_z \rangle + 1) \\ -\gamma\langle a_j \rangle + \mathcal{E}_j^{(a)}\langle \sigma_- \rangle \end{pmatrix}, \quad (15a)$$

$$\frac{d}{dt}\langle b_j \boldsymbol{\sigma} \rangle = \mathbf{M}_j^{(b)}\langle b_j \boldsymbol{\sigma} \rangle + \begin{pmatrix} 0 \\ -\frac{1}{2}\mathcal{E}_j^{(b)}(\langle \sigma_z \rangle + 1) \\ -\gamma\langle b_j \rangle + \mathcal{E}_j^{(b)}\langle \sigma_- \rangle \end{pmatrix}, \quad (15b)$$

where

$$\mathbf{M}_j^{(a)} = \mathbf{M} - \left(\kappa_a + i\omega_j^{(a)}\right)\mathbb{1}, \quad (16a)$$

$$\mathbf{M}_j^{(b)} = \mathbf{M} - \left(\kappa_b + i\omega_j^{(b)}\right)\mathbb{1}; \quad (16b)$$

and

$$\frac{d}{dt}\langle a_j^\dagger \boldsymbol{\sigma} \rangle = \mathbf{M}_j^{(a^\dagger)}\langle a_j^\dagger \boldsymbol{\sigma} \rangle + \begin{pmatrix} -\frac{1}{2}\mathcal{E}_j^{(a)*}(\langle \sigma_z \rangle + 1) \\ 0 \\ -\gamma\langle a_j^\dagger \rangle + \mathcal{E}_j^{(a)*}\langle \sigma_+ \rangle \end{pmatrix}, \quad (17a)$$

$$\frac{d}{dt}\langle b_j^\dagger \boldsymbol{\sigma} \rangle = \mathbf{M}_j^{(b^\dagger)}\langle b_j^\dagger \boldsymbol{\sigma} \rangle + \begin{pmatrix} -\frac{1}{2}\mathcal{E}_j^{(b)*}(\langle \sigma_z \rangle + 1) \\ 0 \\ -\gamma\langle b_j^\dagger \rangle + \mathcal{E}_j^{(b)*}\langle \sigma_+ \rangle \end{pmatrix}, \quad (17b)$$

where

$$\mathbf{M}_j^{(a^\dagger)} = \mathbf{M} - \left(\kappa_a - i\omega_j^{(a)}\right)\mathbb{1}, \quad (18a)$$

$$\mathbf{M}_j^{(b^\dagger)} = \mathbf{M} - \left(\kappa_b - i\omega_j^{(b)}\right)\mathbb{1}, \quad (18b)$$

## 2.4 Second-Order: Cavity Equations

$$\frac{d}{dt}\langle a_j b_k \rangle = -\left(\kappa_a + \kappa_b + i\left(\omega_j^{(a)} + \omega_k^{(b)}\right)\right)\langle a_j b_k \rangle - \mathcal{E}_j^{(a)}\langle b_k \sigma_- \rangle - \mathcal{E}_k^{(b)}\langle a_j \sigma_- \rangle, \quad (19a)$$

$$\frac{d}{dt}\langle a_j^\dagger b_k^\dagger \rangle = -\left(\kappa_a + \kappa_b - i\left(\omega_j^{(a)} + \omega_k^{(b)}\right)\right)\langle a_j^\dagger b_k^\dagger \rangle - \mathcal{E}_j^{(a)*}\langle b_k^\dagger \sigma_+ \rangle - \mathcal{E}_k^{(b)*}\langle a_j^\dagger \sigma_+ \rangle, \quad (19b)$$

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$$\frac{d}{dt}\langle a_j^\dagger a_k \rangle = -\left(2\kappa_a - i\left(\omega_j^{(a)} - \omega_k^{(a)}\right)\right)\langle a_j^\dagger a_k \rangle - \mathcal{E}_j^{(a)*}\langle a_k \sigma_+ \rangle - \mathcal{E}_k^{(a)}\langle a_j^\dagger \sigma_- \rangle, \quad (19c)$$

$$\frac{d}{dt}\langle b_j^\dagger b_k \rangle = -\left(2\kappa_b - i\left(\omega_j^{(b)} - \omega_k^{(b)}\right)\right)\langle b_j^\dagger b_k \rangle - \mathcal{E}_j^{(b)*}\langle b_k \sigma_+ \rangle - \mathcal{E}_k^{(b)}\langle b_j^\dagger \sigma_- \rangle, \quad (19d)$$

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$$\frac{d}{dt}\langle a_j^\dagger b_k \rangle = -\left(\kappa_a + \kappa_b - i\left(\omega_j^{(a)} - \omega_k^{(b)}\right)\right)\langle a_j^\dagger b_k \rangle - \mathcal{E}_j^{(a)*}\langle b_k \sigma_+ \rangle - \mathcal{E}_k^{(b)}\langle a_j^\dagger \sigma_- \rangle, \quad (19e)$$

$$\frac{d}{dt}\langle b_j^\dagger a_k \rangle = -\left(\kappa_a + \kappa_b - i\left(\omega_j^{(b)} - \omega_k^{(a)}\right)\right)\langle b_j^\dagger a_k \rangle - \mathcal{E}_j^{(b)*}\langle a_k \sigma_+ \rangle - \mathcal{E}_k^{(a)}\langle b_j^\dagger \sigma_- \rangle. \quad (19f)$$

## 2.5 Third-Order: Cavity-Atom Coupled Equations

Using the vector notation, we have moment equations

$$\frac{d}{dt}\langle a_j b_k \boldsymbol{\sigma} \rangle = \mathbf{M}_{j,k}^{(ab)}\langle a_j b_k \boldsymbol{\sigma} \rangle + \begin{pmatrix} 0 \\ -\frac{1}{2}\mathcal{E}_j^{(a)}(\langle b_k \sigma_z \rangle + \langle b_k \rangle) - \frac{1}{2}\mathcal{E}_k^{(b)}(\langle a_j \sigma_z \rangle + \langle a_j \rangle) \\ -\gamma\langle a_j b_k \rangle + \mathcal{E}_j^{(a)}\langle b_k \sigma_- \rangle + \mathcal{E}_k^{(b)}\langle a_j \sigma_- \rangle \end{pmatrix}, \quad (20a)$$

$$\frac{d}{dt}\langle a_j^\dagger b_k^\dagger \boldsymbol{\sigma} \rangle = \mathbf{M}_{j,k}^{(a^\dagger b^\dagger)}\langle a_j^\dagger b_k^\dagger \boldsymbol{\sigma} \rangle + \begin{pmatrix} -\frac{1}{2}\mathcal{E}_j^{(a)*}(\langle b_k^\dagger \sigma_z \rangle + \langle b_k^\dagger \rangle) - \frac{1}{2}\mathcal{E}_k^{(b)*}(\langle a_j^\dagger \sigma_z \rangle + \langle a_j^\dagger \rangle) \\ 0 \\ -\gamma\langle a_j^\dagger b_k^\dagger \rangle + \mathcal{E}_j^{(a)*}\langle b_k^\dagger \sigma_+ \rangle + \mathcal{E}_k^{(b)*}\langle a_j^\dagger \sigma_+ \rangle \end{pmatrix}, \quad (20b)$$

where

$$\mathbf{M}_{j,k}^{(ab)} = \mathbf{M} - \left(\kappa_a + \kappa_b + i\left(\omega_j^{(a)} + \omega_k^{(b)}\right)\right)\mathbb{1}, \quad (21a)$$

$$\mathbf{M}_{j,k}^{(a^\dagger b^\dagger)} = \mathbf{M} - \left(\kappa_a + \kappa_b - i\left(\omega_j^{(a)} + \omega_k^{(b)}\right)\right)\mathbb{1}; \quad (21b)$$

and

$$\frac{d}{dt}\langle a_j^\dagger a_k \boldsymbol{\sigma} \rangle = \mathbf{M}_{j,k}^{(a^\dagger a)}\langle a_j^\dagger a_k \boldsymbol{\sigma} \rangle + \begin{pmatrix} -\frac{1}{2}\mathcal{E}_j^{(a)*}(\langle a_k \sigma_z \rangle + \langle a_k \rangle) \\ -\frac{1}{2}\mathcal{E}_k^{(a)}(\langle a_j^\dagger \sigma_z \rangle + \langle a_j^\dagger \rangle) \\ -\gamma\langle a_j^\dagger a_k \rangle + \mathcal{E}_j^{(a)*}\langle a_k \sigma_+ \rangle + \mathcal{E}_k^{(a)}\langle a_j^\dagger \sigma_- \rangle \end{pmatrix}, \quad (22a)$$

$$\frac{d}{dt}\langle b_j^\dagger b_k \boldsymbol{\sigma} \rangle = \mathbf{M}_{j,k}^{(b^\dagger b)}\langle b_j^\dagger b_k \boldsymbol{\sigma} \rangle + \begin{pmatrix} -\frac{1}{2}\mathcal{E}_j^{(b)*}(\langle b_k \sigma_z \rangle + \langle b_k \rangle) \\ -\frac{1}{2}\mathcal{E}_k^{(b)}(\langle b_j^\dagger \sigma_z \rangle + \langle b_j^\dagger \rangle) \\ -\gamma\langle b_j^\dagger b_k \rangle + \mathcal{E}_j^{(b)*}\langle b_k \sigma_+ \rangle + \mathcal{E}_k^{(b)}\langle b_j^\dagger \sigma_- \rangle \end{pmatrix}, \quad (22b)$$

where

$$\mathbf{M}_{j,k}^{(a^\dagger a)} = \mathbf{M} - \left(2\kappa_a - i\left(\omega_j^{(a)} - \omega_k^{(a)}\right)\right)\mathbb{1}, \quad (23a)$$

$$\mathbf{M}_{j,k}^{(b^\dagger b)} = \mathbf{M} - \left(2\kappa_b - i\left(\omega_j^{(b)} - \omega_k^{(b)}\right)\right)\mathbb{1}; \quad (23b)$$

and

$$\frac{d}{dt}\langle a_j^\dagger b_k \boldsymbol{\sigma} \rangle = \mathbf{M}_{j,k}^{(a^\dagger b)}\langle a_j^\dagger b_k \boldsymbol{\sigma} \rangle + \begin{pmatrix} -\frac{1}{2}\mathcal{E}_j^{(a)*}(\langle b_k \sigma_z \rangle + \langle b_k \rangle) \\ -\frac{1}{2}\mathcal{E}_k^{(b)}(\langle a_j^\dagger \sigma_z \rangle + \langle a_j^\dagger \rangle) \\ -\gamma\langle a_j^\dagger b_k \rangle + \mathcal{E}_j^{(a)*}\langle b_k \sigma_+ \rangle + \mathcal{E}_k^{(b)}\langle a_j^\dagger \sigma_- \rangle \end{pmatrix}, \quad (24a)$$

$$\frac{d}{dt}\langle b_j^\dagger a_k \boldsymbol{\sigma} \rangle = \mathbf{M}_{j,k}^{(b^\dagger a)}\langle b_j^\dagger a_k \boldsymbol{\sigma} \rangle + \begin{pmatrix} -\frac{1}{2}\mathcal{E}_j^{(b)*}(\langle a_k \sigma_z \rangle + \langle a_k \rangle) \\ -\frac{1}{2}\mathcal{E}_k^{(a)}(\langle b_j^\dagger \sigma_z \rangle + \langle b_j^\dagger \rangle) \\ -\gamma\langle b_j^\dagger a_k \rangle + \mathcal{E}_j^{(b)*}\langle a_k \sigma_+ \rangle + \mathcal{E}_k^{(a)}\langle b_j^\dagger \sigma_- \rangle \end{pmatrix}, \quad (24b)$$

where

$$\mathbf{M}_{j,k}^{(a^\dagger b)} = \mathbf{M} - \left( \kappa_a + \kappa_b - i \left( \omega_j^{(a)} - \omega_k^{(b)} \right) \right) \mathbb{1}, \quad (25a)$$

$$\mathbf{M}_{j,k}^{(b^\dagger a)} = \mathbf{M} - \left( \kappa_a + \kappa_b - i \left( \omega_j^{(b)} - \omega_k^{(a)} \right) \right) \mathbb{1}; \quad (25b)$$

## 2.6 Third-Order: Cavity Equations

$$\begin{aligned} \frac{d}{dt} \langle a_j^\dagger a_k b_l \rangle &= - \left( 2\kappa_a + \kappa_b - i\omega_j^{(a)} + i \left( \omega_k^{(a)} + \omega_l^{(b)} \right) \right) \langle a_j^\dagger a_k b_l \rangle \\ &\quad - \mathcal{E}_j^{(a)*} \langle a_k b_l \sigma_+ \rangle - \mathcal{E}_k^{(a)} \langle a_j^\dagger b_l \sigma_- \rangle - \mathcal{E}_l^{(b)} \langle a_j^\dagger a_k \sigma_- \rangle, \end{aligned} \quad (26a)$$

$$\begin{aligned} \frac{d}{dt} \langle b_j^\dagger b_k a_l \rangle &= - \left( \kappa_a + 2\kappa_b - i\omega_j^{(b)} + i \left( \omega_k^{(b)} + \omega_l^{(a)} \right) \right) \langle b_j^\dagger b_k a_l \rangle \\ &\quad - \mathcal{E}_j^{(b)*} \langle b_k a_l \sigma_+ \rangle - \mathcal{E}_k^{(b)} \langle b_j^\dagger a_l \sigma_- \rangle - \mathcal{E}_l^{(a)} \langle b_j^\dagger b_k \sigma_- \rangle, \end{aligned} \quad (26b)$$

and

$$\begin{aligned} \frac{d}{dt} \langle b_j^\dagger a_k^\dagger a_l \rangle &= - \left( 2\kappa_a + \kappa_b - i \left( \omega_j^{(b)} + \omega_k^{(a)} \right) + i\omega_l^{(a)} \right) \langle b_j^\dagger a_k^\dagger a_l \rangle \\ &\quad - \mathcal{E}_j^{(b)*} \langle a_k^\dagger a_l \sigma_+ \rangle - \mathcal{E}_k^{(a)*} \langle b_j^\dagger a_l \sigma_+ \rangle - \mathcal{E}_l^{(a)} \langle b_j^\dagger a_k^\dagger \sigma_- \rangle, \end{aligned} \quad (27a)$$

$$\begin{aligned} \frac{d}{dt} \langle a_j^\dagger b_k^\dagger b_l \rangle &= - \left( \kappa_a + 2\kappa_b - i \left( \omega_j^{(a)} + \omega_k^{(b)} \right) + i\omega_l^{(b)} \right) \langle a_j^\dagger b_k^\dagger b_l \rangle \\ &\quad - \mathcal{E}_j^{(a)*} \langle b_k^\dagger b_l \sigma_+ \rangle - \mathcal{E}_k^{(b)*} \langle a_j^\dagger b_l \sigma_+ \rangle - \mathcal{E}_l^{(b)} \langle a_j^\dagger b_k^\dagger \sigma_- \rangle. \end{aligned} \quad (27b)$$

## 2.7 Fourth-Order: Cavity-Atom Coupled Equations

Using the vector notation, we have moment equations

$$\frac{d}{dt} \langle a_j^\dagger a_k b_l \boldsymbol{\sigma} \rangle = \mathbf{M}_{j,k,l}^{(a^\dagger ab)} \langle a_j^\dagger a_k b_l \rangle + \begin{pmatrix} -\frac{1}{2} \mathcal{E}_j^{(a)*} (\langle a_k b_l \sigma_z \rangle + \langle a_k b_l \rangle) \\ -\frac{1}{2} \mathcal{E}_k^{(a)} (\langle a_j^\dagger b_l \sigma_z \rangle + \langle a_j^\dagger b_l \rangle) - \frac{1}{2} \mathcal{E}_l^{(b)} (\langle a_j^\dagger a_k \sigma_z \rangle + \langle a_j^\dagger a_k \rangle) \\ -\gamma \langle a_j^\dagger a_k b_l \rangle + \mathcal{E}_j^{(a)*} \langle a_k b_l \sigma_+ \rangle + \mathcal{E}_k^{(a)} \langle a_j^\dagger b_l \sigma_- \rangle + \mathcal{E}_l^{(b)} \langle a_j^\dagger a_k \sigma_- \rangle \end{pmatrix}, \quad (28a)$$

$$\frac{d}{dt} \langle b_j^\dagger b_k a_l \boldsymbol{\sigma} \rangle = \mathbf{M}_{j,k,l}^{(b^\dagger ba)} \langle b_j^\dagger b_k a_l \rangle + \begin{pmatrix} -\frac{1}{2} \mathcal{E}_j^{(b)*} (\langle b_k a_l \sigma_z \rangle + \langle b_k a_l \rangle) \\ -\frac{1}{2} \mathcal{E}_k^{(b)} (\langle b_j^\dagger a_l \sigma_z \rangle + \langle b_j^\dagger a_l \rangle) - \frac{1}{2} \mathcal{E}_l^{(a)} (\langle b_j^\dagger b_k \sigma_z \rangle + \langle b_j^\dagger b_k \rangle) \\ -\gamma \langle b_j^\dagger b_k a_l \rangle + \mathcal{E}_j^{(b)*} \langle b_k a_l \sigma_+ \rangle + \mathcal{E}_k^{(b)} \langle b_j^\dagger a_l \sigma_- \rangle + \mathcal{E}_l^{(a)} \langle b_j^\dagger b_k \sigma_- \rangle \end{pmatrix}, \quad (28b)$$

where

$$\mathbf{M}_{j,k,l}^{(a^\dagger ab)} = \mathbf{M} - \left( 2\kappa_a + \kappa_b - i\omega_j^{(a)} + i \left( \omega_k^{(a)} + \omega_l^{(b)} \right) \right), \quad (29a)$$

$$\mathbf{M}_{j,k,l}^{(b^\dagger ba)} = \mathbf{M} - \left( \kappa_a + 2\kappa_b - i\omega_j^{(b)} + i \left( \omega_k^{(b)} + \omega_l^{(a)} \right) \right); \quad (29b)$$

and

$$\frac{d}{dt} \langle b_j^\dagger a_k^\dagger a_l \boldsymbol{\sigma} \rangle = \mathbf{M}_{j,k,l}^{(b^\dagger a^\dagger a)} \langle b_j^\dagger a_k^\dagger a_l \rangle + \begin{pmatrix} -\frac{1}{2} \mathcal{E}_j^{(b)*} (\langle a_k^\dagger a_l \sigma_z \rangle + \langle a_k^\dagger a_l \rangle) - \frac{1}{2} \mathcal{E}_k^{(a)*} (\langle b_j^\dagger a_l \sigma_z \rangle + \langle b_j^\dagger a_l \rangle) \\ -\frac{1}{2} \mathcal{E}_l^{(a)} (\langle b_j^\dagger a_k^\dagger \sigma_z \rangle + \langle b_j^\dagger a_k^\dagger \rangle) \\ -\gamma \langle b_j^\dagger a_k^\dagger a_l \rangle + \mathcal{E}_j^{(b)*} \langle a_k^\dagger a_l \sigma_+ \rangle + \mathcal{E}_k^{(a)*} \langle b_j^\dagger a_l \sigma_+ \rangle + \mathcal{E}_l^{(a)} \langle b_j^\dagger a_k^\dagger \sigma_- \rangle \end{pmatrix}, \quad (30a)$$

$$\frac{d}{dt} \langle a_j^\dagger b_k^\dagger b_l \boldsymbol{\sigma} \rangle = \mathbf{M}_{j,k,l}^{(a^\dagger b^\dagger b)} \langle a_j^\dagger b_k^\dagger b_l \rangle + \begin{pmatrix} -\frac{1}{2} \mathcal{E}_j^{(a)*} (\langle b_k^\dagger b_l \sigma_z \rangle + \langle b_k^\dagger b_l \rangle) - \frac{1}{2} \mathcal{E}_k^{(b)*} (\langle a_j^\dagger b_l \sigma_z \rangle + \langle a_j^\dagger b_l \rangle) \\ -\frac{1}{2} \mathcal{E}_l^{(b)} (\langle a_j^\dagger b_k^\dagger \sigma_z \rangle + \langle a_j^\dagger b_k^\dagger \rangle) \\ -\gamma \langle a_j^\dagger b_k^\dagger b_l \rangle + \mathcal{E}_j^{(a)*} \langle b_k^\dagger b_l \sigma_+ \rangle + \mathcal{E}_k^{(b)*} \langle a_j^\dagger b_l \sigma_+ \rangle + \mathcal{E}_l^{(b)} \langle a_j^\dagger b_k^\dagger \sigma_- \rangle \end{pmatrix}, \quad (30b)$$

where

$$\mathbf{M}_{j,k,l}^{(b^\dagger a^\dagger a)} = \mathbf{M} - \left( 2\kappa_a + \kappa_b - i \left( \omega_j^{(b)} + \omega_k^{(a)} \right) - i\omega_l^{(a)} \right), \quad (31a)$$

$$\mathbf{M}_{j,k,l}^{(a^\dagger b^\dagger b)} = \mathbf{M} - \left( \kappa_a + 2\kappa_b - i \left( \omega_j^{(a)} + \omega_k^{(b)} \right) - i\omega_l^{(b)} \right). \quad (31b)$$

## 2.8 Fourth-Order: Cavity Equation

Finally, we have

$$\begin{aligned} \frac{d}{dt} \langle a_j^\dagger b_k^\dagger b_l a_m \rangle = & - \left( 2\kappa_a + 2\kappa_b - i \left( \omega_j^{(a)} + \omega_k^{(b)} \right) + i \left( \omega_l^{(b)} + \omega_m^{(a)} \right) \right) \langle a_j^\dagger b_k^\dagger b_l a_m \rangle \\ & - \mathcal{E}_j^{(a)*} \langle b_k^\dagger b_l a_m \sigma_+ \rangle - \mathcal{E}_k^{(b)*} \langle a_j^\dagger a_m b_l \sigma_+ \rangle \\ & - \mathcal{E}_l^{(b)} \langle b_k^\dagger a_j^\dagger a_m \sigma_- \rangle - \mathcal{E}_m^{(a)} \langle a_j^\dagger b_k^\dagger b_l \sigma_- \rangle. \end{aligned} \quad (32)$$

## 3 Second-Order Cross-Correlation Function

The second-order correlation function for the filtered output field in the steady state is given by

$$G_{\text{cross}}^{(2)}(\tau) = \langle A^\dagger(0) B^\dagger B(\tau) A(0) \rangle = \sum_{j,k=-N}^N \langle A^\dagger(0) b_j^\dagger b_k(\tau) A(0) \rangle, \quad (33)$$

with the normalised second-order correlation function given by

$$g_{\text{cross}}^{(2)}(\tau) = \frac{G_{\text{cross}}^{(2)}(\tau)}{\langle A^\dagger A \rangle_{ss} \langle B^\dagger B \rangle_{ss}}. \quad (34)$$

Using the Quantum Regression Theorem, we can solve for the second-order correlation function by solving the following moment equations

$$\frac{d}{d\tau} \langle A^\dagger(0) \boldsymbol{\sigma} A(0) \rangle = \mathbf{M} \langle A^\dagger(0) \boldsymbol{\sigma} A(0) \rangle + \mathbf{B}, \quad (35a)$$

$$\frac{d}{d\tau} \langle A^\dagger(0) b_j^\dagger(\tau) A(0) \rangle = - \left( \kappa_b - i\omega_j^{(b)} \right) \langle A^\dagger(0) b_j^\dagger(\tau) A(0) \rangle - \mathcal{E}_j^{(b)*} \langle A^\dagger(0) \sigma_+(\tau) A(0) \rangle, \quad (35b)$$

$$\frac{d}{d\tau} \langle A^\dagger(0) b_j(\tau) A(0) \rangle = - \left( \kappa_b + i\omega_j^{(b)} \right) \langle A^\dagger(0) b_j(\tau) A(0) \rangle - \mathcal{E}_j^{(b)} \langle A^\dagger(0) \sigma_-(\tau) A(0) \rangle, \quad (35c)$$

with

$$\langle \boldsymbol{\sigma} \rangle = \begin{pmatrix} \langle A^\dagger(0) \sigma_-(\tau) A(0) \rangle \\ \langle A^\dagger(0) \sigma_+(\tau) A(0) \rangle \\ \langle A^\dagger(0) \sigma_z(\tau) A(0) \rangle \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ -\gamma \langle A^\dagger A \rangle_{ss} \end{pmatrix}. \quad (36)$$

We also need to solve

$$\frac{d}{d\tau} \langle A^\dagger(0) b_j \boldsymbol{\sigma}(\tau) A(0) \rangle = \mathbf{M}_j^{(b)} \langle A^\dagger(0) b_j \boldsymbol{\sigma}(\tau) A(0) \rangle + \begin{pmatrix} 0 \\ -\frac{1}{2} \mathcal{E}_j^{(b)} (\langle A^\dagger(0) \sigma_z(\tau) A(0) \rangle + \langle A^\dagger A \rangle_{ss}) \\ -\gamma \langle A^\dagger(0) b_j(\tau) A(0) \rangle + \mathcal{E}_j^{(b)} \langle A^\dagger(0) \sigma_-(\tau) A(0) \rangle \end{pmatrix}, \quad (37a)$$

and

$$\frac{d}{d\tau} \langle A^\dagger(0) b_j^\dagger \boldsymbol{\sigma}(\tau) A(0) \rangle = \mathbf{M}_j^{(b^\dagger)} \langle A^\dagger(0) b_j^\dagger \boldsymbol{\sigma}(\tau) A(0) \rangle + \begin{pmatrix} -\frac{1}{2} \mathcal{E}_j^{(b)*} (\langle A^\dagger(0) \sigma_z(\tau) A(0) \rangle + \langle A^\dagger A \rangle_{ss}) \\ 0 \\ -\gamma \langle A^\dagger(0) b_j^\dagger(\tau) A(0) \rangle + \mathcal{E}_j^{(b)*} \langle A^\dagger(0) \sigma_+(\tau) A(0) \rangle \end{pmatrix}. \quad (37b)$$

where

$$\mathbf{M}_j^{(b)} = \mathbf{M} - \left( \kappa_b + i\omega_j^{(b)} \right) \mathbb{1}, \quad (38a)$$

$$\mathbf{M}_j^{(b^\dagger)} = \mathbf{M} - \left( \kappa_b - i\omega_j^{(b)} \right) \mathbb{1}. \quad (38b)$$

Finally, we will also need to solve

$$\begin{aligned} \frac{d}{d\tau} \langle A^\dagger(0) b_k^\dagger b_l(\tau) A(0) \rangle &= - \left( 2\kappa_b - i \left( \omega_k^{(b)} - \omega_l^{(b)} \right) \right) \langle A^\dagger(0) b_k^\dagger b_l(\tau) A(0) \rangle \\ &\quad - \mathcal{E}_k^{(b)*} \langle A^\dagger(0) b_l \sigma_+(\tau) A(0) \rangle - \mathcal{E}_l^{(b)} \langle A^\dagger(0) b_k^\dagger \sigma_-(\tau) A(0) \rangle. \end{aligned} \quad (39)$$