Three-Level Atom: One Multi-Mode Filter Moment Equations

Jacob Ngaha

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1 Hamiltonian and Master Equation

The Hamiltonian is

$$H = -\hbar \left(\frac{\alpha}{2} + \delta \right) |e\rangle \langle e| - 2\hbar\delta |f\rangle \langle f| + \hbar \frac{\Omega}{2} \left(\Sigma_{+} + \Sigma_{-} \right) + \hbar \sum_{j=-N}^{N} \omega_j a_j^{\dagger} a_j + \frac{i\hbar}{2} \sum_{j=-N}^{N} \left(\mathcal{E}_j^{*} a_j \Sigma_{+} - \mathcal{E}_j a_j^{\dagger} \Sigma_{-} \right) \tag{1}$$

where Ω is the Rabi frequency,

$$\alpha = \omega_{fe} - \omega_{eg}, \tag{2}$$

is the atomic anharmonicity, δ is given by

$$2\delta = 2\omega_d - \omega_{fg}, \tag{3}$$

is the *drive detuning from two-photon resonance*

$$\Sigma_{-} = \sigma_{-}^{eg} + \xi \sigma_{-}^{fe} = |g\rangle \langle e| + \xi |e\rangle \langle f|, \quad \Sigma_{+} = \Sigma_{-}^{\dagger}, \tag{4}$$

is the atomic raising (lowering) operator, ω_0 is the resonance frequency of the cavity mode, $a^{\dagger}$ (a) is the cavity photon creation (annihilation) operator, N is the number of modes either side of the central mode ($2N + 1$ total modes),

$$\omega_j = \omega_0 + j\delta\omega \tag{5}$$

is the resonance frequency of the j^{th} mode with mode frequency spacing $\delta\omega$, and

$$\mathcal{E}_j = \sqrt{\frac{\epsilon \Gamma \kappa}{2N + 1}} e^{im\varphi_j}, \tag{6}$$

is the cascaded systems coupling where Γ is the atomic decay rate, κ is the cavity decay rate, and ϵ is the percentage of fluorescence sent to the filter,

$$m\varphi_j = \frac{mj\pi}{N}, \tag{7}$$

sets the size of the frequency dependent time delay, with integer m. The master equation is

$$\begin{aligned} \frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{1}{i\hbar}[H, \rho] &+ \frac{\Gamma}{2}(1 - \epsilon) \left(2\Sigma_{-}\rho\Sigma_{+} - \Sigma_{+}\Sigma_{-}\rho - \rho\Sigma_{+}\Sigma_{-} \right) \\ &+ \frac{\kappa}{2} \sum_{j=-N}^{N} \left(2a_j \rho a_j^{\dagger} - a_j^{\dagger} a_j \rho - \rho a_j^{\dagger} a_j \right) \\ &+ \frac{1}{2} \sum_{j=-N}^{N} \left(2C_j \rho C_j^{\dagger} - C_j^{\dagger} C_j \rho - \rho C_j^{\dagger} C_j \right), \end{aligned} \tag{8}$$

where

$$C_j = \sqrt{\frac{\epsilon\gamma}{2N+1}} e^{im\varphi_j} \sigma_- + \sqrt{\kappa} a_j, \tag{9}$$

is the cascaded systems decay operator. Expanding the master equation out and simplifying it, we arrive at a more compact form:

$$\begin{aligned}
\frac{\mathrm{d}\rho}{\mathrm{d}t} = {} & i\left(\frac{\alpha}{2}+\delta\right)\left(\sigma^{ee}\rho - \rho\sigma^{ee}\right) + 2i\delta\left(\sigma^{ff}\rho - \rho\sigma^{ff}\right) - i\frac{\Omega}{2}\left(\Sigma_+\rho - \rho\Sigma_+\right) \\
& - i\frac{\Omega}{2}\left(\Sigma_-\rho - \rho\Sigma_-\right) + \frac{\Gamma}{2}\left(2\Sigma_-\rho\Sigma_+ - \Sigma_+\Sigma_-\rho - \rho\Sigma_+\Sigma_-\right) \\
& - i\sum_{j=-N}^{N}\omega_j\left(a_j^\dagger a_j\rho - \rho a_j^\dagger a_j\right) + \kappa\sum_{j=-N}^{N}\left(2a_j\rho a_j^\dagger - a_j^\dagger a_j\rho - \rho a_j^\dagger a_j\right) \\
& - \sum_{j=-N}^{N}\mathcal{E}_j\left(a_j^\dagger\Sigma_-\rho - \Sigma_-\rho a_j^\dagger\right) - \sum_{j=-N}^{N}\mathcal{E}_j^*\left(\rho\Sigma_+a_j - a_j\rho\Sigma_+\right).
\end{aligned} \tag{10}$$

2 Operator Averages

2.1 First-Order: Atomic Equations

Rearranging the density operator equations, we can write the atomic moment equations in matrix form

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\boldsymbol{\sigma}\rangle = \boldsymbol{M}\langle\boldsymbol{\sigma}\rangle + \boldsymbol{B}, \tag{11}$$

where

$$\langle\boldsymbol{\sigma}\rangle = \begin{pmatrix} \langle\sigma^{gg}\rangle \\ \langle\sigma_-^{eg}\rangle \\ \langle\sigma_+^{eg}\rangle \\ \langle\sigma^{ee}\rangle \\ \langle\sigma_-^{fe}\rangle \\ \langle\sigma_+^{fe}\rangle \\ \langle\sigma_-^{fg}\rangle \\ \langle\sigma_+^{fg}\rangle \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \Gamma\xi^2 \\ i\xi\frac{\Omega}{2} \\ -i\xi\frac{\Omega}{2} \\ 0 \\ 0 \end{pmatrix}, \tag{12}$$

and

$$\boldsymbol{M} = \begin{pmatrix}
0 & -i\frac{\Omega}{2} & i\frac{\Omega}{2} & \Gamma & 0 & 0 & 0 & 0 \\
-i\frac{\Omega}{2} & -\left[\frac{\Gamma}{2}-i\left(\frac{\alpha}{2}+\delta\right)\right] & 0 & i\frac{\Omega}{2} & \Gamma\xi & 0 & -i\xi\frac{\Omega}{2} & 0 \\
i\frac{\Omega}{2} & 0 & -\left[\frac{\Gamma}{2}+i\left(\frac{\alpha}{2}+\delta\right)\right] & -i\frac{\Omega}{2} & 0 & \Gamma\xi & 0 & i\xi\frac{\Omega}{2} \\
-\Gamma\xi^2 & i\frac{\Omega}{2} & -i\frac{\Omega}{2} & -\Gamma\left(1+\xi^2\right) & -i\xi\frac{\Omega}{2} & i\xi\frac{\Omega}{2} & 0 & 0 \\
-i\xi\frac{\Omega}{2} & 0 & 0 & -i\xi\Omega & -\left[\frac{\Gamma}{2}\left(1+\xi^2\right)+i\left(\frac{\alpha}{2}-\delta\right)\right] & 0 & i\frac{\Omega}{2} & 0 \\
i\xi\frac{\Omega}{2} & 0 & 0 & i\xi\Omega & 0 & -\left[\frac{\Gamma}{2}\left(1+\xi^2\right)-i\left(\frac{\alpha}{2}-\delta\right)\right] & 0 & -i\frac{\Omega}{2} \\
0 & -i\xi\frac{\Omega}{2} & 0 & 0 & i\frac{\Omega}{2} & 0 & -\left[\frac{\Gamma}{2}\xi^2-2i\delta\right] & 0 \\
0 & 0 & i\xi\frac{\Omega}{2} & 0 & 0 & -i\frac{\Omega}{2} & 0 & -\left[\frac{\Gamma}{2}\xi^2+2i\delta\right]
\end{pmatrix}. \tag{13}$$

This differential equation has solution

$$\langle\boldsymbol{\sigma}(t)\rangle = e^{\boldsymbol{M}t}\langle\boldsymbol{\sigma}(0)\rangle + \left(1 - e^{\boldsymbol{M}t}\right)\langle\boldsymbol{\sigma}\rangle_{ss}, \tag{14}$$

where

$$\langle\boldsymbol{\sigma}\rangle_{ss} = -\boldsymbol{M}^{-1}\boldsymbol{B}. \tag{15}$$

2.2 First-Order: Cavity Equations

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j\rangle &= -\left(\kappa + i\omega_j\right)\langle a_j\rangle - \mathcal{E}_j\left(\langle\sigma_-^{eg}\rangle + \xi\langle\sigma_-^{fe}\rangle\right), && \text{(16a)}\\
\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger\rangle &= -\left(\kappa - i\omega_j\right)\langle a_j^\dagger\rangle - \mathcal{E}_j\left(\langle\sigma_+^{eg}\rangle + \xi\langle\sigma_+^{fe}\rangle\right) && \text{(16b)}
\end{aligned}
$$

2.3 Second-Order: Cavity-Atom Coupled Equations

Using the vector notation, we have moment equations

$$
\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j\boldsymbol{\sigma}\rangle = \boldsymbol{M}_j^{(a)}\langle a_j\boldsymbol{\sigma}\rangle + \begin{pmatrix} -\mathcal{E}_j\langle\sigma_-^{eg}\rangle \\ -\mathcal{E}_j\xi\langle\sigma_-^{fg}\rangle \\ -\mathcal{E}_j\langle\sigma^{ee}\rangle \\ \Gamma\xi^2\langle a_j\rangle - \mathcal{E}_j\xi\langle\sigma_-^{fe}\rangle \\ i\xi\frac{\Omega}{2}\langle a_j\rangle \\ -i\xi\frac{\Omega}{2}\langle a_j\rangle - \mathcal{E}_j\xi\left(1 - \langle\sigma^{gg}\rangle - \langle\sigma^{ee}\rangle\right) \\ 0 \\ -\mathcal{E}_j\langle\sigma_+^{fe}\rangle \end{pmatrix}, \tag{17a}
$$

and

$$
\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger\boldsymbol{\sigma}\rangle = \boldsymbol{M}_j^{(a^\dagger)}\langle a_j^\dagger\boldsymbol{\sigma}\rangle + \begin{pmatrix} -\mathcal{E}_j^*\langle\sigma_+^{eg}\rangle \\ -\mathcal{E}_j^*\langle\sigma^{ee}\rangle \\ -\mathcal{E}_j^*\xi\langle\sigma_+^{fg}\rangle \\ \Gamma\xi^2\langle a_j^\dagger\rangle - \mathcal{E}_j^*\xi\langle\sigma_+^{fe}\rangle \\ i\xi\frac{\Omega}{2}\langle a_j^\dagger\rangle - \mathcal{E}_j^*\xi\left(1 - \langle\sigma^{gg}\rangle - \langle\sigma^{ee}\rangle\right) \\ -i\xi\frac{\Omega}{2}\langle a_j^\dagger\rangle \\ -\mathcal{E}_j^*\langle\sigma_-^{fe}\rangle \\ 0 \end{pmatrix}, \tag{17b}
$$

where

$$
\begin{aligned}
\boldsymbol{M}_j^{(a)} &= \boldsymbol{M} - \left(\kappa + i\omega_j\right)\mathbb{1}, && \text{(18a)}\\
\boldsymbol{M}_j^{(a^\dagger)} &= \boldsymbol{M} - \left(\kappa - i\omega_j\right)\mathbb{1}. && \text{(18b)}
\end{aligned}
$$

2.4 Second-Order: Cavity Equations

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j a_k\rangle &= -\left(2\kappa + i\left(\omega_j + \omega_k\right)\right)\langle a_j a_k\rangle - \mathcal{E}_j\left(\langle a_k\sigma_-^{eg}\rangle + \xi\langle a_k\sigma_-^{fe}\rangle\right) - \mathcal{E}_k\left(\langle a_j\sigma_-^{eg}\rangle + \xi\langle a_j\sigma_-^{fe}\rangle\right), && \text{(19a)}\\
\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger a_k^\dagger\rangle &= -\left(2\kappa - i\left(\omega_j + \omega_k\right)\right)\langle a_j^\dagger a_k^\dagger\rangle - \mathcal{E}_j^*\left(\langle a_k^\dagger\sigma_+^{eg}\rangle + \langle a_k^\dagger\sigma_+^{fe}\rangle\right) - \mathcal{E}_k^*\left(\langle a_j^\dagger\sigma_+^{eg}\rangle + \langle a_j^\dagger\sigma_+^{fe}\rangle\right), && \text{(19b)}\\
\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger a_k\rangle &= -\left(2\kappa - i\left(\omega_j - \omega_k\right)\right) - \mathcal{E}_j^*\left(\langle a_k\sigma_+^{eg}\rangle + \xi\langle a_k\sigma_+^{fe}\rangle\right) - \mathcal{E}_k\left(\langle a_j^\dagger\sigma_-^{eg}\rangle + \xi\langle a_j^\dagger\sigma_-^{fe}\rangle\right). && \text{(19c)}
\end{aligned}
$$

2.5 Third-Order: Cavity-Atom Coupled Equations

For the third-order moment equations we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j a_k \boldsymbol{\sigma}\rangle = \boldsymbol{M}_{j,k}^{(aa)}\langle a_j a_k \boldsymbol{\sigma}\rangle + \begin{pmatrix} -\mathcal{E}_j\langle a_k\sigma_-^{eg}\rangle - \mathcal{E}_k\langle a_j\sigma_-^{eg}\rangle \\ -\mathcal{E}_j\xi\langle a_k\sigma_-^{fg}\rangle - \mathcal{E}_k\xi\langle a_j\sigma_-^{fg}\rangle \\ -\mathcal{E}_j\langle a_k\sigma^{ee}\rangle - \mathcal{E}_k\langle a_j\sigma^{ee}\rangle \\ \Gamma\xi\langle a_j a_k\rangle - \mathcal{E}_j\xi\langle a_k\sigma_-^{fe}\rangle - \mathcal{E}_k\xi\langle a_j\sigma_-^{fe}\rangle \\ i\xi\frac{\Omega}{2}\langle a_j a_k\rangle \\ -i\xi\frac{\Omega}{2}\langle a_j a_k\rangle - \mathcal{E}_j\xi\left(\langle a_k\rangle - \langle a_k\sigma^{gg}\rangle - \langle a_k\sigma^{ee}\rangle\right) - \mathcal{E}_k\xi\left(\langle a_j\rangle - \langle a_j\sigma^{gg}\rangle - \langle a_j\sigma^{ee}\rangle\right) \\ 0 \\ -\mathcal{E}_j\langle a_k\sigma_+^{fe}\rangle - \mathcal{E}_k\langle a_j\sigma_+^{fe}\rangle \end{pmatrix}, \tag{20a}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger a_k^\dagger \boldsymbol{\sigma}\rangle = \boldsymbol{M}_{j,k}^{(a^\dagger a^\dagger)}\langle a_j^\dagger a_k^\dagger \boldsymbol{\sigma}\rangle + \begin{pmatrix} -\mathcal{E}_j^*\langle a_k^\dagger\sigma_+^{eg}\rangle - \mathcal{E}_k^*\langle a_j^\dagger\sigma_+^{eg}\rangle \\ -\mathcal{E}_j^*\langle a_k^\dagger\sigma^{ee}\rangle - \mathcal{E}_k^*\langle a_j^\dagger\sigma^{ee}\rangle \\ -\mathcal{E}_j^*\xi\langle a_k^\dagger\sigma_+^{fg}\rangle - \mathcal{E}_k^*\xi\langle a_j^\dagger\sigma_+^{fg}\rangle \\ \Gamma\xi^2\langle a_j^\dagger a_k^\dagger\rangle - \mathcal{E}_j^*\xi\langle a_k^\dagger\sigma_+^{fe}\rangle - \mathcal{E}_k^*\xi\langle a_j^\dagger\sigma_+^{fe}\rangle \\ i\xi\frac{\Omega}{2}\langle a_j^\dagger a_k^\dagger\rangle - \mathcal{E}_j^*\xi\left(\langle a_k^\dagger\rangle - \langle a_k^\dagger\sigma^{gg}\rangle - \langle a_k^\dagger\sigma^{ee}\rangle\right) - \mathcal{E}_k^*\xi\left(\langle a_j^\dagger\rangle - \langle a_j^\dagger\sigma^{gg}\rangle - \langle a_j^\dagger\sigma^{ee}\rangle\right) \\ -i\xi\frac{\Omega}{2}\langle a_j^\dagger a_k^\dagger\rangle \\ -\mathcal{E}_j^*\langle a_k^\dagger\sigma_-^{fe}\rangle - \mathcal{E}_k^*\langle a_j^\dagger\sigma_-^{fe}\rangle \\ 0 \end{pmatrix}, \tag{20b}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger a_k \boldsymbol{\sigma}\rangle = \boldsymbol{M}_{j,k}^{(a^\dagger a)}\langle a_j^\dagger a_k \boldsymbol{\sigma}\rangle + \begin{pmatrix} -\mathcal{E}_j^*\langle a_k\sigma_+^{eg}\rangle - \mathcal{E}_k\langle a_j^\dagger\sigma_-^{eg}\rangle \\ -\mathcal{E}_j^*\langle a_k\sigma^{ee}\rangle - \mathcal{E}_k\xi\langle a_j^\dagger\sigma_-^{fg}\rangle \\ -\mathcal{E}_j^*\xi\langle a_k\sigma_+^{fg}\rangle - \mathcal{E}_k\langle a_j^\dagger\sigma^{ee}\rangle \\ \Gamma\xi^2\langle a_j^\dagger a_k\rangle - \mathcal{E}_j^*\xi\langle a_k\sigma_+^{fe}\rangle - \mathcal{E}_k\xi\langle a_k^\dagger\sigma_-^{fe}\rangle \\ i\xi\frac{\Omega}{2}\langle a_j^\dagger a_k\rangle - \mathcal{E}_j^*\xi\left(\langle a_k\rangle - \langle a_k\sigma^{gg}\rangle - \langle a_k\sigma^{ee}\rangle\right) \\ -i\xi\frac{\Omega}{2}\langle a_j^\dagger a_k\rangle - \mathcal{E}_k\xi\left(\langle a_j^\dagger\rangle - \langle a_j^\dagger\sigma^{gg}\rangle - \langle a_j^\dagger\sigma^{ee}\rangle\right) \\ -\mathcal{E}_j^*\langle a_k\sigma_-^{fe}\rangle \\ -\mathcal{E}_k\langle a_j^\dagger\sigma_+^{fe}\rangle \end{pmatrix}, \tag{21}$$

where

$$\boldsymbol{M}_{j,k}^{(aa)} = \boldsymbol{M} - \left(2\kappa + i\left(\omega_j + \omega_k\right)\right)\mathbb{1}, \tag{22a}$$

$$\boldsymbol{M}_{j,k}^{(a^\dagger a^\dagger)} = \boldsymbol{M} - \left(2\kappa - i\left(\omega_j + \omega_k\right)\right)\mathbb{1}, \tag{22b}$$

$$\boldsymbol{M}_{j,k}^{(a^\dagger a)} = \boldsymbol{M} - \left(2\kappa - i\omega_j + i\omega_k\right)\mathbb{1}. \tag{22c}$$

2.6 Third-Order: Cavity Equations

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger a_k a_l\rangle =& -\left(3\kappa - i\left(\omega_j - \omega_k - \omega_l\right)\right)\langle a_j^\dagger a_k a_l\rangle \\
& - \mathcal{E}_j^*\left(\langle a_k a_l \sigma_+^{eg}\rangle + \xi\langle a_k a_l \sigma_+^{fe}\rangle\right) \\
& - \mathcal{E}_k\left(\langle a_j^\dagger a_l \sigma_-^{eg}\rangle + \xi\langle a_j^\dagger a_l \sigma_-^{fe}\rangle\right) \\
& - \mathcal{E}_l\left(\langle a_j^\dagger a_k \sigma_-^{eg}\rangle + \xi\langle a_j^\dagger a_k \sigma_-^{fe}\rangle\right), \qquad (23a)\\
\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger a_k^\dagger a_l\rangle =& -\left(3\kappa - i\left(\omega_j + \omega_k - \omega_l\right)\right)\langle a_j^\dagger a_k^\dagger a_l\rangle \\
& - \mathcal{E}_j^*\left(\langle a_k^\dagger a_l \sigma_+^{eg}\rangle + \xi\langle a_k^\dagger a_l \sigma_+^{fe}\rangle\right) \\
& - \mathcal{E}_k^*\left(\langle a_j^\dagger a_l \sigma_+^{eg}\rangle + \xi\langle a_j^\dagger a_l \sigma_+^{fe}\rangle\right) \\
& - \mathcal{E}_l\left(\langle a_j^\dagger a_k^\dagger \sigma_-^{eg}\rangle + \xi\langle a_j^\dagger a_k^\dagger \sigma_-^{fe}\rangle\right), \qquad (23b)
\end{aligned}$$

2.7 Fourth-Order: Cavity-Atom Coupled Equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger a_k a_l \boldsymbol{\sigma}\rangle = \boldsymbol{M}_{j,k,l}^{(a^\dagger a a)}\langle a_j^\dagger a_k a_l\rangle + \begin{pmatrix}
-\mathcal{E}_j^*\langle a_k a_l \sigma_+^{eg}\rangle - \mathcal{E}_k\langle a_j^\dagger a_l \sigma_-^{eg}\rangle - \mathcal{E}_l\langle a_j^\dagger a_k \sigma_-^{eg}\rangle \\
-\mathcal{E}_j^*\langle a_k a_l \sigma^{ee}\rangle - \mathcal{E}_k\xi\langle a_j^\dagger a_l \sigma_-^{fg}\rangle - \mathcal{E}_l\xi\langle a_j^\dagger a_k \sigma_-^{fg}\rangle \\
-\mathcal{E}_j^*\xi\langle a_k a_l \sigma_+^{fg}\rangle - \mathcal{E}_k\langle a_j^\dagger a_l \sigma^{ee}\rangle - \mathcal{E}_l\langle a_j^\dagger a_k \sigma^{ee}\rangle \\
\Gamma\xi^2\langle a_j^\dagger a_k a_l\rangle - \mathcal{E}_j^*\xi\langle a_k a_l \sigma_+^{fe}\rangle - \mathcal{E}_k\xi\langle a_j^\dagger a_l \sigma_-^{fe}\rangle - \mathcal{E}_l\xi\langle a_j^\dagger a_k \sigma_-^{fe}\rangle \\
i\xi\frac{\Omega}{2}\langle a_j^\dagger a_k a_l\rangle - \mathcal{E}_j^*\xi\left(\langle a_k a_l\rangle - \langle a_k a_l \sigma^{gg}\rangle - \langle a_k a_l \sigma^{ee}\rangle\right) \\
-i\xi\frac{\Omega}{2}\langle a_j^\dagger a_k a_l\rangle - \mathcal{E}_k\xi\left(\langle a_j^\dagger a_k\rangle - \langle a_j^\dagger a_l \sigma^{gg}\rangle - \langle a_j^\dagger a_l \sigma^{ee}\rangle\right) - \mathcal{E}_l\xi\left(\langle a_j^\dagger a_l\rangle - \langle a_j^\dagger a_k \sigma^{gg}\rangle - \langle a_j^\dagger a_k \sigma^{ee}\rangle\right) \\
-\mathcal{E}_j^*\langle a_k a_l \sigma_-^{fe}\rangle \\
-\mathcal{E}_k\langle a_j^\dagger a_l \sigma_+^{fe}\rangle - \mathcal{E}_l\langle a_j^\dagger a_k \sigma_-^{fe}\rangle
\end{pmatrix}, \qquad (24a)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger a_k^\dagger a_l \boldsymbol{\sigma}\rangle = \boldsymbol{M}_{j,k,l}^{(a^\dagger a^\dagger a)}\langle a_j^\dagger a_k^\dagger a_l\rangle + \begin{pmatrix}
-\mathcal{E}_j^*\langle a_k^\dagger a_l \sigma_+^{eg}\rangle - \mathcal{E}_k^*\langle a_j^\dagger a_l \sigma_+^{eg}\rangle - \mathcal{E}_l\langle a_j^\dagger a_k^\dagger \sigma_-^{eg}\rangle \\
-\mathcal{E}_j^*\langle a_k^\dagger a_l \sigma^{ee}\rangle - \mathcal{E}_k^*\langle a_j^\dagger a_l \sigma^{ee}\rangle - \mathcal{E}_l\xi\langle a_j^\dagger a_k^\dagger \sigma_-^{fg}\rangle \\
-\mathcal{E}_j^*\xi\langle a_k^\dagger a_l \sigma_+^{fg}\rangle - \mathcal{E}_k^*\xi\langle a_j^\dagger a_l \sigma_+^{fg}\rangle - \mathcal{E}_l\langle a_j^\dagger a_k^\dagger \sigma^{ee}\rangle \\
\Gamma\xi^2\langle a_j^\dagger a_k^\dagger a_l\rangle - \mathcal{E}_j^*\xi\langle a_k^\dagger a_l \sigma_+^{fe}\rangle - \mathcal{E}_k^*\xi\langle a_j^\dagger a_l \sigma_+^{fe}\rangle - \mathcal{E}_l\xi\langle a_j^\dagger a_k^\dagger \sigma_-^{fe}\rangle \\
i\xi\frac{\Omega}{2}\langle a_j^\dagger a_k^\dagger a_l\rangle - \mathcal{E}_j^*\xi\left(\langle a_k^\dagger a_l\rangle - \langle a_k^\dagger a_l \sigma^{gg}\rangle - \langle a_k^\dagger a_l \sigma^{ee}\rangle\right) - \mathcal{E}_j^*\xi\left(\langle a_j^\dagger a_l\rangle - \langle a_j^\dagger a_l \sigma^{gg}\rangle - \langle a_j^\dagger a_l \sigma^{ee}\rangle\right) \\
-i\xi\frac{\Omega}{2}\langle a_j^\dagger a_k^\dagger a_l\rangle - \mathcal{E}_l\xi\left(\langle a_j^\dagger a_k^\dagger\rangle - \langle a_j^\dagger a_k^\dagger \sigma^{gg}\rangle - \langle a_j^\dagger a_k^\dagger \sigma^{ee}\rangle\right) \\
-\mathcal{E}_j^*\langle a_k^\dagger a_l \sigma_-^{fe}\rangle - \mathcal{E}_k^*\langle a_j^\dagger a_l \sigma_-^{fe}\rangle \\
-\mathcal{E}_l\langle a_j^\dagger a_k^\dagger \sigma_+^{fe}\rangle
\end{pmatrix}, \qquad (24b)$$

where

$$\boldsymbol{M}^{(a^{\dagger}aa)}_{j,k,l} = \boldsymbol{M} - \left(3\kappa - i\omega_{j} + i \left(\omega_{k} + \omega_{l} \right) \right), \tag{25a}$$

$$\boldsymbol{M}^{(a^{\dagger}a^{\dagger}a)}_{j,k,l} = \boldsymbol{M} - \left(3\kappa - i \left(\omega_{j} + \omega_{k} \right) - i\omega_{l} \right). \tag{25b}$$

2.8 Fourth-Order: Cavity Equation

Finally, for the fourth-order cavity moment equation, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \langle a^{\dagger}_{j} a^{\dagger}_{k} a_{l} a_{m} \rangle = & - \left(4\kappa - i \left(\omega_{j} + \omega_{k} \right) + i \left(\omega_{l} + \omega_{m} \right) \right) \langle a^{\dagger}_{j} a^{\dagger}_{k} a_{l} a_{m} \rangle \\ & - \mathcal{E}^{*}_{j} \left(\langle a^{\dagger}_{k} a_{l} a_{m} \sigma^{eg}_{+} \rangle + \xi \langle a^{\dagger}_{k} a_{l} a_{m} \sigma^{fe}_{+} \rangle \right) \\ & - \mathcal{E}^{*}_{k} \left(\langle a^{\dagger}_{j} a_{l} a_{m} \sigma^{eg}_{+} \rangle + \xi \langle a^{\dagger}_{j} a_{l} a_{m} \sigma^{fe}_{+} \rangle \right) \\ & - \mathcal{E}_{l} \left(\langle a^{\dagger}_{j} a^{\dagger}_{k} a_{m} \sigma^{eg}_{-} \rangle + \xi \langle a^{\dagger}_{j} a^{\dagger}_{k} a_{m} \sigma^{fe}_{-} \rangle \right) \\ & - \mathcal{E}_{m} \left(\langle a^{\dagger}_{j} a^{\dagger}_{k} a_{l} \sigma^{eg}_{-} \rangle + \xi \langle a^{\dagger}_{j} a^{\dagger}_{k} a_{l} \sigma^{fe}_{-} \rangle \right). \end{aligned} \tag{26}$$

3 First-Order Correlation Function

The first-order correlation function for the filtered output field in the steady state is given by

$$G^{(1)}(\tau) = \langle A^{\dagger}(\tau) A(0) \rangle = \sum_{j=-N}^{N} \langle a^{\dagger}_{j}(\tau) A(0) \rangle. \tag{27}$$

Moving into the steady state, we use the Quantum Regression Theorem to solve for this with the moment equations

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \langle \boldsymbol{\sigma}(\tau) A(0) \rangle = \boldsymbol{M} \langle \boldsymbol{\sigma}(\tau) A(0) \rangle + \boldsymbol{B}, \tag{28a}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \langle a^{\dagger}_{j}(\tau) A(0) \rangle = - \left(\kappa - i\omega_{j} \right) \langle a^{\dagger}_{j} A(0) \rangle - \mathcal{E}^{*}_{j} \left(\langle \sigma^{eg}_{+}(\tau) A(0) \rangle + \xi \langle \sigma^{fe}_{+}(\tau) A(0) \rangle \right), \tag{28b}$$

where

$$\langle \boldsymbol{\sigma} \rangle = \begin{pmatrix} \langle \sigma^{gg}(\tau) A(0) \rangle \\ \langle \sigma^{eg}_{-}(\tau) A(0) \rangle \\ \langle \sigma^{eg}_{+}(\tau) A(0) \rangle \\ \langle \sigma^{ee}(\tau) A(0) \rangle \\ \langle \sigma^{fe}_{-}(\tau) A(0) \rangle \\ \langle \sigma^{fe}_{+}(\tau) A(0) \rangle \\ \langle \sigma^{fg}_{-}(\tau) A(0) \rangle \\ \langle \sigma^{fg}_{+}(\tau) A(0) \rangle \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \Gamma \xi^{2} \langle A \rangle_{ss} \\ i\xi \frac{\Omega}{2} \langle A \rangle_{ss} \\ -i\xi \frac{\Omega}{2} \langle A \rangle_{ss} \\ 0 \\ 0 \end{pmatrix}, \tag{29}$$

4 Second-Order Correlation Function

The second-order correlation function for the filtered output field in the steady state is given by

$$G^{(2)}(\tau) = \langle A^{\dagger}(0) A^{\dagger} A(\tau) A(0) \rangle = \sum_{k,l=-N}^{N} \langle A^{\dagger}(0) a^{\dagger}_{k} a_{l}(\tau) A(0) \rangle, \tag{30}$$

with the normalised second-order correlation function given by

$$g^{(2)}(\tau) = \frac{G^{(2)}(\tau)}{\langle A^\dagger A\rangle_{ss}^2}. \tag{31}$$

Using the Quantum Regression Theorem, we can solve for the second-order correlation function by solving the following moment equations

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\langle A^\dagger(0)\boldsymbol{\sigma} A(0)\rangle = \boldsymbol{M}\langle A^\dagger(0)\boldsymbol{\sigma} A(0)\rangle + \boldsymbol{B}, \tag{32a}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\langle A^\dagger(0)a_j^\dagger(\tau)A(0)\rangle = -\left(\kappa - i\omega_j\right)\langle A^\dagger(0)a_j^\dagger(\tau)A(0)\rangle - \mathcal{E}_j^*\left(\langle A^\dagger(0)\sigma_+^{eg}(\tau)A(0)\rangle + \xi\langle A^\dagger(0)\sigma_+^{fe}A(0)\rangle\right), \tag{32b}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\langle A^\dagger(0)a_j(\tau)A(0)\rangle = -\left(\kappa + i\omega_j\right)\langle A^\dagger(0)a_j(\tau)A(0)\rangle - \mathcal{E}_j\left(\langle A^\dagger(0)\sigma_-^{eg}(\tau)A(0)\rangle + \xi\langle A^\dagger(0)\sigma_-^{fe}A(0)\rangle\right), \tag{32c}$$

with

$$\langle\boldsymbol{\sigma}\rangle = \begin{pmatrix} \langle A^\dagger(0)\sigma^{gg}(\tau)A(0)\rangle \\ \langle A^\dagger(0)\sigma_-^{eg}(\tau)A(0)\rangle \\ \langle A^\dagger(0)\sigma_+^{eg}(\tau)A(0)\rangle \\ \langle A^\dagger(0)\sigma^{ee}(\tau)A(0)\rangle \\ \langle A^\dagger(0)\sigma_-^{fe}(\tau)A(0)\rangle \\ \langle A^\dagger(0)\sigma_+^{fe}(\tau)A(0)\rangle \\ \langle A^\dagger(0)\sigma_-^{fg}(\tau)A(0)\rangle \\ \langle A^\dagger(0)\sigma_+^{fg}(\tau)A(0)\rangle \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \Gamma\xi^2\langle A^\dagger A\rangle_{ss} \\ i\xi\frac{\Omega}{2}\langle A^\dagger A\rangle_{ss} \\ -i\xi\frac{\Omega}{2}\langle A^\dagger A\rangle_{ss} \\ 0 \\ 0 \end{pmatrix}. \tag{33}$$

We also need to solve

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\langle A^\dagger(0)a_j\boldsymbol{\sigma}(\tau)A(0)\rangle &= \boldsymbol{M}_j^{(a)}\langle A^\dagger(0)a_j\boldsymbol{\sigma}(\tau)A(0)\rangle \\ &+ \begin{pmatrix} -\mathcal{E}_j\langle A^\dagger(0)\sigma_-^{eg}(\tau)A(0)\rangle \\ -\mathcal{E}_j\xi\langle A^\dagger(0)\sigma_-^{fg}(\tau)A(0)\rangle \\ -\mathcal{E}_j\langle A^\dagger(0)\sigma^{ee}(\tau)A(0)\rangle \\ \Gamma\xi^2\langle A^\dagger(0)a_j(\tau)A(0)\rangle - \mathcal{E}_j\xi\langle A^\dagger(0)\sigma_-^{fe}(\tau)A(0)\rangle \\ i\xi\frac{\Omega}{2}\langle A^\dagger(0)a_j(\tau)A(0)\rangle \\ -i\xi\frac{\Omega}{2}\langle A^\dagger(0)a_j(\tau)A(0)\rangle - \mathcal{E}_j\xi\left(\langle A^\dagger A\rangle_{ss} - \langle A^\dagger(0)\sigma^{gg}(\tau)A(0)\rangle - \langle A^\dagger(0)\sigma^{ee}(\tau)A(0)\rangle\right) \\ 0 \\ -\mathcal{E}_j\langle A^\dagger(0)\sigma_+^{fe}(\tau)A(0)\rangle \end{pmatrix},\end{aligned} \tag{34a}$$

and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\langle A^\dagger(0)a_j^\dagger\boldsymbol{\sigma}(\tau)A(0)\rangle &= \boldsymbol{M}_j^{(a^\dagger)}\langle A^\dagger(0)a_j^\dagger\boldsymbol{\sigma}(\tau)A(0)\rangle \\ &+ \begin{pmatrix} -\mathcal{E}_j^*\langle A^\dagger(0)\sigma_+^{eg}(\tau)A(0)\rangle \\ -\mathcal{E}_j^*\langle A^\dagger(0)\sigma^{ee}(\tau)A(0)\rangle \\ -\mathcal{E}_j^*\xi\langle A^\dagger(0)\sigma_+^{fg}(\tau)A(0)\rangle \\ \Gamma\xi^2\langle A^\dagger(0)a_j^\dagger(\tau)A(0)\rangle - \mathcal{E}_j^*\xi\langle A^\dagger(0)\sigma_+^{fe}(\tau)A(0)\rangle \\ i\xi\frac{\Omega}{2}\langle A^\dagger(0)a_j^\dagger(\tau)A(0)\rangle - \mathcal{E}_j^*\xi\left(\langle A^\dagger A\rangle_{ss} - \langle A^\dagger(0)\sigma^{gg}(\tau)A(0)\rangle - \langle A^\dagger(0)\sigma^{ee}(\tau)A(0)\rangle\right) \\ -i\xi\frac{\Omega}{2}\langle A^\dagger(0)a_j^\dagger(\tau)A(0)\rangle \\ -\mathcal{E}_j^*\langle A^\dagger(0)\sigma_-^{fe}(\tau)A(0)\rangle \\ 0 \end{pmatrix},\end{aligned} \tag{34b}$$

(34c)

where

$$\boldsymbol{M}_j^{(a)} = \boldsymbol{M} - (\kappa + i\omega_j)\,\mathbb{1}, \tag{35a}$$

$$\boldsymbol{M}_j^{(a^\dagger)} = \boldsymbol{M} - (\kappa - i\omega_j)\,\mathbb{1}. \tag{35b}$$

Finally, we will also need to solve

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\langle A^\dagger(0)a_k^\dagger a_l(\tau)A(0)\rangle = &-(2\kappa - i\omega_k + i\omega_l)\,\langle A^\dagger(0)a_k^\dagger a_l(\tau)A(0)\rangle \\ &- \mathcal{E}_k^*\left(\langle A^\dagger(0)a_l\sigma_+^{eg}(\tau)A(0)\rangle + \xi\langle A^\dagger(0)a_l\sigma_+^{fe}(\tau)A(0)\rangle\right) \\ &- \mathcal{E}_l\left(\langle A^\dagger(0)a_k^\dagger\sigma_-^{eg}(\tau)A(0)\rangle + \xi\langle A^\dagger(0)a_k^\dagger\sigma_-^{fe}(\tau)A(0)\rangle\right).\end{aligned} \tag{36}$$