Multi-Mode Three-Level Atom

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March 23, 2021

1 Hamiltonian and Master Equation

The Hamiltonian is

$$H = -\hbar\left(\frac{\alpha}{2}+\delta\right)|e\rangle\langle e| - 2\hbar\delta\,|f\rangle\langle f| + \hbar\frac{\Omega}{2}\left(\Sigma_{+}+\Sigma_{-}\right) + \hbar\sum_{j=-N}^{N}\omega_j a_j^{\dagger}a_j + \frac{i\hbar}{2}\sum_{j=-N}^{N}\left(\mathcal{E}_j^{*}a_j\Sigma_{+} - \mathcal{E}_j a_j^{\dagger}\Sigma_{-}\right) \tag{1}$$

where Ω is the Rabi frequency,

$$\alpha = \omega_{fe} - \omega_{eg}, \tag{2}$$

is the atomic anharmonicity, δ is given by

$$2\delta = 2\omega_d - \omega_{fg}, \tag{3}$$

is the *drive detuning from two-photon resonance*

$$\Sigma_{-} = \sigma_{-}^{eg} + \xi\sigma_{-}^{fe} = |g\rangle\langle e| + \xi\,|e\rangle\langle f|\,, \quad \Sigma_{+} = \Sigma_{-}^{\dagger}, \tag{4}$$

is the atomic raising (lowering) operator, ω_0 is the resonance frequency of the cavity mode, $a^{\dagger}$ (a) is the cavity photon creation (annihilation) operator, N is the number of modes either side of the central mode ($2N+1$ total modes),

$$\omega_j = \omega_0 + j\delta\omega \tag{5}$$

is the resonance frequency of the j^{th} mode with mode frequency spacing $\delta\omega$, and

$$\mathcal{E}_j = \sqrt{\frac{\epsilon\Gamma\kappa}{2N+1}}e^{im\varphi_j}, \tag{6}$$

is the cascaded systems coupling where Γ is the atomic decay rate, κ is the cavity decay rate, and ϵ is the percentage of fluorescence sent to the filter,

$$m\varphi_j = \frac{mj\pi}{N}, \tag{7}$$

sets the size of the frequency dependent time delay, with integer m. The master equation is

$$\begin{aligned}\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{1}{i\hbar}[H,\rho] &+ \frac{\Gamma}{2}(1-\epsilon)\left(2\Sigma_{-}\rho\Sigma_{+} - \Sigma_{+}\Sigma_{-}\rho - \rho\Sigma_{+}\Sigma_{-}\right)\\ &+ \frac{\kappa}{2}\sum_{j=-N}^{N}\left(2a_j\rho a_j^{\dagger} - a_j^{\dagger}a_j\rho - \rho a_j^{\dagger}a_j\right)\\ &+ \frac{1}{2}\sum_{j=-N}^{N}\left(2C_j\rho C_j^{\dagger} - C_j^{\dagger}C_j\rho - \rho C_j^{\dagger}C_j\right),\end{aligned} \tag{8}$$

where

$$C_j = \sqrt{\frac{\epsilon\gamma}{2N+1}} e^{im\varphi_j} \sigma_- + \sqrt{\kappa} a_j, \tag{9}$$

is the cascaded systems decay operator. Expanding the master equation out and simplifying it, we arrive at a more compact form:

$$\begin{aligned}
\frac{\mathrm{d}\rho}{\mathrm{d}t} = {} & i\left(\frac{\alpha}{2} + \delta\right)\left(\sigma^{ee}\rho - \rho\sigma^{ee}\right) + 2i\delta\left(\sigma^{ff}\rho - \rho\sigma^{ff}\right) - i\frac{\Omega}{2}\left(\Sigma_+\rho - \rho\Sigma_+\right) \\
& - i\frac{\Omega}{2}\left(\Sigma_-\rho - \rho\Sigma_-\right) + \frac{\Gamma}{2}\left(2\Sigma_-\rho\Sigma_+ - \Sigma_+\Sigma_-\rho - \rho\Sigma_+\Sigma_-\right) \\
& - i\sum_{j=-N}^{N}\omega_j\left(a_j^\dagger a_j\rho - \rho a_j^\dagger a_j\right) + \kappa\sum_{j=-N}^{N}\left(2a_j\rho a_j^\dagger - a_j^\dagger a_j\rho - \rho a_j^\dagger a_j\right) \\
& - \sum_{j=-N}^{N}\mathcal{E}_j\left(a_j^\dagger\Sigma_-\rho - \Sigma_-\rho a_j^\dagger\right) - \sum_{j=-N}^{N}\mathcal{E}_j^*\left(\rho\Sigma_+ a_j - a_j\rho\Sigma_+\right).
\end{aligned} \tag{10}$$

1.1 Operator relations

Here's a list of some atomic operator relations to make things easier

$$\sigma^{gg}\Sigma_- = \sigma_-^{eg}, \quad \Sigma_+\sigma^{gg} = \sigma_+^{eg}, \tag{11a}$$

$$\sigma_-^{eg}\Sigma_- = \xi\sigma_-^{fg}, \quad \Sigma_+\sigma_-^{eg} = \sigma^{ee}, \tag{11b}$$

$$\sigma_+^{eg}\Sigma_- = \sigma^{ee}, \quad \Sigma_+\sigma_+^{eg} = \xi\sigma_+^{fg}, \tag{11c}$$

$$\sigma^{ee}\Sigma_- = \xi\sigma_-^{fe}, \quad \Sigma_+\sigma^{ee} = \xi\sigma_+^{fe}, \tag{11d}$$

$$\sigma_-^{fe}\Sigma_- = 0, \quad \Sigma_+\sigma_-^{fe} = \xi\left(1 - \sigma^{gg} - \sigma^{ee}\right), \tag{11e}$$

$$\sigma_+^{fe}\Sigma_- = \xi\left(1 - \sigma^{gg} - \sigma^{ee}\right), \quad \Sigma_+\sigma_+^{fe} = 0, \tag{11f}$$

$$\sigma_-^{fg}\Sigma_- = 0, \quad \Sigma_+\sigma_-^{fg} = \sigma_-^{fe}, \tag{11g}$$

$$\sigma_+^{fg}\Sigma_- = \sigma_+^{fe}, \quad \Sigma_+\sigma_+^{fg} = 0, \tag{11h}$$

2 Operator Averages

2.1 First-Order: Atomic Equations

Rearranging the density operator equations, we can write the atomic moment equations in matrix form

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\boldsymbol{\sigma}\rangle = \boldsymbol{M}\langle\boldsymbol{\sigma}\rangle + \boldsymbol{B}, \tag{12}$$

where

$$\langle\boldsymbol{\sigma}\rangle = \begin{pmatrix} \langle\sigma^{gg}\rangle \\ \langle\sigma_-^{eg}\rangle \\ \langle\sigma_+^{eg}\rangle \\ \langle\sigma^{ee}\rangle \\ \langle\sigma_-^{fe}\rangle \\ \langle\sigma_+^{fe}\rangle \\ \langle\sigma_-^{fg}\rangle \\ \langle\sigma_+^{fg}\rangle \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \Gamma\xi^2 \\ i\xi\frac{\Omega}{2} \\ -i\xi\frac{\Omega}{2} \\ 0 \\ 0 \end{pmatrix}, \tag{13}$$

and

$$\boldsymbol{M} = \begin{pmatrix} 0 & -i\frac{\Omega}{2} & i\frac{\Omega}{2} & \Gamma & 0 & 0 & 0 & 0 \\ -i\frac{\Omega}{2} & -\left[\frac{\Gamma}{2}-i\left(\frac{\alpha}{2}+\delta\right)\right] & 0 & i\frac{\Omega}{2} & \Gamma\xi & 0 & -i\xi\frac{\Omega}{2} & 0 \\ i\frac{\Omega}{2} & 0 & -\left[\frac{\Gamma}{2}+i\left(\frac{\alpha}{2}+\delta\right)\right] & -i\frac{\Omega}{2} & 0 & \Gamma\xi & 0 & i\xi\frac{\Omega}{2} \\ -\Gamma\xi^2 & i\frac{\Omega}{2} & -i\frac{\Omega}{2} & -\Gamma\left(1+\xi^2\right) & -i\xi\frac{\Omega}{2} & i\xi\frac{\Omega}{2} & 0 & 0 \\ -i\xi\frac{\Omega}{2} & 0 & 0 & -i\xi\Omega & -\left[\frac{\Gamma}{2}\left(1+\xi^2\right)+i\left(\frac{\alpha}{2}-\delta\right)\right] & 0 & i\frac{\Omega}{2} & 0 \\ i\xi\frac{\Omega}{2} & 0 & 0 & i\xi\Omega & 0 & -\left[\frac{\Gamma}{2}\left(1+\xi^2\right)-i\left(\frac{\alpha}{2}-\delta\right)\right] & 0 & -i\frac{\Omega}{2} \\ 0 & -i\xi\frac{\Omega}{2} & 0 & 0 & i\frac{\Omega}{2} & 0 & -\left[\frac{\Gamma}{2}\xi^2-2i\delta\right] & 0 \\ 0 & 0 & i\xi\frac{\Omega}{2} & 0 & 0 & -i\frac{\Omega}{2} & 0 & -\left[\frac{\Gamma}{2}\xi^2+2i\delta\right] \end{pmatrix}. \tag{14}$$

This differential equation has solution

$$\langle\boldsymbol{\sigma}(t)\rangle = e^{\boldsymbol{M}t}\langle\boldsymbol{\sigma}(0)\rangle + \left(1 - e^{\boldsymbol{M}t}\right)\langle\boldsymbol{\sigma}\rangle_{ss}, \tag{15}$$

where

$$\langle\boldsymbol{\sigma}\rangle_{ss} = -\boldsymbol{M}^{-1}\boldsymbol{B}. \tag{16}$$

2.2 First-Order: Cavity Equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j\rangle = -\left(\kappa + i\omega_j\right)\langle a_j\rangle - \mathcal{E}_j\left(\langle\sigma_-^{eg}\rangle + \xi\langle\sigma_-^{fe}\rangle\right), \tag{17a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger\rangle = -\left(\kappa - i\omega_j\right)\langle a_j^\dagger\rangle - \mathcal{E}_j\left(\langle\sigma_+^{eg}\rangle + \xi\langle\sigma_+^{fe}\rangle\right) \tag{17b}$$

2.3 Second-Order: Cavity-Atom Coupled Equations

Using the vector notation, we have moment equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j\boldsymbol{\sigma}\rangle = \boldsymbol{M}^{(j)}\langle a_j\boldsymbol{\sigma}\rangle + \begin{pmatrix} -\mathcal{E}_j\langle\sigma_-^{eg}\rangle \\ -\mathcal{E}_j\xi\langle\sigma_-^{fg}\rangle \\ -\mathcal{E}_j\langle\sigma^{ee}\rangle \\ \Gamma\xi^2\langle a_j\rangle - \mathcal{E}_j\xi\langle\sigma_-^{fe}\rangle \\ i\xi\frac{\Omega}{2}\langle a_j\rangle \\ -i\xi\frac{\Omega}{2}\langle a_j\rangle - \mathcal{E}_j\xi\left(1 - \langle\sigma^{gg}\rangle - \langle\sigma^{ee}\rangle\right) \\ 0 \\ -\mathcal{E}_j\langle\sigma_+^{fe}\rangle \end{pmatrix}, \tag{18a}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger\boldsymbol{\sigma}\rangle = \boldsymbol{M}^{(j^*)}\langle a_j^\dagger\boldsymbol{\sigma}\rangle + \begin{pmatrix} -\mathcal{E}_j^*\langle\sigma_+^{eg}\rangle \\ -\mathcal{E}_j^*\langle\sigma^{ee}\rangle \\ -\mathcal{E}_j^*\xi\langle\sigma_+^{fg}\rangle \\ \Gamma\xi^2\langle a_j^\dagger\rangle - \mathcal{E}_j^*\xi\langle\sigma_+^{fe}\rangle \\ i\xi\frac{\Omega}{2}\langle a_j^\dagger\rangle - \mathcal{E}_j^*\xi\left(1 - \langle\sigma^{gg}\rangle - \langle\sigma^{ee}\rangle\right) \\ -i\xi\frac{\Omega}{2}\langle a_j^\dagger\rangle \\ -\mathcal{E}_j^*\langle\sigma_-^{fe}\rangle \\ 0 \end{pmatrix}, \tag{18b}$$

where

$$\boldsymbol{M}^{(j)} = \boldsymbol{M} - \left(\kappa + i\omega_j\right)\mathbb{1}, \quad \boldsymbol{M}^{(j^*)} = \boldsymbol{M} - \left(\kappa - i\omega_j\right)\mathbb{1}. \tag{19}$$

2.4 Second-Order: Cavity Equations

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j a_k\rangle =& -\left(2\kappa + i\left(\omega_j+\omega_k\right)\right)\langle a_j a_k\rangle - \mathcal{E}_j\langle a_k\Sigma_-\rangle - \mathcal{E}_k\langle a_j\Sigma_-\rangle \\
& -\left(2\kappa + i\left(\omega_j+\omega_k\right)\right)\langle a_j a_k\rangle - \mathcal{E}_j\left(\langle a_k\sigma_-^{eg}\rangle + \xi\langle a_k\sigma_-^{fe}\rangle\right) - \mathcal{E}_k\left(\langle a_j\sigma_-^{eg}\rangle + \xi\langle a_j\sigma_-^{fe}\rangle\right), \qquad (20a)\\
\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger a_k^\dagger\rangle =& -\left(2\kappa - i\left(\omega_j+\omega_k\right)\right)\langle a_j^\dagger a_k^\dagger\rangle - \mathcal{E}_j^*\langle a_k^\dagger\Sigma_+\rangle - \mathcal{E}_k^*\langle a_j^\dagger\Sigma_+\rangle \\
& -\left(2\kappa - i\left(\omega_j+\omega_k\right)\right)\langle a_j^\dagger a_k^\dagger\rangle - \mathcal{E}_j^*\left(\langle a_k^\dagger\sigma_+^{eg}\rangle + \langle a_k^\dagger\sigma_+^{fe}\rangle\right) - \mathcal{E}_k^*\left(\langle a_j^\dagger\sigma_+^{eg}\rangle + \langle a_j^\dagger\sigma_+^{fe}\rangle\right), \qquad (20b)\\
\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger a_k\rangle =& -\left(2\kappa - i\left(\omega_j-\omega_k\right)\right) - \mathcal{E}_j^*\langle a_k\Sigma_+\rangle - \mathcal{E}_k\langle a_j^\dagger\Sigma_-\rangle \\
& -\left(2\kappa - i\left(\omega_j-\omega_k\right)\right) - \mathcal{E}_j^*\left(\langle a_k\sigma_+^{eg}\rangle + \xi\langle a_k\sigma_+^{fe}\rangle\right) - \mathcal{E}_k\left(\langle a_j^\dagger\sigma_-^{eg}\rangle + \xi\langle a_j^\dagger\sigma_-^{fe}\rangle\right) \qquad (20c)
\end{aligned}
$$

2.5 Third-Order: Cavity-Atom Coupled Equations

For the third-order moment equations we have

$$
\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j a_k\boldsymbol{\sigma}\rangle = \boldsymbol{M}^{(j,k)}\langle a_j a_k\boldsymbol{\sigma}\rangle + \begin{pmatrix}
-\mathcal{E}_j\langle a_k\sigma_-^{eg}\rangle - \mathcal{E}_k\langle a_j\sigma_-^{eg}\rangle \\
-\mathcal{E}_j\xi\langle a_k\sigma_-^{fg}\rangle - \mathcal{E}_k\xi\langle a_j\sigma_-^{fg}\rangle \\
-\mathcal{E}_j\langle a_k\sigma^{ee}\rangle - \mathcal{E}_k\langle a_j\sigma^{ee}\rangle \\
\Gamma\xi\langle a_j a_k\rangle - \mathcal{E}_j\xi\langle a_k\sigma_-^{fe}\rangle - \mathcal{E}_k\xi\langle a_j\sigma_-^{fe}\rangle \\
i\xi\frac{\Omega}{2}\langle a_j a_k\rangle \\
-i\xi\frac{\Omega}{2}\langle a_j a_k\rangle - \mathcal{E}_j\xi\left(\langle a_k\rangle - \langle a_k\sigma^{gg}\rangle - \langle a_k\sigma^{ee}\rangle\right) - \mathcal{E}_k\xi\left(\langle a_j\rangle - \langle a_j\sigma^{gg}\rangle - \langle a_j\sigma^{ee}\rangle\right) \\
0 \\
-\mathcal{E}_j\langle a_k\sigma_+^{fe}\rangle - \mathcal{E}_k\langle a_j\sigma_+^{fe}\rangle
\end{pmatrix}, \tag{21a}
$$

and

$$
\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger a_k^\dagger\boldsymbol{\sigma}\rangle = \boldsymbol{M}^{(j^*,k^*)}\langle a_j^\dagger a_k^\dagger\boldsymbol{\sigma}\rangle + \begin{pmatrix}
-\mathcal{E}_j^*\langle a_k^\dagger\sigma_+^{eg}\rangle - \mathcal{E}_k^*\langle a_j^\dagger\sigma_+^{eg}\rangle \\
-\mathcal{E}_j^*\langle a_k^\dagger\sigma^{ee}\rangle - \mathcal{E}_k^*\langle a_j^\dagger\sigma^{ee}\rangle \\
-\mathcal{E}_j^*\xi\langle a_k^\dagger\sigma_+^{fg}\rangle - \mathcal{E}_k^*\xi\langle a_j^\dagger\sigma_+^{fg}\rangle \\
\Gamma\xi^2\langle a_j^\dagger a_k^\dagger\rangle - \mathcal{E}_j^*\xi\langle a_k^\dagger\sigma_+^{fe}\rangle - \mathcal{E}_k^*\xi\langle a_j^\dagger\sigma_+^{fe}\rangle \\
i\xi\frac{\Omega}{2}\langle a_j^\dagger a_k^\dagger\rangle - \mathcal{E}_j^*\xi\left(\langle a_k^\dagger\rangle - \langle a_k^\dagger\sigma^{gg}\rangle - \langle a_k^\dagger\sigma^{ee}\rangle\right) - \mathcal{E}_k^*\xi\left(\langle a_j^\dagger\rangle - \langle a_j^\dagger\sigma^{gg}\rangle - \langle a_j^\dagger\sigma^{ee}\rangle\right) \\
-i\xi\frac{\Omega}{2}\langle a_j^\dagger a_k^\dagger\rangle \\
-\mathcal{E}_j^*\langle a_k^\dagger\sigma_-^{fe}\rangle - \mathcal{E}_k^*\langle a_j^\dagger\sigma_-^{fe}\rangle \\
0
\end{pmatrix}, \tag{21b}
$$

and

$$
\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger a_k\boldsymbol{\sigma}\rangle = \boldsymbol{M}^{(j^*,k)}\langle a_j^\dagger a_k\boldsymbol{\sigma}\rangle + \begin{pmatrix}
-\mathcal{E}_j^*\langle a_k\sigma_+^{eg}\rangle - \mathcal{E}_k\langle a_j^\dagger\sigma_-^{eg}\rangle \\
-\mathcal{E}_j^*\langle a_k\sigma^{ee}\rangle - \mathcal{E}_k\xi\langle a_j^\dagger\sigma_-^{fg}\rangle \\
-\mathcal{E}_j^*\xi\langle a_k\sigma_+^{fg}\rangle - \mathcal{E}_k\langle a_j^\dagger\sigma^{ee}\rangle \\
\Gamma\xi^2\langle a_j^\dagger a_k\rangle - \mathcal{E}_j^*\xi\langle a_k\sigma_+^{fe}\rangle - \mathcal{E}_k\xi\langle a_k^\dagger\sigma_-^{fe}\rangle \\
i\xi\frac{\Omega}{2}\langle a_j^\dagger a_k\rangle - \mathcal{E}_j^*\xi\left(\langle a_k\rangle - \langle a_k\sigma^{gg}\rangle - \langle a_k\sigma^{ee}\rangle\right) \\
-i\xi\frac{\Omega}{2}\langle a_j^\dagger a_k\rangle - \mathcal{E}_k\xi\left(\langle a_j^\dagger\rangle - \langle a_j^\dagger\sigma^{gg}\rangle - \langle a_j^\dagger\sigma^{ee}\rangle\right) \\
-\mathcal{E}_j^*\langle a_k\sigma_-^{fe}\rangle \\
-\mathcal{E}_k\langle a_j^\dagger\sigma_+^{fe}\rangle
\end{pmatrix}, \tag{22}
$$

where

$$
\begin{aligned}
\boldsymbol{M}^{(j,k)} &= \boldsymbol{M} - \left(2\kappa + i\left(\omega_j + \omega_k\right)\right)\mathbb{1}, && \text{(23a)}\\
\boldsymbol{M}^{(j^*,k^*)} &= \boldsymbol{M} - \left(2\kappa - i\left(\omega_j + \omega_k\right)\right)\mathbb{1}, && \text{(23b)}\\
\boldsymbol{M}^{(j^*,k)} &= \boldsymbol{M} - \left(2\kappa - i\omega_j + i\omega_k\right)\mathbb{1}. && \text{(23c)}
\end{aligned}
$$

2.6 Third-Order: Cavity Equations

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger a_k a_l\rangle =& -\left(3\kappa - i\left(\omega_j - \omega_k - \omega_l\right)\right)\langle a_j^\dagger a_k a_l\rangle \\
& - \mathcal{E}_j^*\left(\langle a_k a_l \sigma_+^{eg}\rangle + \xi\langle a_k a_l \sigma_+^{fe}\rangle\right) \\
& - \mathcal{E}_k\left(\langle a_j^\dagger a_l \sigma_-^{eg}\rangle + \xi\langle a_j^\dagger a_l \sigma_-^{fe}\rangle\right) \\
& - \mathcal{E}_l\left(\langle a_j^\dagger a_k \sigma_-^{eg}\rangle + \xi\langle a_j^\dagger a_k \sigma_-^{fe}\rangle\right), && \text{(24a)}\\
\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger a_k^\dagger a_l\rangle =& -\left(3\kappa - i\left(\omega_j + \omega_k - \omega_l\right)\right)\langle a_j^\dagger a_k^\dagger a_l\rangle \\
& - \mathcal{E}_j^*\left(\langle a_k^\dagger a_l \sigma_+^{eg}\rangle + \xi\langle a_k^\dagger a_l \sigma_+^{fe}\rangle\right) \\
& - \mathcal{E}_k^*\left(\langle a_j^\dagger a_l \sigma_+^{eg}\rangle + \xi\langle a_j^\dagger a_l \sigma_+^{fe}\rangle\right) \\
& - \mathcal{E}_l\left(\langle a_j^\dagger a_k^\dagger \sigma_-^{eg}\rangle + \xi\langle a_j^\dagger a_k^\dagger \sigma_-^{fe}\rangle\right), && \text{(24b)}
\end{aligned}
$$

2.7 Fourth-Order: Cavity-Atom Coupled Equations

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger a_k a_l \boldsymbol{\sigma}\rangle = \boldsymbol{M}^{(j^*,k,l)}\langle a_j^\dagger a_k a_l\rangle + \\
&\begin{pmatrix}
-\mathcal{E}_j^*\langle a_k a_l \sigma_+^{eg}\rangle - \mathcal{E}_k\langle a_j^\dagger a_l \sigma_-^{eg}\rangle - \mathcal{E}_l\langle a_j^\dagger a_k \sigma_-^{eg}\rangle \\
-\mathcal{E}_j^*\langle a_k a_l \sigma^{ee}\rangle - \mathcal{E}_k\xi\langle a_j^\dagger a_l \sigma_-^{fg}\rangle - \mathcal{E}_l\xi\langle a_j^\dagger a_k \sigma_-^{fg}\rangle \\
-\mathcal{E}_j^*\xi\langle a_k a_l \sigma_+^{fg}\rangle - \mathcal{E}_k\langle a_j^\dagger a_l \sigma^{ee}\rangle - \mathcal{E}_l\langle a_j^\dagger a_k \sigma^{ee}\rangle \\
\Gamma\xi^2\langle a_j^\dagger a_k a_l\rangle - \mathcal{E}_j^*\xi\langle a_k a_l \sigma_+^{fe}\rangle - \mathcal{E}_k\xi\langle a_j^\dagger a_l \sigma_-^{fe}\rangle - \mathcal{E}_l\xi\langle a_j^\dagger a_k \sigma_-^{fe}\rangle \\
i\xi\frac{\Omega}{2}\langle a_j^\dagger a_k a_l\rangle - \mathcal{E}_j^*\xi\left(\langle a_k a_l\rangle - \langle a_k a_l \sigma^{gg}\rangle - \langle a_k a_l \sigma^{ee}\rangle\right) \\
-i\xi\frac{\Omega}{2}\langle a_j^\dagger a_k a_l\rangle - \mathcal{E}_k\xi\left(\langle a_j^\dagger a_k\rangle - \langle a_j^\dagger a_l \sigma^{gg}\rangle - \langle a_j^\dagger a_l \sigma^{ee}\rangle\right) - \mathcal{E}_l\xi\left(\langle a_j^\dagger a_l\rangle - \langle a_j^\dagger a_k \sigma^{gg}\rangle - \langle a_j^\dagger a_k \sigma^{ee}\rangle\right) \\
-\mathcal{E}_j^*\langle a_k a_l \sigma_-^{fe}\rangle \\
-\mathcal{E}_k\langle a_j^\dagger a_l \sigma_+^{fe}\rangle - \mathcal{E}_l\langle a_j^\dagger a_k \sigma_-^{fe}\rangle
\end{pmatrix},
\end{aligned}
\tag{25a}
$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger a_k^\dagger a_l \boldsymbol{\sigma}\rangle = \boldsymbol{M}^{(j^*,k^*,l)}\langle a_j^\dagger a_k^\dagger a_l\rangle + \begin{pmatrix} -\mathcal{E}_j^*\langle a_k^\dagger a_l \sigma_+^{eg}\rangle - \mathcal{E}_k^*\langle a_j^\dagger a_l \sigma_+^{eg}\rangle - \mathcal{E}_l\langle a_j^\dagger a_k^\dagger \sigma_-^{eg}\rangle \\ -\mathcal{E}_j^*\langle a_k^\dagger a_l \sigma^{ee}\rangle - \mathcal{E}_k^*\langle a_j^\dagger a_l \sigma^{ee}\rangle - \mathcal{E}_l\xi\langle a_j^\dagger a_k^\dagger \sigma_-^{fg}\rangle \\ -\mathcal{E}_j^*\xi\langle a_k^\dagger a_l \sigma_+^{fg}\rangle - \mathcal{E}_k^*\xi\langle a_j^\dagger a_l \sigma_+^{fg}\rangle - \mathcal{E}_l\langle a_j^\dagger a_k^\dagger \sigma^{ee}\rangle \\ \Gamma\xi^2\langle a_j^\dagger a_k^\dagger a_l\rangle - \mathcal{E}_j^*\xi\langle a_k^\dagger a_l \sigma_+^{fe}\rangle - \mathcal{E}_k^*\xi\langle a_j^\dagger a_l \sigma_+^{fe}\rangle - \mathcal{E}_l\xi\langle a_j^\dagger a_k^\dagger \sigma_-^{fe}\rangle \\ i\xi\frac{\Omega}{2}\langle a_j^\dagger a_k^\dagger a_l\rangle - \mathcal{E}_j^*\xi\left(\langle a_k^\dagger a_l\rangle - \langle a_k^\dagger a_l\sigma^{gg}\rangle - \langle a_k^\dagger a_l\sigma^{ee}\rangle\right) - \mathcal{E}_j^*\xi\left(\langle a_j^\dagger a_l\rangle - \langle a_j^\dagger a_l\sigma^{gg}\rangle - \langle a_j^\dagger a_l\sigma^{ee}\rangle\right) \\ -i\xi\frac{\Omega}{2}\langle a_j^\dagger a_k^\dagger a_l\rangle - \mathcal{E}_l\xi\left(\langle a_j^\dagger a_k^\dagger\rangle - \langle a_j^\dagger a_k^\dagger\sigma^{gg}\rangle - \langle a_j^\dagger a_k^\dagger\sigma^{ee}\rangle\right) \\ -\mathcal{E}_j^*\langle a_k^\dagger a_l \sigma_-^{fe}\rangle - \mathcal{E}_k^*\langle a_j^\dagger a_l \sigma_-^{fe}\rangle \\ -\mathcal{E}_l\langle a_j^\dagger a_k^\dagger \sigma_+^{fe}\rangle \end{pmatrix}, \tag{25b}$$

where

$$\boldsymbol{M}^{(j^*,k,l)} = \boldsymbol{M} - \left(3\kappa - i\omega_j + i\left(\omega_k + \omega_l\right)\right), \tag{26a}$$

$$\boldsymbol{M}^{(j^*,k^*,l)} = \boldsymbol{M} - \left(3\kappa - i\left(\omega_j + \omega_k\right) - i\omega_l\right). \tag{26b}$$

2.8 Fourth-Order: Cavity Equation

Finally, for the fourth-order cavity moment equation, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger a_k^\dagger a_l a_m\rangle = &- \left(4\kappa - i\left(\omega_j + \omega_k\right) + i\left(\omega_l + \omega_m\right)\right)\langle a_j^\dagger a_k^\dagger a_l a_m\rangle \\ &- \mathcal{E}_j^*\left(\langle a_k^\dagger a_l a_m \sigma_+^{eg}\rangle + \xi\langle a_k^\dagger a_l a_m \sigma_+^{fe}\rangle\right) \\ &- \mathcal{E}_k^*\left(\langle a_j^\dagger a_l a_m \sigma_+^{eg}\rangle + \xi\langle a_j^\dagger a_l a_m \sigma_+^{fe}\rangle\right) \\ &- \mathcal{E}_l\left(\langle a_j^\dagger a_k^\dagger a_m \sigma_-^{eg}\rangle + \xi\langle a_j^\dagger a_k^\dagger a_m \sigma_-^{fe}\rangle\right) \\ &- \mathcal{E}_m\left(\langle a_j^\dagger a_k^\dagger a_l \sigma_-^{eg}\rangle + \xi\langle a_j^\dagger a_k^\dagger a_l \sigma_-^{fe}\rangle\right). \end{aligned} \tag{27}$$

3 First-Order Correlation Function

The first-order correlation function for the filtered output field is given by

$$G^{(1)}(t,\tau) = \langle A^\dagger(t+\tau)A(\tau)\rangle = \sum_{j=-N}^{N}\langle a_j^\dagger(t+\tau)A(t)\rangle. \tag{28}$$

Moving into the steady state, we use the Quantum Regression Theorem to solve for this with the moment equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\boldsymbol{\sigma}(\tau)A(0)\rangle = \boldsymbol{M}\langle\boldsymbol{\sigma}(\tau)A(0)\rangle + \boldsymbol{B}, \tag{29a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger(\tau)A(0)\rangle = -\left(\kappa - i\omega_j\right)\langle a_j^\dagger A(0)\rangle - \mathcal{E}_j^*\left(\langle\sigma_+^{eg}(\tau)A(0)\rangle + \xi\langle\sigma_+^{fe}(\tau)A(0)\rangle\right), \tag{29b}$$

where

$$\langle\boldsymbol{\sigma}\rangle = \begin{pmatrix} \langle\sigma^{gg}(\tau)A(0)\rangle \\ \langle\sigma^{eg}_{-}(\tau)A(0)\rangle \\ \langle\sigma^{eg}_{+}(\tau)A(0)\rangle \\ \langle\sigma^{ee}(\tau)A(0)\rangle \\ \langle\sigma^{fe}_{-}(\tau)A(0)\rangle \\ \langle\sigma^{fe}_{+}(\tau)A(0)\rangle \\ \langle\sigma^{fg}_{-}(\tau)A(0)\rangle \\ \langle\sigma^{fg}_{+}(\tau)A(0)\rangle \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \Gamma\xi^2\langle A\rangle_{ss} \\ i\xi\frac{\Omega}{2}\langle A\rangle_{ss} \\ -i\xi\frac{\Omega}{2}\langle A\rangle_{ss} \\ 0 \\ 0 \end{pmatrix}, \tag{30}$$

4 Second-Order Correlation Function

The second-order correlation function for the filtered output field is given by

$$G^{(2)}(t,\tau) = \langle A^{\dagger}(t)A^{\dagger}A(t+\tau)A(t)\rangle = \sum_{k,l=-N}^{N}\langle A^{\dagger}(0)a^{\dagger}_{k}a_{l}(\tau)A(0)\rangle, \tag{31}$$

with the normalised second-order correlation function given by

$$g^{(2)}(\tau) = \frac{G^{(2)}(\tau)}{\langle A^{\dagger}A\rangle^2_{ss}}. \tag{32}$$

Using the Quantum Regression Theorem, we can solve for the second-order correlation function by solving the following moment equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle A^{\dagger}(0)\boldsymbol{\sigma}A(0)\rangle = \boldsymbol{M}\langle A^{\dagger}(0)\boldsymbol{\sigma}A(0)\rangle + \boldsymbol{B}, \tag{33a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle A^{\dagger}(0)a^{\dagger}_{j}(\tau)A(0)\rangle = -\left(\kappa - i\omega_j\right)\langle A^{\dagger}(0)a^{\dagger}_{j}(\tau)A(0)\rangle - \mathcal{E}^{*}_{j}\left(\langle A^{\dagger}(0)\sigma^{eg}_{+}(\tau)A(0)\rangle + \xi\langle A^{\dagger}(0)\sigma^{fe}_{+}A(0)\rangle\right), \tag{33b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle A^{\dagger}(0)a_{j}(\tau)A(0)\rangle = -\left(\kappa + i\omega_j\right)\langle A^{\dagger}(0)a_{j}(\tau)A(0)\rangle - \mathcal{E}_{j}\left(\langle A^{\dagger}(0)\sigma^{eg}_{-}(\tau)A(0)\rangle + \xi\langle A^{\dagger}(0)\sigma^{fe}_{-}A(0)\rangle\right), \tag{33c}$$

with

$$\langle\boldsymbol{\sigma}\rangle = \begin{pmatrix} \langle A^{\dagger}(0)\sigma^{gg}(\tau)A(0)\rangle \\ \langle A^{\dagger}(0)\sigma^{eg}_{-}(\tau)A(0)\rangle \\ \langle A^{\dagger}(0)\sigma^{eg}_{+}(\tau)A(0)\rangle \\ \langle A^{\dagger}(0)\sigma^{ee}(\tau)A(0)\rangle \\ \langle A^{\dagger}(0)\sigma^{fe}_{-}(\tau)A(0)\rangle \\ \langle A^{\dagger}(0)\sigma^{fe}_{+}(\tau)A(0)\rangle \\ \langle A^{\dagger}(0)\sigma^{fg}_{-}(\tau)A(0)\rangle \\ \langle A^{\dagger}(0)\sigma^{fg}_{+}(\tau)A(0)\rangle \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \Gamma\xi^2\langle A^{\dagger}A\rangle_{ss} \\ i\xi\frac{\Omega}{2}\langle A^{\dagger}A\rangle_{ss} \\ -i\xi\frac{\Omega}{2}\langle A^{\dagger}A\rangle_{ss} \\ 0 \\ 0 \end{pmatrix}. \tag{34}$$

We also need to solve

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle A^{\dagger}(0)a_{j}\boldsymbol{\sigma}(\tau)A(0)\rangle = \boldsymbol{M}^{(j)}\langle A^{\dagger}(0)a_{j}\boldsymbol{\sigma}(\tau)A(0)\rangle + \begin{pmatrix} -\mathcal{E}_j\langle A^{\dagger}(0)\sigma^{eg}_{-}(\tau)A(0)\rangle \\ -\mathcal{E}_j\xi\langle A^{\dagger}(0)\sigma^{fg}_{-}(\tau)A(0)\rangle \\ -\mathcal{E}_j\langle A^{\dagger}(0)\sigma^{ee}(\tau)A(0)\rangle \\ \Gamma\xi^2\langle A^{\dagger}(0)a_j(\tau)A(0)\rangle - \mathcal{E}_j\xi\langle A^{\dagger}(0)\sigma^{fe}_{-}(\tau)A(0)\rangle \\ i\xi\frac{\Omega}{2}\langle A^{\dagger}(0)a_j(\tau)A(0)\rangle \\ -i\xi\frac{\Omega}{2}\langle A^{\dagger}(0)a_j(\tau)A(0)\rangle - \mathcal{E}_j\xi\left(\langle A^{\dagger}A\rangle_{ss} - \langle A^{\dagger}(0)\sigma^{gg}(\tau)A(0)\rangle - \langle A^{\dagger}(0)\sigma^{ee}(\tau)A(0)\rangle\right) \\ 0 \\ -\mathcal{E}_j\langle A^{\dagger}(0)\sigma^{fe}_{+}(\tau)A(0)\rangle \end{pmatrix}, \tag{35a}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle A^\dagger(0)a_j^\dagger\boldsymbol{\sigma}(\tau)A(0)\rangle = \boldsymbol{M}^{(j^*)}\langle A^\dagger(0)a_j^\dagger\boldsymbol{\sigma}(\tau)A(0)\rangle + \begin{pmatrix} -\mathcal{E}_j^*\langle A^\dagger(0)\sigma_+^{eg}(\tau)A(0)\rangle \\ -\mathcal{E}_j^*\langle A^\dagger(0)\sigma^{ee}(\tau)A(0)\rangle \\ -\mathcal{E}_j^*\xi\langle A^\dagger(0)\sigma_+^{fg}(\tau)A(0)\rangle \\ \Gamma\xi^2\langle A^\dagger(0)a_j^\dagger(\tau)A(0)\rangle - \mathcal{E}_j^*\xi\langle A^\dagger(0)\sigma_+^{fe}(\tau)A(0)\rangle \\ i\xi\frac{\Omega}{2}\langle A^\dagger(0)a_j^\dagger(\tau)A(0)\rangle - \mathcal{E}_j^*\xi\left(\langle A^\dagger A\rangle_{ss} - \langle A^\dagger(0)\sigma^{gg}(\tau)A(0)\rangle - \langle A^\dagger(0)\sigma^{ee}(\tau)A(0)\rangle\right) \\ -i\xi\frac{\Omega}{2}\langle A^\dagger(0)a_j^\dagger(\tau)A(0)\rangle \\ -\mathcal{E}_j^*\langle A^\dagger(0)\sigma_-^{fe}(\tau)A(0)\rangle \\ 0 \end{pmatrix}, \tag{35b}$$

(35c)

where

$$\boldsymbol{M}^{(j)} = \boldsymbol{M} - (\kappa + i\omega_j)\,\mathbb{1}, \quad \boldsymbol{M}^{(j^*)} = \boldsymbol{M} - (\kappa - i\omega_j)\,\mathbb{1}. \tag{36}$$

Finally, we will also need to solve

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle A^\dagger(0)a_k^\dagger a_l(\tau)A(0)\rangle = &-(2\kappa - i\omega_k + i\omega_l)\langle A^\dagger(0)a_k^\dagger a_l(\tau)A(0)\rangle \\ &-\mathcal{E}_k^*\left(\langle A^\dagger(0)a_l\sigma_+^{eg}(\tau)A(0)\rangle + \xi\langle A^\dagger(0)a_l\sigma_+^{fe}(\tau)A(0)\rangle\right) \\ &-\mathcal{E}_l\left(\langle A^\dagger(0)a_k^\dagger\sigma_-^{eg}(\tau)A(0)\rangle + \xi\langle A^\dagger(0)a_k^\dagger\sigma_-^{fe}(\tau)A(0)\rangle\right).\end{aligned} \tag{37}$$