

# Three-Level Atom: One Multi-Mode Filter Moment Equations

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## 1 Hamiltonian and Master Equation

The Hamiltonian is

$$H = -\hbar \left( \frac{\alpha}{2} + \delta \right) |e\rangle \langle e| - 2\hbar\delta |f\rangle \langle f| + \hbar \frac{\Omega}{2} (\Sigma_+ + \Sigma_-) + \hbar \sum_{j=-N}^N \omega_j a_j^\dagger a_j + \frac{i\hbar}{2} \sum_{j=-N}^N \left( \mathcal{E}_j^* a_j \Sigma_+ - \mathcal{E}_j a_j^\dagger \Sigma_- \right) \quad (1)$$

where  $\Omega$  is the Rabi frequency,

$$\alpha = \omega_{fe} - \omega_{eg}, \quad (2)$$

is the atomic anharmonicity,  $\delta$  is given by

$$2\delta = 2\omega_d - \omega_{fg}, \quad (3)$$

is the *drive detuning from two-photon resonance*

$$\Sigma_- = \sigma_-^{eg} + \xi \sigma_-^{fe} = |g\rangle \langle e| + \xi |e\rangle \langle f|, \quad \Sigma_+ = \Sigma_-^\dagger, \quad (4)$$

is the atomic raising (lowering) operator,  $\omega_0$  is the resonance frequency of the cavity mode,  $a^\dagger$  ( $a$ ) is the cavity photon creation (annihilation) operator,  $N$  is the number of modes either side of the central mode ( $2N + 1$  total modes),

$$\omega_j = \omega_0 + j\delta\omega \quad (5)$$

is the resonance frequency of the  $j^{\text{th}}$  mode with mode frequency spacing  $\delta\omega$ , and

$$\mathcal{E}_j = \sqrt{\frac{\epsilon\Gamma\kappa}{2N+1}} e^{im\varphi_j}, \quad (6)$$

is the cascaded systems coupling where  $\Gamma$  is the atomic decay rate,  $\kappa$  is the cavity decay rate, and  $\epsilon$  is the percentage of fluorescence sent to the filter,

$$m\varphi_j = \frac{mj\pi}{N}, \quad (7)$$

sets the size of the frequency dependent time delay, with integer  $m$ . The master equation is

$$\begin{aligned} \frac{d\rho}{dt} = & \frac{1}{i\hbar} [H, \rho] + \frac{\Gamma}{2} (1 - \epsilon) (2\Sigma_- \rho \Sigma_+ - \Sigma_+ \Sigma_- \rho - \rho \Sigma_+ \Sigma_-) \\ & + \frac{\kappa}{2} \sum_{j=-N}^N \left( 2a_j \rho a_j^\dagger - a_j^\dagger a_j \rho - \rho a_j^\dagger a_j \right) \\ & + \frac{1}{2} \sum_{j=-N}^N \left( 2C_j \rho C_j^\dagger - C_j^\dagger C_j \rho - \rho C_j^\dagger C_j \right), \end{aligned} \quad (8)$$

where

$$C_j = \sqrt{\frac{\epsilon\gamma}{2N+1}} e^{im\varphi_j} \sigma_- + \sqrt{\kappa} a_j, \quad (9)$$

is the cascaded systems decay operator. Expanding the master equation out and simplifying it, we arrive at a more compact form:

$$\begin{aligned} \frac{d\rho}{dt} = & i \left( \frac{\alpha}{2} + \delta \right) (\sigma^{ee} \rho - \rho \sigma^{ee}) + 2i\delta (\sigma^{ff} \rho - \rho \sigma^{ff}) - i \frac{\Omega}{2} (\Sigma_+ \rho - \rho \Sigma_+) \\ & - i \frac{\Omega}{2} (\Sigma_- \rho - \rho \Sigma_-) + \frac{\Gamma}{2} (2\Sigma_- \rho \Sigma_+ - \Sigma_+ \Sigma_- \rho - \rho \Sigma_+ \Sigma_-) \\ & - i \sum_{j=-N}^N \omega_j (a_j^\dagger a_j \rho - \rho a_j^\dagger a_j) + \kappa \sum_{j=-N}^N (2a_j \rho a_j^\dagger - a_j^\dagger a_j \rho - \rho a_j^\dagger a_j) \\ & - \sum_{j=-N}^N \mathcal{E}_j (a_j^\dagger \Sigma_- \rho - \Sigma_- \rho a_j^\dagger) - \sum_{j=-N}^N \mathcal{E}_j^* (\rho \Sigma_+ a_j - a_j \rho \Sigma_+). \end{aligned} \quad (10)$$

## 2 Operator Averages

### 2.1 First-Order: Atomic Equations

Rearranging the density operator equations, we can write the atomic moment equations in matrix form

$$\frac{d}{dt} \langle \sigma \rangle = \mathbf{M} \langle \sigma \rangle + \mathbf{B}, \quad (11)$$

where

$$\langle \sigma \rangle = \begin{pmatrix} \langle \sigma^{gg} \rangle \\ \langle \sigma_-^{eg} \rangle \\ \langle \sigma_+^{eg} \rangle \\ \langle \sigma^{ee} \rangle \\ \langle \sigma_-^{fe} \rangle \\ \langle \sigma_+^{fe} \rangle \\ \langle \sigma_-^{fg} \rangle \\ \langle \sigma_+^{fg} \rangle \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \Gamma \xi^2 \\ i \xi \frac{\Omega}{2} \\ -i \xi \frac{\Omega}{2} \\ 0 \\ 0 \end{pmatrix}, \quad (12)$$

and

$$\mathbf{M} = \begin{pmatrix} 0 & -i \frac{\Omega}{2} & i \frac{\Omega}{2} & \Gamma & 0 & 0 & 0 & 0 \\ -i \frac{\Omega}{2} & -[\frac{\Gamma}{2} - i(\frac{\alpha}{2} + \delta)] & 0 & i \frac{\Omega}{2} & \Gamma \xi & 0 & -i \xi \frac{\Omega}{2} & 0 \\ i \frac{\Omega}{2} & 0 & -[\frac{\Gamma}{2} + i(\frac{\alpha}{2} + \delta)] & -i \frac{\Omega}{2} & 0 & \Gamma \xi & 0 & i \xi \frac{\Omega}{2} \\ -\Gamma \xi^2 & i \frac{\Omega}{2} & -i \frac{\Omega}{2} & -\Gamma(1 + \xi^2) & -i \xi \frac{\Omega}{2} & i \xi \frac{\Omega}{2} & 0 & 0 \\ -i \xi \frac{\Omega}{2} & 0 & 0 & -i \xi \Omega & -[\frac{\Gamma}{2}(1 + \xi^2) + i(\frac{\alpha}{2} - \delta)] & 0 & i \frac{\Omega}{2} & 0 \\ i \xi \frac{\Omega}{2} & 0 & 0 & i \xi \Omega & 0 & -[\frac{\Gamma}{2}(1 + \xi^2) - i(\frac{\alpha}{2} - \delta)] & 0 & -i \frac{\Omega}{2} \\ 0 & -i \xi \frac{\Omega}{2} & 0 & 0 & i \frac{\Omega}{2} & 0 & -[\frac{\Gamma}{2} \xi^2 - 2i\delta] & 0 \\ 0 & 0 & i \xi \frac{\Omega}{2} & 0 & 0 & -i \frac{\Omega}{2} & 0 & -[\frac{\Gamma}{2} \xi^2 + 2i\delta] \end{pmatrix}. \quad (13)$$

This differential equation has solution

$$\langle \sigma(t) \rangle = e^{\mathbf{M}t} \langle \sigma(0) \rangle + (1 - e^{\mathbf{M}t}) \langle \sigma \rangle_{ss}, \quad (14)$$

where

$$\langle \sigma \rangle_{ss} = -\mathbf{M}^{-1} \mathbf{B}. \quad (15)$$

## 2.2 First-Order: Cavity Equations

$$\frac{d}{dt}\langle a_j \rangle = -(\kappa + i\omega_j)\langle a_j \rangle - \mathcal{E}_j \left( \langle \sigma_-^{eg} \rangle + \xi \langle \sigma_-^{fe} \rangle \right), \quad (16a)$$

$$\frac{d}{dt}\langle a_j^\dagger \rangle = -(\kappa - i\omega_j)\langle a_j^\dagger \rangle - \mathcal{E}_j \left( \langle \sigma_+^{eg} \rangle + \xi \langle \sigma_+^{fe} \rangle \right) \quad (16b)$$

## 2.3 Second-Order: Cavity-Atom Coupled Equations

Using the vector notation, we have moment equations

$$\frac{d}{dt}\langle a_j \boldsymbol{\sigma} \rangle = \mathbf{M}_j^{(a)} \langle a_j \boldsymbol{\sigma} \rangle + \begin{pmatrix} -\mathcal{E}_j \langle \sigma_-^{eg} \rangle \\ -\mathcal{E}_j \xi \langle \sigma_-^{fg} \rangle \\ -\mathcal{E}_j \langle \sigma_-^{ee} \rangle \\ \Gamma \xi^2 \langle a_j \rangle - \mathcal{E}_j \xi \langle \sigma_-^{fe} \rangle \\ i\xi \frac{\Omega}{2} \langle a_j \rangle \\ -i\xi \frac{\Omega}{2} \langle a_j \rangle - \mathcal{E}_j \xi (1 - \langle \sigma^{gg} \rangle - \langle \sigma^{ee} \rangle) \\ 0 \\ -\mathcal{E}_j \langle \sigma_+^{fe} \rangle \end{pmatrix}, \quad (17a)$$

and

$$\frac{d}{dt}\langle a_j^\dagger \boldsymbol{\sigma} \rangle = \mathbf{M}_j^{(a^\dagger)} \langle a_j^\dagger \boldsymbol{\sigma} \rangle + \begin{pmatrix} -\mathcal{E}_j^* \langle \sigma_+^{eg} \rangle \\ -\mathcal{E}_j^* \langle \sigma_+^{ee} \rangle \\ -\mathcal{E}_j^* \xi \langle \sigma_+^{fg} \rangle \\ \Gamma \xi^2 \langle a_j^\dagger \rangle - \mathcal{E}_j^* \xi \langle \sigma_+^{fe} \rangle \\ i\xi \frac{\Omega}{2} \langle a_j^\dagger \rangle - \mathcal{E}_j^* \xi (1 - \langle \sigma^{gg} \rangle - \langle \sigma^{ee} \rangle) \\ -i\xi \frac{\Omega}{2} \langle a_j^\dagger \rangle \\ -\mathcal{E}_j^* \langle \sigma_-^{fe} \rangle \\ 0 \end{pmatrix}, \quad (17b)$$

where

$$\mathbf{M}_j^{(a)} = \mathbf{M} - (\kappa + i\omega_j) \mathbf{1}, \quad (18a)$$

$$\mathbf{M}_j^{(a^\dagger)} = \mathbf{M} - (\kappa - i\omega_j) \mathbf{1}. \quad (18b)$$

## 2.4 Second-Order: Cavity Equations

$$\frac{d}{dt}\langle a_j a_k \rangle = -(2\kappa + i(\omega_j + \omega_k))\langle a_j a_k \rangle - \mathcal{E}_j \left( \langle a_k \sigma_-^{eg} \rangle + \xi \langle a_k \sigma_-^{fe} \rangle \right) - \mathcal{E}_k \left( \langle a_j \sigma_-^{eg} \rangle + \xi \langle a_j \sigma_-^{fe} \rangle \right), \quad (19a)$$

$$\frac{d}{dt}\langle a_j^\dagger a_k^\dagger \rangle = -(2\kappa - i(\omega_j + \omega_k))\langle a_j^\dagger a_k^\dagger \rangle - \mathcal{E}_j^* \left( \langle a_k^\dagger \sigma_+^{eg} \rangle + \langle a_k^\dagger \sigma_+^{fe} \rangle \right) - \mathcal{E}_k^* \left( \langle a_j^\dagger \sigma_+^{eg} \rangle + \langle a_j^\dagger \sigma_+^{fe} \rangle \right), \quad (19b)$$

$$\frac{d}{dt}\langle a_j^\dagger a_k \rangle = -(2\kappa - i(\omega_j - \omega_k))\langle a_j^\dagger a_k \rangle - \mathcal{E}_j^* \left( \langle a_k \sigma_+^{eg} \rangle + \xi \langle a_k \sigma_+^{fe} \rangle \right) - \mathcal{E}_k \left( \langle a_j^\dagger \sigma_-^{eg} \rangle + \xi \langle a_j^\dagger \sigma_-^{fe} \rangle \right). \quad (19c)$$

## 2.5 Third-Order: Cavity-Atom Coupled Equations

For the third-order moment equations we have

$$\frac{d}{dt}\langle a_j a_k \sigma \rangle = \mathbf{M}_{j,k}^{(aa)} \langle a_j a_k \sigma \rangle + \begin{pmatrix} -\mathcal{E}_j \langle a_k \sigma_-^{eg} \rangle - \mathcal{E}_k \langle a_j \sigma_-^{eg} \rangle \\ -\mathcal{E}_j \xi \langle a_k \sigma_-^{fg} \rangle - \mathcal{E}_k \xi \langle a_j \sigma_-^{fg} \rangle \\ -\mathcal{E}_j \langle a_k \sigma_-^{ee} \rangle - \mathcal{E}_k \langle a_j \sigma_-^{ee} \rangle \\ \Gamma \xi \langle a_j a_k \rangle - \mathcal{E}_j \xi \langle a_k \sigma_-^{fe} \rangle - \mathcal{E}_k \xi \langle a_j \sigma_-^{fe} \rangle \\ i\xi \frac{\Omega}{2} \langle a_j a_k \rangle \\ -i\xi \frac{\Omega}{2} \langle a_j a_k \rangle - \mathcal{E}_j \xi (\langle a_k \rangle - \langle a_k \sigma^{gg} \rangle - \langle a_k \sigma^{ee} \rangle) - \mathcal{E}_k \xi (\langle a_j \rangle - \langle a_j \sigma^{gg} \rangle - \langle a_j \sigma^{ee} \rangle) \\ 0 \\ -\mathcal{E}_j \langle a_k \sigma_+^{fe} \rangle - \mathcal{E}_k \langle a_j \sigma_+^{fe} \rangle \end{pmatrix}, \quad (20a)$$

and

$$\frac{d}{dt}\langle a_j^\dagger a_k^\dagger \sigma \rangle = \mathbf{M}_{j,k}^{(a^\dagger a^\dagger)} \langle a_j^\dagger a_k^\dagger \sigma \rangle + \begin{pmatrix} -\mathcal{E}_j^* \langle a_k^\dagger \sigma_+^{eg} \rangle - \mathcal{E}_k^* \langle a_j^\dagger \sigma_+^{eg} \rangle \\ -\mathcal{E}_j^* \langle a_k^\dagger \sigma_+^{ee} \rangle - \mathcal{E}_k^* \langle a_j^\dagger \sigma_+^{ee} \rangle \\ -\mathcal{E}_j^* \xi \langle a_k^\dagger \sigma_+^{fg} \rangle - \mathcal{E}_k^* \xi \langle a_j^\dagger \sigma_+^{fg} \rangle \\ \Gamma \xi^2 \langle a_j^\dagger a_k^\dagger \rangle - \mathcal{E}_j^* \xi \langle a_k^\dagger \sigma_+^{fe} \rangle - \mathcal{E}_k^* \xi \langle a_j^\dagger \sigma_+^{fe} \rangle \\ i\xi \frac{\Omega}{2} \langle a_j^\dagger a_k^\dagger \rangle - \mathcal{E}_j^* \xi (\langle a_k^\dagger \rangle - \langle a_k^\dagger \sigma^{gg} \rangle - \langle a_k^\dagger \sigma^{ee} \rangle) - \mathcal{E}_k^* \xi (\langle a_j^\dagger \rangle - \langle a_j^\dagger \sigma^{gg} \rangle - \langle a_j^\dagger \sigma^{ee} \rangle) \\ -i\xi \frac{\Omega}{2} \langle a_j^\dagger a_k^\dagger \rangle \\ -\mathcal{E}_j^* \langle a_k^\dagger \sigma_-^{fe} \rangle - \mathcal{E}_k^* \langle a_j^\dagger \sigma_-^{fe} \rangle \\ 0 \end{pmatrix}, \quad (20b)$$

and

$$\frac{d}{dt}\langle a_j^\dagger a_k \sigma \rangle = \mathbf{M}_{j,k}^{(a^\dagger a)} \langle a_j^\dagger a_k \sigma \rangle + \begin{pmatrix} -\mathcal{E}_j^* \langle a_k \sigma_+^{eg} \rangle - \mathcal{E}_k \langle a_j^\dagger \sigma_-^{eg} \rangle \\ -\mathcal{E}_j^* \langle a_k \sigma_+^{ee} \rangle - \mathcal{E}_k \xi \langle a_j^\dagger \sigma_-^{fg} \rangle \\ -\mathcal{E}_j^* \xi \langle a_k \sigma_+^{fg} \rangle - \mathcal{E}_k \langle a_j^\dagger \sigma_-^{ee} \rangle \\ \Gamma \xi^2 \langle a_j^\dagger a_k \rangle - \mathcal{E}_j^* \xi \langle a_k \sigma_+^{fe} \rangle - \mathcal{E}_k \xi \langle a_j^\dagger \sigma_-^{fe} \rangle \\ i\xi \frac{\Omega}{2} \langle a_j^\dagger a_k \rangle - \mathcal{E}_j^* \xi (\langle a_k \rangle - \langle a_k \sigma^{gg} \rangle - \langle a_k \sigma^{ee} \rangle) \\ -i\xi \frac{\Omega}{2} \langle a_j^\dagger a_k \rangle - \mathcal{E}_k \xi (\langle a_j^\dagger \rangle - \langle a_j^\dagger \sigma^{gg} \rangle - \langle a_j^\dagger \sigma^{ee} \rangle) \\ -\mathcal{E}_j^* \langle a_k \sigma_-^{fe} \rangle \\ -\mathcal{E}_k \langle a_j^\dagger \sigma_+^{fe} \rangle \end{pmatrix}, \quad (21)$$

where

$$\mathbf{M}_{j,k}^{(aa)} = \mathbf{M} - (2\kappa + i(\omega_j + \omega_k)) \mathbb{1}, \quad (22a)$$

$$\mathbf{M}_{j,k}^{(a^\dagger a^\dagger)} = \mathbf{M} - (2\kappa - i(\omega_j + \omega_k)) \mathbb{1}, \quad (22b)$$

$$\mathbf{M}_{j,k}^{(a^\dagger a)} = \mathbf{M} - (2\kappa - i\omega_j + i\omega_k) \mathbb{1}. \quad (22c)$$

## 2.6 Third-Order: Cavity Equations

$$\begin{aligned}
\frac{d}{dt}\langle a_j^\dagger a_k a_l \rangle &= -(3\kappa - i(\omega_j - \omega_k - \omega_l))\langle a_j^\dagger a_k a_l \rangle \\
&\quad - \mathcal{E}_j^* \left( \langle a_k a_l \sigma_+^{eg} \rangle + \xi \langle a_k a_l \sigma_+^{fe} \rangle \right) \\
&\quad - \mathcal{E}_k \left( \langle a_j^\dagger a_l \sigma_-^{eg} \rangle + \xi \langle a_j^\dagger a_l \sigma_-^{fe} \rangle \right) \\
&\quad - \mathcal{E}_l \left( \langle a_j^\dagger a_k \sigma_-^{eg} \rangle + \xi \langle a_j^\dagger a_k \sigma_-^{fe} \rangle \right), \tag{23a}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt}\langle a_j^\dagger a_k^\dagger a_l \rangle &= -(3\kappa - i(\omega_j + \omega_k - \omega_l))\langle a_j^\dagger a_k^\dagger a_l \rangle \\
&\quad - \mathcal{E}_j^* \left( \langle a_k^\dagger a_l \sigma_+^{eg} \rangle + \xi \langle a_k^\dagger a_l \sigma_+^{fe} \rangle \right) \\
&\quad - \mathcal{E}_k^* \left( \langle a_j^\dagger a_l \sigma_+^{eg} \rangle + \xi \langle a_j^\dagger a_l \sigma_+^{fe} \rangle \right) \\
&\quad - \mathcal{E}_l \left( \langle a_j^\dagger a_k^\dagger \sigma_-^{eg} \rangle + \xi \langle a_j^\dagger a_k^\dagger \sigma_-^{fe} \rangle \right), \tag{23b}
\end{aligned}$$

## 2.7 Fourth-Order: Cavity-Atom Coupled Equations

$$\begin{aligned}
\frac{d}{dt}\langle a_j^\dagger a_k a_l \sigma \rangle &= \mathbf{M}_{j,k,l}^{(a^\dagger a a)} \langle a_j^\dagger a_k a_l \sigma \rangle + \\
&\quad \left( \begin{aligned}
& -\mathcal{E}_j^* \langle a_k a_l \sigma_+^{eg} \rangle - \mathcal{E}_k \langle a_j^\dagger a_l \sigma_-^{eg} \rangle - \mathcal{E}_l \langle a_j^\dagger a_k \sigma_-^{eg} \rangle \\
& -\mathcal{E}_j^* \langle a_k a_l \sigma^{ee} \rangle - \mathcal{E}_k \xi \langle a_j^\dagger a_l \sigma_-^{fg} \rangle - \mathcal{E}_l \xi \langle a_j^\dagger a_k \sigma_-^{fg} \rangle \\
& -\mathcal{E}_j^* \xi \langle a_k a_l \sigma_+^{fg} \rangle - \mathcal{E}_k \langle a_j^\dagger a_l \sigma^{ee} \rangle - \mathcal{E}_l \langle a_j^\dagger a_k \sigma^{ee} \rangle \\
& \Gamma \xi^2 \langle a_j^\dagger a_k a_l \rangle - \mathcal{E}_j^* \xi \langle a_k a_l \sigma_+^{fe} \rangle - \mathcal{E}_k \xi \langle a_j^\dagger a_l \sigma_-^{fe} \rangle - \mathcal{E}_l \xi \langle a_j^\dagger a_k \sigma_-^{fe} \rangle \\
& i\xi \frac{\Omega}{2} \langle a_j^\dagger a_k a_l \rangle - \mathcal{E}_j^* \xi (\langle a_k a_l \rangle - \langle a_k a_l \sigma^{gg} \rangle - \langle a_k a_l \sigma^{ee} \rangle) \\
& -i\xi \frac{\Omega}{2} \langle a_j^\dagger a_k a_l \rangle - \mathcal{E}_k \xi (\langle a_j^\dagger a_k \rangle - \langle a_j^\dagger a_l \sigma^{gg} \rangle - \langle a_j^\dagger a_l \sigma^{ee} \rangle) - \mathcal{E}_l \xi (\langle a_j^\dagger a_l \rangle - \langle a_j^\dagger a_k \sigma^{gg} \rangle - \langle a_j^\dagger a_k \sigma^{ee} \rangle) \\
& -\mathcal{E}_j^* \langle a_k a_l \sigma_-^{fe} \rangle \\
& -\mathcal{E}_k \langle a_j^\dagger a_l \sigma_+^{fe} \rangle - \mathcal{E}_l \langle a_j^\dagger a_k \sigma_-^{fe} \rangle
\end{aligned} \right), \tag{24a}
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{dt}\langle a_j^\dagger a_k^\dagger a_l \sigma \rangle &= \mathbf{M}_{j,k,l}^{(a^\dagger a^\dagger a)} \langle a_j^\dagger a_k^\dagger a_l \sigma \rangle + \\
&\quad \left( \begin{aligned}
& -\mathcal{E}_j^* \langle a_k^\dagger a_l \sigma_+^{eg} \rangle - \mathcal{E}_k^* \langle a_j^\dagger a_l \sigma_+^{eg} \rangle - \mathcal{E}_l \langle a_j^\dagger a_k^\dagger \sigma_-^{eg} \rangle \\
& -\mathcal{E}_j^* \langle a_k^\dagger a_l \sigma^{ee} \rangle - \mathcal{E}_k^* \langle a_j^\dagger a_l \sigma^{ee} \rangle - \mathcal{E}_l \xi \langle a_j^\dagger a_k^\dagger \sigma_-^{fg} \rangle \\
& -\mathcal{E}_j^* \xi \langle a_k^\dagger a_l \sigma_+^{fg} \rangle - \mathcal{E}_k^* \xi \langle a_j^\dagger a_l \sigma_+^{fg} \rangle - \mathcal{E}_l \langle a_j^\dagger a_k^\dagger \sigma^{ee} \rangle \\
& \Gamma \xi^2 \langle a_j^\dagger a_k^\dagger a_l \rangle - \mathcal{E}_j^* \xi \langle a_k^\dagger a_l \sigma_+^{fe} \rangle - \mathcal{E}_k^* \xi \langle a_j^\dagger a_l \sigma_+^{fe} \rangle - \mathcal{E}_l \xi \langle a_j^\dagger a_k^\dagger \sigma_-^{fe} \rangle \\
& i\xi \frac{\Omega}{2} \langle a_j^\dagger a_k^\dagger a_l \rangle - \mathcal{E}_j^* \xi (\langle a_k^\dagger a_l \rangle - \langle a_k^\dagger a_l \sigma^{gg} \rangle - \langle a_k^\dagger a_l \sigma^{ee} \rangle) - \mathcal{E}_k^* \xi (\langle a_j^\dagger a_l \rangle - \langle a_j^\dagger a_l \sigma^{gg} \rangle - \langle a_j^\dagger a_l \sigma^{ee} \rangle) \\
& -i\xi \frac{\Omega}{2} \langle a_j^\dagger a_k^\dagger a_l \rangle - \mathcal{E}_l \xi (\langle a_j^\dagger a_k^\dagger \rangle - \langle a_j^\dagger a_k^\dagger \sigma^{gg} \rangle - \langle a_j^\dagger a_k^\dagger \sigma^{ee} \rangle) \\
& -\mathcal{E}_j^* \langle a_k^\dagger a_l \sigma_-^{fe} \rangle - \mathcal{E}_k^* \langle a_j^\dagger a_l \sigma_-^{fe} \rangle \\
& -\mathcal{E}_l \langle a_j^\dagger a_k^\dagger \sigma_+^{fe} \rangle
\end{aligned} \right), \tag{24b}
\end{aligned}$$

where

$$\mathbf{M}_{j,k,l}^{(a^\dagger aa)} = \mathbf{M} - (3\kappa - i\omega_j + i(\omega_k + \omega_l)) \mathbb{1}, \quad (25a)$$

$$\mathbf{M}_{j,k,l}^{(a^\dagger a^\dagger a)} = \mathbf{M} - (3\kappa - i(\omega_j + \omega_k) - i\omega_l) \mathbb{1}. \quad (25b)$$

## 2.8 Fourth-Order: Cavity Equation

Finally, for the fourth-order cavity moment equation, we have

$$\begin{aligned} \frac{d}{dt} \langle a_j^\dagger a_k^\dagger a_l a_m \rangle = & -(4\kappa - i(\omega_j + \omega_k) + i(\omega_l + \omega_m)) \langle a_j^\dagger a_k^\dagger a_l a_m \rangle \\ & - \mathcal{E}_j^* \left( \langle a_k^\dagger a_l a_m \sigma_+^{eg} \rangle + \xi \langle a_k^\dagger a_l a_m \sigma_+^{fe} \rangle \right) \\ & - \mathcal{E}_k^* \left( \langle a_j^\dagger a_l a_m \sigma_+^{eg} \rangle + \xi \langle a_j^\dagger a_l a_m \sigma_+^{fe} \rangle \right) \\ & - \mathcal{E}_l \left( \langle a_j^\dagger a_k^\dagger a_m \sigma_-^{eg} \rangle + \xi \langle a_j^\dagger a_k^\dagger a_m \sigma_-^{fe} \rangle \right) \\ & - \mathcal{E}_m \left( \langle a_j^\dagger a_k^\dagger a_l \sigma_-^{eg} \rangle + \xi \langle a_j^\dagger a_k^\dagger a_l \sigma_-^{fe} \rangle \right). \end{aligned} \quad (26)$$

## 3 First-Order Correlation Function

The first-order correlation function for the filtered output field in the steady state is given by

$$G^{(1)}(\tau) = \langle A^\dagger(\tau) A(0) \rangle = \sum_{j=-N}^N \langle a_j^\dagger(\tau) A(0) \rangle. \quad (27)$$

Moving into the steady state, we use the Quantum Regression Theorem to solve for this with the moment equations

$$\frac{d}{d\tau} \langle \boldsymbol{\sigma}(\tau) A(0) \rangle = \mathbf{M} \langle \boldsymbol{\sigma}(\tau) A(0) \rangle + \mathbf{B}, \quad (28a)$$

$$\frac{d}{d\tau} \langle a_j^\dagger(\tau) A(0) \rangle = -(\kappa - i\omega_j) \langle a_j^\dagger A(0) \rangle - \mathcal{E}_j^* \left( \langle \sigma_+^{eg}(\tau) A(0) \rangle + \xi \langle \sigma_+^{fe}(\tau) A(0) \rangle \right), \quad (28b)$$

where

$$\langle \boldsymbol{\sigma} \rangle = \begin{pmatrix} \langle \sigma^{gg}(\tau) A(0) \rangle \\ \langle \sigma_-^{eg}(\tau) A(0) \rangle \\ \langle \sigma_+^{eg}(\tau) A(0) \rangle \\ \langle \sigma^{ee}(\tau) A(0) \rangle \\ \langle \sigma_-^{fe}(\tau) A(0) \rangle \\ \langle \sigma_+^{fe}(\tau) A(0) \rangle \\ \langle \sigma_-^{fg}(\tau) A(0) \rangle \\ \langle \sigma_+^{fg}(\tau) A(0) \rangle \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \Gamma \xi^2 \langle A \rangle_{ss} \\ i\xi \frac{\Omega}{2} \langle A \rangle_{ss} \\ -i\xi \frac{\Omega}{2} \langle A \rangle_{ss} \\ 0 \\ 0 \end{pmatrix}, \quad (29)$$

## 4 Second-Order Correlation Function

The second-order correlation function for the filtered output field in the steady state is given by

$$G^{(2)}(\tau) = \langle A^\dagger(0) A^\dagger(\tau) A(0) \rangle = \sum_{k,l=-N}^N \langle A^\dagger(0) a_k^\dagger a_l(\tau) A(0) \rangle, \quad (30)$$

with the normalised second-order correlation function given by

$$g^{(2)}(\tau) = \frac{G^{(2)}(\tau)}{\langle A^\dagger A \rangle_{ss}^2}. \quad (31)$$

Using the Quantum Regression Theorem, we can solve for the second-order correlation function by solving the following moment equations

$$\frac{d}{d\tau} \langle A^\dagger(0) \boldsymbol{\sigma} A(0) \rangle = \mathbf{M} \langle A^\dagger(0) \boldsymbol{\sigma} A(0) \rangle + \mathbf{B}, \quad (32a)$$

$$\frac{d}{d\tau} \langle A^\dagger(0) a_j^\dagger(\tau) A(0) \rangle = -(\kappa - i\omega_j) \langle A^\dagger(0) a_j^\dagger(\tau) A(0) \rangle - \mathcal{E}_j^* \left( \langle A^\dagger(0) \sigma_+^{eg}(\tau) A(0) \rangle + \xi \langle A^\dagger(0) \sigma_+^{fe} A(0) \rangle \right), \quad (32b)$$

$$\frac{d}{d\tau} \langle A^\dagger(0) a_j(\tau) A(0) \rangle = -(\kappa + i\omega_j) \langle A^\dagger(0) a_j(\tau) A(0) \rangle - \mathcal{E}_j \left( \langle A^\dagger(0) \sigma_-^{eg}(\tau) A(0) \rangle + \xi \langle A^\dagger(0) \sigma_-^{fe} A(0) \rangle \right), \quad (32c)$$

with

$$\langle \boldsymbol{\sigma} \rangle = \begin{pmatrix} \langle A^\dagger(0) \sigma^{gg}(\tau) A(0) \rangle \\ \langle A^\dagger(0) \sigma_-^{eg}(\tau) A(0) \rangle \\ \langle A^\dagger(0) \sigma_+^{eg}(\tau) A(0) \rangle \\ \langle A^\dagger(0) \sigma^{ee}(\tau) A(0) \rangle \\ \langle A^\dagger(0) \sigma_-^{fe}(\tau) A(0) \rangle \\ \langle A^\dagger(0) \sigma_+^{fe}(\tau) A(0) \rangle \\ \langle A^\dagger(0) \sigma_-^{fg}(\tau) A(0) \rangle \\ \langle A^\dagger(0) \sigma_+^{fg}(\tau) A(0) \rangle \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \Gamma \xi^2 \langle A^\dagger A \rangle_{ss} \\ i\xi \frac{\Omega}{2} \langle A^\dagger A \rangle_{ss} \\ -i\xi \frac{\Omega}{2} \langle A^\dagger A \rangle_{ss} \\ 0 \\ 0 \end{pmatrix}. \quad (33)$$

We also need to solve

$$\begin{aligned} \frac{d}{d\tau} \langle A^\dagger(0) a_j \boldsymbol{\sigma}(\tau) A(0) \rangle &= \mathbf{M}_j^{(a)} \langle A^\dagger(0) a_j \boldsymbol{\sigma}(\tau) A(0) \rangle \\ &+ \begin{pmatrix} -\mathcal{E}_j \langle A^\dagger(0) \sigma_-^{eg}(\tau) A(0) \rangle \\ -\mathcal{E}_j \xi \langle A^\dagger(0) \sigma_-^{fg}(\tau) A(0) \rangle \\ -\mathcal{E}_j \langle A^\dagger(0) \sigma^{ee}(\tau) A(0) \rangle \\ \Gamma \xi^2 \langle A^\dagger(0) a_j(\tau) A(0) \rangle - \mathcal{E}_j \xi \langle A^\dagger(0) \sigma_-^{fe}(\tau) A(0) \rangle \\ i\xi \frac{\Omega}{2} \langle A^\dagger(0) a_j(\tau) A(0) \rangle \\ -i\xi \frac{\Omega}{2} \langle A^\dagger(0) a_j(\tau) A(0) \rangle - \mathcal{E}_j \xi (\langle A^\dagger A \rangle_{ss} - \langle A^\dagger(0) \sigma^{gg}(\tau) A(0) \rangle - \langle A^\dagger(0) \sigma^{ee}(\tau) A(0) \rangle) \\ 0 \\ -\mathcal{E}_j \langle A^\dagger(0) \sigma_+^{fe}(\tau) A(0) \rangle \end{pmatrix}, \end{aligned} \quad (34a)$$

and

$$\begin{aligned} \frac{d}{d\tau} \langle A^\dagger(0) a_j^\dagger \boldsymbol{\sigma}(\tau) A(0) \rangle &= \mathbf{M}_j^{(a^\dagger)} \langle A^\dagger(0) a_j^\dagger \boldsymbol{\sigma}(\tau) A(0) \rangle \\ &+ \begin{pmatrix} -\mathcal{E}_j^* \langle A^\dagger(0) \sigma_+^{eg}(\tau) A(0) \rangle \\ -\mathcal{E}_j^* \langle A^\dagger(0) \sigma^{ee}(\tau) A(0) \rangle \\ -\mathcal{E}_j^* \xi \langle A^\dagger(0) \sigma_+^{fg}(\tau) A(0) \rangle \\ \Gamma \xi^2 \langle A^\dagger(0) a_j^\dagger(\tau) A(0) \rangle - \mathcal{E}_j^* \xi \langle A^\dagger(0) \sigma_+^{fe}(\tau) A(0) \rangle \\ i\xi \frac{\Omega}{2} \langle A^\dagger(0) a_j^\dagger(\tau) A(0) \rangle - \mathcal{E}_j^* \xi (\langle A^\dagger A \rangle_{ss} - \langle A^\dagger(0) \sigma^{gg}(\tau) A(0) \rangle - \langle A^\dagger(0) \sigma^{ee}(\tau) A(0) \rangle) \\ -i\xi \frac{\Omega}{2} \langle A^\dagger(0) a_j^\dagger(\tau) A(0) \rangle \\ -\mathcal{E}_j^* \langle A^\dagger(0) \sigma_-^{fe}(\tau) A(0) \rangle \\ 0 \end{pmatrix}, \end{aligned} \quad (34b)$$

$$(34c)$$

where

$$\mathbf{M}_j^{(a)} = \mathbf{M} - (\kappa + i\omega_j) \mathbb{1}, \quad (35a)$$

$$\mathbf{M}_j^{(a^\dagger)} = \mathbf{M} - (\kappa - i\omega_j) \mathbb{1}. \quad (35b)$$

Finally, we will also need to solve

$$\begin{aligned} \frac{d}{d\tau} \langle A^\dagger(0) a_k^\dagger a_l(\tau) A(0) \rangle &= - (2\kappa - i\omega_k + i\omega_l) \langle A^\dagger(0) a_k^\dagger a_l(\tau) A(0) \rangle \\ &\quad - \mathcal{E}_k^* \left( \langle A^\dagger(0) a_l \sigma_+^{eg}(\tau) A(0) \rangle + \xi \langle A^\dagger(0) a_l \sigma_+^{fe}(\tau) A(0) \rangle \right) \\ &\quad - \mathcal{E}_l \left( \langle A^\dagger(0) a_k^\dagger \sigma_-^{eg}(\tau) A(0) \rangle + \xi \langle A^\dagger(0) a_k^\dagger \sigma_-^{fe}(\tau) A(0) \rangle \right). \end{aligned} \quad (36)$$