Three-Level Atom: Two Multi-Mode Filters Moment Equations

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1 Hamiltonian and Master Equation

The Hamiltonian is

$$\begin{aligned} H = -\hbar\left(\frac{\alpha}{2}+\delta\right)|e\rangle\langle e| - 2\hbar\delta\,|f\rangle\langle f| + \hbar\frac{\Omega}{2}\left(\Sigma_{+}+\Sigma_{-}\right) + \hbar\sum_{j=-N}^{N}\left(\omega_{j}^{(a)}a_{j}^{\dagger}a_{j} + \omega_{j}^{(b)}b_{j}^{\dagger}b_{j}\right) \\ + \frac{i\hbar}{2}\sum_{j=-N}^{N}\left(\mathcal{E}_{j}^{(a)*}a_{j}\Sigma_{+} - \mathcal{E}_{j}^{(a)}\Sigma_{-}a_{j}^{\dagger}\right) + \frac{i\hbar}{2}\sum_{j=-N}^{N}\left(\mathcal{E}_{j}^{(b)*}b_{j}\Sigma_{+} - \mathcal{E}_{j}^{(b)}\Sigma_{-}b_{j}^{\dagger}\right) \end{aligned} \tag{1}$$

where Ω is the Rabi frequency,

$$\alpha = \omega_{fe} - \omega_{eg}, \tag{2}$$

is the atomic anharmonicity, δ is given by

$$2\delta = 2\omega_{d} - \omega_{fg}, \tag{3}$$

is the *drive detuning from two-photon resonance*

$$\Sigma_{-} = \sigma_{-}^{eg} + \xi\sigma_{-}^{fe} = |g\rangle\langle e| + \xi\,|e\rangle\langle f|\,, \quad \Sigma_{+} = \Sigma_{-}^{\dagger}, \tag{4}$$

is the atomic raising (lowering) operator, $a^{\dagger}$ ($b^{\dagger}$) and a (b) are the cavity photon creation and annihilation operators for filter A (B), N is the number of modes either side of the central mode ($2N+1$ total modes),

$$\omega_{j}^{(a)} = \omega_{0}^{(a)} + j\delta\omega^{(a)}, \quad \omega_{j}^{(b)} = \omega_{0}^{(b)} + j\delta\omega^{(b)} \tag{5}$$

is the resonance frequency of the j^{th}mode, with central frequency $\omega_{0}^{(a/b)}$ and mode frequency spacing $\delta\omega^{(a/b)}$, and

$$\mathcal{E}_{j}^{(a)} = \sqrt{\frac{\epsilon\gamma\kappa_{a}}{2\left(2N+1\right)}}e^{im\varphi_{j}}, \quad \mathcal{E}_{j}^{(b)} = \sqrt{\frac{\epsilon\gamma\kappa_{b}}{2\left(2N+1\right)}}e^{im\varphi_{j}}, \tag{6}$$

is the cascaded systems coupling of the j^{th}mode for filter A (B), where γ is the atomic decay rate, κ_{a} (κ_{b}) is the cavity decay rate for filter A (B), ϵ is the percentage of fluorescence sent to the filter, and

$$\varphi_{j} = \frac{j\pi}{N}, \tag{7}$$

sets the size of the frequency dependent time delay, with integer m.

The master equation is

$$\begin{aligned} \frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{1}{i\hbar}[H,\rho] + \frac{\gamma}{2}(1-\epsilon)\left(2\sigma_{-}\rho\sigma_{+} - \sigma_{+}\sigma_{-}\rho - \rho\sigma_{+}\sigma_{-}\right) \\ + \frac{\kappa_{a}}{2}\sum_{j=-N}^{N}\left(2a_{j}\rho a_{j}^{\dagger} - a_{j}^{\dagger}a_{j}\rho - \rho a_{j}^{\dagger}a_{j}\right) \end{aligned} \tag{8}$$

$$
\begin{aligned}
&+ \frac{1}{2} \sum_{j=-N}^{N} \left(2C_j^{(a)} \rho C_j^{(a)\dagger} - C_j^{(a)\dagger} C_j^{(a)} \rho - \rho C_j^{(a)\dagger} C_j^{(a)} \right) \\
&+ \frac{\kappa_b}{2} \sum_{j=-N}^{N} \left(2b_j \rho b_j^\dagger - b_j^\dagger b_j \rho - \rho b_j^\dagger b_j \right) \\
&+ \frac{1}{2} \sum_{j=-N}^{N} \left(2C_j^{(b)} \rho C_j^{(b)\dagger} - C_j^{(b)\dagger} C_j^{(b)} \rho - \rho C_j^{(b)\dagger} C_j^{(b)} \right),
\end{aligned} \tag{9}
$$

where

$$
C_j = \sqrt{\frac{\epsilon \gamma}{2N+1}} \sigma_- + \sqrt{\kappa} e^{im\varphi_j} a_j, \tag{10}
$$

is the cascaded systems decay operator. Expanding the master equation out into neat terms, we arrive at:

$$
\begin{aligned}
\frac{\mathrm{d}\rho}{\mathrm{d}t} = &\, i \left(\frac{\alpha}{2} + \delta \right) \left(\sigma^{ee} \rho - \rho \sigma^{ee} \right) + 2i\delta \left(\sigma^{ff} \rho - \rho \sigma^{ff} \right) - i \frac{\Omega}{2} \left(\Sigma_+ \rho - \rho \Sigma_+ \right) \\
&- i \frac{\Omega}{2} \left(\Sigma_- \rho - \rho \Sigma_- \right) + \frac{\Gamma}{2} \left(2\Sigma_- \rho \Sigma_+ - \Sigma_+ \Sigma_- \rho - \rho \Sigma_+ \Sigma_- \right) \\
&- i \sum_{j=-N}^{N} \omega_j^{(a)} \left(a_j^\dagger a_j \rho - \rho a_j^\dagger a_j \right) + \kappa_a \sum_{j=-N}^{N} \left(2a_j \rho a_j^\dagger - a_j^\dagger a_j \rho - \rho a_j^\dagger a_j \right) \\
&- \sum_{j=-N}^{N} \mathcal{E}_j^{(a)} \left(a_j^\dagger \sigma_- \rho - \sigma_- \rho a_j^\dagger \right) - \sum_{j=-N}^{N} \mathcal{E}_j^{(a)*} \left(\rho \sigma_+ a_j - a_j \rho \sigma_+ \right) \\
&- i \sum_{j=-N}^{N} \omega_j^{(b)} \left(b_j^\dagger b_j \rho - \rho b_j^\dagger b_j \right) + \kappa_b \sum_{j=-N}^{N} \left(2b_j \rho b_j^\dagger - b_j^\dagger b_j \rho - \rho b_j^\dagger b_j \right) \\
&- \sum_{j=-N}^{N} \mathcal{E}_j^{(b)} \left(b_j^\dagger \sigma_- \rho - \sigma_- \rho b_j^\dagger \right) - \sum_{j=-N}^{N} \mathcal{E}_j^{(b)*} \left(\rho \sigma_+ b_j - b_j \rho \sigma_+ \right).
\end{aligned} \tag{11}
$$

2 Operator Averages

2.1 First-Order: Atomic Equations

Rearranging the density operator equations, we can write the atomic moment equations in matrix form

$$
\frac{\mathrm{d}}{\mathrm{d}t} \langle \boldsymbol{\sigma} \rangle = \boldsymbol{M} \langle \boldsymbol{\sigma} \rangle + \boldsymbol{B}, \tag{12}
$$

where

$$
\langle \boldsymbol{\sigma} \rangle = \begin{pmatrix} \langle \sigma^{gg} \rangle \\ \langle \sigma_-^{eg} \rangle \\ \langle \sigma_+^{eg} \rangle \\ \langle \sigma^{ee} \rangle \\ \langle \sigma_-^{fe} \rangle \\ \langle \sigma_+^{fe} \rangle \\ \langle \sigma_-^{fg} \rangle \\ \langle \sigma_+^{fg} \rangle \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \Gamma \xi^2 \\ i\xi \frac{\Omega}{2} \\ -i\xi \frac{\Omega}{2} \\ 0 \\ 0 \end{pmatrix}, \tag{13}
$$

and

$$\boldsymbol{M} = \begin{pmatrix} 0 & -i\frac{\Omega}{2} & i\frac{\Omega}{2} & \Gamma & 0 & 0 & 0 & 0 \\ -i\frac{\Omega}{2} & -\left[\frac{\Gamma}{2}-i\left(\frac{\alpha}{2}+\delta\right)\right] & 0 & i\frac{\Omega}{2} & \Gamma\xi & 0 & -i\xi\frac{\Omega}{2} & 0 \\ i\frac{\Omega}{2} & 0 & -\left[\frac{\Gamma}{2}+i\left(\frac{\alpha}{2}+\delta\right)\right] & -i\frac{\Omega}{2} & 0 & \Gamma\xi & 0 & i\xi\frac{\Omega}{2} \\ -\Gamma\xi^2 & i\frac{\Omega}{2} & -i\frac{\Omega}{2} & -\Gamma\left(1+\xi^2\right) & -i\xi\frac{\Omega}{2} & i\xi\frac{\Omega}{2} & 0 & 0 \\ -i\xi\frac{\Omega}{2} & 0 & 0 & -i\xi\Omega & -\left[\frac{\Gamma}{2}\left(1+\xi^2\right)+i\left(\frac{\alpha}{2}-\delta\right)\right] & 0 & i\frac{\Omega}{2} & 0 \\ i\xi\frac{\Omega}{2} & 0 & 0 & i\xi\Omega & 0 & -\left[\frac{\Gamma}{2}\left(1+\xi^2\right)-i\left(\frac{\alpha}{2}-\delta\right)\right] & 0 & -i\frac{\Omega}{2} \\ 0 & -i\xi\frac{\Omega}{2} & 0 & 0 & i\frac{\Omega}{2} & 0 & -\left[\frac{\Gamma}{2}\xi^2-2i\delta\right] & 0 \\ 0 & 0 & i\xi\frac{\Omega}{2} & 0 & 0 & -i\frac{\Omega}{2} & 0 & -\left[\frac{\Gamma}{2}\xi^2+2i\delta\right] \end{pmatrix}. \tag{14}$$

This differential equation has solution

$$\langle\boldsymbol{\sigma}(t)\rangle = e^{\boldsymbol{M}t}\langle\boldsymbol{\sigma}(0)\rangle + \left(1 - e^{\boldsymbol{M}t}\right)\langle\boldsymbol{\sigma}\rangle_{ss}, \tag{15}$$

where

$$\langle\boldsymbol{\sigma}\rangle_{ss} = -\boldsymbol{M}^{-1}\boldsymbol{B}. \tag{16}$$

2.2 First-Order: Cavity Equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j\rangle = -\left(\kappa + i\omega_j\right)\langle a_j\rangle - \mathcal{E}_j\left(\langle\sigma_-^{eg}\rangle + \xi\langle\sigma_-^{fe}\rangle\right), \tag{17a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger\rangle = -\left(\kappa - i\omega_j\right)\langle a_j^\dagger\rangle - \mathcal{E}_j\left(\langle\sigma_+^{eg}\rangle + \xi\langle\sigma_+^{fe}\rangle\right), \tag{17b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle b_j\rangle = -\left(\kappa_b + i\omega_j^{(b)}\right)\langle b_j\rangle - \mathcal{E}_j^{(b)}\left(\langle\sigma_-^{eg}\rangle + \xi\langle\sigma_-^{fe}\rangle\right), \tag{17c}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle b_j^\dagger\rangle = -\left(\kappa_b - i\omega_j^{(b)}\right)\langle b_j^\dagger\rangle - \mathcal{E}_j^{(b)*}\left(\langle\sigma_+^{eg}\rangle + \xi\langle\sigma_+^{fe}\rangle\right). \tag{17d}$$

2.3 Second-Order: Cavity-Atom Coupled Equations

Using the vector notation, we have moment equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j\boldsymbol{\sigma}\rangle = \boldsymbol{M}_j^{(a)}\langle a_j\boldsymbol{\sigma}\rangle + \begin{pmatrix} -\mathcal{E}_j^{(a)}\langle\sigma_-^{eg}\rangle \\ -\mathcal{E}_j^{(a)}\xi\langle\sigma_-^{fg}\rangle \\ -\mathcal{E}_j^{(a)}\langle\sigma^{ee}\rangle \\ \Gamma\xi^2\langle a_j\rangle - \mathcal{E}_j^{(a)}\xi\langle\sigma_-^{fe}\rangle \\ i\xi\frac{\Omega}{2}\langle a_j\rangle \\ -i\xi\frac{\Omega}{2}\langle a_j\rangle - \mathcal{E}_j^{(a)}\xi\left(1 - \langle\sigma^{gg}\rangle - \langle\sigma^{ee}\rangle\right) \\ 0 \\ -\mathcal{E}_j^{(a)}\langle\sigma_+^{fe}\rangle \end{pmatrix}, \tag{18a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle b_j\boldsymbol{\sigma}\rangle = \boldsymbol{M}_j^{(b)}\langle b_j\boldsymbol{\sigma}\rangle + \begin{pmatrix} -\mathcal{E}_j^{(b)}\langle\sigma_-^{eg}\rangle \\ -\mathcal{E}_j^{(b)}\xi\langle\sigma_-^{fg}\rangle \\ -\mathcal{E}_j^{(b)}\langle\sigma^{ee}\rangle \\ \Gamma\xi^2\langle b_j\rangle - \mathcal{E}_j^{(b)}\xi\langle\sigma_-^{fe}\rangle \\ i\xi\frac{\Omega}{2}\langle b_j\rangle \\ -i\xi\frac{\Omega}{2}\langle b_j\rangle - \mathcal{E}_j^{(b)}\xi\left(1 - \langle\sigma^{gg}\rangle - \langle\sigma^{ee}\rangle\right) \\ 0 \\ -\mathcal{E}_j^{(b)}\langle\sigma_+^{fe}\rangle \end{pmatrix}, \tag{18b}$$

where

$$\boldsymbol{M}_j^{(a)} = \boldsymbol{M} - \left(\kappa_a + i\omega_j^{(a)}\right)\mathbb{1}, \tag{19a}$$

$$\boldsymbol{M}_j^{(b)} = \boldsymbol{M} - \left(\kappa_b + i\omega_j^{(b)}\right)\mathbb{1}; \tag{19b}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger \boldsymbol{\sigma}\rangle = \boldsymbol{M}_j^{(a^\dagger)}\langle a_j^\dagger \boldsymbol{\sigma}\rangle + \begin{pmatrix} -\mathcal{E}_j^{(a)*}\langle\sigma_+^{eg}\rangle \\ -\mathcal{E}_j^{(a)*}\langle\sigma^{ee}\rangle \\ -\mathcal{E}_j^{(a)*}\xi\langle\sigma_+^{fg}\rangle \\ \Gamma\xi^2\langle a_j^\dagger\rangle - \mathcal{E}_j^{(a)*}\xi\langle\sigma_+^{fe}\rangle \\ i\xi\frac{\Omega}{2}\langle a_j^\dagger\rangle - \mathcal{E}_j^{(a)*}\xi\left(1 - \langle\sigma^{gg}\rangle - \langle\sigma^{ee}\rangle\right) \\ -i\xi\frac{\Omega}{2}\langle a_j^\dagger\rangle \\ -\mathcal{E}_j^{(a)*}\langle\sigma_-^{fe}\rangle \\ 0 \end{pmatrix}, \tag{20a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle b_j^\dagger \boldsymbol{\sigma}\rangle = \boldsymbol{M}_j^{(b^\dagger)}\langle b_j^\dagger \boldsymbol{\sigma}\rangle + \begin{pmatrix} -\mathcal{E}_j^{(b)*}\langle\sigma_+^{eg}\rangle \\ -\mathcal{E}_j^{(b)*}\langle\sigma^{ee}\rangle \\ -\mathcal{E}_j^{(b)*}\xi\langle\sigma_+^{fg}\rangle \\ \Gamma\xi^2\langle b_j^\dagger\rangle - \mathcal{E}_j^{(b)*}\xi\langle\sigma_+^{fe}\rangle \\ i\xi\frac{\Omega}{2}\langle b_j^\dagger\rangle - \mathcal{E}_j^{(b)*}\xi\left(1 - \langle\sigma^{gg}\rangle - \langle\sigma^{ee}\rangle\right) \\ -i\xi\frac{\Omega}{2}\langle b_j^\dagger\rangle \\ -\mathcal{E}_j^{(b)*}\langle\sigma_-^{fe}\rangle \\ 0 \end{pmatrix}, \tag{20b}$$

$$\tag{20c}$$

where

$$\boldsymbol{M}_j^{(a^\dagger)} = \boldsymbol{M} - \left(\kappa_a - i\omega_j^{(a)}\right)\mathbb{1}, \tag{21a}$$

$$\boldsymbol{M}_j^{(b^\dagger)} = \boldsymbol{M} - \left(\kappa_b - i\omega_j^{(b)}\right)\mathbb{1}. \tag{21b}$$

2.4 Second-Order: Cavity Equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j b_k\rangle = -\left(\kappa_a + \kappa_b + i\left(\omega_j^{(a)} + \omega_k^{(b)}\right)\right)\langle a_j b_k\rangle - \mathcal{E}_j^{(a)}\langle b_k\sigma_-\rangle - \mathcal{E}_k^{(b)}\langle a_j\sigma_-\rangle, \tag{22a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger b_k^\dagger\rangle = -\left(\kappa_a + \kappa_b - i\left(\omega_j^{(a)} + \omega_k^{(b)}\right)\right)\langle a_j^\dagger b_k^\dagger\rangle - \mathcal{E}_j^{(a)*}\langle b_k^\dagger\sigma_+\rangle - \mathcal{E}_k^{(b)*}\langle a_j^\dagger\sigma_+\rangle, \tag{22b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger a_k\rangle = -\left(2\kappa_a - i\left(\omega_j^{(a)} - \omega_k^{(a)}\right)\right)\langle a_j^\dagger a_k\rangle - \mathcal{E}_j^{(a)*}\langle a_k\sigma_+\rangle - \mathcal{E}_k^{(a)}\langle a_j^\dagger\sigma_-\rangle, \tag{22c}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle b_j^\dagger b_k\rangle = -\left(2\kappa_b - i\left(\omega_j^{(b)} - \omega_k^{(b)}\right)\right)\langle b_j^\dagger b_k\rangle - \mathcal{E}_j^{(b)*}\langle b_k\sigma_+\rangle - \mathcal{E}_k^{(b)}\langle b_j^\dagger\sigma_-\rangle, \tag{22d}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger b_k\rangle = -\left(\kappa_a + \kappa_b - i\left(\omega_j^{(a)} - \omega_k^{(b)}\right)\right)\langle a_j^\dagger b_k\rangle - \mathcal{E}_j^{(a)*}\langle b_k\sigma_+\rangle - \mathcal{E}_k^{(b)}\langle a_j^\dagger\sigma_-\rangle, \tag{22e}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle b_j^\dagger a_k\rangle = -\left(\kappa_a + \kappa_b - i\left(\omega_j^{(b)} - \omega_k^{(a)}\right)\right)\langle b_j^\dagger a_k\rangle - \mathcal{E}_j^{(b)*}\langle a_k\sigma_+\rangle - \mathcal{E}_k^{(a)}\langle b_j^\dagger\sigma_-\rangle. \tag{22f}$$

2.5 Third-Order: Cavity-Atom Coupled Equations

Using the vector notation, we have moment equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j a_k \boldsymbol{\sigma}\rangle = \boldsymbol{M}^{(j,k)}\langle a_j a_k \boldsymbol{\sigma}\rangle + \begin{pmatrix} 0 \\ -\frac{1}{2}\mathcal{E}_j\left(\langle a_k\sigma_z\rangle + \langle a_k\rangle\right) - \frac{1}{2}\mathcal{E}_k\left(\langle a_j\sigma_z\rangle + \langle a_j\rangle\right) \\ -\gamma\langle a_j a_k\rangle + \mathcal{E}_j\langle a_k\sigma_-\rangle + \mathcal{E}_k\langle a_j\sigma_-\rangle \end{pmatrix}, \tag{23a}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger a_k^\dagger \boldsymbol{\sigma}\rangle = \boldsymbol{M}^{(j^*,k^*)}\langle a_j^\dagger a_k^\dagger \boldsymbol{\sigma}\rangle + \begin{pmatrix} -\frac{1}{2}\mathcal{E}_j^*\left(\langle a_k^\dagger\sigma_z\rangle + \langle a_k^\dagger\rangle\right) - \frac{1}{2}\mathcal{E}_k^*\left(\langle a_j^\dagger\sigma_z\rangle + \langle a_j^\dagger\rangle\right) \\ 0 \\ -\gamma\langle a_j^\dagger a_k^\dagger\rangle + \mathcal{E}_j^*\langle a_k^\dagger\sigma_+\rangle + \mathcal{E}_k^*\langle a_j^\dagger\sigma_+\rangle \end{pmatrix}, \tag{23b}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger a_k \boldsymbol{\sigma}\rangle = \boldsymbol{M}^{(j^*,k)}\langle a_j^\dagger a_k \boldsymbol{\sigma}\rangle + \begin{pmatrix} -\frac{1}{2}\mathcal{E}_j^*\left(\langle a_k\sigma_z\rangle + \langle a_k\rangle\right) \\ -\frac{1}{2}\mathcal{E}_k\left(\langle a_j^\dagger\sigma_z\rangle + \langle a_j^\dagger\rangle\right) \\ -\gamma\langle a_j^\dagger a_k\rangle + \mathcal{E}_j^*\langle a_k\sigma_+\rangle + \mathcal{E}_k\langle a_j^\dagger\sigma_-\rangle \end{pmatrix}, \tag{23c}$$

where

$$\boldsymbol{M}^{(j,k)} = \boldsymbol{M} - \left(2\kappa + i\left(\omega_j + \omega_k\right)\right)\mathbb{1}, \tag{24a}$$
$$\boldsymbol{M}^{(j^*,k^*)} = \boldsymbol{M} - \left(2\kappa - i\left(\omega_j + \omega_k\right)\right)\mathbb{1}, \tag{24b}$$
$$\boldsymbol{M}^{(j^*,k)} = \boldsymbol{M} - \left(2\kappa - i\left(\omega_j - \omega_k\right)\right)\mathbb{1}. \tag{24c}$$

2.6 Third-Order: Cavity Equations

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger a_k b_l\rangle = &- \left(2\kappa_a + \kappa_b - i\omega_j^{(a)} + i\left(\omega_k^{(a)} + \omega_l^{(b)}\right)\right)\langle a_j^\dagger a_k b_l\rangle \\ &- \mathcal{E}_j^{(a)*}\langle a_k b_l\sigma_+\rangle - \mathcal{E}_k^{(a)}\langle a_j^\dagger b_l\sigma_-\rangle - \mathcal{E}_l^{(b)}\langle a_j^\dagger a_k\sigma_-\rangle,\end{aligned} \tag{25a}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle b_j^\dagger b_k a_l\rangle = &- \left(\kappa_a + 2\kappa_b - i\omega_j^{(b)} + i\left(\omega_k^{(b)} + \omega_l^{(a)}\right)\right)\langle b_j^\dagger b_k a_l\rangle \\ &- \mathcal{E}_j^{(b)*}\langle b_k a_l\sigma_+\rangle - \mathcal{E}_k^{(b)}\langle b_j^\dagger a_l\sigma_-\rangle - \mathcal{E}_l^{(a)}\langle b_j^\dagger b_k\sigma_-\rangle,\end{aligned} \tag{25b}$$

and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle b_j^\dagger a_k^\dagger a_l\rangle = &- \left(2\kappa_a + \kappa_b - i\left(\omega_j^{(b)} + \omega_k^{(a)}\right) + i\omega_l^{(a)}\right)\langle b_j^\dagger a_k^\dagger a_l\rangle \\ &- \mathcal{E}_j^{(b)*}\langle a_k^\dagger a_l\sigma_+\rangle - \mathcal{E}_k^{(a)*}\langle b_j^\dagger a_l\sigma_+\rangle - \mathcal{E}_l^{(a)}\langle b_j^\dagger a_k^\dagger\sigma_-\rangle,\end{aligned} \tag{26a}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger b_k^\dagger b_l\rangle = &- \left(\kappa_a + 2\kappa_b - i\left(\omega_j^{(a)} + \omega_k^{(b)}\right) + i\omega_l^{(b)}\right)\langle a_j^\dagger b_k^\dagger b_l\rangle \\ &- \mathcal{E}_j^{(a)*}\langle b_k^\dagger b_l\sigma_+\rangle - \mathcal{E}_k^{(b)*}\langle a_j^\dagger b_l\sigma_+\rangle - \mathcal{E}_l^{(b)}\langle a_j^\dagger b_k^\dagger\sigma_-\rangle.\end{aligned} \tag{26b}$$

2.7 Fourth-Order: Cavity-Atom Coupled Equations

Using the vector notation, we have moment equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger a_k b_l \boldsymbol{\sigma}\rangle = \boldsymbol{M}_a^{(j^*,k,l)}\langle a_j^\dagger a_k b_l\rangle + \begin{pmatrix} -\frac{1}{2}\mathcal{E}_j^{(a)*}\left(\langle a_k b_l\sigma_z\rangle + \langle a_k b_l\rangle\right) \\ -\frac{1}{2}\mathcal{E}_k^{(a)}\left(\langle a_j^\dagger b_l\sigma_z\rangle + \langle a_j^\dagger b_l\rangle\right) - \frac{1}{2}\mathcal{E}_l^{(b)}\left(\langle a_j^\dagger a_k\sigma_z\rangle + \langle a_j^\dagger a_k\rangle\right) \\ -\gamma\langle a_j^\dagger a_k b_l\rangle + \mathcal{E}_j^{(a)*}\langle a_k b_l\sigma_+\rangle + \mathcal{E}_k^{(a)}\langle a_j^\dagger b_l\sigma_-\rangle + \mathcal{E}_l^{(b)}\langle a_j^\dagger a_k\sigma_-\rangle \end{pmatrix}, \tag{27a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle b_j^\dagger b_k a_l \boldsymbol{\sigma}\rangle = \boldsymbol{M}_b^{(j^*,k,l)}\langle b_j^\dagger b_k a_l\rangle + \begin{pmatrix} -\frac{1}{2}\mathcal{E}_j^{(b)*}\left(\langle b_k a_l \sigma_z\rangle + \langle b_k a_l\rangle\right) \\ -\frac{1}{2}\mathcal{E}_k^{(b)}\left(\langle b_j^\dagger a_l \sigma_z\rangle + \langle b_j^\dagger a_l\rangle\right) - \frac{1}{2}\mathcal{E}_l^{(a)}\left(\langle b_j^\dagger b_k \sigma_z\rangle + \langle b_j^\dagger b_k\rangle\right) \\ -\gamma\langle b_j^\dagger a_k a_l\rangle + \mathcal{E}_j^{(b)*}\langle b_k a_l \sigma_+\rangle + \mathcal{E}_k^{(b)}\langle b_j^\dagger a_l \sigma_-\rangle + \mathcal{E}_l^{(a)}\langle b_j^\dagger b_k \sigma_-\rangle \end{pmatrix}, \tag{27b}$$

where

$$\boldsymbol{M}_a^{(j^*,k,l)} = \boldsymbol{M} - \left(2\kappa_a + \kappa_b - i\omega_j^{(a)} + i\left(\omega_k^{(a)} + \omega_l^{(b)}\right)\right), \tag{28a}$$

$$\boldsymbol{M}_b^{(j^*,k,l)} = \boldsymbol{M} - \left(\kappa_a + 2\kappa_b - i\omega_j^{(b)} + i\left(\omega_k^{(b)} + \omega_l^{(a)}\right)\right); \tag{28b}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle b_j^\dagger a_k^\dagger a_l \boldsymbol{\sigma}\rangle = \boldsymbol{M}_a^{(j^*,k^*,l)}\langle b_j^\dagger a_k^\dagger a_l\rangle + \begin{pmatrix} -\frac{1}{2}\mathcal{E}_j^{(b)*}\left(\langle a_k^\dagger a_l \sigma_z\rangle + \langle a_k^\dagger a_l\rangle\right) - \frac{1}{2}\mathcal{E}_k^{(a)*}\left(\langle b_j^\dagger a_l \sigma_z\rangle + \langle b_j^\dagger a_l\rangle\right) \\ -\frac{1}{2}\mathcal{E}_l^{(a)}\left(\langle b_j^\dagger a_k^\dagger \sigma_z\rangle + \langle b_j^\dagger a_k^\dagger\rangle\right) \\ -\gamma\langle b_j^\dagger a_k^\dagger a_l\rangle + \mathcal{E}_j^{(b)*}\langle a_k^\dagger a_l \sigma_+\rangle + \mathcal{E}_k^{(a)*}\langle b_j^\dagger a_l \sigma_+\rangle + \mathcal{E}_l^{(a)}\langle b_j^\dagger a_k^\dagger \sigma_-\rangle \end{pmatrix}, \tag{29a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger b_k^\dagger b_l \boldsymbol{\sigma}\rangle = \boldsymbol{M}_b^{(j^*,k^*,l)}\langle a_j^\dagger b_k^\dagger b_l\rangle + \begin{pmatrix} -\frac{1}{2}\mathcal{E}_j^{(a)*}\left(\langle b_k^\dagger b_l \sigma_z\rangle + \langle b_k^\dagger b_l\rangle\right) - \frac{1}{2}\mathcal{E}_k^{(b)*}\left(\langle a_j^\dagger b_l \sigma_z\rangle + \langle a_j^\dagger b_l\rangle\right) \\ -\frac{1}{2}\mathcal{E}_l^{(b)}\left(\langle a_j^\dagger b_k^\dagger \sigma_z\rangle + \langle a_j^\dagger b_k^\dagger\rangle\right) \\ -\gamma\langle a_j^\dagger b_k^\dagger b_l\rangle + \mathcal{E}_j^{(a)*}\langle b_k^\dagger b_l \sigma_+\rangle + \mathcal{E}_k^{(b)*}\langle a_j^\dagger b_l \sigma_+\rangle + \mathcal{E}_l^{(b)}\langle a_j^\dagger b_k^\dagger \sigma_-\rangle \end{pmatrix}, \tag{29b}$$

where

$$\boldsymbol{M}_a^{(j^*,k^*,l)} = \boldsymbol{M} - \left(2\kappa_a + \kappa_b - i\left(\omega_j^{(b)} + \omega_k^{(a)}\right) - i\omega_l^{(a)}\right), \tag{30a}$$

$$\boldsymbol{M}_b^{(j^*,k^*,l)} = \boldsymbol{M} - \left(\kappa_a + 2\kappa_b - i\left(\omega_j^{(a)} + \omega_k^{(b)}\right) - i\omega_l^{(b)}\right). \tag{30b}$$

2.8 Fourth-Order: Cavity Equation

Finally, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger b_k^\dagger b_l a_m\rangle = &-\left(2\kappa_a + 2\kappa_b - i\left(\omega_j^{(a)} + \omega_k^{(b)}\right) + i\left(\omega_l^{(b)} + \omega_m^{(a)}\right)\right)\langle a_j^\dagger b_k^\dagger b_l a_m\rangle \\ &- \mathcal{E}_j^{(a)*}\langle b_k^\dagger b_l a_m \sigma_+\rangle - \mathcal{E}_k^{(b)*}\langle a_j^\dagger a_m b_l \sigma_+\rangle \\ &- \mathcal{E}_l^{(b)}\langle b_k^\dagger a_j^\dagger a_m \sigma_-\rangle - \mathcal{E}_m^{(a)}\langle a_j^\dagger b_k^\dagger b_l \sigma_-\rangle. \end{aligned} \tag{31}$$

3 Second-Order Cross-Correlation Function

The second-order correlation function for the filtered output field in the steady state is given by

$$G^{(2)}_{\text{cross}}(\tau) = \langle A^\dagger(0) B^\dagger B(\tau) A(0)\rangle = \sum_{j,k=-N}^{N}\langle A^\dagger(0) b_j^\dagger b_k(\tau) A(0)\rangle, \tag{32}$$

with the normalised second-order correlation function given by

$$g^{(2)}_{\text{cross}}(\tau) = \frac{G^{(2)}_{\text{cross}}(\tau)}{\langle A^\dagger A\rangle_{ss}\langle B^\dagger B\rangle_{ss}}. \tag{33}$$

Using the Quantum Regression Theorem, we can solve for the second-order correlation function by solving the following moment equations

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\langle A^\dagger(0)\boldsymbol{\sigma}A(0)\rangle = \boldsymbol{M}\langle A^\dagger(0)\boldsymbol{\sigma}A(0)\rangle + \boldsymbol{B}, \tag{34a}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\langle A^\dagger(0)b_j^\dagger(\tau)A(0)\rangle = -\left(\kappa_b - i\omega_j^{(b)}\right)\langle A^\dagger(0)b_j^\dagger(\tau)A(0)\rangle - \mathcal{E}_j^{(b)*}\langle A^\dagger(0)\sigma_+(\tau)A(0)\rangle, \tag{34b}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\langle A^\dagger(0)b_j(\tau)A(0)\rangle = -\left(\kappa_b + i\omega_j^{(b)}\right)\langle A^\dagger(0)b_j(\tau)A(0)\rangle - \mathcal{E}_j^{(b)}\langle A^\dagger(0)\sigma_-(\tau)A(0)\rangle, \tag{34c}$$

with

$$\langle\boldsymbol{\sigma}\rangle = \begin{pmatrix}\langle A^\dagger(0)\sigma_-(\tau)A(0)\rangle\\ \langle A^\dagger(0)\sigma_+(\tau)A(0)\rangle\\ \langle A^\dagger(0)\sigma_z(\tau)A(0)\rangle\end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix}0\\ 0\\ -\gamma\langle A^\dagger A\rangle_{ss}\end{pmatrix}. \tag{35}$$

We also need to solve

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\langle A^\dagger(0)b_j\boldsymbol{\sigma}(\tau)A(0)\rangle = \boldsymbol{M}^{(j)}\langle A^\dagger(0)b_j\boldsymbol{\sigma}(\tau)A(0)\rangle + \begin{pmatrix}0\\ -\frac{1}{2}\mathcal{E}_j^{(b)}\left(\langle A^\dagger(0)\sigma_z(\tau)A(0)\rangle + \langle A^\dagger A\rangle_{ss}\right)\\ -\gamma\langle A^\dagger(0)b_j(\tau)A(0)\rangle + \mathcal{E}_j^{(b)}\langle A^\dagger(0)\sigma_-(\tau)A(0)\rangle\end{pmatrix}, \tag{36a}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\langle A^\dagger(0)b_j^\dagger\boldsymbol{\sigma}(\tau)A(0)\rangle = \boldsymbol{M}^{(j^*)}\langle A^\dagger(0)b_j^\dagger\boldsymbol{\sigma}(\tau)A(0)\rangle + \begin{pmatrix}-\frac{1}{2}\mathcal{E}_j^{(b)*}\left(\langle A^\dagger(0)\sigma_z(\tau)A(0)\rangle + \langle A^\dagger A\rangle_{ss}\right)\\ 0\\ -\gamma\langle A^\dagger(0)b_j^\dagger(\tau)A(0)\rangle + \mathcal{E}_j^*\langle A^\dagger(0)\sigma_+(\tau)A(0)\rangle\end{pmatrix}. \tag{36b}$$

where

$$\boldsymbol{M}^{(j)} = \boldsymbol{M} - \left(\kappa_b + i\omega_j^{(b)}\right)\mathbb{1}, \tag{37a}$$

$$\boldsymbol{M}^{(j^*)} = \boldsymbol{M} - \left(\kappa_b - i\omega_j^{(b)}\right)\mathbb{1}. \tag{37b}$$

Finally, we will also need to solve

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\langle A^\dagger(0)b_k^\dagger b_l(\tau)A(0)\rangle = &-\left(2\kappa_b - i\left(\omega_k^{(b)} - \omega_l^{(b)}\right)\right)\langle A^\dagger(0)b_k^\dagger b_l(\tau)A(0)\rangle\\ &- \mathcal{E}_k^{(b)*}\langle A^\dagger(0)b_l\sigma_+(\tau)A(0)\rangle - \mathcal{E}_l^{(b)}\langle A^\dagger(0)b_k^\dagger\sigma_-(\tau)A(0)\rangle.\end{aligned} \tag{38}$$