

Multi-Mode Filter Equations

Jacob Ngaha

April 14, 2021

1 Hamiltonian and Master Equation

The Hamiltonian is

$$\begin{aligned}
 H = \hbar \frac{\Omega}{2} (\sigma_+ + \sigma_-) + \hbar \sum_{j=-N}^N \left(\omega_j^{(a)} a_j^\dagger a_j + \omega_j^{(b)} b_j^\dagger b_j \right) + \frac{i\hbar}{2} \sqrt{\frac{\epsilon\gamma\kappa_a}{2(2N+1)}} \sum_{j=-N}^N \left(e^{-im\varphi_j} a_j \sigma_+ - e^{im\varphi_j} \sigma_- a_j^\dagger \right) \\
 + \frac{i\hbar}{2} \sqrt{\frac{\epsilon\gamma\kappa_b}{2(2N+1)}} \sum_{j=-N}^N \left(e^{-im\varphi_j} b_j \sigma_+ - e^{im\varphi_j} \sigma_- b_j^\dagger \right)
 \end{aligned} \tag{1}$$

where Ω is the Rabi frequency, σ_+ and σ_- are the atomic raising and lowering operators, a^\dagger (b^\dagger) and a (b) are the cavity photon creation and annihilation operators for filter A (B), N is the number of modes either side of the central mode ($2N+1$ total modes),

$$\omega_j^{(a)} = \omega_0^{(a)} + j\delta\omega^{(a)}, \quad \omega_j^{(b)} = \omega_0^{(b)} + j\delta\omega^{(b)} \tag{2}$$

is the resonance frequency of the j^{th} mode, with central frequency $\omega_0^{(a/b)}$ and mode frequency spacing $\delta\omega^{(a/b)}$, and

$$\mathcal{E}_j^{(a)} = \sqrt{\frac{\epsilon\gamma\kappa_a}{2N+1}} e^{im\varphi_j}, \quad \mathcal{E}_j^{(b)} = \sqrt{\frac{\epsilon\gamma\kappa_b}{2N+1}} e^{im\varphi_j}, \tag{3}$$

is the cascaded systems coupling of the j^{th} mode for filter A (B), where γ is the atomic decay rate, κ_a (κ_b) is the cavity decay rate for filter A (B), ϵ is the percentage of fluorescence sent to the filter, and

$$\varphi_j = \frac{j\pi}{N}, \tag{4}$$

sets the size of the frequency dependent time delay, with integer m .

The master equation is

$$\begin{aligned}
 \frac{d\rho}{dt} = \frac{1}{i\hbar} [H, \rho] + \frac{\gamma}{2} (1 - \epsilon) (2\sigma_- \rho \sigma_+ - \sigma_+ \sigma_- \rho - \rho \sigma_+ \sigma_-) \\
 + \frac{\kappa_a}{2} \sum_{j=-N}^N \left(2a_j \rho a_j^\dagger - a_j^\dagger a_j \rho - \rho a_j^\dagger a_j \right)
 \end{aligned} \tag{5}$$

$$\begin{aligned}
 + \frac{1}{2} \sum_{j=-N}^N \left(2C_j^{(a)} \rho C_j^{(a)\dagger} - C_j^{(a)\dagger} C_j^{(a)} \rho - \rho C_j^{(a)\dagger} C_j^{(a)} \right) \\
 + \frac{\kappa_b}{2} \sum_{j=-N}^N \left(2b_j \rho b_j^\dagger - b_j^\dagger b_j \rho - \rho b_j^\dagger b_j \right)
 \end{aligned} \tag{6}$$

$$+ \frac{1}{2} \sum_{j=-N}^N \left(2C_j^{(b)} \rho C_j^{(b)\dagger} - C_j^{(b)\dagger} C_j^{(b)} \rho - \rho C_j^{(b)\dagger} C_j^{(b)} \right),$$

where

$$C_j = \sqrt{\frac{\epsilon\gamma}{2N+1}} \sigma_- + \sqrt{\kappa} e^{im\varphi_j} a_j, \quad (7)$$

is the cascaded systems decay operator. Expanding the master equation out into neat terms, we arrive at:

$$\begin{aligned} \frac{d\rho}{dt} = & -i\frac{\Omega}{2} (\sigma_+ \rho - \rho \sigma_+) - i\frac{\Omega}{2} (\sigma_- \rho - \rho \sigma_-) + \frac{\gamma}{2} (2\sigma_- \rho \sigma_+ - \sigma_+ \sigma_- \rho - \rho \sigma_+ \sigma_-) \\ & - i \sum_{j=-N}^N \omega_j^{(a)} \left(a_j^\dagger a_j \rho - \rho a_j^\dagger a_j \right) + \kappa_a \sum_{j=-N}^N \left(2a_j \rho a_j^\dagger - a_j^\dagger a_j \rho - \rho a_j^\dagger a_j \right) \\ & - \sum_{j=-N}^N \mathcal{E}_j^{(a)} \left(a_j^\dagger \sigma_- \rho - \sigma_- \rho a_j^\dagger \right) - \sum_{j=-N}^N \mathcal{E}_j^{(a)*} (\rho \sigma_+ a_j - a_j \rho \sigma_+) \\ & - i \sum_{j=-N}^N \omega_j^{(b)} \left(b_j^\dagger b_j \rho - \rho b_j^\dagger b_j \right) + \kappa_b \sum_{j=-N}^N \left(2b_j \rho b_j^\dagger - b_j^\dagger b_j \rho - \rho b_j^\dagger b_j \right) \\ & - \sum_{j=-N}^N \mathcal{E}_j^{(b)} \left(b_j^\dagger \sigma_- \rho - \sigma_- \rho b_j^\dagger \right) - \sum_{j=-N}^N \mathcal{E}_j^{(b)*} (\rho \sigma_+ b_j - b_j \rho \sigma_+). \end{aligned} \quad (8)$$

2 Operator Averages

2.1 First-Order: Atomic Equations

We may write the Bloch equations in matrix form:

$$\frac{d}{dt} \langle \boldsymbol{\sigma} \rangle = \mathbf{M} \langle \boldsymbol{\sigma} \rangle + \mathbf{M}, \quad (9)$$

where

$$\langle \boldsymbol{\sigma} \rangle = \begin{pmatrix} \langle \sigma_- \rangle \\ \langle \sigma_+ \rangle \\ \langle \sigma_z \rangle \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ -\gamma \end{pmatrix}, \quad (10)$$

and

$$\mathbf{M} = \begin{pmatrix} -\frac{\gamma}{2} & 0 & i\frac{\Omega}{2} \\ 0 & -\frac{\gamma}{2} & -i\frac{\Omega}{2} \\ i\Omega & -i\Omega & -\gamma \end{pmatrix}. \quad (11)$$

This differential equation has the solution

$$\langle \boldsymbol{\sigma}(t) \rangle = e^{\mathbf{M}t} (\langle \boldsymbol{\sigma}(0) \rangle - \langle \boldsymbol{\sigma} \rangle_{ss}) + \langle \boldsymbol{\sigma} \rangle_{ss}, \quad (12)$$

where

$$\langle \boldsymbol{\sigma} \rangle_{ss} = -\mathbf{M}^{-1} \mathbf{B} = \frac{1}{2\Omega^2 + \gamma^2} \begin{pmatrix} -i\gamma\Omega \\ i\gamma\Omega \\ -\gamma^2 \end{pmatrix}. \quad (13)$$

2.2 First-Order: Cavity Equations

$$\frac{d}{dt}\langle a_j \rangle = -\left(\kappa_a + i\omega_j^{(a)}\right)\langle a_j \rangle - \mathcal{E}_j^{(a)}\langle \sigma_- \rangle, \quad (14a)$$

$$\frac{d}{dt}\langle a_j^\dagger \rangle = -\left(\kappa_a - i\omega_j^{(a)}\right)\langle a_j^\dagger \rangle - \mathcal{E}_j^{(a)*}\langle \sigma_+ \rangle, \quad (14b)$$

$$\frac{d}{dt}\langle b_j \rangle = -\left(\kappa_b + i\omega_j^{(b)}\right)\langle b_j \rangle - \mathcal{E}_j^{(b)}\langle \sigma_- \rangle, \quad (14c)$$

$$\frac{d}{dt}\langle b_j^\dagger \rangle = -\left(\kappa_b - i\omega_j^{(b)}\right)\langle b_j^\dagger \rangle - \mathcal{E}_j^{(b)*}\langle \sigma_+ \rangle. \quad (14d)$$

2.3 Second-Order: Cavity-Atom Coupled Equations

Using the vector notation, we have moment equations

$$\frac{d}{dt}\langle a_j \boldsymbol{\sigma} \rangle = \mathbf{M}_j^{(a)}\langle a_j \boldsymbol{\sigma} \rangle + \begin{pmatrix} 0 \\ -\frac{1}{2}\mathcal{E}_j^{(a)}(\langle \sigma_z \rangle + 1) \\ -\gamma\langle a_j \rangle + \mathcal{E}_j^{(a)}\langle \sigma_- \rangle \end{pmatrix}, \quad (15a)$$

and

$$\frac{d}{dt}\langle a_j^\dagger \boldsymbol{\sigma} \rangle = \mathbf{M}_{j*}^{(a)}\langle a_j^\dagger \boldsymbol{\sigma} \rangle + \begin{pmatrix} -\frac{1}{2}\mathcal{E}_j^{(a)*}(\langle \sigma_z \rangle + 1) \\ 0 \\ -\gamma\langle a_j^\dagger \rangle + \mathcal{E}_j^{(a)*}\langle \sigma_+ \rangle \end{pmatrix}. \quad (15b)$$

where

$$\mathbf{M}_j^{(a)} = \mathbf{M} - \left(\kappa_a + i\omega_j^{(a)}\right)\mathbb{1}, \quad \mathbf{M}_{j*}^{(a)} = \mathbf{M} - \left(\kappa_a - i\omega_j^{(a)}\right)\mathbb{1}, \quad (16a)$$

$$\mathbf{M}_j^{(b)} = \mathbf{M} - \left(\kappa_b + i\omega_j^{(b)}\right)\mathbb{1}, \quad \mathbf{M}_{j*}^{(b)} = \mathbf{M} - \left(\kappa_b - i\omega_j^{(b)}\right)\mathbb{1}, \quad (16b)$$

$$(16c)$$

2.4 Second-Order: Cavity Equations

$$\frac{d}{dt}\langle a_j b_k \rangle = -\left(\kappa_a + \kappa_b + i\left(\omega_j^{(a)} + \omega_k^{(b)}\right)\right)\langle a_j b_k \rangle - \mathcal{E}_j^{(a)}\langle b_k \sigma_- \rangle - \mathcal{E}_k^{(b)}\langle a_j \sigma_- \rangle, \quad (17a)$$

$$\frac{d}{dt}\langle a_j^\dagger b_k^\dagger \rangle = -\left(\kappa_a + \kappa_b - i\left(\omega_j^{(a)} + \omega_k^{(b)}\right)\right)\langle a_j^\dagger b_k^\dagger \rangle - \mathcal{E}_j^{(a)*}\langle b_k^\dagger \sigma_+ \rangle - \mathcal{E}_k^{(b)*}\langle a_j^\dagger \sigma_+ \rangle, \quad (17b)$$

$$\frac{d}{dt}\langle a_j^\dagger a_k \rangle = -\left(2\kappa_a - i\left(\omega_j^{(a)} - \omega_k^{(a)}\right)\right)\langle a_j^\dagger a_k \rangle - \mathcal{E}_j^{(a)*}\langle a_k \sigma_+ \rangle - \mathcal{E}_k^{(a)}\langle a_j^\dagger \sigma_- \rangle, \quad (17c)$$

$$\frac{d}{dt}\langle b_j^\dagger b_k \rangle = -\left(2\kappa_b - i\left(\omega_j^{(b)} - \omega_k^{(b)}\right)\right)\langle b_j^\dagger b_k \rangle - \mathcal{E}_j^{(b)*}\langle b_k \sigma_+ \rangle - \mathcal{E}_k^{(b)}\langle b_j^\dagger \sigma_- \rangle, \quad (17d)$$

$$\frac{d}{dt}\langle a_j^\dagger b_k \rangle = -\left(\kappa_a + \kappa_b - i\left(\omega_j^{(a)} - \omega_k^{(b)}\right)\right)\langle a_j^\dagger b_k \rangle - \mathcal{E}_j^{(a)*}\langle b_k \sigma_+ \rangle - \mathcal{E}_k^{(b)}\langle a_j^\dagger \sigma_- \rangle, \quad (17e)$$

$$\frac{d}{dt}\langle b_j^\dagger a_k \rangle = -\left(\kappa_a + \kappa_b - i\left(\omega_j^{(b)} - \omega_k^{(a)}\right)\right)\langle b_j^\dagger a_k \rangle - \mathcal{E}_j^{(b)*}\langle a_k \sigma_+ \rangle - \mathcal{E}_k^{(a)}\langle b_j^\dagger \sigma_- \rangle. \quad (17f)$$

2.5 Third-Order: Cavity-Atom Coupled Equations

Using the vector notation, we have moment equations

$$\frac{d}{dt}\langle a_j a_k \sigma \rangle = \mathbf{M}^{(j,k)} \langle a_j a_k \sigma \rangle + \begin{pmatrix} 0 \\ -\frac{1}{2}\mathcal{E}_j (\langle a_k \sigma_z \rangle + \langle a_k \rangle) - \frac{1}{2}\mathcal{E}_k (\langle a_j \sigma_z \rangle + \langle a_j \rangle) \\ -\gamma \langle a_j a_k \rangle + \mathcal{E}_j \langle a_k \sigma_- \rangle + \mathcal{E}_k \langle a_j \sigma_- \rangle \end{pmatrix}, \quad (18a)$$

and

$$\frac{d}{dt}\langle a_j^\dagger a_k^\dagger \sigma \rangle = \mathbf{M}^{(j^*,k^*)} \langle a_j^\dagger a_k^\dagger \sigma \rangle + \begin{pmatrix} -\frac{1}{2}\mathcal{E}_j^* (\langle a_k^\dagger \sigma_z \rangle + \langle a_k^\dagger \rangle) - \frac{1}{2}\mathcal{E}_k^* (\langle a_j^\dagger \sigma_z \rangle + \langle a_j^\dagger \rangle) \\ 0 \\ -\gamma \langle a_j^\dagger a_k^\dagger \rangle + \mathcal{E}_j^* \langle a_k^\dagger \sigma_+ \rangle + \mathcal{E}_k^* \langle a_j^\dagger \sigma_+ \rangle \end{pmatrix}, \quad (18b)$$

and

$$\frac{d}{dt}\langle a_j^\dagger a_k \sigma \rangle = \mathbf{M}^{(j^*,k)} \langle a_j^\dagger a_k \sigma \rangle + \begin{pmatrix} -\frac{1}{2}\mathcal{E}_j^* (\langle a_k \sigma_z \rangle + \langle a_k \rangle) \\ -\frac{1}{2}\mathcal{E}_k (\langle a_j^\dagger \sigma_z \rangle + \langle a_j^\dagger \rangle) \\ -\gamma \langle a_j^\dagger a_k \rangle + \mathcal{E}_j^* \langle a_k \sigma_+ \rangle + \mathcal{E}_k \langle a_j^\dagger \sigma_- \rangle \end{pmatrix}, \quad (18c)$$

where

$$\mathbf{M}^{(j,k)} = \mathbf{M} - (2\kappa + i(\omega_j + \omega_k)) \mathbb{1}, \quad (19a)$$

$$\mathbf{M}^{(j^*,k^*)} = \mathbf{M} - (2\kappa - i(\omega_j + \omega_k)) \mathbb{1}, \quad (19b)$$

$$\mathbf{M}^{(j^*,k)} = \mathbf{M} - (2\kappa - i(\omega_j - \omega_k)) \mathbb{1}. \quad (19c)$$

2.6 Third-Order: Cavity Equations

$$\begin{aligned} \frac{d}{dt}\langle a_j^\dagger a_k b_l \rangle &= - \left(2\kappa_a + \kappa_b - i\omega_j^{(a)} + i(\omega_k^{(a)} + \omega_l^{(b)}) \right) \langle a_j^\dagger a_k b_l \rangle \\ &\quad - \mathcal{E}_j^{(a)*} \langle a_k b_l \sigma_+ \rangle - \mathcal{E}_k^{(a)} \langle a_j^\dagger b_l \sigma_- \rangle - \mathcal{E}_l^{(b)} \langle a_j^\dagger a_k \sigma_- \rangle, \end{aligned} \quad (20a)$$

$$\begin{aligned} \frac{d}{dt}\langle b_j^\dagger b_k a_l \rangle &= - \left(\kappa_a + 2\kappa_b - i\omega_j^{(b)} + i(\omega_k^{(b)} + \omega_l^{(a)}) \right) \langle b_j^\dagger b_k a_l \rangle \\ &\quad - \mathcal{E}_j^{(b)*} \langle b_k a_l \sigma_+ \rangle - \mathcal{E}_k^{(b)} \langle b_j^\dagger a_l \sigma_- \rangle - \mathcal{E}_l^{(a)} \langle b_j^\dagger b_k \sigma_- \rangle, \end{aligned} \quad (20b)$$

and

$$\begin{aligned} \frac{d}{dt}\langle b_j^\dagger a_k^\dagger a_l \rangle &= - \left(2\kappa_a + \kappa_b - i(\omega_j^{(b)} + \omega_k^{(a)}) + i\omega_l^{(a)} \right) \langle b_j^\dagger a_k^\dagger a_l \rangle \\ &\quad - \mathcal{E}_j^{(b)*} \langle a_k^\dagger a_l \sigma_+ \rangle - \mathcal{E}_k^{(a)*} \langle b_j^\dagger a_l \sigma_+ \rangle - \mathcal{E}_l^{(a)} \langle b_j^\dagger a_k^\dagger \sigma_- \rangle, \end{aligned} \quad (21a)$$

$$\begin{aligned} \frac{d}{dt}\langle a_j^\dagger b_k^\dagger b_l \rangle &= - \left(\kappa_a + 2\kappa_b - i(\omega_j^{(a)} + \omega_k^{(b)}) + i\omega_l^{(b)} \right) \langle a_j^\dagger b_k^\dagger b_l \rangle \\ &\quad - \mathcal{E}_j^{(a)*} \langle b_k^\dagger b_l \sigma_+ \rangle - \mathcal{E}_k^{(b)*} \langle a_j^\dagger b_l \sigma_+ \rangle - \mathcal{E}_l^{(b)} \langle a_j^\dagger b_k^\dagger \sigma_- \rangle. \end{aligned} \quad (21b)$$

2.7 Fourth-Order: Cavity-Atom Coupled Equations

Using the vector notation, we have moment equations

$$\frac{d}{dt}\langle a_j^\dagger a_k b_l \sigma \rangle = \mathbf{M}_a^{(j^*,k,l)} \langle a_j^\dagger a_k b_l \sigma \rangle + \begin{pmatrix} -\frac{1}{2}\mathcal{E}_j^{(a)*} (\langle a_k b_l \sigma_z \rangle + \langle a_k b_l \rangle) \\ -\frac{1}{2}\mathcal{E}_k^{(a)} (\langle a_j^\dagger b_l \sigma_z \rangle + \langle a_j^\dagger b_l \rangle) - \frac{1}{2}\mathcal{E}_l^{(b)} (\langle a_j^\dagger a_k \sigma_z \rangle + \langle a_j^\dagger a_k \rangle) \\ -\gamma \langle a_j^\dagger a_k b_l \rangle + \mathcal{E}_j^{(a)*} \langle a_k b_l \sigma_+ \rangle + \mathcal{E}_k^{(a)} \langle a_j^\dagger b_l \sigma_- \rangle + \mathcal{E}_l^{(b)} \langle a_j^\dagger a_k \sigma_- \rangle \end{pmatrix}, \quad (22a)$$

$$\frac{d}{dt}\langle b_j^\dagger b_k a_l \sigma \rangle = \mathbf{M}_b^{(j^*, k, l)} \langle b_j^\dagger b_k a_l \rangle + \begin{pmatrix} -\frac{1}{2} \mathcal{E}_j^{(b)*} (\langle b_k a_l \sigma_z \rangle + \langle b_k a_l \rangle) \\ -\frac{1}{2} \mathcal{E}_k^{(b)} (\langle b_j^\dagger a_l \sigma_z \rangle + \langle b_j^\dagger a_l \rangle) - \frac{1}{2} \mathcal{E}_l^{(a)} (\langle b_j^\dagger b_k \sigma_z \rangle + \langle b_j^\dagger b_k \rangle) \\ -\gamma \langle b_j^\dagger a_k a_l \rangle + \mathcal{E}_j^{(b)*} \langle b_k a_l \sigma_+ \rangle + \mathcal{E}_k^{(b)} \langle b_j^\dagger a_l \sigma_- \rangle + \mathcal{E}_l^{(a)} \langle b_j^\dagger b_k \sigma_- \rangle \end{pmatrix}, \quad (22b)$$

where

$$\mathbf{M}_a^{(j^*, k, l)} = \mathbf{M} - \left(2\kappa_a + \kappa_b - i\omega_j^{(a)} + i(\omega_k^{(a)} + \omega_l^{(b)}) \right), \quad (23a)$$

$$\mathbf{M}_b^{(j^*, k, l)} = \mathbf{M} - \left(\kappa_a + 2\kappa_b - i\omega_j^{(b)} + i(\omega_k^{(b)} + \omega_l^{(a)}) \right); \quad (23b)$$

and

$$\frac{d}{dt}\langle b_j^\dagger a_k^\dagger a_l \sigma \rangle = \mathbf{M}_a^{(j^*, k^*, l)} \langle b_j^\dagger a_k^\dagger a_l \rangle + \begin{pmatrix} -\frac{1}{2} \mathcal{E}_j^{(b)*} (\langle a_k^\dagger a_l \sigma_z \rangle + \langle a_k^\dagger a_l \rangle) - \frac{1}{2} \mathcal{E}_k^{(a)*} (\langle b_j^\dagger a_l \sigma_z \rangle + \langle b_j^\dagger a_l \rangle) \\ -\frac{1}{2} \mathcal{E}_l^{(a)} (\langle b_j^\dagger a_k^\dagger \sigma_z \rangle + \langle b_j^\dagger a_k^\dagger \rangle) \\ -\gamma \langle b_j^\dagger a_k^\dagger a_l \rangle + \mathcal{E}_j^{(b)*} \langle a_k^\dagger a_l \sigma_+ \rangle + \mathcal{E}_k^{(a)*} \langle b_j^\dagger a_l \sigma_+ \rangle + \mathcal{E}_l^{(a)} \langle b_j^\dagger a_k^\dagger \sigma_- \rangle \end{pmatrix}, \quad (24a)$$

$$\frac{d}{dt}\langle a_j^\dagger b_k^\dagger b_l \sigma \rangle = \mathbf{M}_b^{(j^*, k^*, l)} \langle a_j^\dagger b_k^\dagger b_l \rangle + \begin{pmatrix} -\frac{1}{2} \mathcal{E}_j^{(a)*} (\langle b_k^\dagger b_l \sigma_z \rangle + \langle b_k^\dagger b_l \rangle) - \frac{1}{2} \mathcal{E}_k^{(b)*} (\langle a_j^\dagger b_l \sigma_z \rangle + \langle a_j^\dagger b_l \rangle) \\ -\frac{1}{2} \mathcal{E}_l^{(b)} (\langle a_j^\dagger b_k^\dagger \sigma_z \rangle + \langle a_j^\dagger b_k^\dagger \rangle) \\ -\gamma \langle a_j^\dagger b_k^\dagger b_l \rangle + \mathcal{E}_j^{(a)*} \langle b_k^\dagger b_l \sigma_+ \rangle + \mathcal{E}_k^{(b)*} \langle a_j^\dagger b_l \sigma_+ \rangle + \mathcal{E}_l^{(b)} \langle a_j^\dagger b_k^\dagger \sigma_- \rangle \end{pmatrix}, \quad (24b)$$

where

$$\mathbf{M}_a^{(j^*, k^*, l)} = \mathbf{M} - \left(2\kappa_a + \kappa_b - i(\omega_j^{(b)} + \omega_k^{(a)}) - i\omega_l^{(a)} \right), \quad (25a)$$

$$\mathbf{M}_b^{(j^*, k^*, l)} = \mathbf{M} - \left(\kappa_a + 2\kappa_b - i(\omega_j^{(a)} + \omega_k^{(b)}) - i\omega_l^{(b)} \right). \quad (25b)$$

2.8 Fourth-Order: Cavity Equation

Finally, we have

$$\begin{aligned} \frac{d}{dt}\langle a_j^\dagger b_k^\dagger b_l a_m \rangle = & - \left(2\kappa_a + 2\kappa_b - i(\omega_j^{(a)} + \omega_k^{(b)}) + i(\omega_l^{(b)} + \omega_m^{(a)}) \right) \langle a_j^\dagger b_k^\dagger b_l a_m \rangle \\ & - \mathcal{E}_j^{(a)*} \langle b_k^\dagger b_l a_m \sigma_+ \rangle - \mathcal{E}_k^{(b)*} \langle a_j^\dagger a_m b_l \sigma_+ \rangle \\ & - \mathcal{E}_l^{(b)} \langle b_k^\dagger a_j^\dagger a_m \sigma_- \rangle - \mathcal{E}_m^{(a)} \langle a_j^\dagger b_k^\dagger b_l \sigma_- \rangle. \end{aligned} \quad (26)$$

3 Second-Order Cross-Correlation Function

The second-order correlation function for the filtered output field in the steady state is given by

$$G_{\text{cross}}^{(2)}(\tau) = \langle A^\dagger(0) B^\dagger B(\tau) A(0) \rangle = \sum_{j,k=-N}^N \langle A^\dagger(0) b_j^\dagger b_k(\tau) A(0) \rangle, \quad (27)$$

with the normalised second-order correlation function given by

$$g_{\text{cross}}^{(2)}(\tau) = \frac{G_{\text{cross}}^{(2)}(\tau)}{\langle A^\dagger A \rangle_{ss} \langle B^\dagger B \rangle_{ss}}. \quad (28)$$

Using the Quantum Regression Theorem, we can solve for the second-order correlation function by solving the following moment equations

$$\frac{d}{d\tau} \langle A^\dagger(0) \sigma A(0) \rangle = \mathbf{M} \langle A^\dagger(0) \sigma A(0) \rangle + \mathbf{B}, \quad (29a)$$

$$\frac{d}{d\tau} \langle A^\dagger(0) b_j^\dagger(\tau) A(0) \rangle = - \left(\kappa_b - i\omega_j^{(b)} \right) \langle A^\dagger(0) b_j^\dagger(\tau) A(0) \rangle - \mathcal{E}_j^{(b)*} \langle A^\dagger(0) \sigma_+(\tau) A(0) \rangle, \quad (29b)$$

$$\frac{d}{d\tau} \langle A^\dagger(0) b_j(\tau) A(0) \rangle = - \left(\kappa_b + i\omega_j^{(b)} \right) \langle A^\dagger(0) b_j(\tau) A(0) \rangle - \mathcal{E}_j^{(b)} \langle A^\dagger(0) \sigma_-(\tau) A(0) \rangle, \quad (29c)$$

with

$$\langle \boldsymbol{\sigma} \rangle = \begin{pmatrix} \langle A^\dagger(0) \sigma_-(\tau) A(0) \rangle \\ \langle A^\dagger(0) \sigma_+(\tau) A(0) \rangle \\ \langle A^\dagger(0) \sigma_z(\tau) A(0) \rangle \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ -\gamma \langle A^\dagger A \rangle_{ss} \end{pmatrix}. \quad (30)$$

We also need to solve

$$\frac{d}{d\tau} \langle A^\dagger(0) b_j \boldsymbol{\sigma}(\tau) A(0) \rangle = \mathbf{M}^{(j)} \langle A^\dagger(0) b_j \boldsymbol{\sigma}(\tau) A(0) \rangle + \begin{pmatrix} 0 \\ -\frac{1}{2} \mathcal{E}_j^{(b)} (\langle A^\dagger(0) \sigma_z(\tau) A(0) \rangle + \langle A^\dagger A \rangle_{ss}) \\ -\gamma \langle A^\dagger(0) b_j(\tau) A(0) \rangle + \mathcal{E}_j^{(b)} \langle A^\dagger(0) \sigma_-(\tau) A(0) \rangle \end{pmatrix}, \quad (31a)$$

and

$$\frac{d}{d\tau} \langle A^\dagger(0) b_j^\dagger \boldsymbol{\sigma}(\tau) A(0) \rangle = \mathbf{M}^{(j*)} \langle A^\dagger(0) b_j^\dagger \boldsymbol{\sigma}(\tau) A(0) \rangle + \begin{pmatrix} -\frac{1}{2} \mathcal{E}_j^{(b)*} (\langle A^\dagger(0) \sigma_z(\tau) A(0) \rangle + \langle A^\dagger A \rangle_{ss}) \\ 0 \\ -\gamma \langle A^\dagger(0) b_j^\dagger(\tau) A(0) \rangle + \mathcal{E}_j^{(b)*} \langle A^\dagger(0) \sigma_+(\tau) A(0) \rangle \end{pmatrix}. \quad (31b)$$

where

$$\mathbf{M}^{(j)} = \mathbf{M} - \left(\kappa_b + i\omega_j^{(b)} \right) \mathbb{1}, \quad (32a)$$

$$\mathbf{M}^{(j*)} = \mathbf{M} - \left(\kappa_b - i\omega_j^{(b)} \right) \mathbb{1}. \quad (32b)$$

Finally, we will also need to solve

$$\begin{aligned} \frac{d}{d\tau} \langle A^\dagger(0) b_k^\dagger b_l(\tau) A(0) \rangle &= - \left(2\kappa_b - i \left(\omega_k^{(b)} - \omega_l^{(b)} \right) \right) \langle A^\dagger(0) b_k^\dagger b_l(\tau) A(0) \rangle \\ &\quad - \mathcal{E}_k^{(b)*} \langle A^\dagger(0) b_l \sigma_+(\tau) A(0) \rangle - \mathcal{E}_l^{(b)} \langle A^\dagger(0) b_k^\dagger \sigma_-(\tau) A(0) \rangle. \end{aligned} \quad (33)$$