Two-Level Atom: Two Multi-Mode Filters Moment Equations

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1 Hamiltonian and Master Equation

The Hamiltonian is

$$
\begin{aligned}
H = \hbar\frac{\Omega}{2}\left(\sigma_{+}+\sigma_{-}\right) + \hbar\sum_{j=-N}^{N}\left(\omega_{j}^{(a)}a_{j}^{\dagger}a_{j} + \omega_{j}^{(b)}b_{j}^{\dagger}b_{j}\right) + \frac{i\hbar}{2}\sum_{j=-N}^{N}\left(\mathcal{E}_{j}^{(a)*}a_{j}\sigma_{+} - \mathcal{E}_{j}^{(a)}\sigma_{-}a_{j}^{\dagger}\right) \\
+ \frac{i\hbar}{2}\sum_{j=-N}^{N}\left(\mathcal{E}_{j}^{(b)*}b_{j}\sigma_{+} - \mathcal{E}_{j}^{(b)}\sigma_{-}b_{j}^{\dagger}\right) \qquad (1)
\end{aligned}
$$

where Ω is the Rabi frequency, σ_{+} and σ_{-} are the atomic raising and lowering operators, $a^{\dagger}$ ($b^{\dagger}$) and a (b) are the cavity photon creation and annihilation operators for filter A (B), N is the number of modes either side of the central mode ($2N+1$ total modes),

$$
\omega_{j}^{(a)} = \omega_{0}^{(a)} + j\delta\omega^{(a)}, \quad \omega_{j}^{(b)} = \omega_{0}^{(b)} + j\delta\omega^{(b)} \qquad (2)
$$

is the resonance frequency of the j^{th}mode, with central frequency $\omega_{0}^{(a/b)}$ and mode frequency spacing $\delta\omega^{(a/b)}$, and

$$
\mathcal{E}_{j}^{(a)} = \sqrt{\frac{\epsilon\gamma\kappa_{a}}{2\left(2N+1\right)}}e^{im\varphi_{j}}, \quad \mathcal{E}_{j}^{(b)} = \sqrt{\frac{\epsilon\gamma\kappa_{a}}{2\left(2N+1\right)}}e^{im\varphi_{j}}, \qquad (3)
$$

is the cascaded systems coupling of the j^{th}mode for filter A (B), where γ is the atomic decay rate, κ_{a} (κ_{b}) is the cavity decay rate for filter A (B), ϵ is the percentage of fluorescence sent to the filter, and

$$
\varphi_{j} = \frac{j\pi}{N}, \qquad (4)
$$

sets the size of the frequency dependent time delay, with integer m.

The master equation is

$$
\begin{aligned}
\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{1}{i\hbar}[H,\rho] &+ \frac{\gamma}{2}(1-\epsilon)\left(2\sigma_{-}\rho\sigma_{+} - \sigma_{+}\sigma_{-}\rho - \rho\sigma_{+}\sigma_{-}\right) \\
&+ \frac{\kappa_{a}}{2}\sum_{j=-N}^{N}\left(2a_{j}\rho a_{j}^{\dagger} - a_{j}^{\dagger}a_{j}\rho - \rho a_{j}^{\dagger}a_{j}\right) \qquad (5) \\
&+ \frac{1}{2}\sum_{j=-N}^{N}\left(2C_{j}^{(a)}\rho C_{j}^{(a)\dagger} - C_{j}^{(a)\dagger}C_{j}^{(a)}\rho - \rho C_{j}^{(a)\dagger}C_{j}^{(a)}\right) \\
&+ \frac{\kappa_{b}}{2}\sum_{j=-N}^{N}\left(2b_{j}\rho b_{j}^{\dagger} - b_{j}^{\dagger}b_{j}\rho - \rho b_{j}^{\dagger}b_{j}\right) \qquad (6)
\end{aligned}
$$

$$+\frac{1}{2}\sum_{j=-N}^{N}\left(2C_j^{(b)}\rho C_j^{(b)\dagger}-C_j^{(b)\dagger}C_j^{(b)}\rho-\rho C_j^{(b)\dagger}C_j^{(b)}\right),$$

where

$$C_j=\sqrt{\frac{\epsilon\gamma}{2N+1}}\sigma_- + \sqrt{\kappa}e^{im\varphi_j}a_j, \tag{7}$$

is the cascaded systems decay operator. Expanding the master equation out into neat terms, we arrive at:

$$\begin{aligned}\frac{\mathrm{d}\rho}{\mathrm{d}t} &= -i\frac{\Omega}{2}\left(\sigma_+\rho-\rho\sigma_+\right)-i\frac{\Omega}{2}\left(\sigma_-\rho-\rho\sigma_-\right)+\frac{\gamma}{2}\left(2\sigma_-\rho\sigma_+-\sigma_+\sigma_-\rho-\rho\sigma_+\sigma_-\right)\\
&-i\sum_{j=-N}^{N}\omega_j^{(a)}\left(a_j^\dagger a_j\rho-\rho a_j^\dagger a_j\right)+\kappa_a\sum_{j=-N}^{N}\left(2a_j\rho a_j^\dagger-a_j^\dagger a_j\rho-\rho a_j^\dagger a_j\right)\\
&-\sum_{j=-N}^{N}\mathcal{E}_j^{(a)}\left(a_j^\dagger\sigma_-\rho-\sigma_-\rho a_j^\dagger\right)-\sum_{j=-N}^{N}\mathcal{E}_j^{(a)*}\left(\rho\sigma_+a_j-a_j\rho\sigma_+\right)\\
&-i\sum_{j=-N}^{N}\omega_j^{(b)}\left(b_j^\dagger b_j\rho-\rho b_j^\dagger b_j\right)+\kappa_b\sum_{j=-N}^{N}\left(2b_j\rho b_j^\dagger-b_j^\dagger b_j\rho-\rho b_j^\dagger b_j\right)\\
&-\sum_{j=-N}^{N}\mathcal{E}_j^{(b)}\left(b_j^\dagger\sigma_-\rho-\sigma_-\rho b_j^\dagger\right)-\sum_{j=-N}^{N}\mathcal{E}_j^{(b)*}\left(\rho\sigma_+b_j-b_j\rho\sigma_+\right).\end{aligned} \tag{8}$$

2 Operator Averages

2.1 First-Order: Atomic Equations

We may write the Bloch equations in matrix form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\boldsymbol{\sigma}\rangle=\boldsymbol{M}\langle\boldsymbol{\sigma}\rangle+\boldsymbol{M}, \tag{9}$$

where

$$\langle\boldsymbol{\sigma}\rangle=\begin{pmatrix}\langle\sigma_-\rangle\\ \langle\sigma_+\rangle\\ \langle\sigma_z\rangle\end{pmatrix},\quad \boldsymbol{B}=\begin{pmatrix}0\\0\\-\gamma\end{pmatrix}, \tag{10}$$

and

$$\boldsymbol{M}=\begin{pmatrix}-\frac{\gamma}{2} & 0 & i\frac{\Omega}{2}\\ 0 & -\frac{\gamma}{2} & -i\frac{\Omega}{2}\\ i\Omega & -i\Omega & -\gamma\end{pmatrix}. \tag{11}$$

This differential equation has the solution

$$\langle\boldsymbol{\sigma}(t)\rangle=e^{\boldsymbol{M}t}\left(\langle\boldsymbol{\sigma}(0)\rangle-\langle\boldsymbol{\sigma}\rangle_{ss}\right)+\langle\boldsymbol{\sigma}\rangle_{ss}, \tag{12}$$

where

$$\langle\boldsymbol{\sigma}\rangle_{ss}=-\boldsymbol{M}^{-1}\boldsymbol{B}=\frac{1}{2\Omega^2+\gamma^2}\begin{pmatrix}-i\gamma\Omega\\ i\gamma\Omega\\ -\gamma^2\end{pmatrix}. \tag{13}$$

2.2 First-Order: Cavity Equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j\rangle = -\left(\kappa_a + i\omega_j^{(a)}\right)\langle a_j\rangle - \mathcal{E}_j^{(a)}\langle\sigma_-\rangle, \tag{14a}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger\rangle = -\left(\kappa_a - i\omega_j^{(a)}\right)\langle a_j^\dagger\rangle - \mathcal{E}_j^{(a)*}\langle\sigma_+\rangle, \tag{14b}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\langle b_j\rangle = -\left(\kappa_b + i\omega_j^{(b)}\right)\langle b_j\rangle - \mathcal{E}_j^{(b)}\langle\sigma_-\rangle, \tag{14c}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\langle b_j^\dagger\rangle = -\left(\kappa_b - i\omega_j^{(b)}\right)\langle b_j^\dagger\rangle - \mathcal{E}_j^{(b)*}\langle\sigma_+\rangle. \tag{14d}$$

2.3 Second-Order: Cavity-Atom Coupled Equations

Using the vector notation, we have moment equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j\boldsymbol{\sigma}\rangle = \boldsymbol{M}_j^{(a)}\langle a_j\boldsymbol{\sigma}\rangle + \begin{pmatrix} 0 \\ -\frac{1}{2}\mathcal{E}_j^{(a)}\left(\langle\sigma_z\rangle + 1\right) \\ -\gamma\langle a_j\rangle + \mathcal{E}_j^{(a)}\langle\sigma_-\rangle \end{pmatrix}, \tag{15a}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\langle b_j\boldsymbol{\sigma}\rangle = \boldsymbol{M}_j^{(b)}\langle b_j\boldsymbol{\sigma}\rangle + \begin{pmatrix} 0 \\ -\frac{1}{2}\mathcal{E}_j^{(b)}\left(\langle\sigma_z\rangle + 1\right) \\ -\gamma\langle b_j\rangle + \mathcal{E}_j^{(b)}\langle\sigma_-\rangle \end{pmatrix}, \tag{15b}$$

where

$$\boldsymbol{M}_j^{(a)} = \boldsymbol{M} - \left(\kappa_a + i\omega_j^{(a)}\right)\mathbb{1}, \tag{16a}$$
$$\boldsymbol{M}_j^{(b)} = \boldsymbol{M} - \left(\kappa_b + i\omega_j^{(b)}\right)\mathbb{1}; \tag{16b}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger\boldsymbol{\sigma}\rangle = \boldsymbol{M}_j^{(a^\dagger)}\langle a_j^\dagger\boldsymbol{\sigma}\rangle + \begin{pmatrix} -\frac{1}{2}\mathcal{E}_j^{(a)*}\left(\langle\sigma_z\rangle + 1\right) \\ 0 \\ -\gamma\langle a_j^\dagger\rangle + \mathcal{E}_j^{(a)*}\langle\sigma_+\rangle \end{pmatrix}, \tag{17a}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\langle b_j^\dagger\boldsymbol{\sigma}\rangle = \boldsymbol{M}^{(b^\dagger)}\langle b_j^\dagger\boldsymbol{\sigma}\rangle + \begin{pmatrix} -\frac{1}{2}\mathcal{E}_j^{(b)*}\left(\langle\sigma_z\rangle + 1\right) \\ 0 \\ -\gamma\langle b_j^\dagger\rangle + \mathcal{E}_j^{(b)*}\langle\sigma_+\rangle \end{pmatrix}, \tag{17b}$$

where

$$\boldsymbol{M}_j^{(a^\dagger)} = \boldsymbol{M} - \left(\kappa_a - i\omega_j^{(a)}\right)\mathbb{1}, \tag{18a}$$
$$\boldsymbol{M}_j^{(b^\dagger)} = \boldsymbol{M} - \left(\kappa_b - i\omega_j^{(b)}\right)\mathbb{1}, \tag{18b}$$

2.4 Second-Order: Cavity Equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j b_k\rangle = -\left(\kappa_a + \kappa_b + i\left(\omega_j^{(a)} + \omega_k^{(b)}\right)\right)\langle a_j b_k\rangle - \mathcal{E}_j^{(a)}\langle b_k\sigma_-\rangle - \mathcal{E}_k^{(b)}\langle a_j\sigma_-\rangle, \tag{19a}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger b_k^\dagger\rangle = -\left(\kappa_a + \kappa_b - i\left(\omega_j^{(a)} + \omega_k^{(b)}\right)\right)\langle a_j^\dagger b_k^\dagger\rangle - \mathcal{E}_j^{(a)*}\langle b_k^\dagger\sigma_+\rangle - \mathcal{E}_k^{(b)*}\langle a_j^\dagger\sigma_+\rangle, \tag{19b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger a_k\rangle = -\left(2\kappa_a - i\left(\omega_j^{(a)} - \omega_k^{(a)}\right)\right)\langle a_j^\dagger a_k\rangle - \mathcal{E}_j^{(a)*}\langle a_k\sigma_+\rangle - \mathcal{E}_k^{(a)}\langle a_j^\dagger\sigma_-\rangle, \tag{19c}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle b_j^\dagger b_k\rangle = -\left(2\kappa_b - i\left(\omega_j^{(b)} - \omega_k^{(b)}\right)\right)\langle b_j^\dagger b_k\rangle - \mathcal{E}_j^{(b)*}\langle b_k\sigma_+\rangle - \mathcal{E}_k^{(b)}\langle b_j^\dagger\sigma_-\rangle, \tag{19d}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger b_k\rangle = -\left(\kappa_a + \kappa_b - i\left(\omega_j^{(a)} - \omega_k^{(b)}\right)\right)\langle a_j^\dagger b_k\rangle - \mathcal{E}_j^{(a)*}\langle b_k\sigma_+\rangle - \mathcal{E}_k^{(b)}\langle a_j^\dagger\sigma_-\rangle, \tag{19e}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle b_j^\dagger a_k\rangle = -\left(\kappa_a + \kappa_b - i\left(\omega_j^{(b)} - \omega_k^{(a)}\right)\right)\langle b_j^\dagger a_k\rangle - \mathcal{E}_j^{(b)*}\langle a_k\sigma_+\rangle - \mathcal{E}_k^{(a)}\langle b_j^\dagger\sigma_-\rangle. \tag{19f}$$

2.5 Third-Order: Cavity-Atom Coupled Equations

Using the vector notation, we have moment equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j b_k\boldsymbol{\sigma}\rangle = \boldsymbol{M}_{j,k}^{(ab)}\langle a_j b_k\boldsymbol{\sigma}\rangle + \begin{pmatrix} 0 \\ -\frac{1}{2}\mathcal{E}_j^{(a)}\left(\langle b_k\sigma_z\rangle + \langle b_k\rangle\right) - \frac{1}{2}\mathcal{E}_k^{(b)}\left(\langle a_j\sigma_z\rangle + \langle a_j\rangle\right) \\ -\gamma\langle a_j b_k\rangle + \mathcal{E}_j^{(a)}\langle b_k\sigma_-\rangle + \mathcal{E}_k^{(b)}\langle a_j\sigma_-\rangle \end{pmatrix}, \tag{20a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger b_k^\dagger\boldsymbol{\sigma}\rangle = \boldsymbol{M}_{j,k}^{(a^\dagger b^\dagger)}\langle a_j^\dagger b_k^\dagger\boldsymbol{\sigma}\rangle + \begin{pmatrix} -\frac{1}{2}\mathcal{E}_j^{(a)*}\left(\langle b_k^\dagger\sigma_z\rangle + \langle b_k^\dagger\rangle\right) - \frac{1}{2}\mathcal{E}_k^{(b)*}\left(\langle a_j^\dagger\sigma_z\rangle + \langle a_j^\dagger\rangle\right) \\ 0 \\ -\gamma\langle a_j^\dagger b_k^\dagger\rangle + \mathcal{E}_j^{(a)*}\langle b_k^\dagger\sigma_+\rangle + \mathcal{E}_k^{(b)*}\langle a_j^\dagger\sigma_+\rangle \end{pmatrix}, \tag{20b}$$

where

$$\boldsymbol{M}_{j,k}^{(ab)} = \boldsymbol{M} - \left(\kappa_a + \kappa_b + i\left(\omega_j^{(a)} + \omega_k^{(b)}\right)\right)\mathbb{1}, \tag{21a}$$

$$\boldsymbol{M}_{j,k}^{(a^\dagger b^\dagger)} = \boldsymbol{M} - \left(\kappa_a + \kappa_b - i\left(\omega_j^{(a)} + \omega_k^{(b)}\right)\right)\mathbb{1}; \tag{21b}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger a_k\boldsymbol{\sigma}\rangle = \boldsymbol{M}_{j,k}^{(a^\dagger a)}\langle a_j^\dagger a_k\boldsymbol{\sigma}\rangle + \begin{pmatrix} -\frac{1}{2}\mathcal{E}_j^{(a)*}\left(\langle a_k\sigma_z\rangle + \langle a_k\rangle\right) \\ -\frac{1}{2}\mathcal{E}_k^{(a)}\left(\langle a_j^\dagger\sigma_z\rangle + \langle a_j^\dagger\rangle\right) \\ -\gamma\langle a_j^\dagger a_k\rangle + \mathcal{E}_j^{(a)*}\langle a_k\sigma_+\rangle + \mathcal{E}_k^{(a)}\langle a_j^\dagger\sigma_-\rangle \end{pmatrix}, \tag{22a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle b_j^\dagger b_k\boldsymbol{\sigma}\rangle = \boldsymbol{M}_{j,k}^{(b^\dagger b)}\langle b_j^\dagger b_k\boldsymbol{\sigma}\rangle + \begin{pmatrix} -\frac{1}{2}\mathcal{E}_j^{(b)*}\left(\langle b_k\sigma_z\rangle + \langle b_k\rangle\right) \\ -\frac{1}{2}\mathcal{E}_k^{(b)}\left(\langle b_j^\dagger\sigma_z\rangle + \langle b_j^\dagger\rangle\right) \\ -\gamma\langle b_j^\dagger b_k\rangle + \mathcal{E}_j^{(b)*}\langle b_k\sigma_+\rangle + \mathcal{E}_k^{(b)}\langle b_j^\dagger\sigma_-\rangle \end{pmatrix}, \tag{22b}$$

where

$$\boldsymbol{M}_{j,k}^{(a^\dagger a)} = \boldsymbol{M} - \left(2\kappa_a - i\left(\omega_j^{(a)} - \omega_k^{(a)}\right)\right)\mathbb{1}, \tag{23a}$$

$$\boldsymbol{M}_{j,k}^{(b^\dagger b)} = \boldsymbol{M} - \left(2\kappa_b - i\left(\omega_j^{(b)} - \omega_k^{(b)}\right)\right)\mathbb{1}; \tag{23b}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger b_k\boldsymbol{\sigma}\rangle = \boldsymbol{M}_{j,k}^{(a^\dagger b)}\langle a_j^\dagger b_k\boldsymbol{\sigma}\rangle + \begin{pmatrix} -\frac{1}{2}\mathcal{E}_j^{(a)*}\left(\langle bk\sigma_z\rangle + \langle b_k\rangle\right) \\ -\frac{1}{2}\mathcal{E}_k^{(b)}\left(\langle a_j^\dagger\sigma_z\rangle + \langle a_j^\dagger\rangle\right) \\ -\gamma\langle a_j^\dagger b_k\rangle + \mathcal{E}_j^{(a)*}\langle b_k\sigma_+\rangle + \mathcal{E}_k^{(b)}\langle a_j^\dagger\sigma_-\rangle \end{pmatrix}, \tag{24a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle b_j^\dagger a_k\boldsymbol{\sigma}\rangle = \boldsymbol{M}_{j,k}^{(b^\dagger a)}\langle b_j^\dagger a_k\boldsymbol{\sigma}\rangle + \begin{pmatrix} -\frac{1}{2}\mathcal{E}_j^{(b)*}\left(\langle a_k\sigma_z\rangle + \langle a_k\rangle\right) \\ -\frac{1}{2}\mathcal{E}_k^{(a)}\left(\langle b_j^\dagger\sigma_z\rangle + \langle b_j^\dagger\rangle\right) \\ -\gamma\langle b_j^\dagger a_k\rangle + \mathcal{E}_j^{(b)*}\langle a_k\sigma_+\rangle + \mathcal{E}_k^{(a)}\langle b_j^\dagger\sigma_-\rangle \end{pmatrix}, \tag{24b}$$

where

$$\boldsymbol{M}_{j,k}^{(a^\dagger b)} = \boldsymbol{M} - \left(\kappa_a + \kappa_b - i\left(\omega_j^{(a)} - \omega_k^{(b)}\right)\right)\mathbb{1}, \tag{25a}$$

$$\boldsymbol{M}_{j,k}^{(b^\dagger a)} = \boldsymbol{M} - \left(\kappa_a + \kappa_b - i\left(\omega_j^{(b)} - \omega_k^{(a)}\right)\right)\mathbb{1}; \tag{25b}$$

2.6 Third-Order: Cavity Equations

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger a_k b_l\rangle = &-\left(2\kappa_a + \kappa_b - i\omega_j^{(a)} + i\left(\omega_k^{(a)} + \omega_l^{(b)}\right)\right)\langle a_j^\dagger a_k b_l\rangle \\ &- \mathcal{E}_j^{(a)*}\langle a_k b_l \sigma_+\rangle - \mathcal{E}_k^{(a)}\langle a_j^\dagger b_l \sigma_-\rangle - \mathcal{E}_l^{(b)}\langle a_j^\dagger a_k \sigma_-\rangle,\end{aligned} \tag{26a}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle b_j^\dagger b_k a_l\rangle = &-\left(\kappa_a + 2\kappa_b - i\omega_j^{(b)} + i\left(\omega_k^{(b)} + \omega_l^{(a)}\right)\right)\langle b_j^\dagger b_k a_l\rangle \\ &- \mathcal{E}_j^{(b)*}\langle b_k a_l \sigma_+\rangle - \mathcal{E}_k^{(b)}\langle b_j^\dagger a_l \sigma_-\rangle - \mathcal{E}_l^{(a)}\langle b_j^\dagger b_k \sigma_-\rangle,\end{aligned} \tag{26b}$$

and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle b_j^\dagger a_k^\dagger a_l\rangle = &-\left(2\kappa_a + \kappa_b - i\left(\omega_j^{(b)} + \omega_k^{(a)}\right) + i\omega_l^{(a)}\right)\langle b_j^\dagger a_k^\dagger a_l\rangle \\ &- \mathcal{E}_j^{(b)*}\langle a_k^\dagger a_l \sigma_+\rangle - \mathcal{E}_k^{(a)*}\langle b_j^\dagger a_l \sigma_+\rangle - \mathcal{E}_l^{(a)}\langle b_j^\dagger a_k^\dagger \sigma_-\rangle,\end{aligned} \tag{27a}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger b_k^\dagger b_l\rangle = &-\left(\kappa_a + 2\kappa_b - i\left(\omega_j^{(a)} + \omega_k^{(b)}\right) + i\omega_l^{(b)}\right)\langle a_j^\dagger b_k^\dagger b_l\rangle \\ &- \mathcal{E}_j^{(a)*}\langle b_k^\dagger b_l \sigma_+\rangle - \mathcal{E}_k^{(b)*}\langle a_j^\dagger b_l \sigma_+\rangle - \mathcal{E}_l^{(b)}\langle a_j^\dagger b_k^\dagger \sigma_-\rangle.\end{aligned} \tag{27b}$$

2.7 Fourth-Order: Cavity-Atom Coupled Equations

Using the vector notation, we have moment equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger a_k b_l \boldsymbol{\sigma}\rangle = \boldsymbol{M}_{j,k,l}^{(a^\dagger ab)}\langle a_j^\dagger a_k b_l \boldsymbol{\sigma}\rangle + \begin{pmatrix} -\frac{1}{2}\mathcal{E}_j^{(a)*}\left(\langle a_k b_l \sigma_z\rangle + \langle a_k b_l\rangle\right) \\ -\frac{1}{2}\mathcal{E}_k^{(a)}\left(\langle a_j^\dagger b_l \sigma_z\rangle + \langle a_j^\dagger b_l\rangle\right) - \frac{1}{2}\mathcal{E}_l^{(b)}\left(\langle a_j^\dagger a_k \sigma_z\rangle + \langle a_j^\dagger a_k\rangle\right) \\ -\gamma\langle a_j^\dagger a_k b_l\rangle + \mathcal{E}_j^{(a)*}\langle a_k b_l \sigma_+\rangle + \mathcal{E}_k^{(a)}\langle a_j^\dagger b_l \sigma_-\rangle + \mathcal{E}_l^{(b)}\langle a_j^\dagger a_k \sigma_-\rangle \end{pmatrix}, \tag{28a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle b_j^\dagger b_k a_l \boldsymbol{\sigma}\rangle = \boldsymbol{M}_{j,k,l}^{(b^\dagger ba)}\langle b_j^\dagger b_k a_l \boldsymbol{\sigma}\rangle + \begin{pmatrix} -\frac{1}{2}\mathcal{E}_j^{(b)*}\left(\langle b_k a_l \sigma_z\rangle + \langle b_k a_l\rangle\right) \\ -\frac{1}{2}\mathcal{E}_k^{(b)}\left(\langle b_j^\dagger a_l \sigma_z\rangle + \langle b_j^\dagger a_l\rangle\right) - \frac{1}{2}\mathcal{E}_l^{(a)}\left(\langle b_j^\dagger b_k \sigma_z\rangle + \langle b_j^\dagger b_k\rangle\right) \\ -\gamma\langle b_j^\dagger a_k a_l\rangle + \mathcal{E}_j^{(b)*}\langle b_k a_l \sigma_+\rangle + \mathcal{E}_k^{(b)}\langle b_j^\dagger a_l \sigma_-\rangle + \mathcal{E}_l^{(a)}\langle b_j^\dagger b_k \sigma_-\rangle \end{pmatrix}, \tag{28b}$$

where

$$\boldsymbol{M}_{j,k,l}^{(a^\dagger ab)} = \boldsymbol{M} - \left(2\kappa_a + \kappa_b - i\omega_j^{(a)} + i\left(\omega_k^{(a)} + \omega_l^{(b)}\right)\right)\mathbb{1}, \tag{29a}$$

$$\boldsymbol{M}_{j,k,l}^{(b^\dagger ba)} = \boldsymbol{M} - \left(\kappa_a + 2\kappa_b - i\omega_j^{(b)} + i\left(\omega_k^{(b)} + \omega_l^{(a)}\right)\right)\mathbb{1}; \tag{29b}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle b_j^\dagger a_k^\dagger a_l \boldsymbol{\sigma}\rangle = \boldsymbol{M}_{j,k,l}^{(b^\dagger a^\dagger a)}\langle b_j^\dagger a_k^\dagger a_l \boldsymbol{\sigma}\rangle + \begin{pmatrix} -\frac{1}{2}\mathcal{E}_j^{(b)*}\left(\langle a_k^\dagger a_l \sigma_z\rangle + \langle a_k^\dagger a_l\rangle\right) - \frac{1}{2}\mathcal{E}_k^{(a)*}\left(\langle b_j^\dagger a_l \sigma_z\rangle + \langle b_j^\dagger a_l\rangle\right) \\ -\frac{1}{2}\mathcal{E}_l^{(a)}\left(\langle b_j^\dagger a_k^\dagger \sigma_z\rangle + \langle b_j^\dagger a_k^\dagger\rangle\right) \\ -\gamma\langle b_j^\dagger a_k^\dagger a_l\rangle + \mathcal{E}_j^{(b)*}\langle a_k^\dagger a_l \sigma_+\rangle + \mathcal{E}_k^{(a)*}\langle b_j^\dagger a_l \sigma_+\rangle + \mathcal{E}_l^{(a)}\langle b_j^\dagger a_k^\dagger \sigma_-\rangle \end{pmatrix}, \tag{30a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger b_k^\dagger b_l \boldsymbol{\sigma}\rangle = \boldsymbol{M}_{j,k,l}^{(a^\dagger b^\dagger b)}\langle a_j^\dagger b_k^\dagger b_l \boldsymbol{\sigma}\rangle + \begin{pmatrix} -\frac{1}{2}\mathcal{E}_j^{(a)*}\left(\langle b_k^\dagger b_l \sigma_z\rangle + \langle b_k^\dagger b_l\rangle\right) - \frac{1}{2}\mathcal{E}_k^{(b)*}\left(\langle a_j^\dagger b_l \sigma_z\rangle + \langle a_j^\dagger b_l\rangle\right) \\ -\frac{1}{2}\mathcal{E}_l^{(b)}\left(\langle a_j^\dagger b_k^\dagger \sigma_z\rangle + \langle a_j^\dagger b_k^\dagger\rangle\right) \\ -\gamma\langle a_j^\dagger b_k^\dagger b_l\rangle + \mathcal{E}_j^{(a)*}\langle b_k^\dagger b_l \sigma_+\rangle + \mathcal{E}_k^{(b)*}\langle a_j^\dagger b_l \sigma_+\rangle + \mathcal{E}_l^{(b)}\langle a_j^\dagger b_k^\dagger \sigma_-\rangle \end{pmatrix}, \tag{30b}$$

where

$$\boldsymbol{M}_{j,k,l}^{(b^\dagger a^\dagger a)} = \boldsymbol{M} - \left(2\kappa_a + \kappa_b - i\left(\omega_j^{(b)} + \omega_k^{(a)}\right) - i\omega_l^{(a)}\right)\mathbb{1}, \tag{31a}$$

$$\boldsymbol{M}_{j,k,l}^{(a^\dagger b^\dagger b)} = \boldsymbol{M} - \left(\kappa_a + 2\kappa_b - i\left(\omega_j^{(a)} + \omega_k^{(b)}\right) - i\omega_l^{(b)}\right)\mathbb{1}. \tag{31b}$$

2.8 Fourth-Order: Cavity Equation

Finally, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger b_k^\dagger b_l a_m\rangle = &-\left(2\kappa_a + 2\kappa_b - i\left(\omega_j^{(a)} + \omega_k^{(b)}\right) + i\left(\omega_l^{(b)} + \omega_m^{(a)}\right)\right)\langle a_j^\dagger b_k^\dagger b_l a_m\rangle \\ &- \mathcal{E}_j^{(a)*}\langle b_k^\dagger b_l a_m \sigma_+\rangle - \mathcal{E}_k^{(b)*}\langle a_j^\dagger a_m b_l \sigma_+\rangle \\ &- \mathcal{E}_l^{(b)}\langle b_k^\dagger a_j^\dagger a_m \sigma_-\rangle - \mathcal{E}_m^{(a)}\langle a_j^\dagger b_k^\dagger b_l \sigma_-\rangle.\end{aligned} \tag{32}$$

3 Second-Order Cross-Correlation Function

The second-order correlation function for the filtered output field in the steady state is given by

$$G_{\mathrm{cross}}^{(2)}(\tau) = \langle A^\dagger(0) B^\dagger B(\tau) A(0)\rangle = \sum_{j,k=-N}^{N}\langle A^\dagger(0) b_j^\dagger b_k(\tau) A(0)\rangle, \tag{33}$$

with the normalised second-order correlation function given by

$$g_{\mathrm{cross}}^{(2)}(\tau) = \frac{G_{\mathrm{cross}}^{(2)}(\tau)}{\langle A^\dagger A\rangle_{ss}\langle B^\dagger B\rangle_{ss}}. \tag{34}$$

Using the Quantum Regression Theorem, we can solve for the second-order correlation function by solving the following moment equations

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\langle A^\dagger(0)\boldsymbol{\sigma} A(0)\rangle = \boldsymbol{M}\langle A^\dagger(0)\boldsymbol{\sigma} A(0)\rangle + \boldsymbol{B}, \tag{35a}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\langle A^\dagger(0) b_j^\dagger(\tau) A(0)\rangle = -\left(\kappa_b - i\omega_j^{(b)}\right)\langle A^\dagger(0) b_j^\dagger(\tau) A(0)\rangle - \mathcal{E}_j^{(b)*}\langle A^\dagger(0)\sigma_+(\tau) A(0)\rangle, \tag{35b}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\langle A^\dagger(0) b_j(\tau) A(0)\rangle = -\left(\kappa_b + i\omega_j^{(b)}\right)\langle A^\dagger(0) b_j(\tau) A(0)\rangle - \mathcal{E}_j^{(b)}\langle A^\dagger(0)\sigma_-(\tau) A(0)\rangle, \tag{35c}$$

with

$$\langle\boldsymbol{\sigma}\rangle = \begin{pmatrix}\langle A^\dagger(0)\sigma_-(\tau)A(0)\rangle \\ \langle A^\dagger(0)\sigma_+(\tau)A(0)\rangle \\ \langle A^\dagger(0)\sigma_z(\tau)A(0)\rangle\end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} 0 \\ 0 \\ -\gamma\langle A^\dagger A\rangle_{ss}\end{pmatrix}. \tag{36}$$

We also need to solve

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\langle A^\dagger(0) b_j \boldsymbol{\sigma}(\tau) A(0)\rangle = \boldsymbol{M}_j^{(b)}\langle A^\dagger(0) b_j\boldsymbol{\sigma}(\tau) A(0)\rangle + \begin{pmatrix} 0 \\ -\frac{1}{2}\mathcal{E}_j^{(b)}\left(\langle A^\dagger(0)\sigma_z(\tau)A(0)\rangle + \langle A^\dagger A\rangle_{ss}\right) \\ -\gamma\langle A^\dagger(0) b_j(\tau) A(0)\rangle + \mathcal{E}_j^{(b)}\langle A^\dagger(0)\sigma_-(\tau)A(0)\rangle\end{pmatrix}, \tag{37a}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\langle A^\dagger(0)b_j^\dagger\boldsymbol{\sigma}(\tau)A(0)\rangle = \boldsymbol{M}_j^{(b^\dagger)}\langle A^\dagger(0)b_j^\dagger\boldsymbol{\sigma}(\tau)A(0)\rangle + \begin{pmatrix} -\frac{1}{2}\mathcal{E}_j^{(b)*}\left(\langle A^\dagger(0)\sigma_z(\tau)A(0)\rangle + \langle A^\dagger A\rangle_{ss}\right) \\ 0 \\ -\gamma\langle A^\dagger(0)b_j^\dagger(\tau)A(0)\rangle + \mathcal{E}_j^{*}\langle A^\dagger(0)\sigma_+(\tau)A(0)\rangle \end{pmatrix}. \tag{37b}$$

where

$$\boldsymbol{M}_j^{(b)} = \boldsymbol{M} - \left(\kappa_b + i\omega_j^{(b)}\right)\mathbb{1}, \tag{38a}$$

$$\boldsymbol{M}_j^{(b^\dagger)} = \boldsymbol{M} - \left(\kappa_b - i\omega_j^{(b)}\right)\mathbb{1}. \tag{38b}$$

Finally, we will also need to solve

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\langle A^\dagger(0)b_k^\dagger b_l(\tau)A(0)\rangle = &-\left(2\kappa_b - i\left(\omega_k^{(b)} - \omega_l^{(b)}\right)\right)\langle A^\dagger(0)b_k^\dagger b_l(\tau)A(0)\rangle \\ &- \mathcal{E}_k^{(b)*}\langle A^\dagger(0)b_l\sigma_+(\tau)A(0)\rangle - \mathcal{E}_l^{(b)}\langle A^\dagger(0)b_k^\dagger\sigma_-(\tau)A(0)\rangle. \end{aligned}\tag{39}$$