Three-Level Atom: Two Multi-Mode Filters Moment Equations

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1 Hamiltonian and Master Equation

The Hamiltonian is

$$\begin{aligned} H = &-\hbar\left(\frac{\alpha}{2}+\delta\right)|e\rangle\langle e| - 2\hbar\delta\,|f\rangle\langle f| + \hbar\frac{\Omega}{2}\left(\Sigma_{+}+\Sigma_{-}\right) + \hbar\sum_{j=-N}^{N}\left(\omega_{j}^{(a)}a_{j}^{\dagger}a_{j} + \omega_{j}^{(b)}b_{j}^{\dagger}b_{j}\right) \\ &+\frac{i\hbar}{2}\sum_{j=-N}^{N}\left(\mathcal{E}_{j}^{(a)*}a_{j}\Sigma_{+} - \mathcal{E}_{j}^{(a)}\Sigma_{-}a_{j}^{\dagger}\right) + \frac{i\hbar}{2}\sum_{j=-N}^{N}\left(\mathcal{E}_{j}^{(b)*}b_{j}\Sigma_{+} - \mathcal{E}_{j}^{(b)}\Sigma_{-}b_{j}^{\dagger}\right) \end{aligned} \tag{1}$$

where Ω is the Rabi frequency,

$$\alpha = \omega_{fe} - \omega_{eg}, \tag{2}$$

is the atomic anharmonicity, δ is given by

$$2\delta = 2\omega_{d} - \omega_{fg}, \tag{3}$$

is the *drive detuning from two-photon resonance*

$$\Sigma_{-} = \sigma_{-}^{eg} + \xi\sigma_{-}^{fe} = |g\rangle\langle e| + \xi\,|e\rangle\langle f|\,, \quad \Sigma_{+} = \Sigma_{-}^{\dagger}, \tag{4}$$

is the atomic raising (lowering) operator, $a^{\dagger}$ ($b^{\dagger}$) and a (b) are the cavity photon creation and annihilation operators for filter A (B), N is the number of modes either side of the central mode ($2N+1$ total modes),

$$\omega_{j}^{(a)} = \omega_{0}^{(a)} + j\delta\omega^{(a)}, \quad \omega_{j}^{(b)} = \omega_{0}^{(b)} + j\delta\omega^{(b)} \tag{5}$$

is the resonance frequency of the j^{th}mode, with central frequency $\omega_{0}^{(a/b)}$ and mode frequency spacing $\delta\omega^{(a/b)}$, and

$$\mathcal{E}_{j}^{(a)} = \sqrt{\frac{\epsilon\gamma\kappa_{a}}{2\left(2N+1\right)}}e^{im\varphi_{j}}, \quad \mathcal{E}_{j}^{(b)} = \sqrt{\frac{\epsilon\gamma\kappa_{b}}{2\left(2N+1\right)}}e^{im\varphi_{j}}, \tag{6}$$

is the cascaded systems coupling of the j^{th}mode for filter A (B), where γ is the atomic decay rate, κ_a (κ_b) is the cavity decay rate for filter A (B), ϵ is the percentage of fluorescence sent to the filter, and

$$\varphi_{j} = \frac{j\pi}{N}, \tag{7}$$

sets the size of the frequency dependent time delay, with integer m.

The master equation is

$$\begin{aligned} \frac{\mathrm{d}\rho}{\mathrm{d}t} = &\frac{1}{i\hbar}[H,\rho] + \frac{\gamma}{2}(1-\epsilon)\left(2\sigma_{-}\rho\sigma_{+} - \sigma_{+}\sigma_{-}\rho - \rho\sigma_{+}\sigma_{-}\right) \\ &+\frac{\kappa_{a}}{2}\sum_{j=-N}^{N}\left(2a_{j}\rho a_{j}^{\dagger} - a_{j}^{\dagger}a_{j}\rho - \rho a_{j}^{\dagger}a_{j}\right) \end{aligned} \tag{8}$$

$$
\begin{aligned}
&+ \frac{1}{2} \sum_{j=-N}^{N} \left(2C_j^{(a)} \rho C_j^{(a)\dagger} - C_j^{(a)\dagger} C_j^{(a)} \rho - \rho C_j^{(a)\dagger} C_j^{(a)} \right) \\
&+ \frac{\kappa_b}{2} \sum_{j=-N}^{N} \left(2b_j \rho b_j^{\dagger} - b_j^{\dagger} b_j \rho - \rho b_j^{\dagger} b_j \right) \\
&+ \frac{1}{2} \sum_{j=-N}^{N} \left(2C_j^{(b)} \rho C_j^{(b)\dagger} - C_j^{(b)\dagger} C_j^{(b)} \rho - \rho C_j^{(b)\dagger} C_j^{(b)} \right),
\end{aligned} \tag{9}
$$

where

$$
C_j = \sqrt{\frac{\epsilon\gamma}{2N+1}} \sigma_- + \sqrt{\kappa} e^{im\varphi_j} a_j, \tag{10}
$$

is the cascaded systems decay operator. Expanding the master equation out into neat terms, we arrive at:

$$
\begin{aligned}
\frac{\mathrm{d}\rho}{\mathrm{d}t} = {}& i \left(\frac{\alpha}{2} + \delta \right) \left(\sigma^{ee} \rho - \rho \sigma^{ee} \right) + 2i\delta \left(\sigma^{ff} \rho - \rho \sigma^{ff} \right) - i \frac{\Omega}{2} \left(\Sigma_+ \rho - \rho \Sigma_+ \right) \\
& - i \frac{\Omega}{2} \left(\Sigma_- \rho - \rho \Sigma_- \right) + \frac{\Gamma}{2} \left(2\Sigma_- \rho \Sigma_+ - \Sigma_+ \Sigma_- \rho - \rho \Sigma_+ \Sigma_- \right) \\
& - i \sum_{j=-N}^{N} \omega_j^{(a)} \left(a_j^{\dagger} a_j \rho - \rho a_j^{\dagger} a_j \right) + \kappa_a \sum_{j=-N}^{N} \left(2a_j \rho a_j^{\dagger} - a_j^{\dagger} a_j \rho - \rho a_j^{\dagger} a_j \right) \\
& - \sum_{j=-N}^{N} \mathcal{E}_j^{(a)} \left(a_j^{\dagger} \sigma_- \rho - \sigma_- \rho a_j^{\dagger} \right) - \sum_{j=-N}^{N} \mathcal{E}_j^{(a)*} \left(\rho \sigma_+ a_j - a_j \rho \sigma_+ \right) \\
& - i \sum_{j=-N}^{N} \omega_j^{(b)} \left(b_j^{\dagger} b_j \rho - \rho b_j^{\dagger} b_j \right) + \kappa_b \sum_{j=-N}^{N} \left(2b_j \rho b_j^{\dagger} - b_j^{\dagger} b_j \rho - \rho b_j^{\dagger} b_j \right) \\
& - \sum_{j=-N}^{N} \mathcal{E}_j^{(b)} \left(b_j^{\dagger} \sigma_- \rho - \sigma_- \rho b_j^{\dagger} \right) - \sum_{j=-N}^{N} \mathcal{E}_j^{(b)*} \left(\rho \sigma_+ b_j - b_j \rho \sigma_+ \right).
\end{aligned} \tag{11}
$$

2 Operator Averages

2.1 First-Order: Atomic Equations

Rearranging the density operator equations, we can write the atomic moment equations in matrix form

$$
\frac{\mathrm{d}}{\mathrm{d}t} \langle \boldsymbol{\sigma} \rangle = \boldsymbol{M} \langle \boldsymbol{\sigma} \rangle + \boldsymbol{B}, \tag{12}
$$

where

$$
\langle \boldsymbol{\sigma} \rangle = \begin{pmatrix} \langle \sigma^{gg} \rangle \\ \langle \sigma_-^{eg} \rangle \\ \langle \sigma_+^{eg} \rangle \\ \langle \sigma^{ee} \rangle \\ \langle \sigma_-^{fe} \rangle \\ \langle \sigma_+^{fe} \rangle \\ \langle \sigma_-^{fg} \rangle \\ \langle \sigma_+^{fg} \rangle \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \Gamma\xi^2 \\ i\xi\frac{\Omega}{2} \\ -i\xi\frac{\Omega}{2} \\ 0 \\ 0 \end{pmatrix}, \tag{13}
$$

and

$$\boldsymbol{M} = \begin{pmatrix} 0 & -i\frac{\Omega}{2} & i\frac{\Omega}{2} & \Gamma & 0 & 0 & 0 & 0 \\ -i\frac{\Omega}{2} & -\left[\frac{\Gamma}{2}-i\left(\frac{\alpha}{2}+\delta\right)\right] & 0 & i\frac{\Omega}{2} & \Gamma\xi & 0 & -i\xi\frac{\Omega}{2} & 0 \\ i\frac{\Omega}{2} & 0 & -\left[\frac{\Gamma}{2}+i\left(\frac{\alpha}{2}+\delta\right)\right] & -i\frac{\Omega}{2} & 0 & \Gamma\xi & 0 & i\xi\frac{\Omega}{2} \\ -\Gamma\xi^2 & i\frac{\Omega}{2} & -i\frac{\Omega}{2} & -\Gamma\left(1+\xi^2\right) & -i\xi\frac{\Omega}{2} & i\xi\frac{\Omega}{2} & 0 & 0 \\ -i\xi\frac{\Omega}{2} & 0 & 0 & -i\xi\Omega & -\left[\frac{\Gamma}{2}\left(1+\xi^2\right)+i\left(\frac{\alpha}{2}-\delta\right)\right] & 0 & i\frac{\Omega}{2} & 0 \\ i\xi\frac{\Omega}{2} & 0 & 0 & i\xi\Omega & 0 & -\left[\frac{\Gamma}{2}\left(1+\xi^2\right)-i\left(\frac{\alpha}{2}-\delta\right)\right] & 0 & -i\frac{\Omega}{2} \\ 0 & -i\xi\frac{\Omega}{2} & 0 & 0 & i\frac{\Omega}{2} & 0 & -\left[\frac{\Gamma}{2}\xi^2-2i\delta\right] & 0 \\ 0 & 0 & i\xi\frac{\Omega}{2} & 0 & 0 & -i\frac{\Omega}{2} & 0 & -\left[\frac{\Gamma}{2}\xi^2+2i\delta\right] \end{pmatrix}. \tag{14}$$

This differential equation has solution

$$\langle\boldsymbol{\sigma}(t)\rangle = e^{\boldsymbol{M}t}\langle\boldsymbol{\sigma}(0)\rangle + \left(1 - e^{\boldsymbol{M}t}\right)\langle\boldsymbol{\sigma}\rangle_{ss}, \tag{15}$$

where

$$\langle\boldsymbol{\sigma}\rangle_{ss} = -\boldsymbol{M}^{-1}\boldsymbol{B}. \tag{16}$$

2.2 First-Order: Cavity Equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j\rangle = -\left(\kappa + i\omega_j\right)\langle a_j\rangle - \mathcal{E}_j\left(\langle\sigma_-^{eg}\rangle + \xi\langle\sigma_-^{fe}\rangle\right), \tag{17a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger\rangle = -\left(\kappa - i\omega_j\right)\langle a_j^\dagger\rangle - \mathcal{E}_j\left(\langle\sigma_+^{eg}\rangle + \xi\langle\sigma_+^{fe}\rangle\right), \tag{17b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle b_j\rangle = -\left(\kappa_b + i\omega_j^{(b)}\right)\langle b_j\rangle - \mathcal{E}_j^{(b)}\left(\langle\sigma_-^{eg}\rangle + \xi\langle\sigma_-^{fe}\rangle\right), \tag{17c}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle b_j^\dagger\rangle = -\left(\kappa_b - i\omega_j^{(b)}\right)\langle b_j^\dagger\rangle - \mathcal{E}_j^{(b)*}\left(\langle\sigma_+^{eg}\rangle + \xi\langle\sigma_+^{fe}\rangle\right). \tag{17d}$$

2.3 Second-Order: Cavity-Atom Coupled Equations

Using the vector notation, we have moment equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j\boldsymbol{\sigma}\rangle = \boldsymbol{M}_j^{(a)}\langle a_j\boldsymbol{\sigma}\rangle + \begin{pmatrix} -\mathcal{E}_j^{(a)}\langle\sigma_-^{eg}\rangle \\ -\mathcal{E}_j^{(a)}\xi\langle\sigma_-^{fg}\rangle \\ -\mathcal{E}_j^{(a)}\langle\sigma^{ee}\rangle \\ \Gamma\xi^2\langle a_j\rangle - \mathcal{E}_j^{(a)}\xi\langle\sigma_-^{fe}\rangle \\ i\xi\frac{\Omega}{2}\langle a_j\rangle \\ -i\xi\frac{\Omega}{2}\langle a_j\rangle - \mathcal{E}_j^{(a)}\xi\left(1 - \langle\sigma^{gg}\rangle - \langle\sigma^{ee}\rangle\right) \\ 0 \\ -\mathcal{E}_j^{(a)}\langle\sigma_+^{fe}\rangle \end{pmatrix}, \tag{18a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle b_j\boldsymbol{\sigma}\rangle = \boldsymbol{M}_j^{(b)}\langle b_j\boldsymbol{\sigma}\rangle + \begin{pmatrix} -\mathcal{E}_j^{(b)}\langle\sigma_-^{eg}\rangle \\ -\mathcal{E}_j^{(b)}\xi\langle\sigma_-^{fg}\rangle \\ -\mathcal{E}_j^{(b)}\langle\sigma^{ee}\rangle \\ \Gamma\xi^2\langle b_j\rangle - \mathcal{E}_j^{(b)}\xi\langle\sigma_-^{fe}\rangle \\ i\xi\frac{\Omega}{2}\langle b_j\rangle \\ -i\xi\frac{\Omega}{2}\langle b_j\rangle - \mathcal{E}_j^{(b)}\xi\left(1 - \langle\sigma^{gg}\rangle - \langle\sigma^{ee}\rangle\right) \\ 0 \\ -\mathcal{E}_j^{(b)}\langle\sigma_+^{fe}\rangle \end{pmatrix}, \tag{18b}$$

where

$$\boldsymbol{M}_j^{(a)} = \boldsymbol{M} - \left(\kappa_a + i\omega_j^{(a)}\right)\mathbb{1}, \tag{19a}$$

$$\boldsymbol{M}_j^{(b)} = \boldsymbol{M} - \left(\kappa_b + i\omega_j^{(b)}\right)\mathbb{1}; \tag{19b}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger \boldsymbol{\sigma}\rangle = \boldsymbol{M}_j^{(a^\dagger)}\langle a_j^\dagger \boldsymbol{\sigma}\rangle + \begin{pmatrix} -\mathcal{E}_j^{(a)*}\langle\sigma_+^{eg}\rangle \\ -\mathcal{E}_j^{(a)*}\langle\sigma^{ee}\rangle \\ -\mathcal{E}_j^{(a)*}\xi\langle\sigma_+^{fg}\rangle \\ \Gamma\xi^2\langle a_j^\dagger\rangle - \mathcal{E}_j^{(a)*}\xi\langle\sigma_+^{fe}\rangle \\ i\xi\frac{\Omega}{2}\langle a_j^\dagger\rangle - \mathcal{E}_j^{(a)*}\xi\left(1 - \langle\sigma^{gg}\rangle - \langle\sigma^{ee}\rangle\right) \\ -i\xi\frac{\Omega}{2}\langle a_j^\dagger\rangle \\ -\mathcal{E}_j^{(a)*}\langle\sigma_-^{fe}\rangle \\ 0 \end{pmatrix}, \tag{20a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle b_j^\dagger \boldsymbol{\sigma}\rangle = \boldsymbol{M}_j^{(b^\dagger)}\langle b_j^\dagger \boldsymbol{\sigma}\rangle + \begin{pmatrix} -\mathcal{E}_j^{(b)*}\langle\sigma_+^{eg}\rangle \\ -\mathcal{E}_j^{(b)*}\langle\sigma^{ee}\rangle \\ -\mathcal{E}_j^{(b)*}\xi\langle\sigma_+^{fg}\rangle \\ \Gamma\xi^2\langle b_j^\dagger\rangle - \mathcal{E}_j^{(b)*}\xi\langle\sigma_+^{fe}\rangle \\ i\xi\frac{\Omega}{2}\langle b_j^\dagger\rangle - \mathcal{E}_j^{(b)*}\xi\left(1 - \langle\sigma^{gg}\rangle - \langle\sigma^{ee}\rangle\right) \\ -i\xi\frac{\Omega}{2}\langle b_j^\dagger\rangle \\ -\mathcal{E}_j^{(b)*}\langle\sigma_-^{fe}\rangle \\ 0 \end{pmatrix}, \tag{20b}$$

$$\tag{20c}$$

where

$$\boldsymbol{M}_j^{(a^\dagger)} = \boldsymbol{M} - \left(\kappa_a - i\omega_j^{(a)}\right)\mathbb{1}, \tag{21a}$$

$$\boldsymbol{M}_j^{(b^\dagger)} = \boldsymbol{M} - \left(\kappa_b - i\omega_j^{(b)}\right)\mathbb{1}. \tag{21b}$$

2.4 Second-Order: Cavity Equations

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j b_k\rangle = &-\left(\kappa_a + \kappa_b + i\left(\omega_j^{(a)} + \omega_k^{(b)}\right)\right)\langle a_j b_k\rangle \\ &- \mathcal{E}_j^{(a)}\left(\langle b_k\sigma_-^{eg}\rangle + \xi\langle b_k\sigma_-^{fe}\rangle\right) - \mathcal{E}_k^{(b)}\left(\langle a_j\sigma_-^{eg}\rangle + \xi\langle a_j\sigma_-^{fe}\rangle\right),\end{aligned} \tag{22a}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger b_k^\dagger\rangle = &-\left(\kappa_a + \kappa_b - i\left(\omega_j^{(a)} + \omega_k^{(b)}\right)\right)\langle a_j^\dagger b_k^\dagger\rangle \\ &- \mathcal{E}_j^{(a)*}\left(\langle b_k^\dagger\sigma_+^{eg}\rangle + \langle b_k^\dagger\sigma_+^{fe}\rangle\right) - \mathcal{E}_k^{(b)*}\left(\langle a_j^\dagger\sigma_+^{eg}\rangle + \langle a_j^\dagger\sigma_+^{fe}\rangle\right),\end{aligned} \tag{22b}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger a_k\rangle = &-\left(2\kappa_a - i\left(\omega_j^{(a)} - \omega_k^{(a)}\right)\right)\langle a_j^\dagger a_k\rangle \\ &- \mathcal{E}_j^{(a)*}\left(\langle a_k\sigma_+^{eg}\rangle + \xi\langle a_k\sigma_+^{fe}\rangle\right) - \mathcal{E}_k^{(a)}\left(\langle a_j^\dagger\sigma_-^{eg}\rangle + \xi\langle a_j^\dagger\sigma_-^{fe}\rangle\right),\end{aligned} \tag{22c}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle b_j^\dagger b_k\rangle = -\left(2\kappa_b - i\left(\omega_j^{(b)} - \omega_k^{(b)}\right)\right)\langle b_j^\dagger b_k\rangle$$

$$- \mathcal{E}_j^{(b)*} \left(\langle b_k \sigma_+^{eg} \rangle + \xi \langle b_k \sigma_+^{fe} \rangle \right) - \mathcal{E}_k^{(b)} \left(\langle b_j^\dagger \sigma_-^{eg} \rangle + \xi \langle b_j^\dagger \sigma_-^{fe} \rangle \right), \tag{22d}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle a_j^\dagger b_k \rangle = - \left(\kappa_a + \kappa_b - i \left(\omega_j^{(a)} - \omega_k^{(b)} \right) \right) \langle a_j^\dagger b_k \rangle$$
$$- \mathcal{E}_j^{(a)*} \left(\langle b_k \sigma_+^{eg} \rangle + \xi \langle b_k \sigma_+^{fe} \rangle \right) - \mathcal{E}_k^{(b)} \left(\langle a_j^\dagger \sigma_-^{eg} \rangle + \xi \langle a_j^\dagger \sigma_-^{fe} \rangle \right), \tag{22e}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle b_j^\dagger a_k \rangle = - \left(\kappa_a + \kappa_b - i \left(\omega_j^{(b)} - \omega_k^{(a)} \right) \right) \langle b_j^\dagger a_k \rangle$$
$$- \mathcal{E}_j^{(b)*} \left(\langle a_k \sigma_+^{eg} \rangle + \xi \langle a_k \sigma_+^{fe} \rangle \right) - \mathcal{E}_k^{(a)} \left(\langle b_j^\dagger \sigma_-^{eg} \rangle + \xi \langle b_j^\dagger \sigma_-^{fe} \rangle \right) \tag{22f}$$

2.5 Third-Order: Cavity-Atom Coupled Equations

Using the vector notation, we have moment equations

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle a_j b_k \boldsymbol{\sigma} \rangle = \boldsymbol{M}_{j,k}^{(ab)} \langle a_j b_k \boldsymbol{\sigma} \rangle + \begin{pmatrix} -\mathcal{E}_j^{(a)} \langle b_k \sigma_-^{eg} \rangle - \mathcal{E}_k^{(b)} \langle a_j \sigma_-^{eg} \rangle \\ -\mathcal{E}_j^{(a)} \xi \langle b_k \sigma_-^{fg} \rangle - \mathcal{E}_k^{(b)} \xi \langle a_j \sigma_-^{fg} \rangle \\ -\mathcal{E}_j^{(a)} \langle b_k \sigma^{ee} \rangle - \mathcal{E}_k^{(b)} \langle a_j \sigma^{ee} \rangle \\ \Gamma \xi \langle a_j b_k \rangle - \mathcal{E}_j^{(a)} \xi \langle b_k \sigma_-^{fe} \rangle - \mathcal{E}_k^{(b)} \xi \langle a_j \sigma_-^{fe} \rangle \\ i \xi \frac{\Omega}{2} \langle a_j b_k \rangle \\ -i \xi \frac{\Omega}{2} \langle a_j b_k \rangle - \mathcal{E}_j^{(a)} \xi \left(\langle b_k \rangle - \langle b_k \sigma^{gg} \rangle - \langle b_k \sigma^{ee} \rangle \right) - \mathcal{E}_k^{(b)} \xi \left(\langle a_j \rangle - \langle a_j \sigma^{gg} \rangle - \langle a_j \sigma^{ee} \rangle \right) \\ 0 \\ -\mathcal{E}_j^{(a)} \langle b_k \sigma_+^{fe} \rangle - \mathcal{E}_k^{(b)} \langle a_j \sigma_+^{fe} \rangle \end{pmatrix}, \tag{23a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle a_j^\dagger b_k^\dagger \boldsymbol{\sigma} \rangle = \boldsymbol{M}_{j,k}^{(a^\dagger b^\dagger)} \langle a_j^\dagger b_k^\dagger \boldsymbol{\sigma} \rangle + \begin{pmatrix} -\mathcal{E}_j^{(a)*} \langle b_k^\dagger \sigma_+^{eg} \rangle - \mathcal{E}_k^{(b)*} \langle a_j^\dagger \sigma_+^{eg} \rangle \\ -\mathcal{E}_j^{(a)*} \langle b_k^\dagger \sigma^{ee} \rangle - \mathcal{E}_k^{(b)*} \langle a_j^\dagger \sigma^{ee} \rangle \\ -\mathcal{E}_j^{(a)*} \xi \langle b_k^\dagger \sigma_+^{fg} \rangle - \mathcal{E}_k^{(b)*} \xi \langle a_j^\dagger \sigma_+^{fg} \rangle \\ \Gamma \xi^2 \langle a_j^\dagger b_k^\dagger \rangle - \mathcal{E}_j^{(a)*} \xi \langle b_k^\dagger \sigma_+^{fe} \rangle - \mathcal{E}_k^{(b)*} \xi \langle a_j^\dagger \sigma_+^{fe} \rangle \\ i \xi \frac{\Omega}{2} \langle a_j^\dagger b_k^\dagger \rangle - \mathcal{E}_j^{(a)*} \xi \left(\langle b_k^\dagger \rangle - \langle b_k^\dagger \sigma^{gg} \rangle - \langle b_k^\dagger \sigma^{ee} \rangle \right) - \mathcal{E}_k^{(b)*} \xi \left(\langle a_j^\dagger \rangle - \langle a_j^\dagger \sigma^{gg} \rangle - \langle a_j^\dagger \sigma^{ee} \rangle \right) \\ -i \xi \frac{\Omega}{2} \langle a_j^\dagger b_k^\dagger \rangle \\ -\mathcal{E}_j^{(a)*} \langle b_k^\dagger \sigma_-^{fe} \rangle - \mathcal{E}_k^{(b)*} \langle a_j^\dagger \sigma_-^{fe} \rangle \\ 0 \end{pmatrix}, \tag{23b}$$

where

$$\boldsymbol{M}_{j,k}^{(ab)} = \boldsymbol{M} - \left(\kappa_a + \kappa_b + i \left(\omega_j^{(a)} + \omega_k^{(b)} \right) \right) \mathbb{1}, \tag{24a}$$
$$\boldsymbol{M}_{j,k}^{(a^\dagger b^\dagger)} = \boldsymbol{M} - \left(\kappa_a + \kappa_b - i \left(\omega_j^{(a)} + \omega_k^{(b)} \right) \right) \mathbb{1}; \tag{24b}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle a_j^\dagger a_k \boldsymbol{\sigma} \rangle = \boldsymbol{M}_{j,k}^{(a^\dagger a)} \langle a_j^\dagger a_k \boldsymbol{\sigma} \rangle + \begin{pmatrix} -\mathcal{E}_j^{(a)*} \langle a_k \sigma_+^{eg} \rangle - \mathcal{E}_k^{(a)} \langle a_j^\dagger \sigma_-^{eg} \rangle \\ -\mathcal{E}_j^{(a)*} \langle a_k \sigma^{ee} \rangle - \mathcal{E}_k^{(a)} \xi \langle a_j^\dagger \sigma_-^{fg} \rangle \\ -\mathcal{E}_j^{(a)*} \xi \langle a_k \sigma_+^{fg} \rangle - \mathcal{E}_k^{(a)} \langle a_j^\dagger \sigma^{ee} \rangle \\ \Gamma \xi^2 \langle a_j^\dagger a_k \rangle - \mathcal{E}_j^{(a)*} \xi \langle a_k \sigma_+^{fe} \rangle - \mathcal{E}_k^{(a)} \xi \langle a_k^\dagger \sigma_-^{fe} \rangle \\ i \xi \frac{\Omega}{2} \langle a_j^\dagger a_k \rangle - \mathcal{E}_j^{(a)*} \xi \left(\langle a_k \rangle - \langle a_k \sigma^{gg} \rangle - \langle a_k \sigma^{ee} \rangle \right) \\ -i \xi \frac{\Omega}{2} \langle a_j^\dagger a_k \rangle - \mathcal{E}_k^{(a)} \xi \left(\langle a_j^\dagger \rangle - \langle a_j^\dagger \sigma^{gg} \rangle - \langle a_j^\dagger \sigma^{ee} \rangle \right) \\ -\mathcal{E}_j^{(a)*} \langle a_k \sigma_-^{fe} \rangle \\ -\mathcal{E}_k^{(a)} \langle a_j^\dagger \sigma_+^{fe} \rangle \end{pmatrix}, \tag{25a}$$

$$
\frac{\mathrm{d}}{\mathrm{d}t}\langle b_j^\dagger b_k \boldsymbol{\sigma}\rangle = \boldsymbol{M}_{j,k}^{(b^\dagger b)}\langle b_j^\dagger b_k \boldsymbol{\sigma}\rangle + \begin{pmatrix}
-\mathcal{E}_j^{(b)*}\langle b_k \sigma_+^{eg}\rangle - \mathcal{E}_k^{(b)}\langle b_j^\dagger \sigma_-^{eg}\rangle \\
-\mathcal{E}_j^{(b)*}\langle b_k \sigma^{ee}\rangle - \mathcal{E}_k^{(b)}\xi\langle b_j^\dagger \sigma_-^{fg}\rangle \\
-\mathcal{E}_j^{(b)*}\xi\langle b_k \sigma_+^{fg}\rangle - \mathcal{E}_k^{(b)}\langle b_j^\dagger \sigma^{ee}\rangle \\
\Gamma\xi^2\langle b_j^\dagger b_k\rangle - \mathcal{E}_j^{(b)*}\xi\langle b_k \sigma_+^{fe}\rangle - \mathcal{E}_k^{(b)}\xi\langle b_k^\dagger \sigma_-^{fe}\rangle \\
i\xi\frac{\Omega}{2}\langle b_j^\dagger b_k\rangle - \mathcal{E}_j^{(b)*}\xi\left(\langle b_k\rangle - \langle b_k\sigma^{gg}\rangle - \langle b_k\sigma^{ee}\rangle\right) \\
-i\xi\frac{\Omega}{2}\langle b_j^\dagger b_k\rangle - \mathcal{E}_k^{(b)}\xi\left(\langle b_j^\dagger\rangle - \langle b_j^\dagger\sigma^{gg}\rangle - \langle b_j^\dagger\sigma^{ee}\rangle\right) \\
-\mathcal{E}_j^{(b)*}\langle b_k \sigma_-^{fe}\rangle \\
-\mathcal{E}_k^{(b)}\langle b_j^\dagger \sigma_+^{fe}\rangle
\end{pmatrix}, \tag{25b}
$$

where

$$
\boldsymbol{M}_{j,k}^{(a^\dagger a)} = \boldsymbol{M} - \left(2\kappa_a - i\left(\omega_j^{(a)} - i\omega_k^{(a)}\right)\right)\mathbb{1}, \tag{26a}
$$

$$
\boldsymbol{M}_{j,k}^{(b^\dagger b)} = \boldsymbol{M} - \left(2\kappa_b - i\left(\omega_j^{(b)} - i\omega_k^{(b)}\right)\right)\mathbb{1}; \tag{26b}
$$

and

$$
\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger b_k \boldsymbol{\sigma}\rangle = \boldsymbol{M}_{j,k}^{(a^\dagger b)}\langle a_j^\dagger b_k \boldsymbol{\sigma}\rangle + \begin{pmatrix}
-\mathcal{E}_j^{(a)*}\langle b_k \sigma_+^{eg}\rangle - \mathcal{E}_k^{(b)}\langle a_j^\dagger \sigma_-^{eg}\rangle \\
-\mathcal{E}_j^{(a)*}\langle b_k \sigma^{ee}\rangle - \mathcal{E}_k^{(b)}\xi\langle a_j^\dagger \sigma_-^{fg}\rangle \\
-\mathcal{E}_j^{(a)*}\xi\langle b_k \sigma_+^{fg}\rangle - \mathcal{E}_k^{(b)}\langle a_j^\dagger \sigma^{ee}\rangle \\
\Gamma\xi^2\langle a_j^\dagger b_k\rangle - \mathcal{E}_j^{(a)*}\xi\langle b_k \sigma_+^{fe}\rangle - \mathcal{E}_k^{(b)}\xi\langle a_k^\dagger \sigma_-^{fe}\rangle \\
i\xi\frac{\Omega}{2}\langle a_j^\dagger b_k\rangle - \mathcal{E}_j^{(a)*}\xi\left(\langle b_k\rangle - \langle b_k\sigma^{gg}\rangle - \langle b_k\sigma^{ee}\rangle\right) \\
-i\xi\frac{\Omega}{2}\langle a_j^\dagger b_k\rangle - \mathcal{E}_k^{(b)}\xi\left(\langle a_j^\dagger\rangle - \langle a_j^\dagger\sigma^{gg}\rangle - \langle a_j^\dagger\sigma^{ee}\rangle\right) \\
-\mathcal{E}_j^{(a)*}\langle b_k \sigma_-^{fe}\rangle \\
-\mathcal{E}_k^{(b)}\langle a_j^\dagger \sigma_+^{fe}\rangle
\end{pmatrix}, \tag{27a}
$$

$$
\frac{\mathrm{d}}{\mathrm{d}t}\langle b_j^\dagger a_k \boldsymbol{\sigma}\rangle = \boldsymbol{M}_{j,k}^{(b^\dagger a)}\langle b_j^\dagger a_k \boldsymbol{\sigma}\rangle + \begin{pmatrix}
-\mathcal{E}_j^{(b)*}\langle a_k \sigma_+^{eg}\rangle - \mathcal{E}_k^{(a)}\langle b_j^\dagger \sigma_-^{eg}\rangle \\
-\mathcal{E}_j^{(b)*}\langle a_k \sigma^{ee}\rangle - \mathcal{E}_k^{(a)}\xi\langle b_j^\dagger \sigma_-^{fg}\rangle \\
-\mathcal{E}_j^{(b)*}\xi\langle a_k \sigma_+^{fg}\rangle - \mathcal{E}_k^{(a)}\langle b_j^\dagger \sigma^{ee}\rangle \\
\Gamma\xi^2\langle b_j^\dagger a_k\rangle - \mathcal{E}_j^{(b)*}\xi\langle a_k \sigma_+^{fe}\rangle - \mathcal{E}_k^{(a)}\xi\langle b_k^\dagger \sigma_-^{fe}\rangle \\
i\xi\frac{\Omega}{2}\langle b_j^\dagger a_k\rangle - \mathcal{E}_j^{(b)*}\xi\left(\langle a_k\rangle - \langle a_k\sigma^{gg}\rangle - \langle a_k\sigma^{ee}\rangle\right) \\
-i\xi\frac{\Omega}{2}\langle b_j^\dagger a_k\rangle - \mathcal{E}_k^{(a)}\xi\left(\langle b_j^\dagger\rangle - \langle b_j^\dagger\sigma^{gg}\rangle - \langle b_j^\dagger\sigma^{ee}\rangle\right) \\
-\mathcal{E}_j^{(b)*}\langle a_k \sigma_-^{fe}\rangle \\
-\mathcal{E}_k^{(a)}\langle b_j^\dagger \sigma_+^{fe}\rangle
\end{pmatrix}, \tag{27b}
$$

where

$$
\boldsymbol{M}_{j,k}^{(a^\dagger b)} = \boldsymbol{M} - \left(\kappa_a + \kappa_b - i\left(\omega_j^{(a)} - i\omega_k^{(b)}\right)\right)\mathbb{1}, \tag{28a}
$$

$$
\boldsymbol{M}_{j,k}^{(b^\dagger a)} = \boldsymbol{M} - \left(\kappa_a + \kappa_b - i\left(\omega_j^{(b)} - i\omega_k^{(a)}\right)\right)\mathbb{1}; \tag{28b}
$$

2.6 Third-Order: Cavity Equations

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger a_k b_l\rangle = &-\left(2\kappa_a + \kappa_b - i\left(\omega_j^{(a)} - \omega_k^{(a)} - \omega_l^{(b)}\right)\right)\langle a_j^\dagger a_k b_l\rangle \\
&- \mathcal{E}_j^{(a)*}\left(\langle a_k b_l \sigma_+^{eg}\rangle + \xi\langle a_k b_l \sigma_+^{fe}\rangle\right) \\
&- \mathcal{E}_k^{(a)}\left(\langle a_j^\dagger b_l \sigma_-^{eg}\rangle + \xi\langle a_j^\dagger b_l \sigma_-^{fe}\rangle\right)
\end{aligned}
$$

$$
\begin{aligned}
&- \mathcal{E}_l^{(b)} \left(\langle a_j^\dagger a_k \sigma_-^{eg} \rangle + \xi \langle a_j^\dagger a_k \sigma_-^{fe} \rangle \right), && \text{(29a)} \\
\frac{\mathrm{d}}{\mathrm{d}t} \langle b_j^\dagger b_k a_l \rangle = &- \left(\kappa_a + 2\kappa_b - i \left(\omega_j^{(b)} - \omega_k^{(b)} - \omega_l^{(a)} \right) \right) \langle b_j^\dagger b_k a_l \rangle \\
&- \mathcal{E}_j^{(b)*} \left(\langle b_k a_l \sigma_+^{eg} \rangle + \xi \langle b_k a_l \sigma_+^{fe} \rangle \right) \\
&- \mathcal{E}_k^{(b)} \left(\langle b_j^\dagger a_l \sigma_-^{eg} \rangle + \xi \langle b_j^\dagger a_l \sigma_-^{fe} \rangle \right) \\
&- \mathcal{E}_l^{(a)} \left(\langle b_j^\dagger b_k \sigma_-^{eg} \rangle + \xi \langle b_j^\dagger b_k \sigma_-^{fe} \rangle \right), && \text{(29b)}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \langle b_j^\dagger a_k^\dagger a_l \rangle = &- \left(2\kappa_a + \kappa_b - i \left(\omega_j^{(b)} + \omega_k^{(a)} - \omega_l^{(a)} \right) \right) \langle b_j^\dagger a_k^\dagger a_l \rangle \\
&- \mathcal{E}_j^{(b)*} \left(\langle a_k^\dagger a_l \sigma_+^{eg} \rangle + \xi \langle a_k^\dagger a_l \sigma_+^{fe} \rangle \right) \\
&- \mathcal{E}_k^{(a)*} \left(\langle b_j^\dagger a_l \sigma_+^{eg} \rangle + \xi \langle b_j^\dagger a_l \sigma_+^{fe} \rangle \right) \\
&- \mathcal{E}_l^{(a)} \left(\langle b_j^\dagger a_k^\dagger \sigma_-^{eg} \rangle + \xi \langle b_j^\dagger a_k^\dagger \sigma_-^{fe} \rangle \right), && \text{(30a)} \\
\frac{\mathrm{d}}{\mathrm{d}t} \langle a_j^\dagger b_k^\dagger b_l \rangle = &- \left(\kappa_a + 2\kappa_b - i \left(\omega_j^{(a)} + \omega_k^{(b)} - \omega_l^{(b)} \right) \right) \langle b_j^\dagger b_k a_l \rangle \\
&- \mathcal{E}_j^{(a)*} \left(\langle b_k^\dagger b_l \sigma_+^{eg} \rangle + \xi \langle b_k^\dagger b_l \sigma_+^{fe} \rangle \right) \\
&- \mathcal{E}_k^{(b)*} \left(\langle a_j^\dagger b_l \sigma_+^{eg} \rangle + \xi \langle a_j^\dagger b_l \sigma_+^{fe} \rangle \right) \\
&- \mathcal{E}_l^{(b)} \left(\langle a_j^\dagger b_k^\dagger \sigma_-^{eg} \rangle + \xi \langle a_j^\dagger b_k^\dagger \sigma_-^{fe} \rangle \right), && \text{(30b)}
\end{aligned}
$$

2.7 Fourth-Order: Cavity-Atom Coupled Equations

Using the vector notation, we have moment equations

$$
\frac{\mathrm{d}}{\mathrm{d}t} \langle a_j^\dagger a_k b_l \boldsymbol{\sigma} \rangle = \boldsymbol{M}_{j,k,l}^{(a^\dagger ab)} \langle a_j^\dagger a_k b_l \boldsymbol{\sigma} \rangle +
\begin{pmatrix}
-\mathcal{E}_j^{(a)*} \langle a_k b_l \sigma_+^{eg} \rangle - \mathcal{E}_k^{(a)} \langle a_j^\dagger b_l \sigma_-^{eg} \rangle - \mathcal{E}_l^{(b)} \langle a_j^\dagger a_k \sigma_-^{eg} \rangle \\
-\mathcal{E}_j^{(a)*} \langle a_k b_l \sigma^{ee} \rangle - \mathcal{E}_k^{(a)} \xi \langle a_j^\dagger b_l \sigma_-^{fg} \rangle - \mathcal{E}_l^{(b)} \xi \langle a_j^\dagger a_k \sigma_-^{fg} \rangle \\
-\mathcal{E}_j^{(a)*} \xi \langle a_k b_l \sigma_+^{fg} \rangle - \mathcal{E}_k^{(a)} \langle a_j^\dagger b_l \sigma^{ee} \rangle - \mathcal{E}_l^{(b)} \langle a_j^\dagger a_k \sigma^{ee} \rangle \\
\Gamma \xi^2 \langle a_j^\dagger a_k b_l \rangle - \mathcal{E}_j^{(a)*} \xi \langle a_k b_l \sigma_+^{fe} \rangle - \mathcal{E}_k^{(a)} \xi \langle a_j^\dagger b_l \sigma_-^{fe} \rangle - \mathcal{E}_l^{(b)} \xi \langle a_j^\dagger a_k \sigma_-^{fe} \rangle \\
i\xi \frac{\Omega}{2} \langle a_j^\dagger a_k b_l \rangle - \mathcal{E}_j^{(a)*} \xi \left(\langle a_k b_l \rangle - \langle a_k b_l \sigma^{gg} \rangle - \langle a_k b_l \sigma^{ee} \rangle \right) \\
-i\xi \frac{\Omega}{2} \langle a_j^\dagger a_k b_l \rangle - \mathcal{E}_k^{(a)} \xi \left(\langle a_j^\dagger a_k \rangle - \langle a_j^\dagger b_l \sigma^{gg} \rangle - \langle a_j^\dagger b_l \sigma^{ee} \rangle \right) - \mathcal{E}_l^{(b)} \xi \left(\langle a_j^\dagger b_l \rangle - \langle a_j^\dagger a_k \sigma^{gg} \rangle - \langle a_j^\dagger a_k \sigma^{ee} \rangle \right) \\
-\mathcal{E}_j^{(a)*} \langle a_k b_l \sigma_-^{fe} \rangle \\
-\mathcal{E}_k^{(a)} \langle a_j^\dagger b_l \sigma_+^{fe} \rangle - \mathcal{E}_l^{(b)} \langle a_j^\dagger a_k \sigma_-^{fe} \rangle
\end{pmatrix},
\tag{31a}
$$

$$
\frac{\mathrm{d}}{\mathrm{d}t} \langle b_j^\dagger b_k a_l \boldsymbol{\sigma} \rangle = \boldsymbol{M}_{j,k,l}^{(b^\dagger ba)} \langle b_j^\dagger b_k a_l \boldsymbol{\sigma} \rangle +
$$

$$\begin{pmatrix} -\mathcal{E}_j^{(b)*}\langle b_k a_l \sigma_+^{eg}\rangle - \mathcal{E}_k^{(b)}\langle b_j^\dagger a_l \sigma_-^{eg}\rangle - \mathcal{E}_l^{(a)}\langle b_j^\dagger b_k \sigma_-^{eg}\rangle \\ -\mathcal{E}_j^{(b)*}\langle b_k a_l \sigma^{ee}\rangle - \mathcal{E}_k^{(b)}\xi\langle b_j^\dagger a_l \sigma_-^{fg}\rangle - \mathcal{E}_l^{(a)}\xi\langle b_j^\dagger b_k \sigma_-^{fg}\rangle \\ -\mathcal{E}_j^{(b)*}\xi\langle b_k a_l \sigma_+^{fg}\rangle - \mathcal{E}_k^{(b)}\langle b_j^\dagger a_l \sigma^{ee}\rangle - \mathcal{E}_l^{(a)}\langle b_j^\dagger b_k \sigma^{ee}\rangle \\ \Gamma\xi^2\langle b_j^\dagger b_k a_l\rangle - \mathcal{E}_j^{(b)*}\xi\langle b_k a_l \sigma_+^{fe}\rangle - \mathcal{E}_k^{(b)}\xi\langle b_j^\dagger a_l \sigma_-^{fe}\rangle - \mathcal{E}_l^{(a)}\xi\langle b_j^\dagger b_k \sigma_-^{fe}\rangle \\ i\xi\frac{\Omega}{2}\langle b_j^\dagger b_k a_l\rangle - \mathcal{E}_j^{(b)*}\xi\left(\langle b_k a_l\rangle - \langle b_k a_l \sigma^{gg}\rangle - \langle b_k a_l \sigma^{ee}\rangle\right) \\ -i\xi\frac{\Omega}{2}\langle b_j^\dagger b_k a_l\rangle - \mathcal{E}_k^{(b)}\xi\left(\langle b_j^\dagger b_k\rangle - \langle b_j^\dagger a_l \sigma^{gg}\rangle - \langle b_j^\dagger a_l \sigma^{ee}\rangle\right) - \mathcal{E}_l^{(a)}\xi\left(\langle b_j^\dagger a_l\rangle - \langle b_j^\dagger b_k \sigma^{gg}\rangle - \langle b_j^\dagger b_k \sigma^{ee}\rangle\right) \\ -\mathcal{E}_j^{(b)*}\langle b_k a_l \sigma_-^{fe}\rangle \\ -\mathcal{E}_k^{(b)}\langle b_j^\dagger a_l \sigma_+^{fe}\rangle - \mathcal{E}_l^{(a)}\langle b_j^\dagger b_k \sigma_-^{fe}\rangle \end{pmatrix}, \tag{31b}$$

where

$$\boldsymbol{M}_{j,k,l}^{(a^\dagger ab)} = \boldsymbol{M} - \left(2\kappa_a + \kappa_b - i\left(\omega_j^{(a)} - \omega_k^{(a)} - \omega_l^{(b)}\right)\right)\mathbb{1}, \tag{32a}$$

$$\boldsymbol{M}_{j,k,l}^{(b^\dagger ba)} = \boldsymbol{M} - \left(\kappa_a + 2\kappa_b - i\left(\omega_j^{(b)} - \omega_k^{(b)} - \omega_l^{(a)}\right)\right)\mathbb{1}; \tag{32b}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle b_j^\dagger a_k^\dagger a_l \boldsymbol{\sigma}\rangle = \boldsymbol{M}_{j,k,l}^{(b^\dagger a^\dagger a)}\langle b_j^\dagger a_k^\dagger a_l \boldsymbol{\sigma}\rangle +$$

$$\begin{pmatrix} -\mathcal{E}_j^{(b)*}\langle a_k^\dagger a_l \sigma_+^{eg}\rangle - \mathcal{E}_k^{(a)*}\langle b_j^\dagger a_l \sigma_+^{eg}\rangle - \mathcal{E}_l^{(a)}\langle b_j^\dagger a_k^\dagger \sigma_-^{eg}\rangle \\ -\mathcal{E}_j^{(b)*}\langle a_k^\dagger a_l \sigma^{ee}\rangle - \mathcal{E}_k^{(a)*}\langle b_j^\dagger a_l \sigma^{ee}\rangle - \mathcal{E}_l^{(a)}\xi\langle b_j^\dagger a_k^\dagger \sigma_-^{fg}\rangle \\ -\mathcal{E}_j^{(b)*}\xi\langle a_k^\dagger a_l \sigma_+^{fg}\rangle - \mathcal{E}_k^{(a)*}\xi\langle b_j^\dagger a_l \sigma_+^{fg}\rangle - \mathcal{E}_l^{(a)}\langle b_j^\dagger a_k^\dagger \sigma^{ee}\rangle \\ \Gamma\xi^2\langle b_j^\dagger a_k^\dagger a_l\rangle - \mathcal{E}_j^{(b)*}\xi\langle a_k^\dagger a_l \sigma_+^{fe}\rangle - \mathcal{E}_k^{(a)*}\xi\langle b_j^\dagger a_l \sigma_+^{fe}\rangle - \mathcal{E}_l^{(a)}\xi\langle b_j^\dagger a_k^\dagger \sigma_-^{fe}\rangle \\ i\xi\frac{\Omega}{2}\langle b_j^\dagger a_k^\dagger a_l\rangle - \mathcal{E}_j^{(b)*}\xi\left(\langle a_k^\dagger a_l\rangle - \langle a_k^\dagger a_l \sigma^{gg}\rangle - \langle a_k^\dagger a_l \sigma^{ee}\rangle\right) - \mathcal{E}_k^{(a)*}\xi\left(\langle b_j^\dagger a_l\rangle - \langle b_j^\dagger a_l \sigma^{gg}\rangle - \langle b_j^\dagger a_l \sigma^{ee}\rangle\right) \\ -i\xi\frac{\Omega}{2}\langle b_j^\dagger a_k^\dagger a_l\rangle - \mathcal{E}_l^{(a)}\xi\left(\langle b_j^\dagger a_k^\dagger\rangle - \langle b_j^\dagger a_k^\dagger \sigma^{gg}\rangle - \langle b_j^\dagger a_k^\dagger \sigma^{ee}\rangle\right) \\ -\mathcal{E}_j^{(b)*}\langle a_k^\dagger a_l \sigma_-^{fe}\rangle - \mathcal{E}_k^{(a)*}\langle b_j^\dagger a_l \sigma_-^{fe}\rangle \\ -\mathcal{E}_l^{(a)}\langle b_j^\dagger a_k^\dagger \sigma_+^{fe}\rangle \end{pmatrix}, \tag{33a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger b_k^\dagger b_l \boldsymbol{\sigma}\rangle = \boldsymbol{M}_{j,k,l}^{(a^\dagger b^\dagger b)}\langle a_j^\dagger b_k^\dagger b_l \boldsymbol{\sigma}\rangle +$$

$$\begin{pmatrix} -\mathcal{E}_j^{(a)*}\langle b_k^\dagger b_l \sigma_+^{eg}\rangle - \mathcal{E}_k^{(b)*}\langle a_j^\dagger b_l \sigma_+^{eg}\rangle - \mathcal{E}_l^{(b)}\langle a_j^\dagger b_k^\dagger \sigma_-^{eg}\rangle \\ -\mathcal{E}_j^{(a)*}\langle b_k^\dagger b_l \sigma^{ee}\rangle - \mathcal{E}_k^{(b)*}\langle a_j^\dagger b_l \sigma^{ee}\rangle - \mathcal{E}_l^{(b)}\xi\langle a_j^\dagger b_k^\dagger \sigma_-^{fg}\rangle \\ -\mathcal{E}_j^{(a)*}\xi\langle b_k^\dagger b_l \sigma_+^{fg}\rangle - \mathcal{E}_k^{(b)*}\xi\langle a_j^\dagger b_l \sigma_+^{fg}\rangle - \mathcal{E}_l^{(b)}\langle a_j^\dagger b_k^\dagger \sigma^{ee}\rangle \\ \Gamma\xi^2\langle a_j^\dagger b_k^\dagger b_l\rangle - \mathcal{E}_j^{(a)*}\xi\langle b_k^\dagger b_l \sigma_+^{fe}\rangle - \mathcal{E}_k^{(b)*}\xi\langle a_j^\dagger b_l \sigma_+^{fe}\rangle - \mathcal{E}_l^{(b)}\xi\langle a_j^\dagger b_k^\dagger \sigma_-^{fe}\rangle \\ i\xi\frac{\Omega}{2}\langle a_j^\dagger b_k^\dagger b_l\rangle - \mathcal{E}_j^{(a)*}\xi\left(\langle b_k^\dagger b_l\rangle - \langle b_k^\dagger b_l \sigma^{gg}\rangle - \langle b_k^\dagger b_l \sigma^{ee}\rangle\right) - \mathcal{E}_k^{(b)*}\xi\left(\langle a_j^\dagger b_l\rangle - \langle a_j^\dagger b_l \sigma^{gg}\rangle - \langle a_j^\dagger b_l \sigma^{ee}\rangle\right) \\ -i\xi\frac{\Omega}{2}\langle a_j^\dagger b_k^\dagger b_l\rangle - \mathcal{E}_l^{(b)}\xi\left(\langle a_j^\dagger b_k^\dagger\rangle - \langle a_j^\dagger b_k^\dagger \sigma^{gg}\rangle - \langle a_j^\dagger b_k^\dagger \sigma^{ee}\rangle\right) \\ -\mathcal{E}_j^{(a)*}\langle b_k^\dagger b_l \sigma_-^{fe}\rangle - \mathcal{E}_k^{(b)*}\langle a_j^\dagger b_l \sigma_-^{fe}\rangle \\ -\mathcal{E}_l^{(b)}\langle a_j^\dagger b_k^\dagger \sigma_+^{fe}\rangle \end{pmatrix}, \tag{33b}$$

where

$$\boldsymbol{M}_{j,k,l}^{(b^\dagger a^\dagger a)} = \boldsymbol{M} - \left(2\kappa_a + \kappa_b - i\left(\omega_j^{(b)} + \omega_k^{(a)} - \omega_l^{(a)}\right)\right)\mathbb{1}, \tag{34a}$$

$$\boldsymbol{M}_{j,k,l}^{(a^\dagger b^\dagger b)} = \boldsymbol{M} - \left(\kappa_a + 2\kappa_b - i\left(\omega_j^{(a)} + \omega_k^{(b)} - \omega_l^{(b)}\right)\right)\mathbb{1}. \tag{34b}$$

2.8 Fourth-Order: Cavity Equation

Finally, we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger b_k^\dagger b_l a_m \rangle = &- \left(2\kappa_a + 2\kappa_b - i\left(\omega_j^{(a)} + \omega_k^{(b)}\right) + i\left(\omega_l^{(b)} + \omega_m^{(a)}\right)\right)\langle a_j^\dagger b_k^\dagger b_l a_m \rangle \\
&- \mathcal{E}_j^{(a)*}\left(\langle b_k^\dagger b_l a_m \sigma_+^{eg}\rangle + \xi\langle b_k^\dagger b_l a_m \sigma_+^{fe}\rangle\right) \\
&- \mathcal{E}_k^{(b)*}\left(\langle a_j^\dagger a_l a_m \sigma_+^{eg}\rangle + \xi\langle a_j^\dagger a_l a_m \sigma_+^{fe}\rangle\right) \\
&- \mathcal{E}_l^{(b)}\left(\langle a_j^\dagger b_k^\dagger a_m \sigma_-^{eg}\rangle + \xi\langle a_j^\dagger b_k^\dagger a_m \sigma_-^{fe}\rangle\right) \\
&- \mathcal{E}_m^{(a)}\left(\langle a_j^\dagger b_k^\dagger b_l \sigma_-^{eg}\rangle + \xi\langle a_j^\dagger b_k^\dagger b_l \sigma_-^{fe}\rangle\right).
\end{aligned} \tag{35}
$$

3 Second-Order Cross-Correlation Function

The second-order correlation function for the filtered output field in the steady state is given by

$$
G_{\mathrm{cross}}^{(2)}(\tau) = \langle A^\dagger(0) B^\dagger B(\tau) A(0)\rangle = \sum_{j,k=-N}^{N} \langle A^\dagger(0) b_j^\dagger b_k(\tau) A(0)\rangle, \tag{36}
$$

with the normalised second-order correlation function given by

$$
g_{\mathrm{cross}}^{(2)}(\tau) = \frac{G_{\mathrm{cross}}^{(2)}(\tau)}{\langle A^\dagger A\rangle_{ss}\langle B^\dagger B\rangle_{ss}}. \tag{37}
$$

Using the Quantum Regression Theorem, we can solve for the second-order correlation function by solving the following moment equations

$$
\frac{\mathrm{d}}{\mathrm{d}\tau}\langle A^\dagger(0)\boldsymbol{\sigma} A(0)\rangle = \boldsymbol{M}\langle A^\dagger(0)\boldsymbol{\sigma} A(0)\rangle + \boldsymbol{B}, \tag{38a}
$$

$$
\frac{\mathrm{d}}{\mathrm{d}\tau}\langle A^\dagger(0) b_j^\dagger(\tau) A(0)\rangle = -\left(\kappa_b - i\omega_j^{(b)}\right)\langle A^\dagger(0) b_j^\dagger(\tau) A(0)\rangle - \mathcal{E}_j^{(b)*}\left(\langle A^\dagger(0)\sigma_+^{eg}(\tau)A(0)\rangle + \xi\langle A^\dagger(0)\sigma_+^{fe}A(0)\rangle\right), \tag{38b}
$$

$$
\frac{\mathrm{d}}{\mathrm{d}\tau}\langle A^\dagger(0) b_j(\tau) A(0)\rangle = -\left(\kappa_b + i\omega_j^{(b)}\right)\langle A^\dagger(0) b_j(\tau) A(0)\rangle - \mathcal{E}_j^{(b)}\left(\langle A^\dagger(0)\sigma_-^{eg}(\tau)A(0)\rangle + \xi\langle A^\dagger(0)\sigma_-^{fe}A(0)\rangle\right), \tag{38c}
$$

with

$$
\langle\boldsymbol{\sigma}\rangle = \begin{pmatrix} \langle A^\dagger(0)\sigma^{gg}(\tau)A(0)\rangle \\ \langle A^\dagger(0)\sigma_-^{eg}(\tau)A(0)\rangle \\ \langle A^\dagger(0)\sigma_+^{eg}(\tau)A(0)\rangle \\ \langle A^\dagger(0)\sigma^{ee}(\tau)A(0)\rangle \\ \langle A^\dagger(0)\sigma_-^{fe}(\tau)A(0)\rangle \\ \langle A^\dagger(0)\sigma_+^{fe}(\tau)A(0)\rangle \\ \langle A^\dagger(0)\sigma_-^{fg}(\tau)A(0)\rangle \\ \langle A^\dagger(0)\sigma_+^{fg}(\tau)A(0)\rangle \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \Gamma\xi^2\langle A^\dagger A\rangle_{ss} \\ i\xi\frac{\Omega}{2}\langle A^\dagger A\rangle_{ss} \\ -i\xi\frac{\Omega}{2}\langle A^\dagger A\rangle_{ss} \\ 0 \\ 0 \end{pmatrix}. \tag{39}
$$

We also need to solve

$$
\frac{\mathrm{d}}{\mathrm{d}\tau}\langle A^\dagger(0) b_j \boldsymbol{\sigma}(\tau) A(0)\rangle = \boldsymbol{M}_j^{(b)}\langle A^\dagger(0) b_j\boldsymbol{\sigma}(\tau)A(0)\rangle
$$

$$
+\begin{pmatrix}
-\mathcal{E}_j^{(b)}\langle A^\dagger(0)\sigma_-^{eg}(\tau)A(0)\rangle \\
-\mathcal{E}_j^{(b)}\xi\langle A^\dagger(0)\sigma_-^{fg}(\tau)A(0)\rangle \\
-\mathcal{E}_j^{(b)}\langle A^\dagger(0)\sigma^{ee}(\tau)A(0)\rangle \\
\Gamma\xi^2\langle A^\dagger(0)b_j(\tau)A(0)\rangle - \mathcal{E}_j^{(b)}\xi\langle A^\dagger(0)\sigma_-^{fe}(\tau)A(0)\rangle \\
i\xi\frac{\Omega}{2}\langle A^\dagger(0)b_j(\tau)A(0)\rangle \\
-i\xi\frac{\Omega}{2}\langle A^\dagger(0)b_j(\tau)A(0)\rangle - \mathcal{E}_j^{(b)}\xi\left(\langle A^\dagger A\rangle_{ss} - \langle A^\dagger(0)\sigma^{gg}(\tau)A(0)\rangle - \langle A^\dagger(0)\sigma^{ee}(\tau)A(0)\rangle\right) \\
0 \\
-\mathcal{E}_j^{(b)}\langle A^\dagger(0)\sigma_+^{fe}(\tau)A(0)\rangle
\end{pmatrix},
\tag{40a}
$$

and

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}\langle A^\dagger(0)b_j^\dagger\boldsymbol{\sigma}(\tau)A(0)\rangle = \boldsymbol{M}_j^{(b^\dagger)}\langle A^\dagger(0)b_j^\dagger\boldsymbol{\sigma}(\tau)A(0)\rangle& \\
+\begin{pmatrix}
-\mathcal{E}_j^{(b)*}\langle A^\dagger(0)\sigma_+^{eg}(\tau)A(0)\rangle \\
-\mathcal{E}_j^{(b)*}\langle A^\dagger(0)\sigma^{ee}(\tau)A(0)\rangle \\
-\mathcal{E}_j^{(b)*}\xi\langle A^\dagger(0)\sigma_+^{fg}(\tau)A(0)\rangle \\
\Gamma\xi^2\langle A^\dagger(0)b_j^\dagger(\tau)A(0)\rangle - \mathcal{E}_j^{(b)*}\xi\langle A^\dagger(0)\sigma_+^{fe}(\tau)A(0)\rangle \\
i\xi\frac{\Omega}{2}\langle A^\dagger(0)b_j^\dagger(\tau)A(0)\rangle - \mathcal{E}_j^{(b)*}\xi\left(\langle A^\dagger A\rangle_{ss} - \langle A^\dagger(0)\sigma^{gg}(\tau)A(0)\rangle - \langle A^\dagger(0)\sigma^{ee}(\tau)A(0)\rangle\right) \\
-i\xi\frac{\Omega}{2}\langle A^\dagger(0)b_j^\dagger(\tau)A(0)\rangle \\
-\mathcal{E}_j^{(b)*}\langle A^\dagger(0)\sigma_-^{fe}(\tau)A(0)\rangle \\
0
\end{pmatrix}&,
\end{aligned}
\tag{40b}
$$

$$
\tag{40c}
$$

where

$$
\boldsymbol{M}_j^{(b)} = \boldsymbol{M} - \left(\kappa_b + i\omega_j^{(b)}\right)\mathbb{1}, \tag{41a}
$$

$$
\boldsymbol{M}_j^{(b^\dagger)} = \boldsymbol{M} - \left(\kappa_b - i\omega_j^{(b)}\right)\mathbb{1}. \tag{41b}
$$

Finally, we will also need to solve

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}\langle A^\dagger(0)b_k^\dagger b_l(\tau)A(0)\rangle = &-\left(2\kappa_b - i\omega_k^{(b)} + i\omega_l^{(b)}\right)\langle A^\dagger(0)b_k^\dagger b_l(\tau)A(0)\rangle \\
&- \mathcal{E}_k^{(b)*}\left(\langle A^\dagger(0)b_l\sigma_+^{eg}(\tau)A(0)\rangle + \xi\langle A^\dagger(0)b_l\sigma_+^{fe}(\tau)A(0)\rangle\right) \\
&- \mathcal{E}_l^{(b)}\left(\langle A^\dagger(0)b_k^\dagger\sigma_-^{eg}(\tau)A(0)\rangle + \xi\langle A^\dagger(0)b_k^\dagger\sigma_-^{fe}(\tau)A(0)\rangle\right).
\end{aligned}
\tag{42}
$$