Multi-Mode Filter Equations

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1 Hamiltonian and Master Equation

The Hamiltonian is

$$H = \hbar \frac{\Omega}{2} \left(\sigma_{+} + \sigma_{-} \right) + \hbar \sum_{j=-N}^{N} \omega_j a_j^{\dagger} a_j + \frac{i\hbar}{2} \sum_{j=-N}^{N} \left(\mathcal{E}_j^{*} a_j \sigma_{+} - \mathcal{E}_j a_j^{\dagger} \sigma_{-} \right), \tag{1}$$

where Ω is the Rabi frequency, σ_{+} and σ_{-} are the atomic raising and lowering operators, $a^{\dagger}$ and a are the cavity photon creation and annihilation operators, N is the number of modes either side of the central mode ($2N + 1$ total modes),

$$\omega_j = \omega_0 + j \delta\omega, \tag{2}$$

is the resonance frequency of the j^{th}mode, with central frequency ω_0 and mode frequency spacing $\delta\omega$, and

$$\mathcal{E}_j = \sqrt{\frac{\epsilon \gamma \kappa}{2N + 1}} e^{i m \varphi_j}, \tag{3}$$

is the cascaded systems coupling of the j^{th}mode, where γ is the atomic decay rate, κ is the cavity decay rate, ϵ is the percentage of fluorescence sent to the filter, and

$$\varphi_j = \frac{j\pi}{N}, \tag{4}$$

sets the size of the frequency dependent time delay, with integer m.

The master equation is

$$\begin{aligned} \frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{1}{i\hbar} [H, \rho] &+ \frac{\gamma}{2} \left(1 - \epsilon \right) \left(2 \sigma_{-} \rho \sigma_{+} - \sigma_{+} \sigma_{-} \rho - \rho \sigma_{+} \sigma_{-} \right) \\ &+ \frac{\kappa}{2} \sum_{j=-N}^{N} \left(2 a_j \rho a_j^{\dagger} - a_j^{\dagger} a_j \rho - \rho a_j^{\dagger} a_j \right) \\ &+ \frac{1}{2} \sum_{j=-N}^{N} \left(2 C_j \rho C_j^{\dagger} - C_j^{\dagger} C_j \rho - \rho C_j^{\dagger} C_j \right), \end{aligned} \tag{5}$$

where

$$C_j = \sqrt{\frac{\epsilon \gamma}{2N + 1}} \sigma_{-} + \sqrt{\kappa} e^{i m \varphi_j} a_j, \tag{6}$$

is the cascaded systems decay operator. Expanding the master equation out into neat terms, we arrive at:

$$\begin{aligned}\frac{\mathrm{d}\rho}{\mathrm{d}t} = &-i\frac{\Omega}{2}\left(\sigma_+\rho - \rho\sigma_+\right) - i\frac{\Omega}{2}\left(\sigma_-\rho - \rho\sigma_-\right) + \frac{\gamma}{2}\left(2\sigma_-\rho\sigma_+ - \sigma_+\sigma_-\rho - \rho\sigma_+\sigma_-\right) \\ &- i\sum_{j=-N}^{N}\omega_j\left(a_j^\dagger a_j\rho - \rho a_j^\dagger a_j\right) + \kappa\sum_{j=-N}^{N}\left(2a_j\rho a_j^\dagger - a_j^\dagger a_j\rho - \rho a_j^\dagger a_j\right) \\ &- \sum_{j=-N}^{N}\mathcal{E}_j\left(a_j^\dagger\sigma_-\rho - \sigma_-\rho a_j^\dagger\right) - \sum_{j=-N}^{N}\mathcal{E}_j^*\left(\rho\sigma_+ a_j - a_j\rho\sigma_+\right).\end{aligned} \tag{7}$$

2 Operator Averages

2.1 First-Order: Atomic Equations

We may write the Bloch equations in matrix form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\boldsymbol{\sigma}\rangle = \boldsymbol{M}\langle\boldsymbol{\sigma}\rangle + \boldsymbol{M}, \tag{8}$$

where

$$\langle\boldsymbol{\sigma}\rangle = \begin{pmatrix}\langle\sigma_-\rangle\\ \langle\sigma_+\rangle\\ \langle\sigma_z\rangle\end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix}0\\0\\-\gamma\end{pmatrix}, \tag{9}$$

and

$$\boldsymbol{M} = \begin{pmatrix}-\frac{\gamma}{2} & 0 & i\frac{\Omega}{2}\\ 0 & -\frac{\gamma}{2} & -i\frac{\Omega}{2}\\ i\Omega & -i\Omega & -\gamma\end{pmatrix}. \tag{10}$$

This differential equation has the solution

$$\langle\boldsymbol{\sigma}(t)\rangle = e^{\boldsymbol{M}t}\left(\langle\boldsymbol{\sigma}(0)\rangle - \langle\boldsymbol{\sigma}\rangle_{ss}\right) + \langle\boldsymbol{\sigma}\rangle_{ss}, \tag{11}$$

where

$$\langle\boldsymbol{\sigma}\rangle_{ss} = -\boldsymbol{M}^{-1}\boldsymbol{B} = \frac{1}{2\Omega^2+\gamma^2}\begin{pmatrix}-i\gamma\Omega\\ i\gamma\Omega\\ -\gamma^2\end{pmatrix}. \tag{12}$$

2.2 First-Order: Cavity Equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j\rangle = -\left(\kappa + i\omega_j\right)\langle a_j\rangle - \mathcal{E}_j\langle\sigma_-\rangle, \tag{13a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger\rangle = -\left(\kappa - i\omega_j\right)\langle a_j^\dagger\rangle - \mathcal{E}_j^*\langle\sigma_+\rangle. \tag{13b}$$

2.3 Second-Order: Cavity-Atom Coupled Equations

Using the vector notation, we have moment equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j\boldsymbol{\sigma}\rangle = \boldsymbol{M}_j\langle a_j\boldsymbol{\sigma}\rangle + \begin{pmatrix}0\\ -\frac{1}{2}\mathcal{E}_j\left(\langle\sigma_z\rangle + 1\right)\\ -\gamma\langle a_j\rangle + \mathcal{E}_j\langle\sigma_-\rangle\end{pmatrix}, \tag{14a}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger \boldsymbol{\sigma}\rangle = \boldsymbol{M}_{j^*}\langle a_j^\dagger \boldsymbol{\sigma}\rangle + \begin{pmatrix} -\frac{1}{2}\mathcal{E}_j^*\left(\langle\sigma_z\rangle + 1\right) \\ 0 \\ -\gamma\langle a_j^\dagger\rangle + \mathcal{E}_j^*\langle\sigma_+\rangle \end{pmatrix}. \tag{14b}$$

where

$$\boldsymbol{M}_j = \boldsymbol{M} - (\kappa + i\omega_j)\,\mathbb{1}, \tag{15a}$$
$$\boldsymbol{M}_{j^*} = \boldsymbol{M} - (\kappa - i\omega_j)\,\mathbb{1}. \tag{15b}$$

2.4 Second-Order: Cavity Equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j a_k\rangle = -\left(2\kappa + i\left(\omega_j + \omega_k\right)\right)\langle a_j a_k\rangle - \mathcal{E}_j\langle a_k\sigma_-\rangle - \mathcal{E}_k\langle a_j\sigma_-\rangle, \tag{16a}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger a_k^\dagger\rangle = -\left(2\kappa - i\left(\omega_j + \omega_k\right)\right)\langle a_j^\dagger a_k^\dagger\rangle - \mathcal{E}_j^*\langle a_k^\dagger\sigma_+\rangle - \mathcal{E}_k^*\langle a_j^\dagger\sigma_+\rangle, \tag{16b}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger a_k\rangle = -\left(2\kappa - i\left(\omega_j - \omega_k\right)\right)\langle a_j^\dagger a_k\rangle - \mathcal{E}_j^*\langle a_k\sigma_+\rangle - \mathcal{E}_k\langle a_j^\dagger\sigma_-\rangle. \tag{16c}$$

2.5 Third-Order: Cavity-Atom Coupled Equations

Using the vector notation, we have moment equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j a_k\boldsymbol{\sigma}\rangle = \boldsymbol{M}_{j,k}\langle a_j a_k\boldsymbol{\sigma}\rangle + \begin{pmatrix} 0 \\ -\frac{1}{2}\mathcal{E}_j\left(\langle a_k\sigma_z\rangle + \langle a_k\rangle\right) - \frac{1}{2}\mathcal{E}_k\left(\langle a_j\sigma_z\rangle + \langle a_j\rangle\right) \\ -\gamma\langle a_j a_k\rangle + \mathcal{E}_j\langle a_k\sigma_-\rangle + \mathcal{E}_k\langle a_j\sigma_-\rangle \end{pmatrix}, \tag{17a}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger a_k^\dagger\boldsymbol{\sigma}\rangle = \boldsymbol{M}_{j^*,k^*}\langle a_j^\dagger a_k^\dagger\boldsymbol{\sigma}\rangle + \begin{pmatrix} -\frac{1}{2}\mathcal{E}_j^*\left(\langle a_k^\dagger\sigma_z\rangle + \langle a_k^\dagger\rangle\right) - \frac{1}{2}\mathcal{E}_k^*\left(\langle a_j^\dagger\sigma_z\rangle + \langle a_j^\dagger\rangle\right) \\ 0 \\ -\gamma\langle a_j^\dagger a_k^\dagger\rangle + \mathcal{E}_j^*\langle a_k^\dagger\sigma_+\rangle + \mathcal{E}_k^*\langle a_j^\dagger\sigma_+\rangle \end{pmatrix}, \tag{17b}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger a_k\boldsymbol{\sigma}\rangle = \boldsymbol{M}_{j^*,k}\langle a_j^\dagger a_k\boldsymbol{\sigma}\rangle + \begin{pmatrix} -\frac{1}{2}\mathcal{E}_j^*\left(\langle a_k\sigma_z\rangle + \langle a_k\rangle\right) \\ -\frac{1}{2}\mathcal{E}_k\left(\langle a_j^\dagger\sigma_z\rangle + \langle a_j^\dagger\rangle\right) \\ -\gamma\langle a_j^\dagger a_k\rangle + \mathcal{E}_j^*\langle a_k\sigma_+\rangle + \mathcal{E}_k\langle a_j^\dagger\sigma_-\rangle \end{pmatrix}, \tag{17c}$$

where

$$\boldsymbol{M}_{j,k} = \boldsymbol{M} - \left(2\kappa + i\left(\omega_j + \omega_k\right)\right)\mathbb{1}, \tag{18a}$$
$$\boldsymbol{M}_{j^*,k^*} = \boldsymbol{M} - \left(2\kappa - i\left(\omega_j + \omega_k\right)\right)\mathbb{1}, \tag{18b}$$
$$\boldsymbol{M}_{j^*,k} = \boldsymbol{M} - \left(2\kappa - i\left(\omega_j - \omega_k\right)\right)\mathbb{1}. \tag{18c}$$

2.6 Third-Order: Cavity Equations

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger a_k a_l\rangle &= -\left(3\kappa - i\omega_j + i\left(\omega_k + \omega_l\right)\right)\langle a_j^\dagger a_k a_l\rangle \\ &\quad - \mathcal{E}_j^*\langle a_k a_l\sigma_+\rangle - \mathcal{E}_k\langle a_j^\dagger a_l\sigma_-\rangle - \mathcal{E}_l\langle a_j^\dagger a_k\sigma_-\rangle,\end{aligned} \tag{19a}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger a_k^\dagger a_l\rangle &= -\left(3\kappa - i\left(\omega_j + \omega_k\right) + i\omega_l\right)\langle a_j^\dagger a_k^\dagger a_l\rangle \\ &\quad - \mathcal{E}_j^*\langle a_k^\dagger a_l\sigma_+\rangle - \mathcal{E}_k^*\langle a_j^\dagger a_l\sigma_+\rangle - \mathcal{E}_l\langle a_j^\dagger a_k^\dagger\sigma_-\rangle.\end{aligned} \tag{19b}$$

2.7 Fourth-Order: Cavity-Atom Coupled Equations

Using the vector notation, we have moment equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger a_k a_l \boldsymbol{\sigma}\rangle = \boldsymbol{M}_{j^*,k,l}\langle a_j^\dagger a_k a_l\rangle + \begin{pmatrix} -\frac{1}{2}\mathcal{E}_j^*\left(\langle a_k a_l \sigma_z\rangle + \langle a_k a_l\rangle\right) \\ -\frac{1}{2}\mathcal{E}_k\left(\langle a_j^\dagger a_l \sigma_z\rangle + \langle a_j^\dagger a_l\rangle\right) - \frac{1}{2}\mathcal{E}_l\left(\langle a_j^\dagger a_k \sigma_z\rangle + \langle a_j^\dagger a_k\rangle\right) \\ -\gamma\langle a_j^\dagger a_k a_l\rangle + \mathcal{E}_j^*\langle a_k a_l \sigma_+\rangle + \mathcal{E}_k\langle a_j^\dagger a_l \sigma_-\rangle + \mathcal{E}_l\langle a_j^\dagger a_k \sigma_-\rangle \end{pmatrix}, \tag{20a}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger a_k^\dagger a_l \boldsymbol{\sigma}\rangle = \boldsymbol{M}_{j^*,k^*,l}\langle a_j^\dagger a_k^\dagger a_l\rangle + \begin{pmatrix} -\frac{1}{2}\mathcal{E}_j^*\left(\langle a_k^\dagger a_l \sigma_z\rangle + \langle a_k^\dagger a_l\rangle\right) - \frac{1}{2}\mathcal{E}_k^*\left(\langle a_j^\dagger a_l \sigma_z\rangle + \langle a_j^\dagger a_l\rangle\right) \\ -\frac{1}{2}\mathcal{E}_l\left(\langle a_j^\dagger a_k^\dagger \sigma_z\rangle + \langle a_j^\dagger a_k^\dagger\rangle\right) \\ -\gamma\langle a_j^\dagger a_k^\dagger a_l\rangle + \mathcal{E}_j^*\langle a_k^\dagger a_l \sigma_+\rangle + \mathcal{E}_k^*\langle a_j^\dagger a_l \sigma_+\rangle + \mathcal{E}_l\langle a_j^\dagger a_k^\dagger \sigma_-\rangle \end{pmatrix}, \tag{20b}$$

where

$$\boldsymbol{M}_{j^*,k,l} = \boldsymbol{M} - \left(3\kappa - i\omega_j + i\left(\omega_k + \omega_l\right)\right), \tag{21a}$$
$$\boldsymbol{M}_{j^*,k^*,l} = \boldsymbol{M} - \left(3\kappa - i\left(\omega_j + \omega_k\right) - i\omega_l\right). \tag{21b}$$

2.8 Fourth-Order: Cavity Equation

Finally, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j^\dagger a_k^\dagger a_l a_m\rangle = &-\left(4\kappa - i\left(\omega_j + \omega_k\right) + i\left(\omega_l + \omega_m\right)\right)\langle a_j^\dagger a_k^\dagger a_l a_m\rangle \\ &- \mathcal{E}_j^*\langle a_k^\dagger a_l a_m \sigma_+\rangle - \mathcal{E}_k^*\langle a_j^\dagger a_l a_m \sigma_+\rangle \\ &- \mathcal{E}_l\langle a_j^\dagger a_k^\dagger a_m \sigma_-\rangle - \mathcal{E}_m\langle a_j^\dagger a_k^\dagger a_l \sigma_-\rangle.\end{aligned} \tag{22}$$

3 First-Order Correlation Function

The first-order correlation function for the filtered output field in the steady state is given by

$$G^{(1)}(t,\tau) = \langle A^\dagger(t+\tau)A(t)\rangle = \sum_{j=-N}^{N}\langle a_j^\dagger(t+\tau)A(t)\rangle. \tag{23}$$

We use the Quantum Regression Theorem to solve for this with the moment equations

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\langle\boldsymbol{\sigma}(t+\tau)A(t)\rangle = \boldsymbol{M}\langle\boldsymbol{\sigma}(t+\tau)A(t)\rangle + \boldsymbol{B}, \tag{24a}$$
$$\frac{\mathrm{d}}{\mathrm{d}\tau}\langle a_j^\dagger(t+\tau)A(t)\rangle = -\left(\kappa - i\omega_j\right)\langle a_j^\dagger(t+\tau)A(t)\rangle - \mathcal{E}_j^*\langle\sigma_+(t+\tau)A(t)\rangle, \tag{24b}$$

where

$$\langle\boldsymbol{\sigma}(t+\tau)A(t)\rangle = \begin{pmatrix}\langle\sigma_-(t+\tau)A(t)\rangle \\ \langle\sigma_+(t+\tau)A(t)\rangle \\ \langle\sigma_z(t+\tau)A(t)\rangle\end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix}0 \\ 0 \\ -\gamma\langle A(t)\rangle\end{pmatrix}. \tag{25}$$

4 Second-Order Correlation Function

The second-order correlation function for the filtered output field in the steady state is given by

$$G^{(2)}(t,\tau) = \langle A^\dagger(t)A^\dagger A(t+\tau)A(t)\rangle = \sum_{k,l=-N}^{N}\langle A^\dagger(t)a_k^\dagger a_l(t+\tau)A(t)\rangle, \tag{26}$$

with the normalised second-order correlation function given by

$$g^{(2)}(t,\tau) = \frac{G^{(2)}(t,\tau)}{\langle A^\dagger A\rangle_{ss}^2}. \tag{27}$$

Using the Quantum Regression Theorem, we can solve for the second-order correlation function by solving the following moment equations

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\langle A^\dagger(t)\boldsymbol{\sigma}(t+\tau)A(t)\rangle = \boldsymbol{M}\langle A^\dagger(t)\boldsymbol{\sigma}(t+\tau)A(t)\rangle + \boldsymbol{B}, \tag{28a}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\langle A^\dagger(t)a_j^\dagger(t+\tau)A(t)\rangle = -\left(\kappa - i\omega_j\right)\langle A^\dagger(t)a_j^\dagger(t+\tau)A(t)\rangle - \mathcal{E}_j^*\langle A^\dagger(t)\sigma_+(t+\tau)A(t)\rangle, \tag{28b}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\langle A^\dagger(t)a_j(t+\tau)A(t)\rangle = -\left(\kappa + i\omega_j\right)\langle A^\dagger(t)a_j(t+\tau)A(t)\rangle - \mathcal{E}_j\langle A^\dagger(t)\sigma_-(t+\tau)A(t)\rangle, \tag{28c}$$

with

$$\langle\boldsymbol{\sigma}\rangle = \begin{pmatrix}\langle A^\dagger(t)\sigma_-(t+\tau)A(t)\rangle \\ \langle A^\dagger(t)\sigma_+(t+\tau)A(t)\rangle \\ \langle A^\dagger(t)\sigma_z(t+\tau)A(t)\rangle\end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix}0 \\ 0 \\ -\gamma\langle A^\dagger A(t)\rangle\end{pmatrix}. \tag{29}$$

We also need to solve

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\langle A^\dagger(t)a_j\boldsymbol{\sigma}(t+\tau)A(t)\rangle = \boldsymbol{M}_j\langle A^\dagger(t)a_j\boldsymbol{\sigma}(t+\tau)A(t)\rangle + \begin{pmatrix}0 \\ -\frac{1}{2}\mathcal{E}_j\left(\langle A^\dagger(t)\sigma_z(t+\tau)A(t)\rangle + \langle A^\dagger A\rangle_{ss}\right) \\ -\gamma\langle A^\dagger(t)a_j(t+\tau)A(t)\rangle + \mathcal{E}_j\langle A^\dagger(t)\sigma_-(t+\tau)A(t)\rangle\end{pmatrix}, \tag{30a}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\langle A^\dagger(t)a_j^\dagger\boldsymbol{\sigma}(t+\tau)A(t)\rangle = \boldsymbol{M}_{j^*}\langle A^\dagger(t)a_j^\dagger\boldsymbol{\sigma}(t+\tau)A(t)\rangle + \begin{pmatrix}-\frac{1}{2}\mathcal{E}_j^*\left(\langle A^\dagger(t)\sigma_z(t+\tau)A(t)\rangle + \langle A^\dagger A(t)\rangle\right) \\ 0 \\ -\gamma\langle A^\dagger(t)a_j^\dagger(t+\tau)A(t)\rangle + \mathcal{E}_j^*\langle A^\dagger(t)\sigma_+(t+\tau)A(t)\rangle\end{pmatrix}. \tag{30b}$$

where

$$\boldsymbol{M}_j = \boldsymbol{M} - \left(\kappa + i\omega_j\right)\mathbb{1}, \tag{31a}$$

$$\boldsymbol{M}_{j^*} = \boldsymbol{M} - \left(\kappa - i\omega_j\right)\mathbb{1}. \tag{31b}$$

Finally, we will also need to solve

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\langle A^\dagger(t)a_k^\dagger a_l(t+\tau)A(t)\rangle = &-\left(2\kappa - i\left(\omega_k - \omega_l\right)\right)\langle A^\dagger(t)a_k^\dagger a_l(t+\tau)A(t)\rangle \\ &- \mathcal{E}_k^*\langle A^\dagger(t)a_l\sigma_+(t+\tau)A(t)\rangle - \mathcal{E}_l\langle A^\dagger(t)a_k^\dagger\sigma_-(t+\tau)A(t)\rangle.\end{aligned} \tag{32}$$