Phase Resetting in the Yamada Model

Jacob Ngaha

8^{th} July, 2024

We compute the phase resetting of an attracting periodic orbit for the Yamada model:

$$\vec{F}\left(\vec{x}\right) = \begin{pmatrix} \dot{G} \\ \dot{Q} \\ \dot{I} \end{pmatrix} = \begin{pmatrix} \gamma\left(A - G - GI\right) \\ \gamma\left(B - Q - aQI\right) \\ \left(1 - G - Q\right) I \end{pmatrix} \tag{1}$$

where G is the gain, Q is the absorption, and I is the intensity. γ, A, B, and a are the system parameters. This is done in a few steps

1. We first calculate an attracting periodic orbit in the region (X) of Fig. 1, for parameters $(A, \gamma) = (7.3757, 0.3540)$.
   (a) First continue the stationary point $x_\star = (A, B, 0)$ until we detect a branching point,
   (b) Switch branches and continue in A until we detect a Hopf bifurcation,
   (c) Continue the Hopf bifurcation in A until $A = 7.3757$,
   (d) Follow a family of periodic orbits emanating from the Hopf bifurcation in γ until $\gamma = 0.3540$,
   (e) "Rotate" the periodic orbit by setting $G(t = 0) = \max\left(G\right)$, and re-verify the solution, with boundary conditions

$$\vec{x}(0) - \vec{x}(1) = 0, \tag{2a}$$
$$\vec{e}_1 \cdot \vec{F}(\vec{x}_0) = 0 \tag{2b}$$

2. Compute the Floquet bundle in the stable direction, with u-vector composed of the state vector $\vec{x}$ and the left eigenvector of the Jacobian $\vec{w}$:

$$\vec{u} = (\vec{x}, \vec{w}) \tag{3a}$$

   with

$$\dot{\vec{x}} = \vec{F}\left(\vec{x}\right), \tag{3b}$$
$$\dot{\vec{w}} = -D_{\vec{x}}^{T} \vec{w}. \tag{3c}$$

   and boundary conditions Eqs. (2a) and

$$\vec{w}(1) - \mu_s \vec{w}(0) = 0, \tag{3d}$$
$$\|\vec{w}(0)\| - w_{\text{norm}} = 0. \tag{3e}$$

(a) Continue solution in the stable floquet multiplier, μ_s, until a branching point is detected at $\mu_s = 1.0$,

(b) Continue solution in the norm of $\vec{w}$, w_{norm}, until $w_{\text{norm}} = 1.0$.

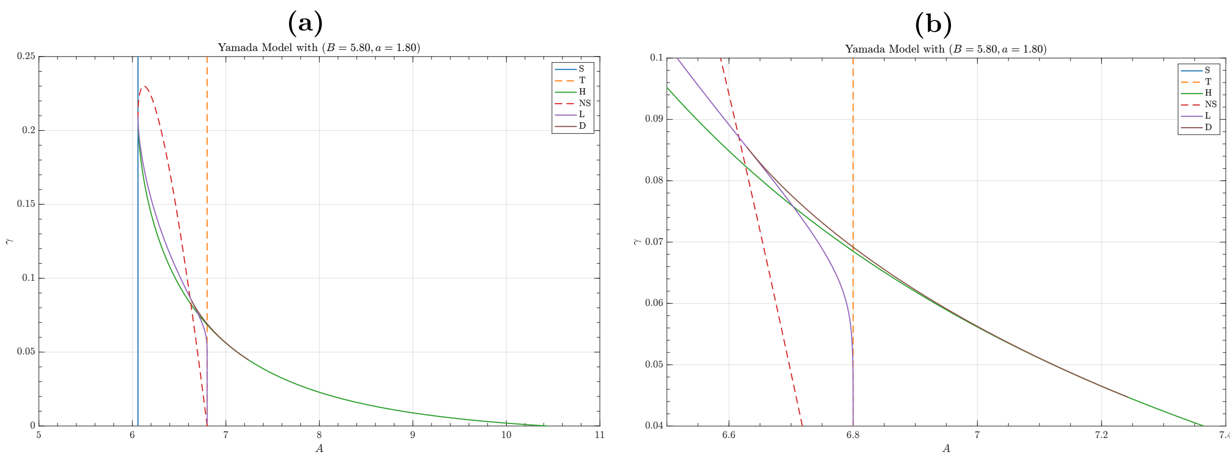

Figure 1: (a) Bifurcation diagram of the Yamada model. (b) Zoom-in of (a).

Phase-Resetting Segments

We calculate the phase-resetting problem as four different segments.

- T is the period of the attracting periodic orbit,
- $\vec{x}_i$ is the state-space vector for segment i,
- $\vec{w}_i$ is the left-eigenvector of the Jacobian for segment i,
- θ_{old} is the phase/point along the periodic orbit where the perturbation is applied,
- θ_{new} is the phase/point long the periodic orbit where the perturbed segment as returned,
- k is the periodicity of the perturbed segment,
- A_{p} is the perturbation amplitude
- $\vec{d}_{\text{p}} = (\cos(\theta_{\text{p}}), 0, \sin(\theta_{\text{p}}))$ is the perturbation directional vector, only applied in the $G - I$ plane.

Segment 1 (`func_seg1`)

$$\vec{u}_1 = (\vec{x}_1, \vec{w}_1)\,, \tag{4a}$$

$$\dot{\vec{x}}_1 = T\theta_{\text{new}}\vec{F}(\vec{x}_1), \tag{4b}$$

$$\dot{\vec{w}}_1 = -T\theta_{\text{new}}D_{\vec{x}_1}\vec{w}_1, \tag{4c}$$

Segment 2 (`func_seg2`)

$$\vec{u}_2 = (\vec{x}_2, \vec{w}_2)\,, \tag{5a}$$
$$\dot{\vec{x}}_2 = T\,(1-\theta_{\text{new}})\,\vec{F}(\vec{x}_2), \tag{5b}$$
$$\dot{\vec{w}}_2 = -T\,(1-\theta_{\text{new}})\,D_{\vec{x}_2}\vec{w}_2, \tag{5c}$$

Segment 3 (`func_seg3`)

$$\vec{u}_3 = \vec{x}_3, \tag{6a}$$
$$\dot{\vec{x}}_3 = T\,(1-\theta_{\text{old}})\,\vec{F}(\vec{x}_3). \tag{6b}$$

Segment 4 (`func_seg4`)

$$\vec{u}_4 = \vec{x}_4, \tag{7a}$$
$$\dot{\vec{x}}_4 = kT\vec{F}(\vec{x}_4). \tag{7b}$$

Boundary conditions
The boundary conditions are separated into three files.
`bcs_seg1_seg2`

$$\vec{x}_1(0) - \vec{x}_2(1) = 0, \tag{8a}$$
$$\vec{x}_1(1) - \vec{x}_2(0) = 0, \tag{8b}$$
$$\hat{\vec{e}}_1 \cdot \vec{F}(\vec{x}_1) = 0, \tag{8c}$$
$$\vec{w}_1(0) - \vec{w}_2(1) = 0, \tag{8d}$$
$$\mu_s\vec{w}_2(0) - \vec{w}_1(1) = 0, \tag{8e}$$
$$\|\vec{w}_2(0)\| - 1 = 0. \tag{8f}$$

`bcs_seg3`

$$\vec{x}_3(1) - \vec{x}_1(0) = 0. \tag{9}$$

`bcs_seg4`

$$\vec{x}_4(0) - \vec{x}_3(0) - A_{\text{p}}\vec{d}_{\text{p}} = 0, \tag{10a}$$
$$(\vec{x}_4(0) - \vec{x}_2(0)) \cdot \vec{w}_2(0) = 0, \tag{10b}$$
$$\|\vec{x}_4(1) - \vec{x}_2(0)\|^2 - \eta = 0. \tag{10c}$$

We write the final boundary condition as the norm-squared to avoid a singularity in the Jacobian :)