Best Estimate d_0

July 9, 2013

Best value Estimate

$$P(d_0|\{d_k = 1, N\}) = \prod_{k=1}^{N} \frac{De^{-(d_k/d_0)^D}}{d_0(k-1)!} (\frac{d_k}{d_0})^{Dk-1} \tag{1}$$

Taking the log of P

$$L = \log(P) = \log\left(\prod_{k=1}^{N} \frac{De^{-(d_k/d_0)^D}}{d_0(k-1)!} (\frac{d_k}{d_0})^{Dk-1}\right) \tag{2}$$

Taking into account that $\log(a \cdot b) = \log(a) + \log(b)$.

$$L = \log(P) = \sum_{k=1}^{N} \log\left(\frac{De^{-(d_k/d_0)^D}}{d_0(k-1)!} (\frac{d_k}{d_0})^{Dk-1}\right) \tag{3}$$

Re ordering terms using log properties.

$$L = \sum_{k=1}^{N} \left(\log(D) - (\frac{d_k}{d_0})^D - \log(d_0) - \log(k-1)! + \log(d_k^{Dk-1}) - \log(d_0^{Dk-1})\right) \tag{4}$$

Now we derive L in order to find the best estimate of d_0

$$\frac{\partial L}{\partial d_0} = \sum_{k=1}^{N} \left[\frac{Dd_k^D}{d_0^{D+1}} - \frac{1}{d_0} - \frac{(Dk-1)d_0^{Dk-2}}{d_0^{Dk-1}}\right] = 0 \tag{5}$$

$$\frac{\partial L}{\partial d_0} = \sum_{k=1}^{N} \left[\frac{Dd_k^D}{d_0^{D+1}} - \frac{1}{d_0} - \frac{(Dk-1)}{d_0}\right] = 0 \tag{6}$$

$$\frac{\partial L}{\partial d_0} = \sum_{k=1}^{N} \left[\frac{Dd_k^D}{d_0^{D+1}} - \frac{Dk}{d_0}\right] = 0 \tag{7}$$

$$\frac{\partial L}{\partial d_0} = \frac{D}{d_0} \sum_{k=1}^{N} \left[\frac{d_k^D}{d_0^D} - k\right] = 0 \tag{8}$$

$$d_0^D = \sum_{k=1}^{N} \frac{d_k^D}{k} \tag{9}$$

$$d_0 = \left(\sum_{k=1}^{N} \frac{d_k^D}{k} \right)^{1/D} \tag{10}$$

Now in order to study the errors we take the second derivative of L respect to d_0 i.e:
with the definition:

$$\frac{\partial^2 L}{\partial d_0^2} = -\sum_{k=1}^{N} \frac{1}{\sigma^2} \tag{11}$$

We derive Eq. (7)

$$\frac{\partial^2 L}{\partial d_0^2} = \sum_{k=1}^{N} -\frac{(D+1)Dd_k^D}{d_0^{D+2}} + \frac{Dk}{d_0^2} = -\sum_{k=1}^{N} \frac{1}{\sigma^2} \tag{12}$$

$$\frac{\partial^2 L}{\partial d_0^2} = \frac{D}{d_0^2} \sum_{k=1}^{N} -\frac{(D+1)d_k^D}{d_0^D} + k = -\sum_{k=1}^{N} \frac{1}{\sigma^2} \tag{13}$$

$$\frac{D}{d_0^2} \sum_{k=1}^{N} -\frac{(D+1)d_k^D}{d_0^D} + k = -\frac{N}{\sigma^2} \tag{14}$$

$$\sigma = \left(-\frac{d_0^2 N}{D \sum_{k=1}^{N} \left[-\frac{(D+1)d_k^D}{d_0^D} + k \right]} \right)^{1/2} \tag{15}$$

if D = 2

$$\sigma = \left(-\frac{d_0^2 N}{2 \sum_{k=1}^{N} \left[-\frac{3d_k^2}{d_0^2} + k \right]} \right)^{1/2} \tag{16}$$

$$\sigma = \left(\frac{d_0^2 N}{2 \sum_{k=1}^{N} \left[\frac{3d_k^2}{d_0^2} - k \right]} \right)^{1/2} \tag{17}$$