Modelling the Milky Way Galaxy & the Large Magellanic Cloud

August, 2015

1 Useful Quantities and definitions

In this section some common quantities useful for describe the density profiles are defined and explained.

1.1 Critical density of the Universe:

The Critial density of the universe is defined as:

$$\rho_c = \frac{3H^2}{8\pi G} \tag{1}$$

Where H is the Hubble parameter and this parameter depends on the cosmological parameters. This density ...

1.2 Virialization

A dark matter halo is virialized when it is in equilibrium, such an equilibrium occurs after the dark matter have collapsed and the force of gravity equals the **relaxation** processes [1]. This happens when the dark matter reach an overdensity value Δ_{vir}. This overdensity corresponds to a radius and a mass r_{vir} & M_{vir} respectively.

$\Delta_{vir} = \frac{\rho_{vir}}{\rho_c}$. For a cosmolgy with $(\Omega_m + \Omega_\Lambda = 1)$ the solution for the **Top Hat** model can be approximated by:

$$\Delta_{vir} = (18\pi^2 + 82x - 39x^2)/\Omega(z) \tag{2}$$

[3, 2] Where $x = \Omega(z) - 1$. For the present time $(z = 0)$ $\Delta_{vir} = 360$.

The behavior of this function is shown in Fig.**??**

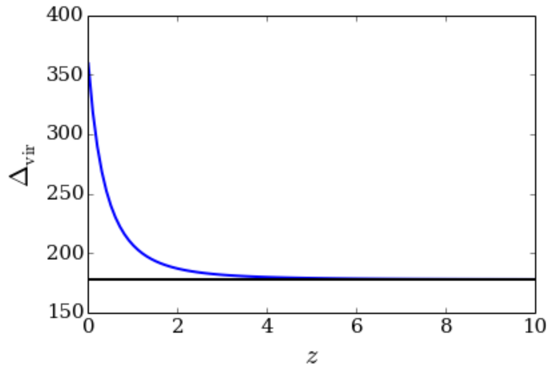


Figure 1: The solid blue line shows the behaviour of Δ_{vir} as function of the redshift. The black line show the value of $\Delta_{vir} = 173$ at $z > 4$

The virial density now can be expressed in terms of Δ_{vir}:

$$\rho_{vir} = \frac{3M_{vir}}{4\pi r_{vir}^3} = \Delta_{vir}\Omega_m\rho_{crit} \tag{3}$$

Where Ω_m is the density parameter that give as the abundance of matter in the Universe, it is define as $\Omega_m = \rho/rho_c$ and the actual value is $\Omega_m \simeq 0.3$ Once the virial density is defined with 3 for a given z then the radius and the virial mass can be related:

$$r_{vir} = \left(\frac{3M_{vir}}{4\pi\Delta_{vir}\Omega_m\rho_{crit}}\right)^{1/3} \tag{4}$$

For example for a halo of mass $M = 1 \times 10^{12} M_\odot$ the corresponding radius is $r_{vir} = 262.4$ Kpc

1.3 r_{200} & M_{200}

There is another radious and mass of particular interest. This is the radius that enclosed a density of 200 times the density of the Universe. M_{200} is defined as:

$$M_{200} = 200\rho_c\frac{4}{3}\pi r_{200}^3 \tag{5}$$

In the same way M_{vir} is defined as:

$$M_{vir} = \Delta_{vir}\Omega_m\rho_c\frac{4}{3}\pi r_{vir}^3 \tag{6}$$

The critial density rho_c is the same for both cases, then it is possible to relate both masses from Eq5 and Eq6 as follows:

$$\frac{M_{200}}{M_{vir}} = \left(\frac{200}{\Delta_{vir}\Omega_m}\right)\left(\frac{r_{200}}{r_{vir}}\right)^3 \tag{7}$$

Here is common to call $q = \left(\frac{200}{\Delta_{vir}\Omega_m}\right)$, at $z = 0$ $q = 2.053$.

$$\frac{M_{200}}{M_{vir}} = q\left(\frac{r_{200}}{r_{vir}}\right)^3 \tag{8}$$

This Eq.8 relates M_{vir} and M_{200} for a given $rvir$ and r_{200}.

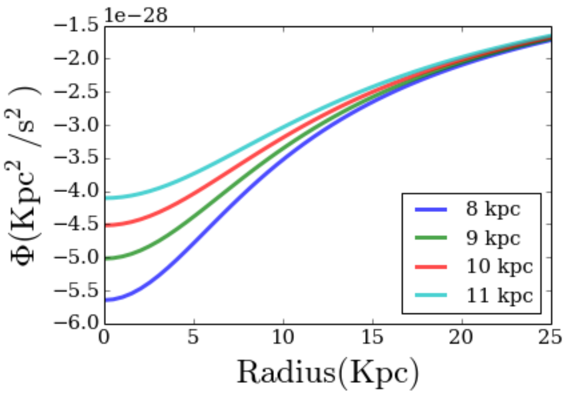
2 Densities profiles

2.1 Plummer

The plumer density profile is one of the simplest models which describes a constant density near the center and falls at large radius.

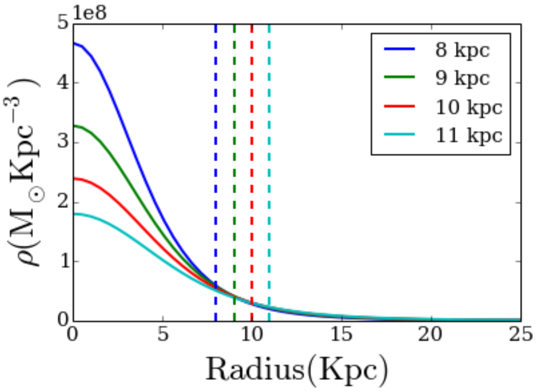
$$\Phi_P(r) = -\frac{GM}{\sqrt{r^2 + a^2}} \tag{9}$$

Where a is call the scale length. The scale length set the length a in which the mayority of the density is enclosed. Note that if a is cero the plummer potential would be exactly as the potential of a point mass. In the other hand if a goes to infty the potential is rewpresenting a very extended mass source. In other words the scale length set up the size of the volume in which the mass M is enclosed.



$$\nabla^2\Phi_P(r) = 4\pi G\rho_P(r) = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi_P(r)}{dr}\right) \tag{10}$$

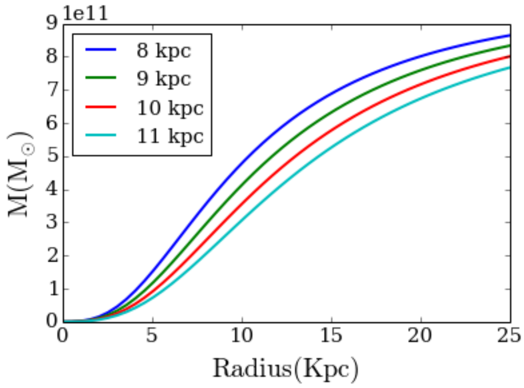
$$\rho_P(r) = \frac{3M}{4\pi a^3}(1 + \frac{r^2}{a^2})^{-5/2} \tag{11}$$



The enclosed mass can be derived from the density by integrating over the volume.

$$M_P(<r) = 4\pi \int_0^r r'^2 \frac{3M}{4\pi a^3}(1+\frac{r'^2}{a^2})^{-5/2}dr' = \frac{3M}{a^3}\left(\frac{a^4 r^3\sqrt{r^2/a^2+1}}{3(r^2+a^2)^2}\right) \tag{12}$$

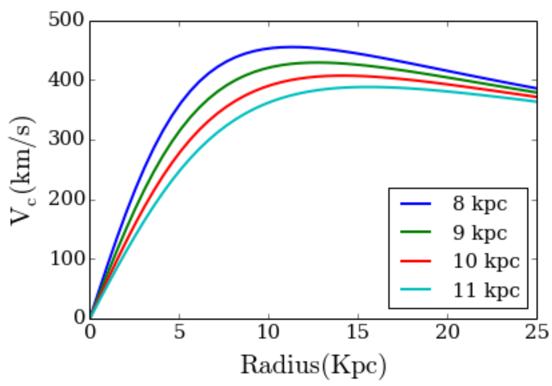
$$M_P(<r) = M\frac{r^3}{(a^2+r^2)^{3/2}} \tag{13}$$



$$F_g = \frac{GmM}{r^2} = ma_c = m\frac{v_c^2}{r} \tag{14}$$

$$v_c = \sqrt{\frac{GM(<r)}{r}} \tag{15}$$

$$v_c = \sqrt{GM(\frac{r^2}{(r^2+a^2)^{3/2}})} \tag{16}$$



2.2 Hernquist profile

The Hernquist profile is derived in such a way that it follows the

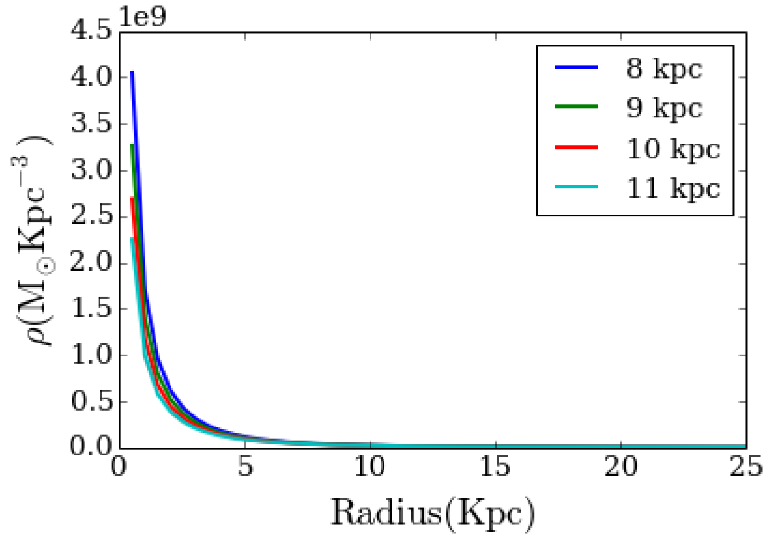
$$\rho_{Hernquist}(r) = \frac{M}{2\pi}\frac{a}{r(r+a)^3} \tag{17}$$

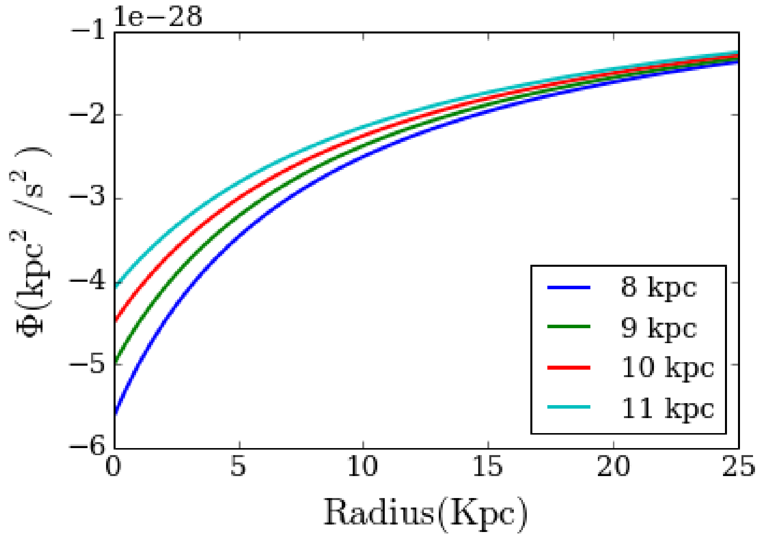
$$M_{Hernquist}(<r) = 2aM\int \frac{r}{(r+a)^3}dr \tag{18}$$

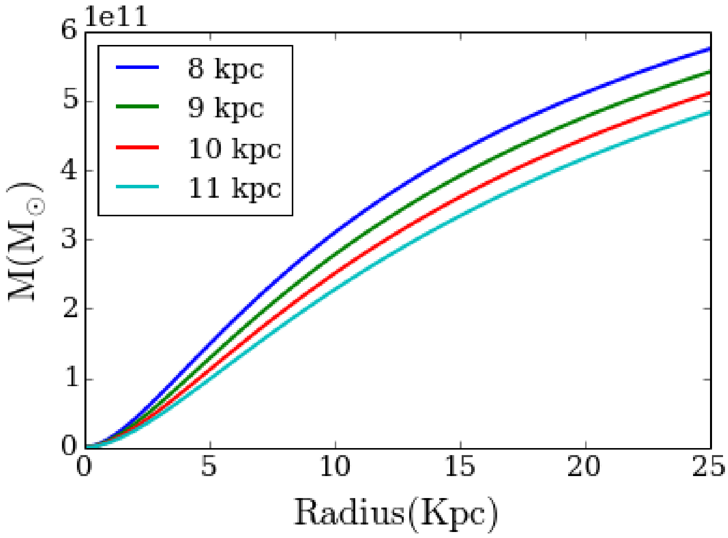
$$M_{Hernquist}(<r) = M\frac{r^2}{(r+a)^2} \tag{19}$$

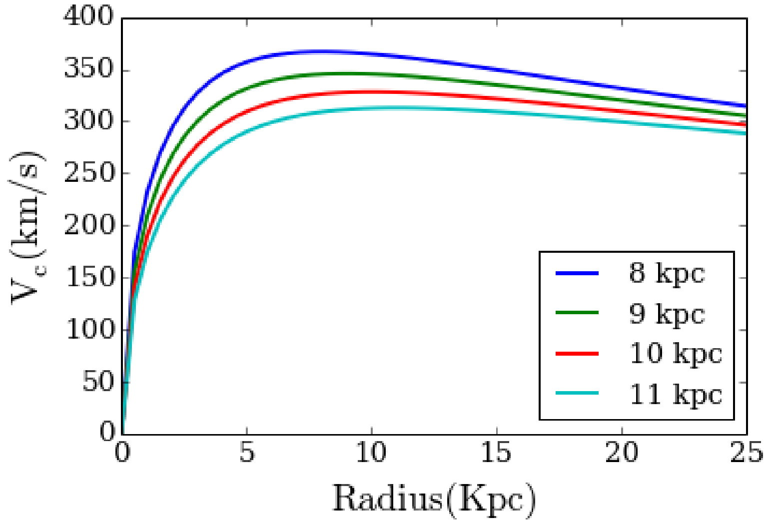
$$\Phi = -\frac{GM}{r+a} \tag{20}$$

$$v_c(r) = GM\frac{r}{(r+a)^2} \tag{21}$$




1e−28
Φ(kpc² /s²)
Radius(Kpc)
8 kpc
9 kpc
10 kpc
11 kpc


1e11
M(M⊙)
Radius(Kpc)
8 kpc
9 kpc
10 kpc
11 kpc


Vc(km/s)
Radius(Kpc)
8 kpc
9 kpc
10 kpc
11 kpc

2.3 Singular Isothermal Sphere

The Singular Isothermal Sphere (**SIS**) describes a system in which the particles follow a Maxwellian density distribution. With this distribution and the Poisson equation the follow density profiles could be derived.

$$\rho_{iso}(r) = \frac{\sigma^2}{2\pi G r^2} \tag{22}$$

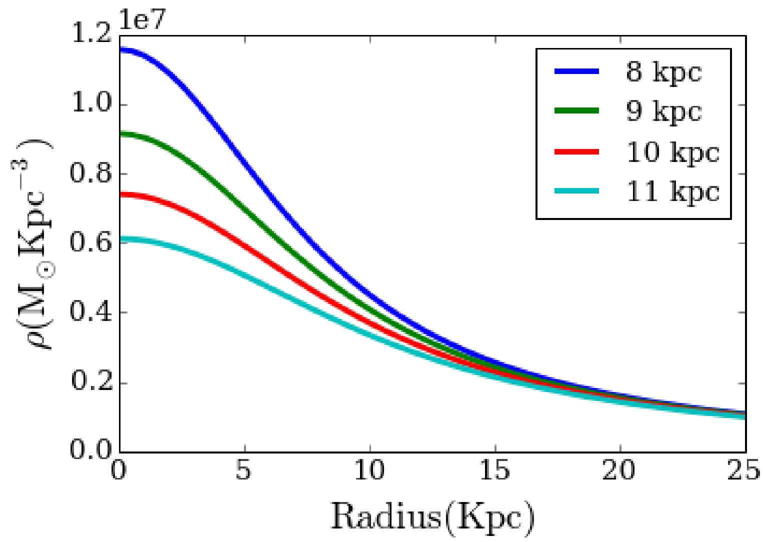
Following the same procedure as with the previuos profiles we find M, Φ and v_c.

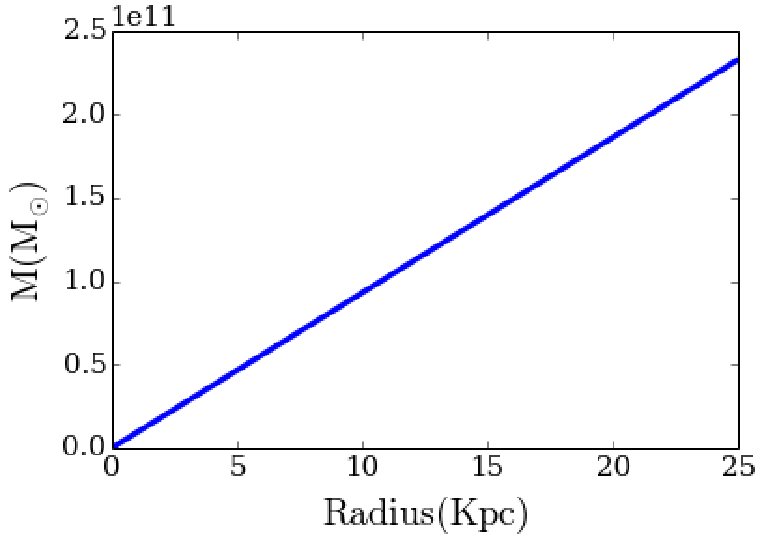
$$M_{iso}(< r) = \frac{2\sigma r}{G} \tag{23}$$

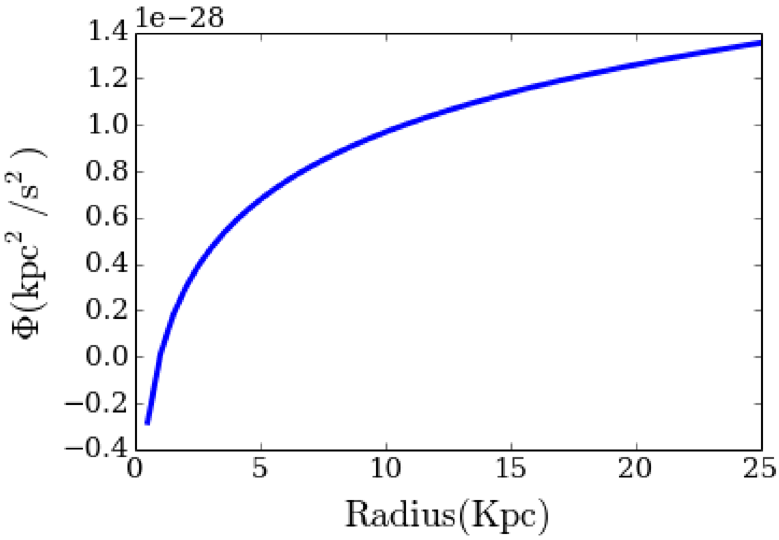
$$\Phi_{iso}(r) = 2\sigma^2 ln(r) + const. \tag{24}$$

$$v_c(r) = \sqrt{2}\sigma \tag{25}$$

This profile is quite different to the previous ones due to the fact that here the input is the velocity instead of the total Mass.







2.4 NFW

$$\rho_{NFW}(r) = \frac{M}{2\pi a^3 (r/a)(1 + r/a)^2} \tag{26}$$

$$M_{NFW}(r) = M\left(ln(1+x) - \frac{x}{1+x}\right) \tag{27}$$

Where $x = r/a$, is useful to define the function $f(x)$ as:

$$f(x) = ln(1+x) - \frac{x}{1+x} \tag{28}$$

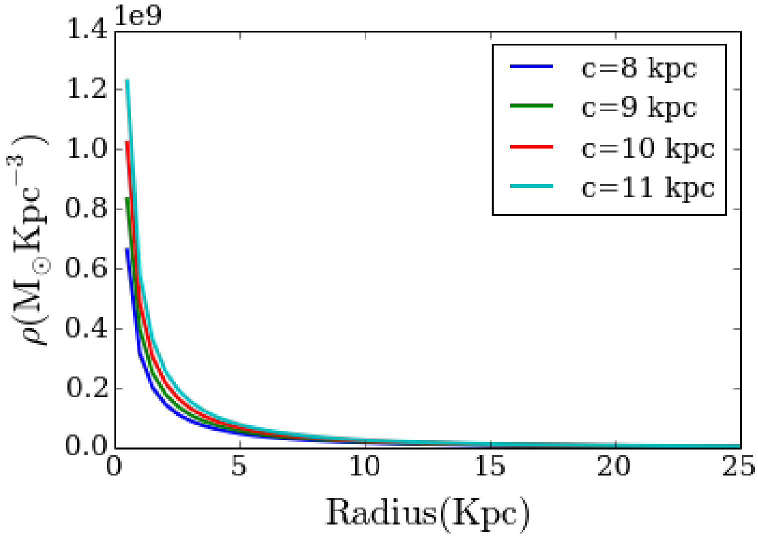
Then 27 can be expresed as:

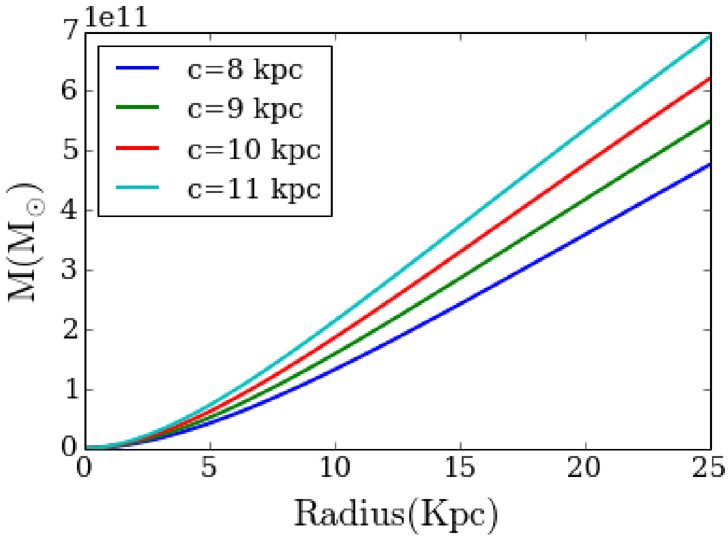
$$M_{NFW} = 4\pi\rho_a a^3 f(x) \tag{29}$$

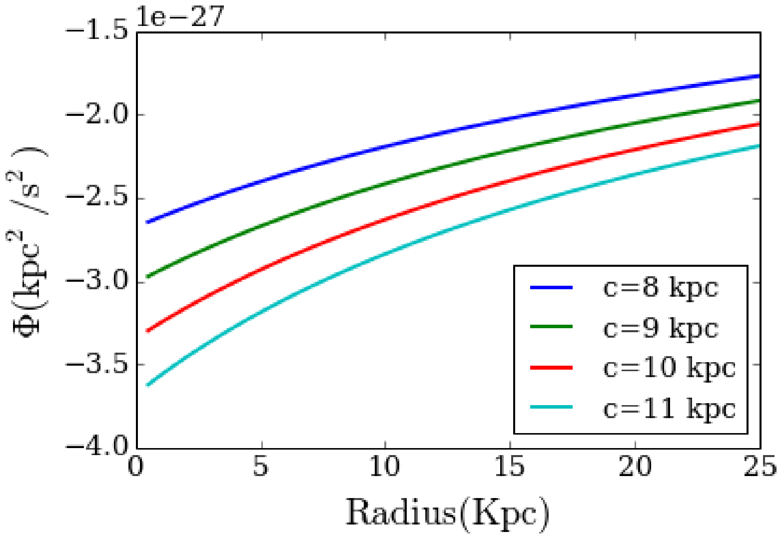
$$\Phi_{NFW} = -4\pi GM \frac{ln(1 + r/a)}{r} \tag{30}$$

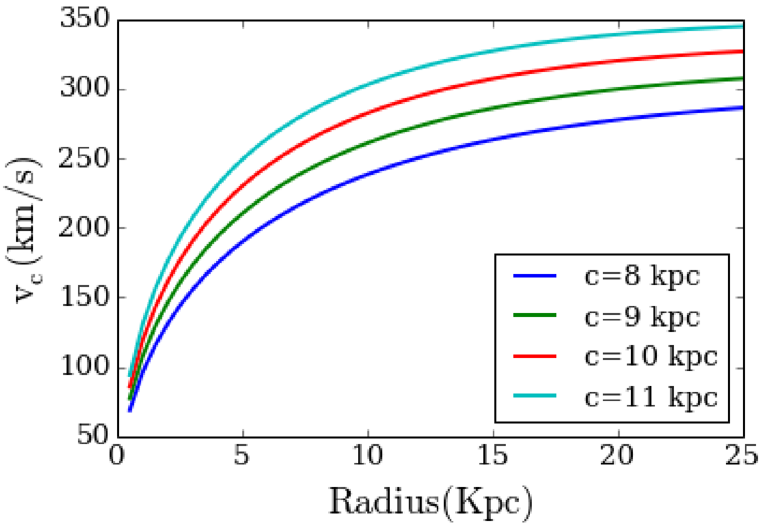
$$c(M_{vir}) = 9.60 \left(\frac{M_{vir}}{10^{12} h^{-1} M_{\odot}} \right)^{-0.075} \tag{31}$$

$$v_c(r) = \sqrt{\left(\frac{M(r)G}{r} \right)} = \sqrt{\left(\frac{2M \left(ln(1 + c) - \frac{c}{1+c} \right)}{r} \right)} \tag{32}$$









3 Conversion from NFW to the Hernquist profile

The average density of the NFW distribution can be expressed as:

$$\bar{\rho}_{NFW}(r) = \frac{3M_{NFW}(r)}{4\pi r^3} \tag{33}$$

And with eq.29 the $\bar{\rho}_{NFW}(r)$ takes de form:

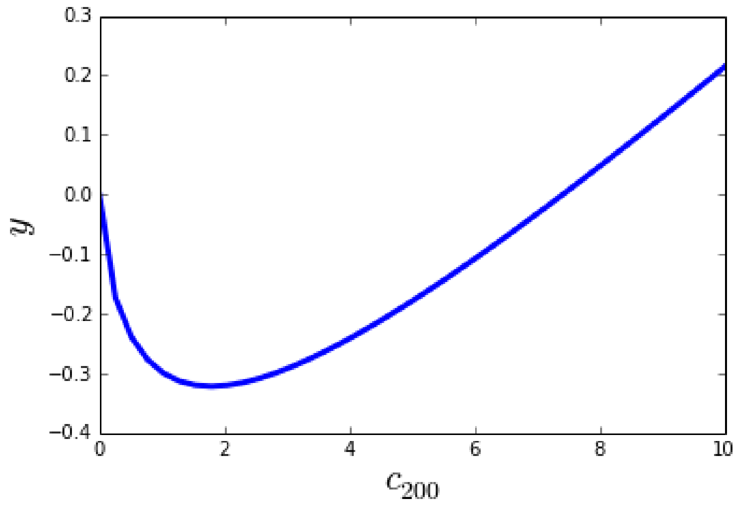
$$\bar{\rho}_{NFW}(r) = 3\rho_a \left(\frac{a}{r}\right)^3 f(x) \tag{34}$$

Now if we want to find the relationship betwee r_{200} and r_{vir} for the NFW profile we have to apply eq8.

$$q = \frac{3\rho_a \frac{a}{r_{200}} f(c_{200})}{3\rho_a \frac{a}{r_{vir}} f(c)} = \frac{c_{200}^3 f(c_{200})}{c_{vir^3} f(c_{vir})} \tag{35}$$

$$\frac{c_{200}}{c_{vir}} = \left(\frac{f(c_{200})}{q f(c_{vir})}\right)^{1/3} \tag{36}$$

For $c_{vir} = 10$ this function is shown in Fig.**??**, where $y = \frac{c_{200}}{c_{vir}} - \left(\frac{f(c_{200})}{q f(c_{vir})}\right)^{1/3}$



Note that the solution of Eq.36 is when $y = 0$, one solution is $c_{200} = 0$ but this is not of particular interest for us.

The other solution is computed analytically using the bisection algorithm. $c_{200} = 7.4$

In order to seek the equivalence between the NFW and the Hernquist profile, We have to match the same enclosed mass of both profiles at a given radius. To his end we have to find M_H in terms of r_s.

$$M_H(r) = M_{NFW}(r) \tag{37}$$

$$\frac{M_H r^2}{a^2 + r^2} = 4\pi\rho_s r_s^3 \left[Ln(1+x) - \frac{x}{1+x}\right] \tag{38}$$

In the limit $r \to 0$

$$M_H = \frac{4\pi\rho_s r_s^3 a^2}{r^2} \left[(x - \frac{x^2}{2}) - x\right] \tag{39}$$

$$M_H = 4\pi\rho_s r_s^3 \frac{a^2}{r^2} \left(-\frac{r^2}{2r_s^2}\right) \tag{40}$$

$$M_H = 2\pi r_s a^2 \tag{41}$$

With this relation is possible now to match both profiles at a given radius $\tilde{r}$

$$M_H(\tilde{r}) = M_{NFW}(\tilde{r}) \tag{42}$$

$$2\pi\rho_s a^2 r_s \frac{\tilde{r}^2}{a^2} \frac{1}{\left(1+\frac{\tilde{r}}{a}\right)^2} = 4\pi\rho_s r_s^3 \left(Ln\left(1+\frac{\tilde{r}}{r_s}\right) - \frac{\tilde{r}}{\tilde{r}+r_s}\right) \tag{43}$$

$$\frac{\tilde{r}^2 a^2}{(a+\tilde{r})^2} = 2r_s^2 \left(Ln\left(1+\frac{\tilde{r}}{r_s}\right) - \frac{\tilde{r}}{\tilde{r}+r_s}\right) \tag{44}$$

$$\frac{\tilde{r}^2 a^2}{(a+\tilde{r})^2} = 2r_s^2 f(\tilde{x}) \tag{45}$$

$$\left(\frac{a}{r_s}\right)^2 = \frac{2}{\tilde{r}^2}(a+\tilde{r})^2 f(\tilde{x}) \tag{46}$$

$$\frac{a}{r_s} = \frac{[2f(\tilde{x})]^{1/2}}{\tilde{r}}(a+\tilde{r}) \tag{47}$$

$$\left(\frac{a}{r_s}\right)\left(1 - \frac{[2f(\tilde{x})]^{1/2}}{\tilde{x}}\right) = [2f(\tilde{x})]^{1/2} \tag{48}$$

$$\frac{a}{r_s} = \frac{[2f(\tilde{x})]^{1/2}}{\left(1 - \frac{[2f(\tilde{x})]^{1/2}}{\tilde{x}}\right)} \tag{49}$$

$$\frac{a}{r_s} = \frac{[2f(\tilde{x})]^{1/2}\tilde{x}}{\tilde{x} - (2f(\tilde{x})^{1/2})} = \frac{1}{\left([2f(\tilde{x})]^{-1/2} - \frac{1}{\tilde{x}}\right)} \tag{50}$$

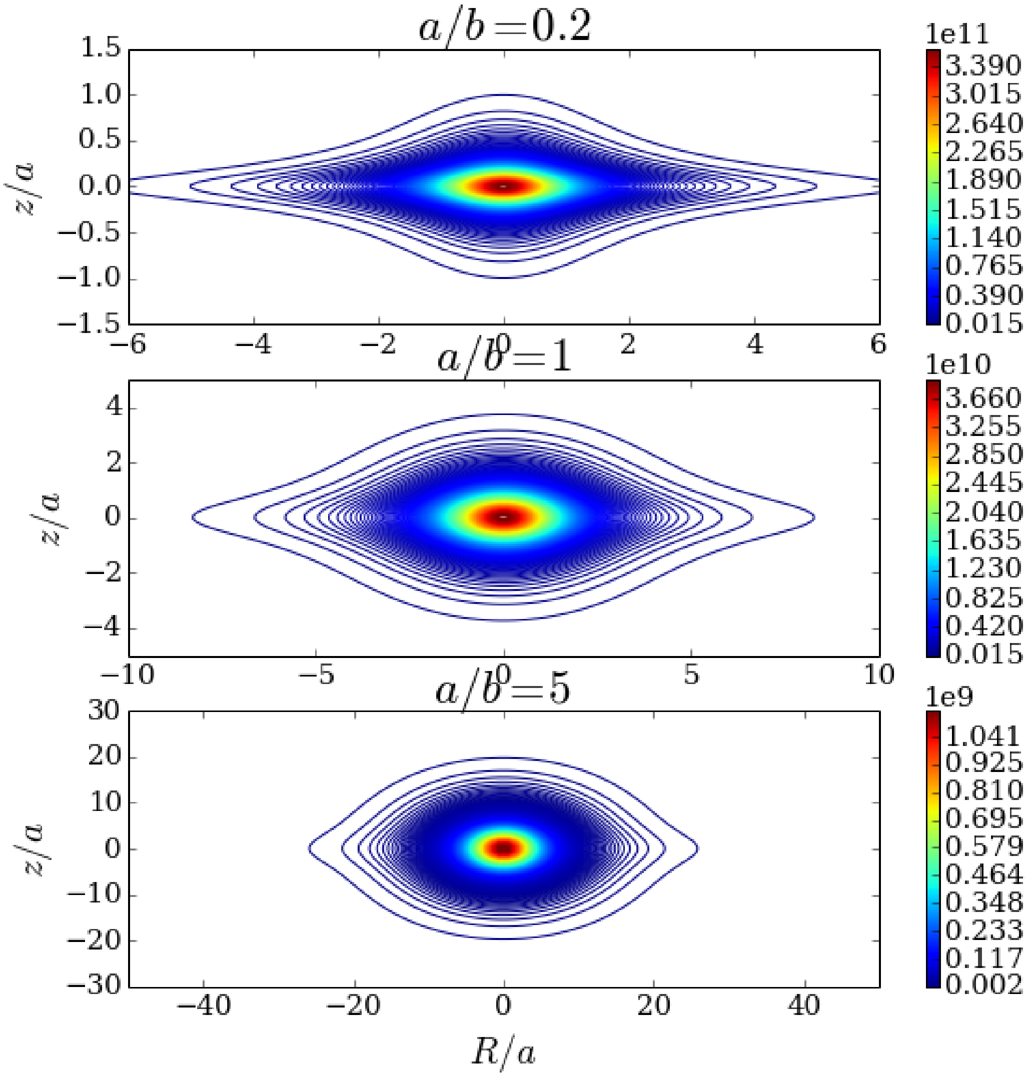
Finally the ratio of the enclosed mass of the Hernquist and the NFW profiles is:

$$\frac{M_H}{M_{vir}} = \frac{2\pi\rho_s a^2 r_s}{4\pi\rho_s r_s^3 f(c_{vir})} = \frac{1}{2f(c_{vir})}\left(\frac{a}{r_s}\right)^2 \tag{51}$$

4 Miyamoto-Nagai Disk

$$\Phi_M(R,z) = -\frac{GM}{\sqrt{R^2 + (a + \sqrt{(}z^2+b^2))^2}} \tag{52}$$

$$\rho_M(R,Z) = \left(\frac{b^2 M}{4\pi}\right) \frac{aR^2 + (a + 3\sqrt{z^2+b^2})(a+\sqrt{z^2+b^2})^2}{[R^2 + (a^2 + \sqrt{z^2+b^2})^2]^{5/2}(z^2+b^2)^{3/2}} \tag{53}$$



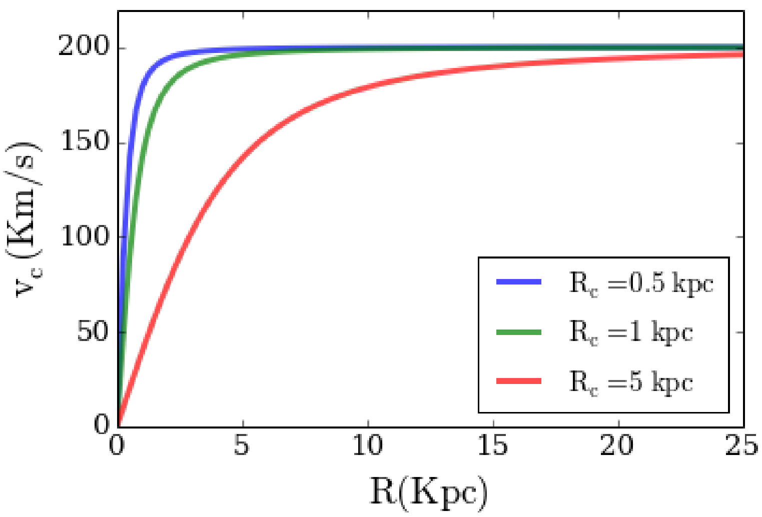
5 Logarithmic Profile

Disc profile

$$\Phi_L(R,z) = \frac{1}{2}v_0^2 ln\left(R_c^2 + R^2 + \frac{z^2}{q_\phi^2}\right) + constant \tag{54}$$

The circular velocity at $z = 0$ is $v_c^2 2(R, z = 0) = r\dfrac{d\Phi}{dR}$:

$$v_c(R, z = 0) = r\frac{d\Phi_L}{dr} = \frac{v_0 R}{\sqrt{r_c^2 + R^2}} \tag{55}$$

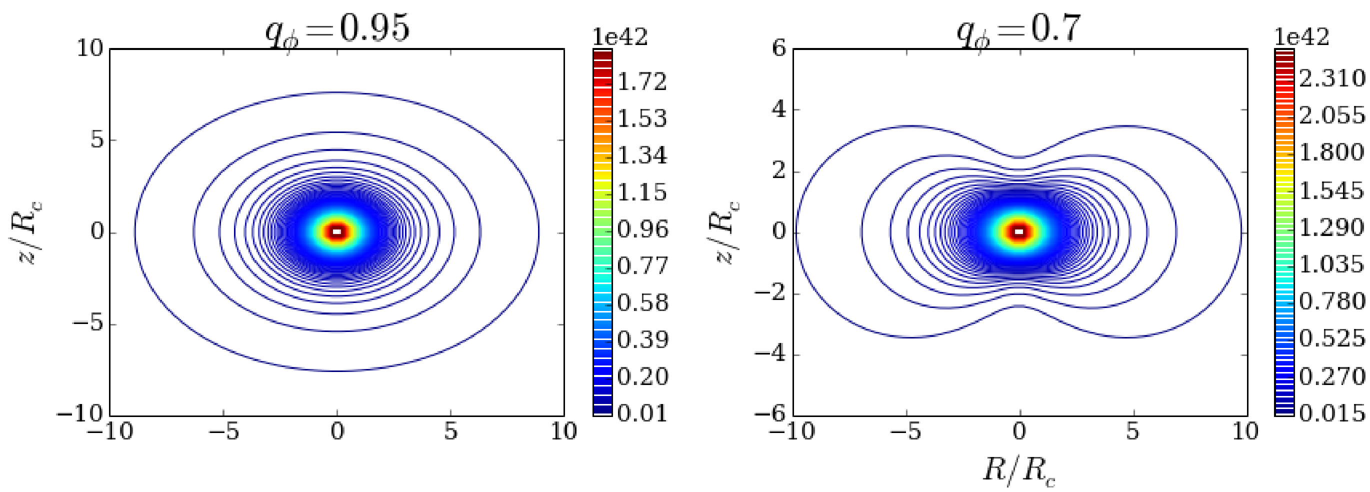


To derive the density we make use of Poisson's equation in cylindrical coordinates:

$$\rho_L(R,z) = \frac{\nabla^2 \Phi_L}{4\pi G} = \frac{1}{4\pi G}\left(\frac{1}{r}\frac{d}{dR}R\frac{d}{dR} + \frac{d^2}{dz^2}\right)\Phi_L \tag{56}$$

$$\rho_L(R,z) = \frac{v_0^2}{8\pi G}\left(\frac{1}{R}\frac{4R(R_c^2+R^2+\frac{z^2}{q_\phi^2}) - 4R^3}{(R_c^2+R^2+\frac{z^2}{q_\phi^2})^2} + \frac{\frac{2}{q_\phi^2}(R_c^2+R^2+\frac{z^2}{q_\phi^2}) - \frac{4z^2}{q_\phi^4}}{(R_c^2+R^2+\frac{z^2}{q_\phi^2})^2}\right) \tag{57}$$

$$\rho_L(R,z) = \frac{v_0^2}{4\pi G q_\phi^2}\frac{(2q_\phi^2+1)R_c^2 + r^2 + (2-q_{\Phi^2})z^2}{(R_c^2+r^2+z^2q_\Phi^{-2})^2} \tag{58}$$



6 Logarithmic profile (Law, Johnston & Majewski)

This profile is almost the same that the one explained in the previous section 5

$$\Phi = v_{halo}^2 ln[r^2 + (z^2/q^2) + d^2] \tag{59}$$

$$v_c = \frac{\sqrt{2}vr}{\sqrt{r^2 + d^2}} \tag{60}$$

with $q = 0.8 - 1.45$, v_{halo} should match $v_{circ,\odot}$, $d = 1 - 20kpc$

7 Logarithmic (Law & Majewski 2010)

`http://iopscience.iop.org/0004-637X/714/1/229/pdf/apj_714_1_229.pdf`

$$\Phi = v_{halo}^2 ln(C_1 x^2 + C_2 y^2 + C_3 xy + (z/q_z)^2 r_{halo}^2) \tag{61}$$

R, r (Cylindrical, spherical)

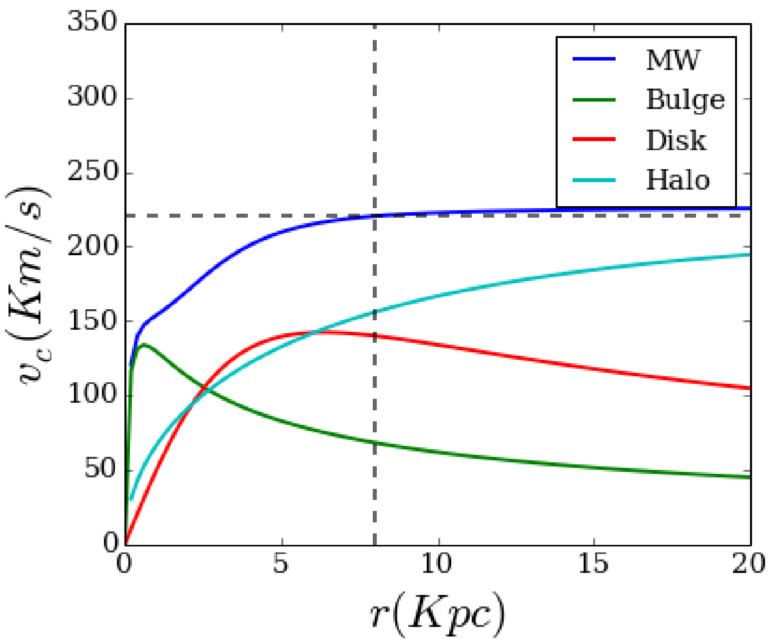
$$C_1 = \left(\frac{cos^2\phi}{q_1^2} + \frac{sin^2\phi}{q_2^2} \right) \tag{62}$$

$$C_2 = \left(\frac{cos^2\phi}{q_2^2} + \frac{sin^2\phi}{q_1^2} \right) \tag{63}$$

$$C_3 = 2sin\phi cos\phi \left(\frac{1}{q_1^2} - \frac{1}{q_2^2} \right) \tag{64}$$

8 Modelling the MW

Component	Besla07	LM2010	Roeland12	Gomez15
Disk Model	Miyamoto-Nagai	Miyamoto-Nagai		
Disk Mass($M_\odot$)	5.5^{10}	1.0×10^{11}		
Disk Param	$R_d = 3.5, z = r_{disk}/5.0$	$\alpha = 1$		$, a = 6.5kpc, b = 0.26Kpc$
Bulge Model	Hernquist	Hernquist		Hernquist
Bulge Mass($M_\odot$)	10^{10}	3.4×10^{10}		
Bulge Param	$0.6kpc$	$3.4 \times 10^{10} M_\odot, c = 0.7kpc$	0.6Kpc	
DM halo Model	NFW	7	Hernquist(NFW)	
DM halo mass($M_\odot$)	10^{12}	$\times 10^{10}$		
Halo Param	$c = 11, r_{vir} = 258Kpc$	$r_{halo} = 12Kpc$		
Solar distance $R_\odot$ (kpc)	8.0			
reference	Besla07	LM2010		



9 TO-DO:

- Study the properties and different parameters (M, $/pho$, v_c, a)
- Put all the plots and explnations in the text
- note: Klypin relation between c and M_{vir} Doesn't take into account adiabatic contraction.
- work in the code that integrates the orbits using the accelerations. (Viernes)

References

1. J. Binney and S. Tremaine. *Galactic Dynamics: Second Edition*. Princeton University Press, 2008.

2. G. L. Bryan and M. L. Norman. Statistical Properties of X-Ray Clusters: Analytic and Numerical Comparisons. , 495:80–99, March 1998.

3. V. R. Eke, S. Cole, and C. S. Frenk. Cluster evolution as a diagnostic for Omega. , 282:263–280, September 1996.