sufficient amount to alter the reaction rates. This phenomenon is generally called *electron screening* in nuclear reactions.

2-4 POLYTROPES

The static configuration of a gaseous sphere held together by self-gravitation must satisfy the condition of hydrostatic equilibrium. The essence of the equation of hydrostatic equilibrium is that the pressure at each point in a stellar interior is sufficient to just balance the weight of the overlying layers of the star. Furthermore, the pressure itself is determined by the equation of state applicable to the local conditions in the stellar interior. These considerations do not in themselves determine the structure of a star. Any specified pressure that may be required to support the overlying layers is obtainable from an unlimited number of combinations of density and temperature at that point. What are clearly needed are more conditions on the density and temperature in a stellar interior that relate to other physical processes that go on there.

It was about the beginning of the twentieth century when several notable physicists, viz., Lane, Ritter, Kelvin, Emden, and Fowler, considered the question of what limitations could be placed on the structure of a star just from the condition of hydrostatic equilibrium alone. They quickly concluded that some other condition relating the physical variables in the stellar interior is necessary in order to be able to specify the structure. The necessary relationships are to be found in the production and transport of thermal energy, subjects to be discussed in subsequent chapters. One explicit auxiliary condition that has been found to correspond to certain idealized physical situations, however, is a pressure expressible in terms of some power of the density only. For historical reasons[1] the assumed pressure-density relationship is written as

$$P = K\rho^{(n+1)/n} \tag{2-283}$$

where the number n is called the polytropic index. Gaseous spheres in hydrostatic equilibrium in which the pressure and density are related by Eq. (2-283) at each point along the radius are called *polytropes*. The constant K depends upon the nature of the polytrope. It was shown by Lane and Emden that if a polytropic pressure-density relation is assumed, the properties of the structure can be computed.

Since, of course, *any* explicit relationship between the pressure and the density would make possible the solution for the structure of a self-gravitating gaseous sphere in hydrostatic equilibrium, one might ask why a relationship of the form

[1] The nomenclature is patterned after quasistatic changes of state of an ideal gas for which a generalized specific heat is held constant. It was found by early workers in kinetic theory that for such changes of state, called *polytropic changes* by R. Emden in his classical treatise "Gaskugeln," B. G. Teubner, Leipzig, 1907, the variables change along a track $P = K\rho^{\gamma'}$, where γ' is determined by C_P, C_V, and C, the specific heat characterizing the process. These matters of historical interest are elegantly summarized in Chandrasekhar, *op. cit.*, chaps. 2 and 4.

of Eq. (2-283) should be chosen for study. The reason lies in the fact that some idealized physical circumstances for a star would lead naturally to equations of that form. To clarify this point, we shall consider examples of stars that can be represented by polytropes.

As a first example, we may follow Kelvin in considering a star that is "boiling" in a state that he described as *adiabatic convective equilibrium.* If the whole interior of a star is completely convective, mass elements are both rising and falling in the interior of a star. A star is said to be in convective equilibrium if any mass element after rising and falling from its initial temperature and density to a new temperature and density finds itself at the same temperature and density as the surroundings. The convective equilibrium is adiabatic if the convective cells move without heat exchange. It will be evident that if some mechanism continuously stirs and mixes the entire interior of a star, it must soon come to a condition of convective equilibrium, for any differences in temperature and density of the surroundings in a star from those of an element that has risen from some lower portion of a star will quickly be eliminated. If radiation pressure is an unimportant determinant in the structure, adiabatic changes are of the form

$$P = K\rho^{\gamma} \tag{2-284}$$

where $\gamma = \frac{5}{3}$ for an ideal monatomic gas. If such a rising or falling element is, furthermore, at the same conditions of temperature, density, and pressure as the surrounding matter at all times, it follows that the run of pressure and density in the star is such that

$$P(r) = K\rho(r)^{5/3} \tag{2-285}$$

It seems, therefore, that a star in convective equilibrium in which radiation pressure is not important is a polytrope of exponent $\gamma = \frac{5}{3}$, which is also a polytrope of index $n = 1.5$. In such a way Kelvin was led quite naturally to at least one physical possibility that would correspond to the structure of a polytrope.

As a second example, consider a star in which radiation pressure is not unimportant. We have defined the quantity β such that

$$P_g = \frac{N_0k}{\mu}\rho T = \beta P \qquad P_r = \tfrac{1}{3}aT^4 = (1-\beta)P \tag{2-286}$$

for a nondegenerate gas.

Equating the values of the pressure from these two equations gives immediately *at each point*

$$T = \left(\frac{N_0k}{\mu}\frac{3}{a}\frac{1-\beta}{\beta}\right)^{1/3}\rho^{1/3} \tag{2-287}$$

Since we also have

$$P = \frac{N_0k}{\mu}\frac{\rho T}{\beta} \tag{2-288}$$

we see that

$$P = \left[\left(\frac{N_0k}{\mu}\right)^4 \frac{3}{a}\frac{1-\beta}{\beta^4}\right]^{1/3}\rho^{4/3} \tag{2-289}$$

This equation is true at each point in the interior of the star we are considering. The ratio of gas pressure to total pressure β does, in general, depend upon the distance from the center of the star. *If,* however, one had a special configuration in which the quantity β was a constant, i.e., such that the gas pressure was a constant fraction of the total pressure throughout the star, then the expression in brackets in Eq. (2-289) reduces to a constant, and one has an equation of the form

$$P = K\rho^{4/3} \tag{2-290}$$

This model star would correspond to a polytrope of polytropic exponent $\frac{4}{3}$ or of index 3. We can see later that this particular polytrope corresponds more closely to stars in radiative equilibrium, i.e., stars for which the energy is transported by radiative transfer rather than by convection. It will, in fact, be shown that the polytrope of index 3 corresponds to that star in radiative equilibrium such that at each distance r from the center of the star, the product of the energy liberated per unit mass from all the material interior to r times the opacity of the gas at the point r is a constant. The properties of a nondegenerate polytrope of index 3 have also been highly developed in the analytical study of gaseous configurations, especially by Eddington. This model star is frequently called the *standard model.* We shall use the standard model often in an attempt to get a first approximation to the runs of temperatures and densities in the interiors of stars.

A third example may be provided by stars supported by the pressure of a completely degenerate electron gas (white dwarfs?). That pressure has been shown to vary as $\rho^{5/3}$ or $\rho^{4/3}$, according to whether the electron velocities are nonrelativistic or relativistic. The corresponding polytropes can provide useful insights into their structure.

These examples give some indication of the physical reasons that lie behind considering the structure of gaseous spheres in hydrostatic equilibrium for which the pressure and density are related by an equation of the form of Eq. (2-283). Motivated by the fact that the density is proportional to T^n in a nondegenerate polytrope of index n, a convenient definition is

$$\rho \equiv \lambda\phi^n \tag{2-291}$$

where λ is a scaling parameter whose value depends upon the definition of the quantity ϕ. This representation for the run of density throughout the star will be convenient for the study of polytropes, where we shall identify the parameter λ with the central density of the star, thereby normalizing the function ϕ to unity at the center. For this representation, the pressure is

$$P = K\rho^{(n+1)/n} = K\lambda^{(n+1)/n}\phi^{n+1} \tag{2-292}$$

The solution for the structure of a polytrope, then, depends upon the coupling of Eq. (2-283) to the condition of hydrostatic equilibrium

$$\frac{dP}{dr} = -\rho \frac{GM_r}{r^2} \qquad \frac{dM_r}{dr} = 4\pi r^2 \rho \tag{2-293}$$

from which it follows that

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = \frac{1}{r^2}\frac{d}{dr}(-GM_r) = -4\pi G\rho \tag{2-294}$$

Substitution of the values of pressure and density for a polytrope of index n into this last equation yields

$$(n+1)K\lambda^{1/n}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = -4\pi G\lambda\phi^n \tag{2-295}$$

This equation can be made more attractive by defining a unit of length

$$a \equiv \left[\frac{(n+1)K\lambda^{(1-n)/n}}{4\pi G}\right]^{\frac{1}{2}} \tag{2-296}$$

and by defining a dimensionless distance variable $\xi = r/a$, whereupon Eq. (2-295) reduces to

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\phi}{d\xi}\right) = -\phi^n \tag{2-297}$$

This equation is called the *Lane-Emden equation* for the structure of a polytrope for index n. The solution for ϕ as a function of ξ completely determines the structure of the polytrope except for the choice of the central density. By setting λ equal to the central density, it is easy to see that the temperaturelike variable ϕ must obey certain boundary conditions at the center of the star, i.e., at $\xi = 0$; viz.,

$$\phi = 1 \qquad \frac{d\phi}{d\xi} = 0 \qquad \text{at } \xi = 0 \tag{2-298}$$

Problem 2-55: Show that $(d\phi/d\xi)_0 = 0$. *Hint:* Expand the equation of hydrostatic equilibrium near the center.

The solution ϕ which satisfies the Lane-Emden equation of index n under these boundary conditions is called the *Lane-Emden function of index n*.

Explicit solutions of the Lane-Emden equation for general values of n apparently do not exist. For values of n other than $n = 0$, 1, and 5, numerical techniques must be employed for the determination of the Lane-Emden function. We first note that if $\phi(\xi)$ is a solution of the equation, then $\phi(-\xi)$ is also a solution. This observation implies that if ϕ is expressed as a power series in ξ, only even powers of ξ appear; that is,

$$\phi(\xi) = C_0 + C_2\xi^2 + C_4\xi^4 + \cdots \tag{2-299}$$

Problem 2-56: By substituting Eq. (2-299) into the Lane-Emden equation, show that the first three terms of the series for ϕ are

$$\phi = 1 - \tfrac{1}{6}\xi^2 + \frac{n}{120}\xi^4 - \cdots \tag{2-300}$$

By taking a sufficient number of terms in this alternating series for ϕ, one can calculate the solution to any desired accuracy for values of $\xi < 1$. For values of $\xi > 1$, this solution can be continued from the differential equation by standard numerical methods. The solutions are found to decrease monotonically from the center and for values of n less than 5 have a zero for some finite value of ξ, say, $\xi = \xi_1$. At $\xi = \xi_1$ it is clear that ϕ's being equal to zero makes the pressure vanish, and the configuration may be said to have a physical boundary at that point. Table 2-5 lists the value of ξ_1 and the derivative of ϕ at $\xi = \xi_1$ for the various Lane-Emden functions of index n. We shall shortly see that the value of ξ_1 and the slope of ϕ at $\xi = \xi_1$ are important for determining large-scale properties of the various gaseous configurations.

It is evident that the structure of each polytrope is specified in terms of the dimensionless length ξ. An inspection of the length a used in forming this dimensionless length shows that its value is determined by two numbers; the first is the constant K, which occurs in Eq. (2-283) relating the pressure to the density, and the second is the parameter λ, which we have taken to be the central density for the solution of this problem. It is apparent, therefore, that each Lane-Emden function ϕ_n represents, for a specified value of the constant K, a one-parameter family of solutions, the parameter being the central density λ. As an example, we may turn to the polytropes that were considered in introducing this whole discussion. In the case of the completely convective polytrope of index 1.5, it is easy to see that the value of K is determined by the particular adiabat of the gas. In adiabatic convective equilibrium the entropy per gram of material is constant, and it specifies the value of K.

Problem 2-57: Ignore the entropy of the radiation and show from Eq. (2-136) that the entropy per gram can be written

$$S = \frac{3}{2}\frac{N_0k}{\mu}\ln\frac{P}{\rho^{5/3}} + \text{const} \tag{2-301}$$

that is, $S = S(K)$.

In the case of the standard model, or polytrope of index 3, we see from Eq. (2-289) that the value of the constant K is given by

$$K = \left[\left(\frac{N_0k}{\mu}\right)^4 \frac{3}{a}\frac{1-\beta}{\beta^4}\right]^{\frac{1}{3}} \tag{2-302}$$

where β represents the ratio of the gas pressure to the total pressure and is a constant throughout the standard model. The selection of a value of K in either of these two instances still allows a complete run of corresponding solutions as determined by the dentral density λ. It is quite clear that considerable leeway still exists for the actual structure of the various polytropes being considered. To understand the way in which these various factors come into play, we need to consider several more large-scale properties of the configuration that can be derived from the material presented so far.

(*1*) *Radius:* The radius of the configuration is by definition determined by the first zero of the Lane-Emden function of order n. Thus

$$R = a\xi_1 = \left[\frac{(n+1)K}{4\pi G}\right]^{\frac{1}{2}} \lambda^{(1-n)/2n}\xi_1 \tag{2-303}$$

(*2*) *Mass:* The mass M interior to the normalized radius ξ is given by

$$M(\xi) = \int_0^{a\xi} 4\pi r^2 \rho \, dr = 4\pi a^3 \int_0^{\xi} \lambda \phi^n \xi^2 \, d\xi \tag{2-304}$$

By using the Lane-Emden equation itself, the integral in Eq. (2-304) can be transformed to

$$\begin{aligned} M(\xi) &= -4\pi a^3 \lambda \int_0^{\xi} \frac{d}{d\xi} \xi^2 \frac{d\phi}{d\xi} \, d\xi \\ &= -4\pi a^3 \lambda \xi^2 \frac{d\phi}{d\xi} \end{aligned} \tag{2-305}$$

Substituting for the unit of length a and evaluating the above expression at $\xi = \xi_1$

Table 2-5 Constants of the Lane-Emden functions†

n	ξ_1	$-\xi_1^2 \left(\frac{d\phi}{d\xi}\right)_{\xi=\xi_1}$	$\frac{\rho_c}{\bar{\rho}}$
0	2.4494	4.8988	1.0000
0.5	2.7528	3.7871	1.8361
1.0	3.14159	3.14159	3.28987
1.5	3.65375	2.71406	5.99071
2.0	4.35287	2.41105	11.40254
2.5	5.35528	2.18720	23.40646
3.0	6.89685	2.01824	54.1825
3.25	8.01894	1.94980	88.153
3.5	9.53581	1.89056	152.884
4.0	14.97155	1.79723	622.408
4.5	31.83646	1.73780	6,189.47
4.9	169.47	1.7355	934,800
5.0	∞	1.73205	∞

† S. Chandrasekhar, "An Introduction to the Study of Stellar Structure," p. 96; reprinted from the Dover Publications edition, Copyright 1939 by The University of Chicago, as reprinted by permission of The University of Chicago.

gives the total mass of the star:

$$\mathfrak{M} = -4\pi \left[\frac{(n+1)K}{4\pi G}\right]^{\frac{3}{2}} \lambda^{(3-n)/2n} \left(\xi^2 \frac{d\phi}{d\xi}\right)_{\xi_1} \tag{2-306}$$

We note that in the case $n = 3$, the mass depends only upon K and is independent of λ. The product of ξ_1^2 times the slope of ϕ evaluated at the first zero, ξ_1, is one of the quantities listed in Table 2-5.

Problem 2-58: Show that the mass of the standard-model polytrope of index 3 is given numerically by

$$\mathfrak{M} = 18.0 \frac{\sqrt{1-\beta}}{\mu^2 \beta^2} \mathfrak{M}_\odot \tag{2-307}$$

For a given composition μ, the mass determines the value of β.

Problem 2-59: Imagine that a white dwarf is a body supported by the pressure of completely degenerate electrons. As the mass of the structure is increased, the central density becomes so high that the degeneracy becomes relativistic at the center, such that (confirm this)

$$P_e \to 1.244 \times 10^{15} \left(\frac{\rho}{\mu_e}\right)^{\frac{4}{3}} \quad \text{dynes/cm}^2 \tag{2-308}$$

and falls off to nonrelativistic degeneracy in the outer portions of the star. As the mass is continually increased, the star shrinks to ever higher densities and ever smaller radius, until the electrons become highly relativistic everywhere. Then Eq. (2-308) is applicable throughout the star. Show that at this point the mass is

$$\mathfrak{M} = \frac{5.80}{\mu_e^2} \mathfrak{M}_\odot \tag{2-309}$$

This mass is called the *Chandrasekhar limit*, since Chandrasekhar showed that this was the maximum mass that could be supported by electron degeneracy. (Other physical effects such as rotation and inverse beta decay have been ignored.) It seems clear that this value must represent a limiting mass because the electrons can be relativistic throughout only if the mass is sufficient to squeeze the volume to a point. Inasmuch as white dwarfs are observed to have nonzero radii, their masses must be less than Eq. (2-309). The question of what happens if the mass exceeds this limit is a difficult one and will not be considered here.

(*3*) *Ratio of mean density to central density:* The mean density of the configuration is given by the total mass of the configuration divided by the volume of the configuration, whereas the central density is equal to λ. Thus, the ratio of the mean density to the central density can be determined from Eqs. (2-303) and (2-306) to be

$$\frac{\bar{\rho}}{\rho_c} = -\frac{3}{\xi_1}\left(\frac{d\phi}{d\xi}\right)_{\xi_1} \tag{2-310}$$

It is evident that the ratio of mean density to central density depends only upon the index of the polytrope. In fact, we may take this to be the main feature of the polytropic index; viz., the extent to which the matter is concentrated toward the center of the star. Although we have not shown it here, the ratio of the central density to the mean density varies between the limits of unity (a star of uniform density) for a polytrope of index zero to value of infinity (a star infinitely concentrated toward the center) for a polytrope of index 5, passing through all intermediate values as n increases from zero to 5.

Problem 2-60: Show that the central density in the standard model exceeds the mean density by the factor $\rho_c/\bar{\rho} = 54.2$.

(4) *The central pressure:* Since ϕ is normalized to unity at $\xi = 0$ by the interpretation of λ, the central pressure may be written, from Eq. (2-292), as

$$P_c = K\lambda^{(n+1)/n} \tag{2-311}$$

To express the central pressure in terms of macroscopic properties, we note that Eq. (2-303) can be written in the form

$$R = \left[\frac{(n+1)}{4\pi G}\,\xi_1^2\right]^{\frac{1}{2}} [K\lambda^{(1-n)/n}]^{\frac{1}{2}} \tag{2-312}$$

from which

$$K\lambda^{(1-n)/n} = \frac{4\pi R^2 G}{(n+1)\xi_1^2} \tag{2-313}$$

Hence, the central pressure is given by

$$\begin{aligned} P_c &= (K\lambda^{(1-n)/n})\lambda^2 = K\lambda^{(1-n)/n}\rho_c^2 \\ &= \frac{4\pi R^2 G}{(n+1)\xi_1^2}\left[\frac{\xi_1}{3}\,\frac{1}{(d\phi/d\xi)_{\xi_1}}\right]^2 \bar{\rho}^2 \\ &= \frac{1}{4\pi(n+1)(d\phi/d\xi)_{\xi_1}^2}\,\frac{G\mathfrak{M}^2}{R^4} \end{aligned} \tag{2-314}$$

Problem 2-61: Show that for the standard model the central pressure is given numerically by

$$P_c = 1.24 \times 10^{17}\left(\frac{\mathfrak{M}}{\mathfrak{M}_\odot}\right)^2\left(\frac{R_\odot}{R}\right)^4 \quad \text{dynes/cm}^2 \tag{2-315}$$

This result gives a larger and more realistic estimate of the central pressure of the sun than the earlier rough arguments because it allows for the central condensation.

(5) *The central temperature:* The central temperature may be computed from the central pressure and the central density by use of the appropriate equation of state. For the case of the ideal ionized nondegenerate gas the relevant relationships are

$$P_g = \frac{N_0 k}{\mu}\,\rho_c T_c = \beta_c P_c \tag{2-316}$$

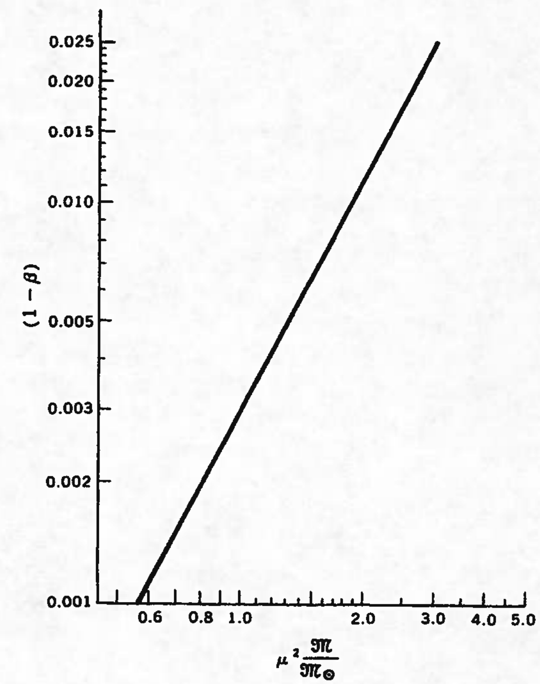

Fig. 2-18 The value of $1 - \beta$ for the standard model of low-mass stars.

where P_c equals the total pressure (gas plus radiation) at the center of the star. Thus the central temperature is given by

$$T_c = \frac{\mu}{N_0 k}\,\frac{\beta_c P_c}{\rho_c} \tag{2-317}$$

Problem 2-62: Show that for the standard model the central temperature is

$$T_c = 4.6 \times 10^6 \mu\beta\left(\frac{\mathfrak{M}}{\mathfrak{M}_\odot}\right)^{\frac{2}{3}}\rho_c^{\frac{1}{3}} \tag{2-318}$$

We may note that the equation above for the central temperature of the standard model contains quantities that are not independent of each other. In Prob. 2-58 it was demonstrated that the mass of the standard model is related to the ratio of gas pressure to total pressure; therefore, for a fixed value of μ it is apparent that β is a function of the mass (or vice versa), although the solution cannot be written explicitly. Figures 2-18 and 2-19 show graphically the dependence of β upon the quantity $\mu^2(\mathfrak{M}/\mathfrak{M}_\odot)$. These figures indicate the growing importance of radiation pressure with increasing mass. Since the mean molecular

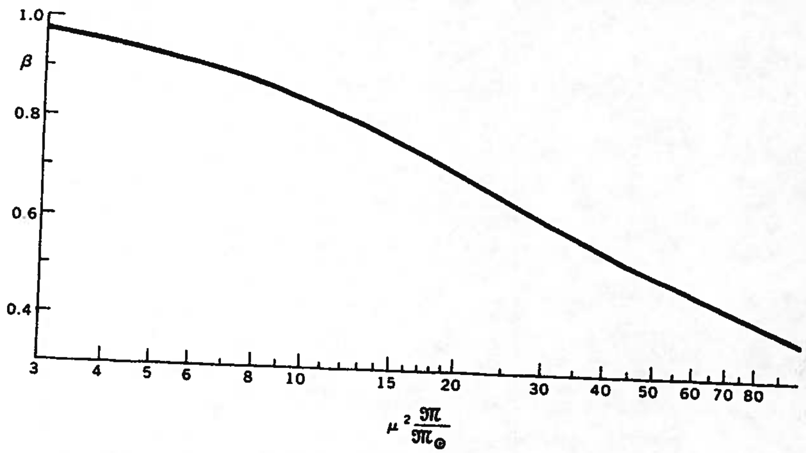

Fig. 2-19 The value of $1 - \beta$ for the standard model of high-mass stars.

weight μ lies between $\frac{1}{2}$ and 2, the quantity $\mu^2(\mathfrak{M}/\mathfrak{M}_\odot)$ is not greatly different from the mass expressed in solar masses. At any rate, this additional constraint must be taken into account when using Eq. (2-318) for the central temperature of the standard model.

We may in fact use Eq. (2-318) at this point to make an estimate of the central temperatures of main-sequence stars. Main-sequence stars certainly are not polytropes of index 3, but we may nonetheless expect to see the trend of central temperatures by representing all main-sequence stars by the standard model. It is clear that the central density is an unknown in Eq. (2-318); however, an earlier problem demonstrated that $\rho_c = 54.2\bar{\rho}$ for the standard model, so that Eq. (2-318) may be altered to read

$$T_c = 17.4 \times 10^6 \mu\beta \left(\frac{\mathfrak{M}}{\mathfrak{M}_\odot}\right)^{2/3} \bar{\rho}^{1/3} \tag{2-319}$$

Problem 2-63: A certain type O star has $\mathfrak{M} = 30\mathfrak{M}_\odot$, $R = 6.6R_\odot$, $X = 0.70$, and $Y = 0.30$. Estimate the importance of radiation pressure and the central temperature by approximating the star by the standard model. A much better calculation[1] on electronic computers yields $\beta_c = 0.77$, $T_c = 3.7 \times 10^7$.

The mean density $\bar{\rho}$ is just the mass divided by the volume, so that we can obtain these properties of main-sequence stars from Table 1-1. In fact, when the properties of Table 1-1 are coupled with the mass-luminosity relationship and with the fact that $\bar{\rho}_\odot = 1.4$ g/cm³, it appears that an approximation to the

[1] R. Stothers, *Astrophys. J.*, **138**:1074 (1963).

average density of main-sequence stars is

$$\bar{\rho} \approx \frac{1.4}{\mathfrak{M}/\mathfrak{M}_\odot} \quad \text{g/cm}^3 \tag{2-320}$$

Problem 2-64: Using the data of Table 1-1, show that a main-sequence star of $6\mathfrak{M}_\odot$ has an average density approximately equal to 0.28 g/cm³. What is the percentage difference between this value and that inferred from Eq. (2-320)? How much of a percentage error in the central temperature would be introduced by that percentage error in the average density?

It is of some interest that the density of main-sequence stars decreases with increasing mass. This fact is a consequence of the virial theorem, which demands higher temperatures for higher values of the potential energy of self-gravitation. These higher temperatures become sufficient to support the star in a more distended configuration of lower density.

It is also clear from Fig. 2-18 that $\beta \approx 1$ for stars of main-sequence mass. The mean molecular weight has a value near 0.7 for the centers of stars that have partially depleted their hydrogen (say, $X \approx 0.5$, $Y \approx 0.5$). By combining these approximations, Eqs. (2-319) and (2-320) yield

$$T_c \approx 14 \times 10^6 \left(\frac{\mathfrak{M}}{\mathfrak{M}_\odot}\right)^{1/3} \tag{2-321}$$

as anticipated central temperatures for main-sequence stars. When the question of thermonuclear reactions is considered, Eq. (2-321) will prove a helpful guide.

Many other interesting physical quantities can be calculated for these model stars called polytropes. An extensive discussion of the mathematical considerations related to this well-developed subject will be found in the monograph on stellar structure by Chandrasekhar. Furthermore, we have considered only uniform polytropes of homogeneous composition, whereas it is possible to divide stars into polytropic shells or mixed polytropes. A great deal of intuitive appreciation for the complexities of the physics of stellar structure may be obtained by an extensive analysis of the structure of polytropes. We have employed only the simplest features in this section as an introduction to the subject. Modern research has shown that the usefulness of polytropes is for the most part limited to this introductory acquaintance. Accurate and detailed models of the structure of real stars may be obtained only from detailed computer calculation. The additional physics needed to make these detailed calculations will constitute the burden of the following chapter.