chapter 2

THERMODYNAMIC STATE OF THE STELLAR INTERIOR

The macroscopic properties of a star are intimately related to the microscopic phenomena occurring in the interior material. These phenomena and their rates depend upon the thermodynamic state of the material. One can calculate that in the interior environment the particles move very short distances compared to distances over which the temperature changes significantly before they collide with other particles. The rates of the fundamental atomic collision processes are, furthermore, very fast in comparison with rates of change of the local thermodynamic state. These facts enable one to assume a very important simplification in the description of the matter, viz., *local thermodynamic equilibrium*. In the state of thermodynamic equilibrium, all properties of matter are calculable in terms of the chemical composition, the density, and the temperature. In his pioneering book "The Internal Constitution of the Stars," Sir Arthur Eddington has given the following vivid description:

> *The inside of a star is a hurly-burly of atoms, electrons, and aether waves. We have to call to aid the most recent discoveries of atomic physics to follow the intricacies of the dance. We started to explore the inside of a star; we soon find ourselves exploring the inside of an atom. Try to picture the tumult! Dishevelled atoms tear along at 50 miles a second with only a few tatters left of the elaborate cloaks of electrons torn from them in the scrimmage. The lost electrons are speeding a hundred times faster to find new resting places. Look out!*

there is nearly a collision as an electron approaches an atomic nucleus; but putting on speed it sweeps round it in a sharp curve. A thousand narrow shaves happen to the electron in 10^{-10} of a second; sometimes there is a side-slip at the curve, but the electron still goes on with increased or decreased energy. Then comes a worse slip than usual; the electron is fairly caught and attached to the atom, and its career of freedom is at an end. But only for an instant. Barely has the atom arranged the new scalp on its girdle when a quantum of aether waves runs into it. With a great explosion the electron is off again for further adventures. Elsewhere two of the atoms are meeting full tilt and rebounding, with further disaster to their scanty remains of vesture.

As we watch the scene we ask ourselves, "Can this be the stately drama of stellar evolution?" . . .[1]

This chaotic situation is reduced to tractable proportions by application of the principles of statistical mechanics. Because thermodynamic equilibrium is quickly achieved on the atomic (but not nuclear) scale, the rates of all atomic (i.e., electromagnetic, but not nuclear) reactions equal those of their inverse reactions. The hurly-burly of the individual electron is replaced by a steady macroscopic state whose properties are embodied in the principles of statistical physics. The functions of state are determined by the chemical composition, density, and temperature. Foremost among these is the pressure $P = P(\rho,T)$, commonly called the *equation of state*, from which the star derives its structural support against gravity. The burden of this chapter will be the discussion of the equation of state and related phenomena.

Near the surface of the stars, the equation of state of the gas is extremely complicated. The atomic constituents of the outer layers are in varying degrees of ionization. Application of the Saha ionization equation reveals that the hydrogen constituent becomes almost completely ionized by the time the temperature has risen to about 10^4 °K, whereas the helium is almost completely ionized by the time the temperature has risen to 10^5 °K, at which temperature the heavier elements have also lost a sizable number of their electrons to the continuum and are in relatively high stages of ionization. For temperatures higher than 10^5 °K, it becomes increasingly more accurate, insofar as the pressure is concerned, to talk of a completely ionized gas. Other important properties of the gas, such as its internal energy and its opacity to radiation, are strongly dependent upon the degree of ionization. For the common stellar composition, in which hydrogen and helium comprise more than 95 percent of the mass, the pressure at temperatures greater than 10^5 °K can be calculated to high accuracy by assuming complete ionization. Significantly, a large fraction of the mass of most stars *does* lie at temperatures higher than 10^5 °K. The bulk of the structure of most stars is determined, therefore, by an equation of state appropriate to completely ionized matter. From an analytical point of view, it is extremely

[1] A. S. Eddington, "The Internal Constitution of the Stars," p. 19, Dover Publications, Inc., New York, 1959.

fortunate that this is so. A completely ionized gas behaves like a perfect gas to extremely high densities. Terrestrial matter reaches a density of only a few grams per cubic centimeter before it begins to resist compression, and the perfect-gas law begins to break down even before that density is reached. The rather large size of atoms and the interatomic forces between the electron clouds of the various atoms set a rather sudden limit to the density of un-ionized matter. The radii of nuclei, on the other hand, are only 10^{-5} of the radii of most atoms. A gas composed of nuclei and electrons, therefore, occupies only about 10^{-15} of the volume occupied by atoms. We may anticipate, therefore, that highly ionized matter can be compressed to extremely high densities before the perfect-gas law will break down as a result of the volume effect.

A *perfect gas* is defined as one in which there are no interactions between the particles of the gas. Although this criterion is never satisfied exactly in real gases, the approximation is physically sound if the average interaction energy between particles is much smaller than their thermal energies. This last condition may be satisfied by a weak interaction or by a sufficiently rarefied gas. In the ionized gas of a stellar interior the real interactions between particles are predominantly the coulomb interactions. It is fortunate that most physical circumstances in the stellar interior are such that the average coulomb energy of particles is much less than their characteristic kinetic energy, which is of the order kT for a nondegenerate gas. For this reason it will be adequate for most applications to use the equation of state of a perfect gas. We shall return later to the question of the real ionized gas and its applications.

2-1 MECHANICAL PRESSURE OF A PERFECT GAS

The microscopic source of pressure in a perfect gas is particle bombardment.[1] The reflection (or absorption) of these particles from a real (or imagined) surface in the gas results in a transfer of momentum to that surface. By Newton's second law ($F = dp/dt$), that momentum transfer exerts a force on the surface. The average force per unit area is called the *pressure*. It is the same mechanical quantity appearing in the statement that the quantity of work performed by the infinitesimal expansion of a contained gas is $dW = P\,dV$. In thermal equilibrium in stellar interiors, the angular distribution of particle momenta is isotropic; i.e., particles are moving with equal probabilities in all directions. When reflected from a surface, those moving normal to the surface will transfer larger amounts of momentum than those that glance off at grazing angles. Imagine that the surface in Fig. 2-1 is one of the surfaces of an evacuated can under particle bombardment. When particles are specularly reflected from that surface, the momentum transferred to the surface is $\Delta p_n = 2p \cos \theta$. Let $F(\theta,p)\,d\theta\,dp$ be the number of particles with momentum p in the range dp striking the surface per unit area per unit time from all directions inclined at angle θ to the normal in the range $d\theta$.

[1] In a nonperfect gas strong forces between the particles will represent an additional source or sink of energy for expansions and will therefore contribute to the pressure.

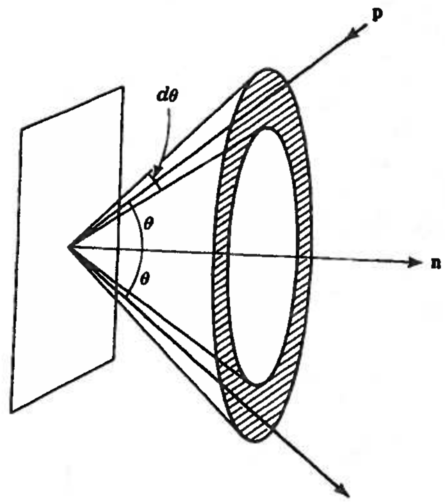


Fig. 2-1 A conical shell defining the set of directions having a spread $d\theta$ about the angle θ to the normal. The number of particles having $|\mathbf{p}| = p$ in the range dp that strike a unit area in unit time within this conical shell of directions is designated $F(\theta,p)\,d\theta\,dp$.

The contribution to the pressure from those particles is

$$dP = 2p \cos\theta\, F(\theta,p)\, d\theta\, dp \tag{2-1}$$

so that the total pressure is

$$P = \int_{\theta=0}^{\pi/2} \int_{p=0}^{\infty} 2p \cos\theta\, F(\theta,p)\, d\theta\, dp \tag{2-2}$$

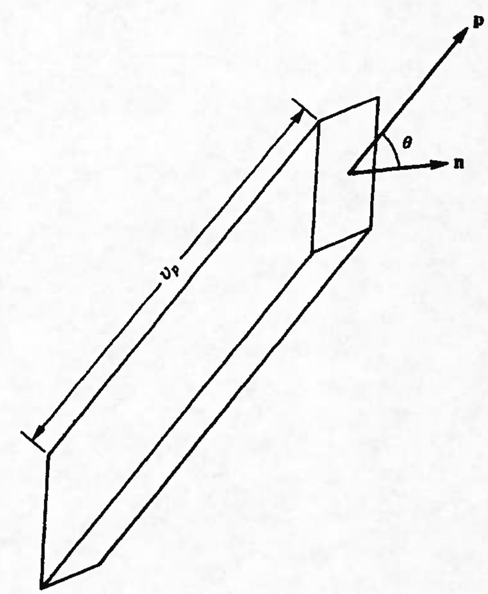
In thermodynamic equilibrium, the angular distribution of momenta is isotropic, whereas the distribution of the magnitudes of the momenta is given by statistical mechanics. The flux $F(\theta,p)\,d\theta\,dp$ may be calculated as the product of the number density of particles with momentum p in the range dp moving in the cone of directions inclined at angle θ in the range $d\theta$ times the volume of such particles capable of passing through the unit surface in unit time. That volume is the volume of the parallelepiped shown in Fig. 2-2 and is equal to the product of v_p, the velocity associated with momentum p, and $\cos\theta$, the cross-sectional area of the column. That is,

$$F(\theta,p)\, d\theta\, dp = v_p \cos\theta\, n(\theta,p)\, d\theta\, dp \tag{2-3}$$

where $n(\theta,p)\,d\theta\,dp$ is the *number density* of particles moving in the prescribed cone. For isotropic radiation, furthermore, the *fraction* of the number of particles moving in the cone of directions at angle θ in the range $d\theta$ is just

$$\frac{n(\theta,p)\, d\theta\, dp}{n(p)\, dp} = \frac{2\pi \sin\theta\, d\theta}{4\pi} \tag{2-4}$$

Fig. 2-2 The parallelepiped whose volume when multiplied by the density of particles about momentum **p** yields the number of particles per unit time of momentum **p** passing through the unit area **n**.



which is the fraction of the total spherical solid angle defined by the conical set of directions. The total number density of particles of momentum p in dp is $n(p)\,dp$. Evidently the gas pressure is

$$P = \int_0^{\pi/2} \int_0^{\infty} 2p \cos\theta\, v_p \cos\theta\, n(p)\, dp\, \tfrac{1}{2} \sin\theta\, d\theta \tag{2-5}$$

The explicit integration over angles is easily performed to yield

$$P = \frac{1}{3} \int_0^{\infty} p v_p n(p)\, dp \qquad \text{perfect gas} \tag{2-6}$$

This pressure integral, valid for a perfect isotropic gas, must be evaluated for several sets of relevant astrophysical circumstances. The relationship of v_p to p depends upon relativistic considerations, whereas the distribution $n(p)$ depends upon the type of particles and the quantum statistics. The simplest perfect gas is the monatomic nondegenerate nonrelativistic one considered in the next subsection, which will be followed by a discussion of the degenerate electron gas and then a discussion of radiation pressure

THE PERFECT MONATOMIC NONDEGENERATE GAS

In the most common case for which the gas density is small enough to be nondegenerate and for which the thermal velocities are nonrelativistic, the pressure of a perfect gas is simply

$$P_g = NkT \tag{2-7}$$

where N is the number of free particles per unit volume.

Problem 2-1: From Chap. 1, the momentum distribution of a nondegenerate nonrelativistic gas in thermal equilibrium is maxwellian:

$$n(p)\,dp = \frac{N4\pi p^2\,dp}{(2\pi mkT)^{3/2}} \exp - \frac{p^2}{2mkT}$$

For a nonrelativistic gas, derive Eq. (2-7) from the pressure integral. The contribution from the several constituents of the gas are additive (Dalton's law of partial pressures). Is Eq. (2-7) also correct for relativistic velocities?

Let the mean molecular weight of the perfect gas be designated by μ. Then the density is

$$\rho = N\mu M_u \tag{2-8}$$

where M_u is the mass of 1 amu. The number of particles per unit volume can then be expressed in terms of the density and the mean molecular weight as

$$N = \frac{\rho}{\mu M_u} = \frac{N_0 \rho}{\mu} \tag{2-9}$$

where $N_0 = 1/M_u$ is Avogadro's number and is equal to 6.0225×10^{23} mole^{-1}. Substitution into Eq. (2-7) gives the pressure of the gas in terms of the density and the mean molecular weight:

$$P_g = \frac{N_0 k}{\mu} \rho T \tag{2-10}$$

The mean molecular weight rather clearly will depend upon the composition of the gas. It is common to let X_Z represent the fraction of the gas *by weight* of element Z; that is, 1 g of gas contains X_Z g of the element of the atomic number Z. It follows that $\Sigma X_Z = 1$. Let us also suppose that each atom of element Z contributes n_Z free particles to the gas. For complete ionization, for instance, it will be true that $n_Z = Z + 1$, Z electrons plus the nucleus. Now let N_Z be the number density of atoms of element Z in the gas. From the definitions of all these quantities it is apparent that

$$N_Z = \frac{\rho_Z}{A_Z} N_0 = \rho \frac{X_Z}{A_Z} N_0 \tag{2-11}$$

Now the total number of free particles per cubic centimeter will be given by

$$N = \sum N_Z n_Z = \rho N_0 \sum \frac{X_Z n_Z}{A_Z} \tag{2-12}$$

where the sum is over all the elements Z. From a comparison of this last equation with Eq. (2-9), the mean molecular weight is given by

$$\frac{1}{\mu} = \sum \frac{X_Z n_Z}{A_Z} \tag{2-13}$$

It is conventional to use a slightly different terminology for the fraction by weight of the two most common elements in the stellar composition. In keeping with this convention, let X be the weight fraction of hydrogen, let Y be the weight fraction of helium, and let $1 - X - Y$ be the weight fraction[1] of all species heavier than helium. Then the mean molecular weight becomes

$$\mu = \left[\frac{Xn_{\mathrm{H}}}{1.008} + \frac{Yn_{\mathrm{He}}}{4.004} + (1 - X - Y)\left\langle\frac{n_Z}{A_Z}\right\rangle\right]^{-1} \tag{2-14}$$

The quantity $\langle n_Z/A_Z\rangle$ is the average of n_Z/A_Z for $Z > 2$, each term being weighted proportional to X_Z.

Equation (2-14) can be further simplified for the case of complete ionization in the inner regions of stars. For complete ionization, the numbers of free particles contributed by the atoms of each element are $n_{\mathrm{H}} = 2$, $n_{\mathrm{He}} = 3$, and $n_Z = Z + 1$. When averaged over the species as they occur in nature, it is a convenient fact that the average atomic weight of element Z is approximately given by $A_Z = 2Z + 2$. The use of that approximation should be adequate in most cases where the fraction by weight of the species heavier than helium is small. With this approximation $\langle n_Z/A_Z\rangle$ in Eq. (2-14) becomes equal to $\frac{1}{2}$:

$$\mu \approx \frac{1}{2X + 3Y/4 + (1 - X - Y)/2} = \frac{2}{1 + 3X + 0.5Y} \tag{2-15}$$

It will also be convenient to have an auxiliary expression for the number density of electrons. Using exactly the same notation as above, we have

$$n_e = \sum N_Z(n_Z - 1) = \rho N_0 \sum \frac{X_Z}{A_Z}(n_Z - 1) \tag{2-16}$$

In the case of complete ionization $n_Z = Z + 1$, so that the number density of electrons becomes

$$n_e = \rho N_0 \sum \frac{X_Z Z}{A_Z} \qquad \text{complete ionization} \tag{2-17}$$

Insertion of the composition-by-weight parameters given above for hydrogen and helium yields

$$n_e = \rho N_0 \left[X + \frac{2Y}{4} + (1 - X - Y)\left\langle\frac{Z}{A_Z}\right\rangle\right] \tag{2-18}$$

where $\langle Z/A_Z\rangle$ is the average for $Z > 2$, the average being taken with respect to X_Z. If the fraction by weight of elements heavier than He is small, it is often

[1] It is common to denote this last weight fraction by Z. To avoid confusion with the nuclear charge in the present discussion, we forego that notation for the moment. We shall use the symbol Z later, however, where the context will make its meaning clear.

adequate to assume $\langle Z/A_Z \rangle \approx \frac{1}{2}$, in which case

$$n_e \approx \tfrac{1}{2}\rho N_0(1 + X) \tag{2-19}$$

It is also common to use a quantity called the *mean molecular weight per electron* μ_e, which is numerically equal to the average number of atomic mass units for each electron in the gas. From Eq. (2-17) it is evident that

$$\frac{1}{\mu_e} = \sum \frac{X_Z Z}{A_Z} \qquad n_e = \frac{\rho N_0}{\mu_e} \tag{2-20}$$

If the ionization is complete, and if $\langle Z/A_Z \rangle \approx \frac{1}{2}$ for $Z > 2$,

$$\mu_e = \frac{2}{1 + X} \tag{2-21}$$

It is advisable for the reader to pause long enough to gain familiarity with the composition parameters and to mentally evaluate the errors in the various approximations.

Problem 2-2: To be sure of understanding the mean molecular weight of the completely ionized gas, calculate and *interpret* the values of μ under the following circumstances: (*a*) all hydrogen, that is, $X = 1$, $Y = 0$; (*b*) all helium, that is, $X = 0$, $Y = 1$; (*c*) all heavy elements, that is, $X = 0$, $Y = 0$. Which of these three values is exactly given by the approximate equation (2-15)?

Problem 2-3: Calculate the mean molecular weight per electron μ_e for completely ionized conditions of all hydrogen ($X = 1$) and for all helium ($Y = 1$). Is Eq. (2-21) exact for $X = Y = 0.5$? Is it exact for $X = Z = 0.5$? What if the Z component is all C^{12} and O^{16}?

Problem 2-4: Show that for conditions under which Eq. (2-15) is valid, the rate of change of the mean molecular weight with respect to the heavy-element content Z, always holding the hydrogen fraction constant, is equal to $\mu^2/4$; that is,

$$\left(\frac{\partial \mu}{\partial Z}\right)_X = \frac{\mu^2}{4}$$

In calculations of stellar structure, and particularly of the structure of evolving stars, a large variety of compositions will be encountered. The statement was made in Chap. 1 that the average composition of the surfaces of population I stars and of the interstellar medium is more or less uniform. It is appropriate, therefore, at this time to present a simplified table of the abundances of the elements (Table 2-1), which are the best that can be inferred for population I objects. Most of the entries are derived from abundances of elements in the solar system, because those are the ones for which the most extensive data exist. The most important exceptions are He and Ne, which are observed only in objects hotter than the sun. It is common to think of the chemical composition of the solar system as a standard, against which other compositions are to be compared. This procedure is no more than a matter of convenience, however, and it must be remembered that the composition of our solar system has no special cosmological significance. A simple calculation reveals that the abundance parameters corresponding to Table 2-1 are

$$X = 0.60 \qquad Y = 0.38 \qquad Z = 0.02$$

These composition parameters may be thought of as characteristic of the majority of population I stars. It must be reemphasized, however, that it is in the deviations of composition from uniformity that some of the most intriguing problems of stellar evolution and nucleosynthesis are to be found.

Problem 2-5: The center of a certain star contains 60 percent hydrogen by weight and 35 percent helium by weight. Evaluate numerically the equation of state. What is the pressure at the center of the star if the density there is 50 g/cm³ and the temperature is 15×10^6 °K?

Of course, some error has been introduced by simplifying assumptions made in obtaining the equation of state. Atoms are never completely ionized, and it is

Table 2-1 Relative abundances of most common species in population I†

		Relative abundance	
Element	*Atomic weight*	*By number*	*By weight*
H	1	1,000	1,000
He‡	4	160	640
O	16	0.90	14
Ne‡	20	0.50	10
C	12	0.40	4.8
N	14	0.11	1.5
Si	28	0.032	0.9
Mg	24	0.025	0.6
S	32	0.022	0.7
A	40	0.008	0.3
Fe	56	0.004	0.2
Na	23	0.002	0.05
Cl	36	0.002	0.07
Al	27	0.002	0.05
Ca	40	0.002	0.08
F	19	0.001	0.02
Ni	59	0.001	0.06
>Ni	>60	$\sim 10^{-4}$	~0.01

† L. H. Aller, "The Abundance of the Elements," Interscience Publishers, Inc., New York, 1961.

‡ Because the sun is a G2 star, its helium abundance is not well known. The value in this table comes from the hotter B stars in the solar neighborhood, which are much younger than the sun. There are some indications that in the sun He/H ≈ 0.10 by number, which is about 60 percent of the amount of He found in B stars. A similar situation occurs for Ne, and it is more likely, but not certain, that in the sun Ne/O ~ 0.1.

the Saha ionization equation that reveals the fraction of any given species that is ionized. In the relatively cool outer portions of a star, the number of free particles will depend upon the temperature. Elaborate techniques have been constructed for calculating a more realistic equation of state applicable to incomplete ionization. The reader who understands the ideas about it presented here, along with its restrictive assumptions, will have little trouble with a more sophisticated treatment of the equation of state.

Other than the lack of complete ionization in the cooler regions of the star, there are two extremely important physical circumstances that cause the equation of state for a perfect nondegenerate monatomic gas to be insufficient: (1) the pressure due to electromagnetic radiation in the interior of the star becomes comparable to the pressure due to particles, and (2) the electron gas becomes degenerate. We shall consider the second of these sets of circumstances, electron degeneracy, first.

ELECTRON DEGENERACY

Because electrons are particles with half-integral spin, the electron gas must obey Fermi-Dirac statistics. The density of electrons having momentum $|\mathbf{p}| = p$ in the range dp is accordingly

$$n_e(p)\,dp = \frac{2}{h^3}\,4\pi p^2\,dp\,P(p) \tag{2-22}$$

where the occupation index for the Fermi gas is

$$P(p) = \left[\exp\left(\alpha + \frac{E}{kT}\right) + 1\right]^{-1} \tag{2-23}$$

That $P(p)$ has a maximum value of unity is an expression of the Pauli exclusion principle, to which electrons must adhere. When $P(p)$ is unity, all the available electronic states of the gas are occupied. It follows that the maximum density of electrons in phase space is

$$[n_e(p)]_{\max} = \frac{2}{h^3}\,4\pi p^2 \tag{2-24}$$

It is this restriction upon the number density of electrons in momentum space which creates *degeneracy pressure*. If one continually increases the electron density, the electrons are forced into high-lying momentum states because the lower states are occupied. These high-momentum electrons will make a large contribution to the pressure integral.

For any given temperature and electron density n_e, the value of the parameter α is determined from the demand that

$$n_e = \int_0^\infty n_e(p)\,dp = n_e(\alpha, T) \tag{2-25}$$

This relationship will be explored quantitatively at a later time, but for the present we note from Eq. (2-23) that if α is a large positive number, $P(p)$ will be

much less than unity for all energies. In this case the Fermi distribution reduces to the maxwellian distribution. As the electron density is increased at constant temperature, the parameter α becomes smaller, going to large negative values at high density.

In the limit of large negative α

$$P(p) = \begin{cases} 1 & \text{for } \dfrac{E}{kT} < |\alpha| \\ 0 & \text{for } \dfrac{E}{kT} > |\alpha| \end{cases} \qquad \text{complete degeneracy} \tag{2-26}$$

This transition occurs smoothly over a range of energy of several kT near the energy $E = |\alpha|kT$. If the energy $-\alpha kT$ is much larger than kT, the distribution function is nearly a step function. This limit is called *complete degeneracy*, and in this limit the quantity $|\alpha kT| = E_f$ is called the *Fermi energy*.

In the following discussion we shall calculate the pressure of a completely degenerate gas. The calculation will first be made for densities such that the energy E_f is nonrelativistic. It will then be repeated for densities high enough for E_f to correspond to relativistic electron velocities. Finally we shall calculate, in the nonrelativistic limit, the pressure of an electron gas for densities such that the distribution function is intermediate to the maxwellian and the completely degenerate distributions.

Complete degeneracy In a completely degenerate gas, the density is high enough so that all the available electron states having energies less than some maximum energy are filled. Since the total number density of electrons is to be finite, the density of states can be filled only up to some limiting value of the electron momentum

$$n_e(p)\,dp = \begin{cases} \dfrac{2}{h^3}\,4\pi p^2\,dp & p < p_0 \\ 0 & p > p_0 \end{cases} \tag{2-27}$$

It is clear that complete degeneracy is the state of minimum kinetic energy, the ground state, so to speak, of a degenerate perfect electron gas. The total number density of electrons in a completely degenerate electron gas is related to the maximum momentum by

$$n_e = \int_0^{p_0} n_e(p)\,dp = \frac{8\pi}{3h^3}\,p_0^3 \tag{2-28}$$

Inversion of this last equation shows that the maximum momentum of a completely degenerate gas is determined by the electron density:

$$p_0 = \left(\frac{3h^3}{8\pi}\,n_e\right)^{\frac{1}{3}} \tag{2-29}$$

The energy associated with the momentum p_0 is the Fermi energy.

The pressure of a completely degenerate perfect electron gas can be computed from the integral of Eq. (2-6) by inserting Eq. (2-27) for $n_e(p)$. Because it is also necessary to insert the velocity of a particle of given momentum, it is common to distinguish between a nonrelativistic and a relativistic degenerate electron gas.

Nonrelativistic complete degeneracy If the energy associated with p_0 is much less than m_ec^2, or 0.51 Mev, then $v_p = p/m$ for all momenta in the degenerate distribution, and the pressure integral is elementary:

$$P_{e,\mathrm{nr}} = \frac{8\pi}{15mh^3}\,p_0^5 \tag{2-30}$$

where nr signifies *nonrelativistic* electrons. Since the maximum momentum of the completely degenerate distribution is related to the electron density by Eq. (2-29), the electron pressure is determined by the electron density:

$$P_{e,\mathrm{nr}} = \frac{h^2}{20m}\left(\frac{3}{\pi}\right)^{2/3} n_e^{5/3} \tag{2-31}$$

The number density of the electrons may be written in terms of the mass density:

$$\begin{aligned} P_{e,\mathrm{nr}} &= \frac{h^2}{20m}\left(\frac{3}{\pi}\right)^{2/3} N_0^{5/3}\left(\frac{\rho}{\mu_e}\right)^{5/3} \\ &= 1.004 \times 10^{13}\left(\frac{\rho}{\mu_e}\right)^{5/3} \qquad \text{dynes/cm}^2 \end{aligned} \tag{2-32}$$

The value of μ_e is generally about 2 unless the gas contains considerable amounts of hydrogen. Inspection of this equation shows that the nonrelativistic-electron pressure varies as the $\frac{5}{3}$ power of the density. Since the pressure of a nondegenerate electron gas varies linearly with the density, it is clear that as the density increases, a point will be reached where the degenerate electron pressure becomes greater than the value that would be given by the formula for the pressure of a nondegenerate gas.

We may thereby define an approximate boundary line in the ρT plane, dividing it into regions of nondegenerate and degenerate gas, respectively, by the condition that the pressures given by the nondegenerate-gas equation and the completely degenerate electron-gas equation be equal. That is, when[1]

$$\frac{N_0k}{\mu_e}\rho T = \frac{h^2}{20m}\left(\frac{3}{\pi}\right)^{2/3}(N_0)^{5/3}\left(\frac{\rho}{\mu_e}\right)^{5/3} \tag{2-33}$$

Numerical evaluation of this equation shows that the completely degenerate electron pressure exceeds the nondegenerate electron pressure when

$$\frac{\rho}{\mu_e} > 2.4 \times 10^{-8}T^{3/2} \qquad \text{g/cm}^3 \tag{2-34}$$

[1] It should perhaps be emphasized that Eq. (2-33) is never "true," since a gas cannot be simultaneously degenerate and nondegenerate. One might say that if ρ and T satisfy this equation, the state of the gas must be intermediate to nondegeneracy and complete degeneracy.

For densities greater than this value, the electron gas must be degenerate. Needless to say, the transition from nondegenerate to degenerate is not sudden and complete. The transition occurs smoothly for densities in the neighborhood of Eq. (2-34). The appropriate equation of state in the transition region will be discussed in the section on partial degeneracy.

It is instructive to apply Eq. (2-34) to two well-known astrophysical environments. At the center of the sun $\rho/\mu_e \approx 10^2$, and $T \approx 10^7$. For these values the inequality of Eq. (2-34) is strong in the opposite direction, so that one will anticipate using the nondegenerate electron pressure at the solar center. White-dwarf densities, on the other hand, are observationally known to be of order $\rho/\mu_e \approx 10^6$, whereas the interior temperatures are of order $T \approx 10^6$. For these values the inequality of Eq. (2-34) is strongly satisfied, and one must expect degeneracy pressure to dominate.

Relativistic complete degeneracy As the electron density is increased, the maximum momentum in a completely degenerate electron gas grows larger. Eventually a density is reached where the most energetic of the electrons in the degenerate distribution become relativistic. When that condition is reached, the substitution $v_p = p/m$ leading to Eq. (2-30) becomes incorrect. The velocity to be associated with the momentum p must be determined by relativistic kinematics.

Before calculating the pressure, let us estimate those densities for which it is necessary that some of the electrons be relativistic. For a relativistic particle, the total energy, which is the sum of the rest-mass energy plus the kinetic energy, forms a right triangle with the rest-mass energy and the momentum times the velocity of light, as illustrated in Fig. 2-3. The right-triangle relationship follows

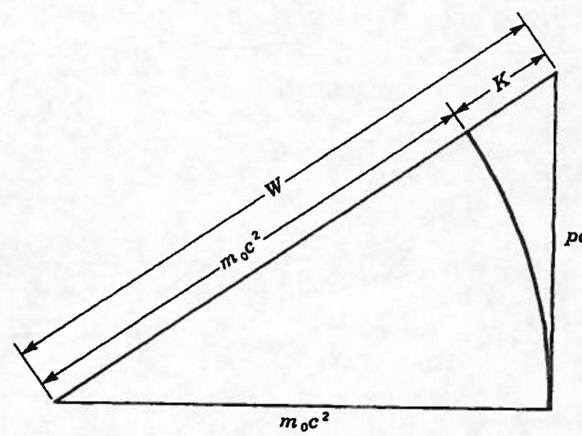


Fig. 2-3 The right triangle representing the relationship of the total energy of a particle to its momentum. The kinetic energy is the excess of the hypotenuse beyond the rest-mass energy.

from the relativistic expression for the total energy of a particle,

$$W^2 = p^2c^2 + m_0^2c^4 \tag{2-35}$$

where m_0 is the rest mass of the particle. The total energy is also given by the square of the velocity of light times the relativistic mass:

$$W = mc^2 = \frac{m_0c^2}{1 - (v/c)^2} \tag{2-36}$$

Equating W^2 in Eq. (2-35) to W^2 in Eq. (2-36) yields

$$pc = \frac{v}{c} W \tag{2-37}$$

What convenient order-of-magnitude criterion will ensure that particles are relativistic? One may say with adequate accuracy that particles become relativistic when v/c approaches unity and when the total energy W becomes appreciably greater than the rest-mass energy. As an order-of-magnitude criterion, it suffices to compute the density at which the electrons of maximum momentum have a total energy equal to, say, twice the rest-mass energy; from Eq. (2-37) the quantity pc will then be approximately $p_0c \sim 2m_0c^2$. On the other hand, the numerical value of p_0c is

$$p_0c = hc\left(\frac{3}{8\pi} n_e\right)^{1/3} = 6.12 \times 10^{-11} n_e^{1/3} \qquad \text{Mev} \tag{2-38}$$

In terms of the density and the mean molecular weight per electron, Eq. (2-38) may be expressed as

$$p_0c = 5.15 \times 10^{-3} \left(\frac{\rho}{\mu_e}\right)^{1/3} \qquad \text{Mev} \tag{2-39}$$

This last equation reveals that $p_0c \approx 2m_0c^2 \approx 1$ Mev when

$$\frac{\rho}{\mu_e} = 7.3 \times 10^6 \text{ g/cm}^3 \qquad \text{relativistic} \tag{2-40}$$

The natural conclusion is that as the density approaches this value, relativistic kinematics must be used in relating the velocity of an electron to its momentum. Densities this large are encountered in astrophysics, in white dwarfs, for instance.

The pressure integral for a completely degenerate gas may be evaluated without difficulty for relativistic particles. Since the momentum of a relativistic particle is given by Eqs. (2-36) and (2-37) as

$$p = \frac{m_0 v}{[1 - (v/c)^2]^{1/2}} \tag{2-41}$$

one can determine by inversion that

$$v = \frac{p/m_0}{[1 + (p/m_0c)^2]^{1/2}} \tag{2-42}$$

Insertion of this value for v_p in the pressure integral yields

$$P_e = \frac{8\pi}{3mh^3} \int_0^{p_0} \frac{p^4\, dp}{[1 + (p/mc)^2]^{1/2}} \tag{2-43}$$

In Eq. (2-43) and those which follow, the electron rest mass is designated simply by m. For evaluation of this integral it is convenient to define a new parameter θ such that

$$\sinh \theta = \frac{p}{mc} \qquad dp = mc \cosh \theta \, d\theta$$

In terms of this new variable the pressure integral becomes

$$P_e = \frac{8\pi m^4c^5}{3h^3} \int_0^{\theta_0} \sinh^4 \theta \, d\theta \tag{2-44}$$

which may be integrated to give

$$P_e = \frac{8\pi m^4c^5}{3h^3} \left(\frac{\sinh^3 \theta_0 \cosh \theta_0}{4} - \frac{3 \sinh 2\theta_0}{16} + \frac{3\theta_0}{8}\right) \tag{2-45}$$

When written in terms of the Fermi momentum,

$$P_e = \frac{\pi m^4c^5}{3h^3} f(x) = 6.003 \times 10^{22} f(x) \qquad \text{dynes/cm}^2 \tag{2-46}$$

where

$$x = \frac{p_0}{mc} = \frac{h}{mc}\left(\frac{3}{8\pi} n_e\right)^{1/3}$$

$$f(x) = x(2x^2 - 3)(x^2 + 1)^{1/2} + 3 \sinh^{-1} x \tag{2-47}$$

The numerical value of the dimensionless parameter x is

$$x = 1.195 \times 10^{-10} n_e^{1/3} = 1.009 \times 10^{-2} \left(\frac{\rho}{\mu_e}\right)^{1/3} \tag{2-48}$$

Problem 2-6: The limit of small x, that is, $p_0 \ll mc$, must correspond to nonrelativistic particles. Show that

$$f(x) \approx \tfrac{8}{5}x^5 - \tfrac{4}{7}x^7 + \cdots \qquad x \to 0$$

and confirm that the pressure obtained from this limiting value of $f(x)$ reduces to the completely degenerate nonrelativistic electron pressure determined previously in Eq. (2-30).

Problem 2-7: The limit of large x must correspond to highly relativistic degeneracy. Show that

$$f(x) \approx 2x^4 - 2x^2 + \cdots \qquad x \to \infty$$

Show that the pressure obtained by inserting this limiting value of $f(x)$ into Eq. (2-46) is identical to that obtained by letting $v_p = c$ in the integral for the pressure given in Eq. (2-6). Does that make sense? Evidently the pressure is proportional to $\rho^{4/3}$ at very high density.

Table 2-2 Pressure of a complete degenerate gas†

x	$f(x)$	x	$f(x)$
0	0	2.0	26.7
0.1	1.60×10^{-5}	2.1	32.9
0.2	5.05×10^{-4}	2.2	40.1
0.3	3.77×10^{-3}	2.3	48.4
0.4	1.55×10^{-2}	2.4	58.0
0.5	4.61×10^{-2}	2.5	68.9
0.6	0.111	2.6	81.2
0.7	0.232	2.7	95.2
0.8	0.436	2.8	110.8
0.9	0.756	2.9	128.3
1.0	1.23	3.0	1.48×10^{2}
1.1	1.90	3.5	2.80×10^{2}
1.2	2.82	4.0	4.85×10^{2}
1.3	4.05	4.5	7.85×10^{2}
1.4	5.63	5.0	1.21×10^{3}
1.5	7.64	5.5	1.78×10^{3}
1.6	10.1	6.0	2.53×10^{3}
1.7	13.2	6.5	3.49×10^{3}
1.8	16.9	7.0	4.71×10^{3}
1.9	21.4	8.0	8.07×10^{3}

† S. Chandrasekhar, "An Introduction to the Study of Stellar Structure," p. 392; reprinted from the Dover Publications edition, Copyright 1939 by The University of Chicago, as reprinted by permission of The University of Chicago.

Table 2-2 lists some numerical values of $f(x)$. From this table and Eq. (2-46) the electron pressure can be evaluated for cases of semirelativistic complete degeneracy. The quantity x is to be evaluated from Eq. (2-48). This result is correct only for a completely degenerate gas. Approximate relativistic expressions for a partially degenerate gas can be obtained if desired.[1] However, densities must exceed 10^6 g/cm^3 for a degenerate gas to be relativistic [Eq. (2-40)], for which the degeneracy will be essentially complete unless $T > 10^9$ °K [Eq. (2-34)]. Densities greater than 10^6 g/cm^3 at a temperature greater than 10^9 °K are probably found only in very late stages of stellar evolution. For all other classes of stars, degeneracy sets in at sufficiently low temperatures so that nonrelativistic kinematics should be adequate for the examination of partial degeneracy.

[1] See, for instance, S. Chandrasekhar, "An Introduction to the Study of Stellar Structure," p. 392, Dover Publications, Inc., New York, 1957, or D. H. Menzel, P. L. Bhatnagar, and H. K. Sen, "Stellar Interiors," p. 35, John Wiley & Sons, Inc., New York, 1963.

Problem 2-8: Show that the kinetic energy per unit volume of a completely degenerate gas is

$$\left(\frac{U}{V}\right)_{\text{kin}} = \frac{\pi m^4 c^5}{3h^3} g(x)$$

where $g(x) = 8x^3[(x^2+1)^{\frac{1}{2}} - 1] - f(x)$. Show also that $U \to \frac{3}{2}PV$ in the limit of small x and $U \to 3PV$ in the limit of large x.

Partial degeneracy The dividing line between degeneracy and nondegeneracy given in Eq. (2-34) defines only the region of the onset of degeneracy in the electron gas. That is, it indicates only the approximate condition under which electron degeneracy is becoming important in the equation of state. Actually, of course, there is a gradual transition from nondegeneracy toward complete degeneracy as the density rises. There is certainly no sharp transition between those extreme conditions. The electron occupation index gradually takes on the shape of a degenerate distribution with increase in density, as illustrated in Fig. 2-4.

The distribution of electron momenta is

$$n_e(p)\,dp = \frac{2}{h^3} \frac{4\pi p^2\,dp}{\exp(\alpha + E/kT) + 1} \tag{2-49}$$

where α is a number that depends upon the electron density and the temperature. That is, α is fixed by the requirement that the total number of electrons equal the electron density n_e:

$$n_e = \int_0^\infty \frac{2}{h^3} \frac{4\pi p^2\,dp}{\exp(\alpha + E/kT) + 1} = n_e(\alpha, T) \tag{2-50}$$

The integral for the pressure of the perfect electron gas becomes

$$P_e = \frac{8\pi}{3h^3} \int_0^\infty \frac{p^3 v_p\,dp}{\exp(\alpha + E/kT) + 1} \tag{2-51}$$

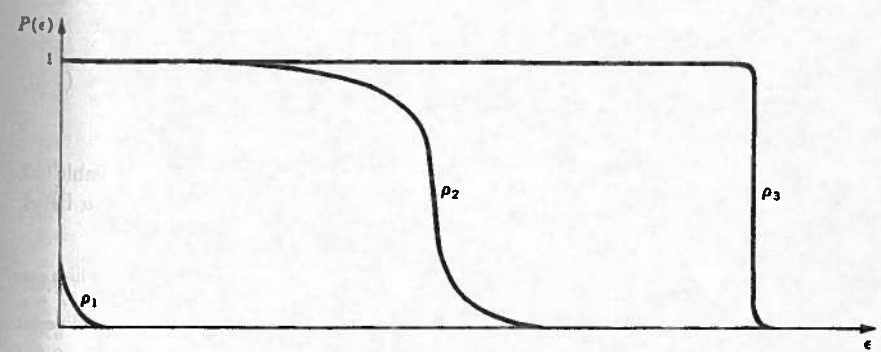


Fig. 2-4 Schematic illustration of the occupation index of an electron gas for three different degrees of degeneracy. In this particular case $\rho_3 > \rho_2 \gg \rho_1$ and $T_2 > T_1$.

As stated at the end of the last section, for temperatures of less than 10^9 °K nonrelativistic electron degeneracy sets in before relativistic degeneracy. Therefore, in considering the partially degenerate gas, we shall restrict ourselves to nonrelativistic kinematics, keeping in mind that the results will be somewhat in error for extremely high temperatures ($T > 10^9$). That is, we shall once again let $v_p = p/m$, whereupon

$$P_e = \frac{8\pi}{3h^3m}\int_0^\infty \frac{p^4\,dp}{\exp\,(\alpha + p^2/2mkT) + 1} \tag{2-52}$$

and

$$n_e = \frac{8\pi}{h^3}\int_0^\infty \frac{p^2\,dp}{\exp\,(\alpha + p^2/2mkT) + 1} \tag{2-53}$$

With the aid of a dimensionless energy $u = p^2/2mkT$, these equations may be written in the form

$$P_e = \frac{8\pi kT}{3h^3}(2mkT)^{3/2}\int_0^\infty \frac{u^{3/2}\,du}{\exp\,(\alpha + u) + 1} \tag{2-54}$$

$$n_e = \frac{4\pi}{h^3}(2mkT)^{3/2}\int_0^\infty \frac{u^{1/2}\,du}{\exp\,(\alpha + u) + 1} \tag{2-55}$$

These two equations constitute a parametric representation of the equation of state, the parameter being the quantity α. The parametric representation is made more explicit by conventionally defining two new functions,[1]

$$F_{3/2}(\alpha) = \int_0^\infty \frac{u^{3/2}\,du}{\exp\,(\alpha + u) + 1}$$
$$F_{1/2}(\alpha) = \int_0^\infty \frac{u^{1/2}\,du}{\exp\,(\alpha + u) + 1} \tag{2-56}$$

in which case the electron pressure and the electron density may be written as

$$P_e = \frac{8\pi kT}{3h^3}(2mkT)^{3/2}F_{3/2}(\alpha)$$
$$n_e = \frac{4\pi}{h^3}(2mkT)^{3/2}F_{1/2}(\alpha) \tag{2-57}$$

The functions $F_{3/2}$ and $F_{1/2}$ have been tabulated for selected values of α in Table 2-3. Their values for other values of α may be interpolated in the range of α listed, and asymptotic values will soon be derived for extreme values of α.

[1] In much of the literature the negative of α is used as the degeneracy parameter, in which case it is usually designated by η or Ψ; or $\Psi = \eta = -\alpha$. Another common notation is $-\alpha kT = \mu$, which is called the *chemical potential*. Many people prefer to normalize the F's in a different way, defining

$$U_n(\alpha) = \frac{1}{\Gamma(n+1)}F_n(\alpha)$$

Table 2-3 Table of Fermi-Dirac functions†

α	$\frac{2}{3}F_{3/2}$	$F_{1/2}$	α	$\frac{2}{3}F_{3/2}$	$F_{1/2}$
4.0	0.016 179	0.016 128	0.0	0.768 536	0.678 094
3.9	0.017 875	0.017 812	−0.1	0.839 082	0.733 403
3.8	0.019 748	0.019 670	−0.2	0.915 332	0.792 181
3.7	0.021 816	0.021 721	−0.3	0.997 637	0.854 521
3.6	0.024 099	0.023 984	−0.4	1.086 358	0.920 505
3.5	0.026 620	0.026 480	−0.5	1.181 862	0.990 209
3.4	0.029 404	0.029 233	−0.6	1.284 526	1.063 694
3.3	0.032 476	0.032 269	−0.7	1.394 729	1.141 015
3.2	0.035 868	0.035 615	−0.8	1.512 858	1.222 215
3.1	0.039 611	0.039 303	−0.9	1.639 302	1.307 327
3.0	0.043 741	0.043 366	−1.0	1.774 455	1.396 375
2.9	0.048 298	0.047 842	−1.1	1.918 709	1.489 372
2.8	0.053 324	0.052 770	−1.2	2.072 461	1.586 323
2.7	0.058 868	0.058 194	−1.3	2.236 106	1.687 226
2.6	0.064 981	0.064 161	−1.4	2.410 037	1.792 068
2.5	0.071 720	0.070 724	−1.5	2.594 650	1.900 833
2.4	0.079 148	0.077 938	−1.6	2.790 334	2.013 496
2.3	0.087 332	0.085 864	−1.7	2.997 478	2.130 027
2.2	0.096 347	0.094 566	−1.8	3.216 467	2.250 391
2.1	0.106 273	0.104 116	−1.9	3.447 683	2.374 548
2.0	0.117 200	0.114 588	−2.0	3.691 502	2.502 458
1.9	0.129 224	0.126 063	−2.1	3.948 298	2.634 072
1.8	0.142 449	0.138 627	−2.2	4.218 438	2.769 344
1.7	0.156 989	0.152 373	−2.3	4.502 287	2.908 224
1.6	0.172 967	0.167 397	−2.4	4.800 202	3.050 659
1.5	0.190 515	0.183 802	−2.5	5.112 536	3.196 598
1.4	0.209 777	0.201 696	−2.6	5.439 637	3.345 988
1.3	0.230 907	0.221 193	−2.7	5.781 847	3.498 775
1.2	0.254 073	0.242 410	−2.8	6.139 503	3.654 905
1.1	0.279 451	0.265 471	−2.9	6.512 937	3.814 326
1.0	0.307 232	0.290 501	−3.0	6.902 476	3.976 985
0.9	0.337 621	0.317 630	−3.1	7.308 441	4.142 831
0.8	0.370 833	0.346 989	−3.2	7.731 147	4.311 811
0.7	0.407 098	0.378 714	−3.3	8.170 906	4.483 876
0.6	0.446 659	0.412 937	−3.4	8.628 023	4.658 977
0.5	0.489 773	0.449 793	−3.5	9.102 801	4.837 066
0.4	0.536 710	0.489 414	−3.6	9.595 535	5.018 095
0.3	0.587 752	0.531 931	−3.7	10.106 516	5.202 020
0.2	0.643 197	0.577 470	−3.8	10.636 034	5.388 795
0.1	0.703 351	0.626 152	−3.9	11.184 369	5.578 378

Table 2-3 Table of Fermi-Dirac functions† (Continued)

α	$\frac{2}{3}F_{3/2}$	$F_{1/2}$	α	$\frac{2}{3}F_{3/2}$	$F_{1/2}$
−4.0	11.751 80	5.770 72	−8.0	52.901 73	15.380 48
−4.1	12.338 60	5.965 80	−8.1	54.453 85	15.662 24
−4.2	12.945 05	6.163 56	−8.2	56.034 24	15.945 80
−4.3	13.571 40	6.363 96	−8.3	57.643 07	16.231 14
−4.4	14.217 93	6.566 98	−8.4	59.280 52	16.518 26
−4.5	14.884 89	6.772 57	−8.5	60.946 78	16.807 14
−4.6	15.572 53	6.980 70	−8.6	62.642 01	17.097 76
−4.7	16.281 11	7.191 34	−8.7	64.366 39	17.390 13
−4.8	17.010 88	7.404 45	−8.8	66.120 09	17.684 23
−4.9	17.762 08	7.620 01	−8.9	67.903 29	17.980 04
−5.0	18.534 96	7.837 97	−9.0	69.716 16	18.277 56
−5.1	19.329 76	8.058 32	−9.1	71.558 86	18.576 77
−5.2	20.146 71	8.281 03	−9.2	73.431 57	18.877 68
−5.3	20.986 04	8.506 06	−9.3	75.334 45	19.180 26
−5.4	21.847 99	8.733 39	−9.4	77.267 68	19.484 51
−5.5	22.732 79	8.962 99	−9.5	79.231 41	19.790 41
−5.6	23.640 67	9.194 85	−9.6	81.225 82	20.097 96
−5.7	24.571 84	9.428 93	−9.7	83.251 06	20.407 15
−5.8	25.526 53	9.665 21	−9.8	85.307 30	20.717 97
−5.9	26.504 95	9.903 67	−9.9	87.394 71	21.030 42
−6.0	27.507 33	10.144 28	−10.0	89.513 44	21.344 47
−6.1	28.533 88	10.387 03	−10.1	91.663 65	21.660 13
−6.2	29.584 81	10.631 90	−10.2	93.845 52	21.977 38
−6.3	30.660 33	10.878 86	−10.3	96.059 18	22.296 22
−6.4	31.760 65	11.127 89	−10.4	98.304 81	22.616 64
−6.5	32.885 98	11.378 98	−10.5	100.582 56	22.938 62
−6.6	34.036 52	11.632 11	−10.6	102.892 59	23.262 17
−6.7	35.212 47	11.887 26	−10.7	105.235 05	23.587 28
−6.8	36.414 04	12.144 40	−10.8	107.610 10	23.913 93
−6.9	37.641 42	12.403 54	−10.9	110.017 89	24.242 12
−7.0	38.894 81	12.664 64	−11.0	112.458 57	24.571 84
−7.1	40.174 41	12.927 69	−11.1	114.932 31	24.903 09
−7.2	41.480 41	13.192 67	−11.2	117.439 24	25.235 86
−7.3	42.813 01	13.459 58	−11.3	119.979 53	25.570 13
−7.4	44.172 39	13.728 39	−11.4	122.553 32	25.905 91
−7.5	45.558 75	13.999 10	−11.5	125.160 76	26.243 19
−7.6	46.972 27	14.271 68	−11.6	127.802 01	26.581 95
−7.7	48.413 15	14.546 12	−11.7	130.477 20	26.922 20
−7.8	49.881 56	14.822 41	−11.8	133.186 50	27.263 93
−7.9	51.377 69	15.100 53	−11.9	135.930 04	27.607 12

Table 2-3 Table of Fermi-Dirac functions† (Continued)

α	$\frac{2}{3}F_{3/2}$	$F_{1/2}$	α	$\frac{2}{3}F_{3/2}$	$F_{1/2}$
−12.0	138.707 97	27.951 78	−14.0	201.709 50	35.142 97
−12.1	141.520 44	28.297 89	−14.1	205.242 49	35.517 00
−12.2	144.367 60	28.645 45	−14.2	208.812 95	35.892 38
−12.3	147.249 58	28.994 46	−14.3	212.421 01	36.269 08
−12.4	150.166 54	29.344 91	−14.4	216.066 81	36.647 12
−12.5	153.118 61	29.696 79	−14.5	219.750 48	37.026 49
−12.6	156.105 94	30.050 09	−14.6	223.472 15	37.407 18
−12.7	159.128 68	30.404 82	−14.7	227.231 96	37.789 18
−12.8	162.186 96	30.760 96	−14.8	231.030 03	38.172 50
−12.9	165.280 92	31.118 51	−14.9	234.866 50	38.557 12
−13.0	168.410 71	31.477 46	−15.0	238.741 50	38.943 04
−13.1	171.576 46	31.837 81	−15.1	242.655 15	39.330 27
−13.2	174.778 31	32.199 56	−15.2	246.607 59	39.718 79
−13.3	178.016 42	32.562 68	−15.3	250.598 95	40.108 59
−13.4	181.290 90	32.927 20	−15.4	254.629 36	40.499 69
−13.5	184.601 90	33.293 08	−15.5	258.698 93	40.892 06
−13.6	187.949 56	33.660 34	−15.6	262.807 81	41.285 71
−13.7	191.334 01	34.028 96	−15.7	266.956 12	41.680 64
−13.8	194.755 40	34.398 94	−15.8	271.143 98	42.076 83
−13.9	198.213 85	34.770 28	−15.9	275.371 53	42.474 29

† Taken from J. McDougall and E. C. Stoner, *Phil. Trans. Roy. Soc.*, **237**:67 (1938).

Problem 2-9: Show that in a perfect nonrelativistic electron gas

$$P_e = \frac{2}{3}\left(\frac{U}{V}\right)_{\text{kin}}$$

for any degree of degeneracy.

Problem 2-10: (*a*) Show that as $\alpha \to \infty$, $F_{3/2}/F_{1/2} \to \frac{3}{2}$, for which case $P_e \to n_e kT$, the pressure of a maxwellian electron gas. (*b*) Show that as $\alpha \to -\infty$, $F_{3/2}/F_{1/2} \to \frac{3}{5}u_0$, for which case $P_e \to (8\pi/15mh^3)p_0^5$, the pressure of a completely degenerate nonrelativistic electron gas.

From Eq. (2-57) it is apparent that

$$P_e = n_e kT\left(\frac{2}{3}\frac{F_{3/2}}{F_{1/2}}\right) \qquad \text{(2-58)}$$

Thus, the factor $2F_{3/2}/3F_{1/2}$ measures the extent to which the electron pressure differs from that of a nondegenerate gas. This multiplication factor is plotted in Fig. 2-5 as a function of the parameter α. It can be seen that the gas pressure is essentially that of a nondegenerate gas for $\alpha > 2$.

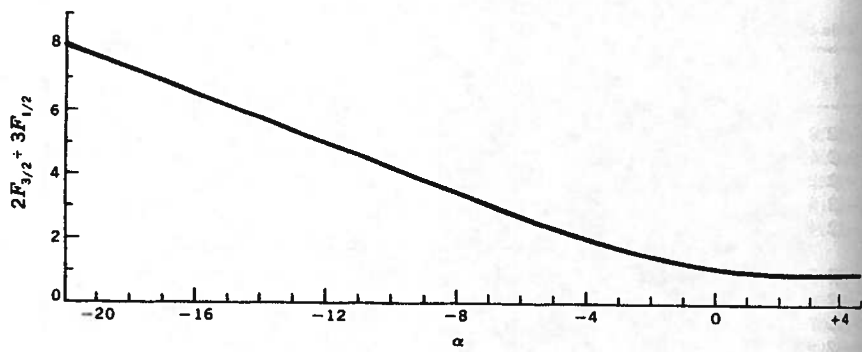


Fig. 2-5 The ratio $2F_{3/2}/3F_{1/2}$ as a function of the degeneracy parameter α. This ratio is equal to the ratio of the pressure of an electron gas to the pressure it would have if it were maxwellian at the same density.

On the other hand, Eq. (2-57) may be written in terms of the mass density,

$$\frac{\rho N_0}{\mu_e} = \frac{4\pi}{h^3}(2mkT)^{3/2}F_{1/2}(\alpha) \tag{2-59}$$

from which it follows that

$$\log\left(\frac{\rho}{\mu_e}T^{-3/2}\right) = \log F_{1/2}(\alpha) - 8.044 \tag{2-60}$$

This equation is plotted in Fig. 2-6, which relates $\log\,[(\rho/\mu_e)T^{-3/2}]$ to the degeneracy parameter α.

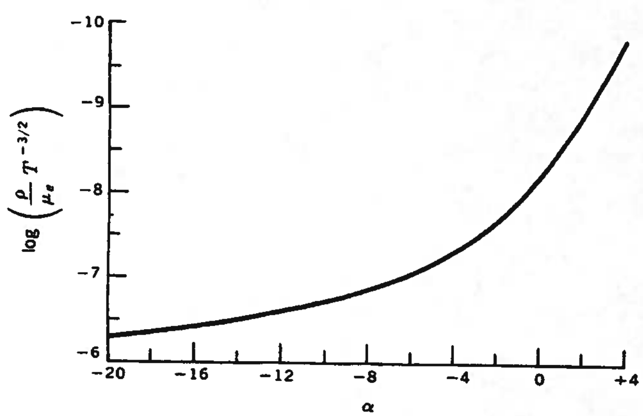


Fig. 2-6 The value of $(\rho/\mu_e)T^{-3/2}$ determines the degeneracy parameter

These equations describe the behavior of the equation of state of an electron gas in the partially degenerate region. For given ρ, T, Eq. (2-59) determines $F_{1/2}(\alpha)$, which in turn allows $F_{3/2}(\alpha)$ to be interpolated from Table 2-3. These calculations have used nonrelativistic kinematics because, in most stars, relativistic degeneracy is important only for such high densities that the degeneracy is essentially complete.

For many problems in nonrelativistic partial degeneracy, however, it is instructive to have appropriate expansions of the equation of state. Expansions that converge rapidly for weak degeneracy (nearly maxwellian) and for strong degeneracy (nearly complete) are easily obtained.

Weak nonrelativistic degeneracy For notational ease, define $\Lambda = \exp(-\alpha)$. Then for $\alpha > 0$, which is seen from Fig. 2-5 to correspond to weak degeneracy, the number Λ is less than unity. Then $F_{1/2}(\Lambda)$ may be expanded:

$$F_{1/2}(\Lambda) = \int_0^\infty \frac{u^{1/2}\,du}{(1/\Lambda)e^u + 1} = \int_0^\infty \Lambda e^{-u}u^{1/2}\frac{1}{1 + \Lambda e^{-u}}\,du$$

$$= \Lambda\int_0^\infty e^{-u}u^{1/2}[1 - \Lambda e^{-u} + (\Lambda e^{-u})^2 - (\Lambda e^{-u})^3 + \cdots]\,du \tag{2-61}$$

which may be integrated term by term to give

$$F_{1/2}(\Lambda) = -\frac{\sqrt{\pi}}{2}\sum_{n=1}^{\infty}\frac{(-1)^n\Lambda^n}{n^{3/2}} \qquad \Lambda < 1$$

or equivalently

$$F_{1/2}(\alpha) = -\frac{\sqrt{\pi}}{2}\sum_{n=1}^{\infty}\frac{(-1)^n e^{-n\alpha}}{n^{3/2}} \qquad \alpha > 0 \tag{2-62}$$

Then Eq. (2-57) becomes

$$n_e = \frac{2(2\pi mkT)^{3/2}}{h^3}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{e^{-n\alpha}}{n^{3/2}} \qquad \alpha > 0 \tag{2-63}$$

Problem 2-11: Show by the same technique used in obtaining Eq. (2-63) that

$$P_e = \frac{2kT(2\pi mkT)^{3/2}}{h^3}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{e^{-n\alpha}}{n^{5/2}} \qquad \alpha > 0 \tag{2-64}$$

Problem 2-12: For large α, the series may be approximated by one term. Show that Eqs. (2-63) and (2-64) then reduce to the maxwellian distribution.

Problem 2-13: Suppose that α is large enough for only the first two terms of the series to be important. Show, then, that

$$P_e \approx n_e kT\left[1 + \frac{n_e h^3}{2^{5/2}(2\pi mkT)^{3/2}} + \cdots\right]$$

Strong nonrelativistic degeneracy The degeneracy becomes strong when α becomes a large negative number or, equivalently, when the parameter Λ becomes a large positive number. The expansion for large Λ employs a lemma due to Sommerfeld, which, as stated by Chandrasekhar,[1] is:

LEMMA: If $\phi(u)$ is a sufficiently regular function which vanishes for $u = 0$, then we have the asymptotic formula

$$\int_0^\infty \frac{du}{(1/\Lambda)e^u + 1} \frac{d\phi(u)}{du} = \phi(u_0) + 2\left[c_2\left(\frac{d^2\phi}{du^2}\right)_{u_0} + c_4\left(\frac{d^4\phi}{du^4}\right)_{u_0} + \cdots\right] \tag{2-65}$$

where $u_0 = \log \Lambda$ and $c_2, c_4, \ldots$ are numerical constants defined by

$$c_\nu = 1 - \frac{1}{2^\nu} + \frac{1}{3^\nu} - \frac{1}{4^\nu} + \cdots$$

The series for the constants c_ν can be summed.[2] For instance,

$$c_2 = \frac{\pi^2}{12} \qquad c_4 = \frac{7\pi^4}{720} \qquad c_6 = \frac{31\pi^6}{30{,}240}$$

Problem 2-14: By applying Sommerfeld's lemma to the integrals $F_{1/2}$ and $F_{3/2}$, show that

$$F_{1/2}(\alpha) = \tfrac{2}{3}(-\alpha)^{3/2}\left(1 + \frac{\pi^2}{8\alpha^2} + \frac{7\pi^4}{640\alpha^4} + \cdots\right)$$
$$F_{3/2}(\alpha) = \tfrac{2}{5}(-\alpha)^{5/2}\left(1 + \frac{5\pi^2}{8\alpha^2} - \frac{7\pi^4}{384\alpha^4} + \cdots\right) \tag{2-66}$$

is a good expansion for $\alpha < -1$. These three-term expansions are accurate to three decimal places for $\alpha < -5.6$ and are quite useful for $\alpha < -3$.

Problem 2-15: Calculate $F_{1/2}(\alpha)$ and $\frac{2}{3}F_{3/2}(\alpha)$ for $\alpha = -3$ and compare the results with the values in Table 2-3.

Since

$$n_e = \frac{4\pi}{h^3}(2mkT)^{3/2}F_{1/2}(\alpha) \tag{2-67}$$

it is evident from Eq. (2-66) that the physical meaning of α in the limit of strong degeneracy is

$$-\alpha \approx \frac{1}{2mkT}\left(\frac{3h^3n_e}{8\pi}\right)^{2/3} \tag{2-68}$$

[1] S. Chandrasekhar, "An Introduction to the Study of Stellar Structure," p. 389; reprinted from the Dover Publications edition, Copyright 1939 by The University of Chicago, as reprinted by permission of The University of Chicago.

[2] See, for instance, H. B. Dwight, "Tables of Integrals and Other Mathematical Data," eq. 47.2, p. 11, The Macmillan Company, New York, 1947.

which from Eq. (2-29) is

$$-\alpha \approx \frac{p_0^2}{2mkT} = \frac{E_f}{kT} \tag{2-69}$$

where E_f is the Fermi energy (the kinetic energy of an electron at the top of the Fermi sea). This result is the same one that was obtained from an inspection of the Fermi distribution function for large negative α. For incomplete degeneracy, however, the energies $|\alpha kT|$ and E_f have different definitions and physical meanings.

If the three-term expansion of $F_{1/2}(\alpha)$ is retained, Eq. (2-59) can be written as an approximate equation relating the value of α to the density and temperature:

$$(-\alpha)^{3/2}\left(1 + \frac{\pi^2}{8\alpha^2} + \frac{7\pi^4}{640\alpha^4} + \cdots\right) = 1.66 \times 10^8 \frac{\rho}{\mu_e} T^{-3/2} \qquad \text{for } \alpha < -3 \tag{2-70}$$

Problem 2-16: Show that the electron pressure is *twice* that of the maxwellian electron-gas formula when $\rho/\mu_e = 5.0 \times 10^{-8}T^{3/2}$. Compare this result with the approximate boundary of Eq. (2-34), which gave the density for which a *completely* degenerate gas formula yields the same pressure as the maxwellian gas formula.

The properties of the equation of state of the perfect electron gas are shown graphically in Fig. 2-7, where the ρT plane is divided into various zones according to the extent of the electron degeneracy. The diagonal line represents the approximate boundary between nondegenerate and degenerate electron gas as given by Eq. (2-34). In the neighborhood of this boundary the equation of state is to be evaluated from the parametric equations (2-57), which apply to partial degeneracy. For densities as high as indicated by Eq. (2-40), an electron gas becomes relativistic. This boundary is shown by the vertical line in Fig. 2-7. In the neighborhood of this line, the pressure of a completely degenerate gas can be evaluated from Eq. (2-46). For very high temperatures ($T > 10^9$) not considered in this discussion, the electron gas can be both relativistic and only partially degenerate. This situation presents a slightly more difficult form of the equation of state. We shall not consider it here. Suffice it to say that the Fermi statistics yield the same expression for the pressure as Eq. (2-53), the difference being that relativistic kinematics are to be used.

Several additional comments concerning a degenerate electron gas are appropriate at this time. With regard to the mechanical pressure which is to support a star, it is clear that the calculations presented here account only for the pressure due to the electrons. The contribution from the particle pressure of the nuclei in the gas must be added. Since nuclei are never degenerate in common stars, the pressure due to them is simply that of a maxwellian gas, whose equations have been developed previously. To calculate the partial pressure of this perfect nuclear gas one must, of course, use the appropriate value of the mean molecular weight. Since the electrons have in this case been accounted for independently, one must use only the mean molecular weight of the remaining ions

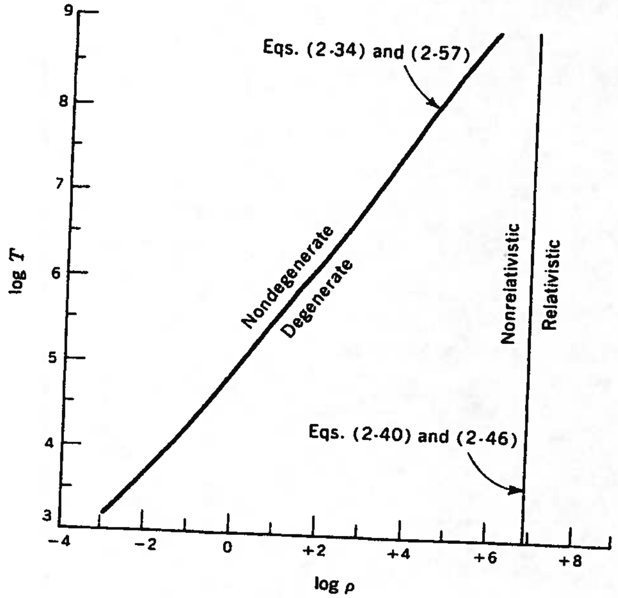


Fig. 2-7 Zones of the equation of state of an electron gas. The nonrelativistic transition region between nondegeneracy and extreme degeneracy is located according to Eq. (2-34), and the pressure is given by Eq. (2-57) in this region. As ρ approaches 10^7 g/cm³, many of the electrons become relativistic, and the distribution becomes highly degenerate, in which case Eq. (2-46) adequately represents the pressure.

and nuclei. Let μ_i designate the mean molecular weight of the ions. The pressure due to particles is then the sum of the electron pressure and the nucleus pressure:

$$P_{gas} = P_e + \frac{N_0 k}{\mu_i} \rho T \tag{2-71}$$

In most practical cases where electron degeneracy does occur, the remaining nuclei are generally those of more advanced phases of stellar structure, consisting of helium nuclei, carbon nuclei, oxygen nuclei, or perhaps even heavier nuclei. In these circumstances the bulk of the pressure will be provided by the degenerate electron gas, the nuclei providing only a small additional term.

Problem 2-17: A gas composed of C^{12} and O^{16} has a density of 2.5×10^5 g/cm³ at 10^8 °K. Is this gas in the degenerate or nondegenerate region of the equation of state? Assuming the degeneracy is complete, is it completely nonrelativistic, partially relativistic, or extremely relativistic? Calculate the electron pressure from Table 2-2. Assuming that the degeneracy is incomplete and nonrelativistic, calculate the electron pressure from Table 2-3. Why is the pressure calculated under assumptions of partial degeneracy greater than the pressure calculated for assumptions of complete degeneracy? Which numerical answer is more correct for the present problem? Why? What is the ratio of the electron pressure to the ion pressure?

Another interesting feature of the pressure of a completely degenerate gas is that it does not depend explicitly upon the temperature. Of course, at any finite temperature the electron gas is never completely degenerate, but in many cases the actual momentum distribution may be closely approximated by complete degeneracy. Whenever the energy associated with the momentum p_0 of the completely degenerate distribution greatly exceeds kT, the distribution of electron momenta will closely resemble that of complete degeneracy. It is in this case that the pressure is approximately independent of the temperature, being absolutely independent of the temperature for complete degeneracy.[1] This fact has the interesting consequence that a small rise in the temperature of an almost completely degenerate electron gas causes almost no change at all in the pressure. This last fact has far-reaching effects on stellar structure and on the evolution of stars. Those stages of stellar structure in which the electron gas is degenerate and is providing the main source of pressure for the gas must admit the possibility of abrupt rises in temperature with no corresponding increase in pressure. This situation actually occurs in certain stages of stellar evolution and leads to runaways in nuclear reaction rates (flash phenomena).

Problem 2-18: Show that the nonrelativistic electron pressure changes with temperature at constant volume according to

$$\left(\frac{\partial P_e}{\partial T}\right)_{n_e} = \frac{8\pi k}{3h^3}(2mkT)^{3/2}\left(\tfrac{5}{2}F_{3/2} - \tfrac{3}{2}F_{1/2}\frac{dF_{3/2}/d\alpha}{dF_{1/2}/d\alpha}\right)$$
$$= \frac{P_e}{T}\left(\frac{5}{2} - \frac{3}{2}\frac{F_{1/2}}{F_{3/2}}\frac{dF_{3/2}/d\alpha}{dF_{1/2}/d\alpha}\right)$$

The quantity in parentheses in the second expression is unity for a nondegenerate gas and zero for a completely degenerate gas. Confirm this by evaluating it with the aid of the appropriate expansions.

Another important feature of the degenerate electron distributions is related to the transport of heat energy in the interiors of stars. The normal processes of energy transport in stellar interiors are altered somewhat when the electron gas becomes degenerate. The most important fact is that heat conductivity, which normally plays a secondary role to radiative transport and to convective transport, becomes important. In the case of nondegeneracy, the mean free path of charged particles is so small that heat conduction is extremely inefficient. When an electron gas is degenerate, however, the mean free path of electrons becomes

[1] Mathematically one shows that $\partial P/\partial T$ is very small by making an expansion of the parametric equation of state and evaluating for noncomplete degeneracy. The reader is referred to Chandrasekhar, *op. cit.*, chap. 10.

quite long. In order for an energetic electron to lose energy, it must fall into a lower-lying cell in momentum space as well as impart a new energy and momentum to the particle from which it scatters. The filling up of the available states in momentum space below a certain level hinders this process and renders energetic electrons quite free to move about in even a partially degenerate electron gas. This very good conductivity will tend to make partially degenerate electron gases isothermal.

White-dwarf stars are, to good approximation, supported by a completely degenerate electron gas. As those stars radiate their thermal energy, becoming increasingly cooler, the nearly degenerate momentum distribution becomes increasingly rectangular. Eventually the thermal energy is radiated away, the temperature falls toward zero, the light goes out, and the object remains an inert mass supported by a dense sea of completely degenerate electrons, or so the story goes. This picture is in keeping with the observed properties of white dwarfs, which, from their observed masses and radii, are known to have densities as large as 10^6 g/cm^3.

Pioneers in stellar structure encountered a subtle paradox in contemplating the above picture, however. Faithful application of the hitherto successful ionization equation seemed to imply that ions and electrons recombine at low temperatures. Since the density of un-ionized matter is at most a few grams per cubic centimeter, it would appear necessary that white dwarf expand as it cools. Yet it could be shown that the thermal energy is, at all stages, insufficient to do the necessary gravitational work. Eddington expressed the paradox as follows:

> *I do not see how a star which has once got into this compressed condition is ever going to get out of it. So far as we know, the close packing of matter is only possible so long as the temperature is great enough to ionize the material. When the star cools down and regains the normal density ordinarily associated with solids, it must expand and do work against gravity.* The star will need energy to cool. *Sirius* comes *on solidifying will have to expand its radius at least tenfold, which means that 90 percent of its lost gravitational energy must be replaced. We can scarcely credit the star with sufficient foresight to retain more than 90 percent in reserve for the difficulty awaiting it. It would seem that the star will be in an awkward predicament when its supply of subatomic energy ultimately fails. Imagine a body continually losing heat but with insufficient energy to grow cold!*[1]

The physical basis for the resolution of this problem is the thermodynamic peculiarity of a degenerate gas: the temperature no longer corresponds to kinetic energy. The electrons in a zero-temperature degenerate gas must still have large kinetic energy if the density is great. The classical ionization equation showed that at high densities atoms become ionized as kT approaches the order of magnitude of the electron binding energy, which is when the kinetic energy of the free-

[1] *Op. cit.*, p. 172.

electron gas approaches the kinetic energy of the bound electrons. The same approximate result applies in degenerate circumstances. Atoms are in an ionized state when the kinetic energy of the electron gas exceeds the kinetic energy of a bound electron.

The approximate truth of this statement can be seen from the following considerations. In a completely degenerate gas, all available electron states with momentum less than p_0 are occupied. The exclusion principle thus forbids the presence of bound electrons unless they are bound so tightly that their momentum exceeds p_0, for otherwise there would be "too many" electrons in a momentum interval. Whereas a rigorous description of quantum statistics is considerably more complicated than this simple argument, the physical necessity of the result is evident.

The physical idea is also similar to that of the band structure of electronic states in solids. Ignore considerations of temperature completely for the moment. When the interatomic separations of atoms are large, the energy levels of electrons are just those associated with isolated atoms. Each energy level possesses a degeneracy equal to that of the atomic level times the total number of atoms. When the interatomic separation is decreased to the point where electronic levels of adjacent atoms overlap, however, a new feature is introduced by the exclusion principle. Since electrons are identical fermions, the mutual wave function of overlapping electrons must be antisymmetric in the electron coordinates. This antisymmetrization introduces a sharing of the indistinguishable electrons by all the atoms. In order that the electrons not be in exactly the same state, the many degenerate atomic energy levels of discrete energy regroup into a *continuous band* of energies for which each electron is shared by all atoms. The wave functions of those electrons in the band can be expressed by wave functions analogous to free electrons. This is what happens in a metal, for instance, for which the continuous band of *quasifree* electrons provides the source of electric conductivity. The same feature is carried to extremes at the densities of stellar interiors. Careful analysis shows that atoms are completely ionized by this mechanism for densities greater than about 10^3 g/cm^3 *independent of the temperature.* This physical effect has come to be called *pressure ionization*, and it resolves in a natural manner the paradox stated by Eddington.

This completes the introduction to the perfect electron gas. We have attempted to focus attention onto the physical principles rather than on the mathematical details. The serious student of stellar structure who has grasped these ideas may turn to more complete treatments for appropriate formulas applicable to the computation of physical problems.

THE PHOTON GAS

Particles are not the only source of mechanical pressure in a perfect gas. Pressure is also exerted by the radiation field in the interior of the star. By the radiation field we mean electromagnetic radiation, the omnipresent flux of photons inside a thermal enclosure. The pressure of the photon gas results from the fact