3. In this problem we consider how radio observations of free-free emission provide diagnostics of the physical conditions of H II regions. (Analogous considerations apply to X-ray observations of hot gas in galaxy clusters.) Assume an H II region to have a uniform electron temperature T and density n_e which we would like to determine by observational means. Since the free-free emission associated with the thermal distribution of electrons occurs under conditions of LTE, satisfy yourself that equation (3.18) of the text yields the solution for the equation of transfer:

$$I_\nu = I_\nu(0)e^{-\tau_\nu} + B_\nu(T)\left(1 - e^{-\tau_\nu}\right). \tag{11}$$

For radio observations spanning, say, $\lambda \sim 100\,\mathrm{cm}$ to $1\,\mathrm{mm}$, show that $h\nu \ll kT$ for all likely values of T. Thus it represents a good approximation to replace $B_\nu(T)$ by its Rayleigh-Jeans limit:

$$\text{Rayleigh} - \text{Jeans}: \qquad B_\nu(T) = \frac{2\nu^2}{c^2}kT.$$

Motivated by this simplification, radio astronomers then like to express the specific intensity I_ν in terms of a *brightness temperature* T_b defined through

$$T_b \equiv \frac{c^2}{2\nu^2 k}I_\nu. \tag{12}$$

Show now that equation (11) can be rewritten as

$$T_b = T_b(0)e^{-\tau_\nu} + T\left(1 - e^{-\tau_\nu}\right), \tag{13}$$

which applies not only to free-free radiation, but wherever (a) we may ignore the effects of scattering, and (b) we may assume that the source function has an LTE value at a uniform temperature T throughout the region being observed. (Notice that we have not yet assumed uniformity of density.)

Applied to free-free emission, τ_ν has the form:

$$\tau_\nu = \int \rho\kappa_\nu^{\text{ff}} ds,$$

where the integral is taken along the line of sight through the H II region and $\rho\kappa_\nu^{\text{ff}}$ is given by equation (15.29) of the text. Assume for simplicity a pure hydrogen plasma, expand the exponential in the correction for stimulated emission, $1 - e^{-h\nu/kT}$, for small $h\nu/kT$, and show that

$$\rho\kappa_\nu^{\text{ff}} = C n_e^2 T^{-3/2} \nu^{-2} \bar{g}_\nu^{\text{ff}},$$

where C is a constant coefficient,

$$C \equiv \left(\frac{2m_e}{3\pi k}\right)^{1/2} \left[\frac{4\pi e^6}{3m_e^2 ck}\right].$$

The Gaunt factor $\bar{g}_\nu^{\text{ff}}$ in the radio regime reads

$$\bar{g}_\nu^{\text{ff}} = \frac{\sqrt{3}}{2\pi}\left[\ln\left(\frac{8k^3T^3}{\pi^2 e^4 m_e \nu^2}\right) - 5\gamma\right],$$

where $\gamma = 0.5772\ldots$ is Euler's constant. Compute $\bar{g}_\nu^{\text{ff}}$ for $\nu = 10^9$ Hz and $T = 10^4$ K, and show that, unlike the optical case, $\bar{g}_\nu^{\text{ff}}$ should not be approximated by unity here. Notice also that Planck's constant h has dropped out of all equations, so that the considerations are purely classical.

Define the *emission measure* as the integral

$$\text{EM} \equiv \int n_e^2 ds,$$

and show that τ_ν can be expressed as

$$\tau_\nu = (\text{EM}) C T^{-3/2} \nu^{-2} \bar{g}_\nu^{\text{ff}}. \tag{14}$$

At low frequencies, $\tau_\nu \gg 1$, whereas at high frequencies, $\tau_\nu \ll 1$. With no background source, show that this implies $T_b \approx T$ at low frequencies, while $T_b \approx T\tau_\nu$ at high frequencies.

For a spherical H II region with radius R_S, show that the observed flux (measured in Janskys $= 10^{-26}$ watts m^{-2} Hz^{-1} by radio astronomers) $F_\nu = \pi I_\nu (R_S^2/r^2)$ where r is the distance to the source. The size R_S can be

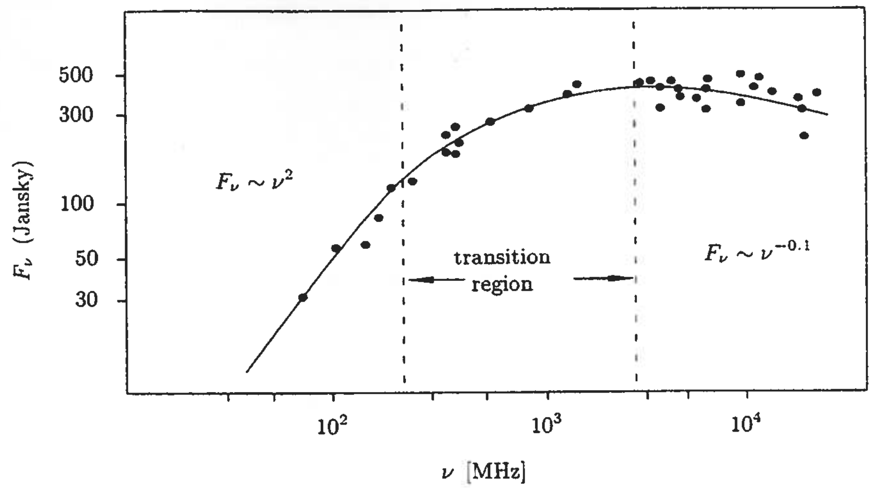

FIGURE P4.1
Observations of free-free emission from the Orion nebula.

determined if the source is angularly resolved and its distance known. Show now that

$$F_\nu = \frac{2\pi k}{c^2}\left(\frac{R_S}{r}\right)^2 \nu^2 T_b \tag{15}$$

will be proportional to ν^2 at low frequencies and to $\bar{g}_\nu^{\rm ff}$ (a nearly flat function $\propto \nu^{-0.1}$) at high (radio) frequencies. Describe qualitatively how this information could be used to deduce T and EM if the spectrum on both sides of the turnover frequency ν_c (where $\tau_\nu = 1$) can be measured. (A better way in the radio to obtain the electron temperature is to measure the strength of the H109α recombination line.)

Figure P4.1 contains observations of an H II region with size $R_S \approx 0.6$ pc in the Orion nebula, which lies at a distance $r \approx 500$ pc. Compute approximate values for T and n_e from this data. Quoted values in the literature are $T \approx 8{,}000$ K, and $n_e \approx 2{,}000\,\mathrm{cm}^{-3}$. Check to see at what frequency $\tau_\nu = 1$ for your results. Your derived temperature will not agree with the value 8,000 K, because the low-frequency measurements have larger effective beam sizes than the high-frequency measurements; thus corrections need to be applied to obtain the true underlying ν^2 dependence at low ν (see the discussion in Osterbrock 1989, pp. 128–130). This fact should provide a warning against the naive use of free-free radiation to deduce the electron temperatures of H II regions, and partially explains why radio astronomers prefer to use recombination lines for this purpose.

Is the size $R_S = 0.6$ pc consistent with Stromgren's theory of H II regions (see Problem Set 3, but note the density difference) if the exciting source is an O5 star? There's actually some indication that the Orion H II region is *density-bounded* (runs out of gas) rather than *ionization-bounded* (runs out of ultraviolet photons).