Astr 545 – Astrophysics of Stars and Accretion – Fall 2015

Homework #1

Due: Tues. Sept. 8, 1:59PM in class, or earlier in SO mailbox (Youdin)

1. Material Derivatives of Fluid:
Consider a Eulerian description in terms of position $\boldsymbol{x}$ and time, t, and a Lagrangian description in terms of initial position $\boldsymbol{x}_\circ$ and time t'. Time is the same in both frames, $t' = t$ and the current position of a fluid element in the Lagrangian frame is $\boldsymbol{x}'(t', \boldsymbol{x}_\circ)$. Any fluid property F must have the same value in both descriptions at the same position and time: $F(\boldsymbol{x}, t) = F(\boldsymbol{x}'(t', \boldsymbol{x}_\circ), t') = F(\boldsymbol{x}_\circ, t')$.

(a) Use the chain rule and the fluid velocity $\boldsymbol{u}$ to express the Lagrangian or material derivative in Eulierian coordinates

$$\frac{DF}{Dt} \equiv \left.\frac{\partial F}{\partial t'}\right|_{\boldsymbol{x}_\circ} = \frac{\partial F}{\partial t} = \boldsymbol{u} \cdot \frac{\partial F}{\partial \boldsymbol{x}}\,, \tag{1}$$

explaining all steps. Recall that the chain rule for two coordinate systems, y_i and y_i' is $\partial f/\partial y_i = (\partial f/\partial y_k')(\partial y_k'/\partial y_i)$ with summation over k.

(b) Now we consider the material derivative of a volume integral

$$\frac{D}{Dt}\int_{V(t)} F(\boldsymbol{x}, t)dV \tag{2}$$

where the volume of the boundary moves with the fluid. From calculus, Leibniz's theorem gives

$$\frac{d}{dt}\int_{V(t)} F(\boldsymbol{x}, t)dV = \int_V \frac{\partial F}{\partial t} + \int_A F d\boldsymbol{A} \cdot \boldsymbol{u}_A \tag{3}$$

where the final term is a intgral over the surface area, A, of the volume, V and the boundary moves at speed $\boldsymbol{u}_A$. Use the fact that the boundary moves at the fluid velocity and Gauss's theorem to show that

$$\frac{D}{Dt}\int_V F(\boldsymbol{x}, t)dV = \int_V \left[\frac{\partial F}{\partial t} + \frac{\partial}{\partial x_i}(F u_i)\right] dV \tag{4}$$

(c) Now use the continuity equation for the density ρ to show that

$$\frac{D}{Dt}\int_V \rho G(\boldsymbol{x}, t)dV = \int_V \rho \frac{DG}{Dt} dV \tag{5}$$

Convince yourself (and show) that since the volume is arbitrary in size and shape (but must move with the fluid), that $\rho D\boldsymbol{u}/Dt = \boldsymbol{f}$ is the correct momentum equation, where $\boldsymbol{f}$ is the force per unit volume on the fluid. [Note (but you do not need to show) that part (b) shows that the momentum equation can also be written in "conservation form" as $\partial(\rho u_i)/\partial t + \partial/\partial x_j(\rho u_j u_i) = f_i$.]

2. Rotating Frame: Consider an inertial frame (I) and a rotating frame (R) with uniform angular velocity $\boldsymbol{\Omega}$.

(a) A vector $\boldsymbol{A}$ that is fixed in the rotating frame will rotate in the inertial frame (unless perfectly aligned so that $\boldsymbol{A} \times \boldsymbol{\Omega} = 0$). Show that the inertial observer sees

$$\left(\frac{d\boldsymbol{A}}{dt}\right)_{\mathrm{I}} = \boldsymbol{\Omega} \times \boldsymbol{A} \tag{6}$$

It helps to consider an angle θ between $\boldsymbol{\Omega}$ and $\boldsymbol{A}$ (so that $|\boldsymbol{\Omega} \times \boldsymbol{A}| = ??$) and an infinitesimal rotation by $\Delta\phi$. Also note that the unit vector, $\hat{n}$ describing the direction of change of $\boldsymbol{A}$ is perpendicular to both $\boldsymbol{\Omega}$ and $\boldsymbol{A}$ (since $\boldsymbol{A}$ isn't changing in length). Thus $\hat{n} = \boldsymbol{\Omega} \times \boldsymbol{A}/|\boldsymbol{\Omega} \times \boldsymbol{A}|$.

(b) Now consider a general vector $\boldsymbol{B}$ which may change in the rotating frame as well. Show that the changes in the inertial and rotating frames are related by

$$\left(\frac{d\boldsymbol{B}}{dt}\right)_{\mathrm{I}} = \left(\frac{d\boldsymbol{B}}{dt}\right)_{\mathrm{R}} + \boldsymbol{\Omega} \times \boldsymbol{B} \tag{7}$$

Hint: express $\boldsymbol{B} = B_j \hat{\imath}_j$ in terms of Cartesian unit vectors $\hat{\imath}_j$ in the rotating frame, applying the result from (a) to the rotation of these unit vectors in the inertial frame.

(c) To derive the momentum equation in a rotating frame, first show that fluid velocities transform as

$$\boldsymbol{u}_{\mathrm{I}} = \boldsymbol{u}_{\mathrm{R}} + \boldsymbol{\Omega} \times \boldsymbol{x} \tag{8}$$

in terms of the position vector $\boldsymbol{x}$ of a fluid element. Then show that the rate of change of velocity transforms as

$$\begin{aligned} \left(\frac{D\boldsymbol{u}_{\mathrm{I}}}{Dt}\right)_{\mathrm{I}} &= \left(\frac{D\boldsymbol{u}_{\mathrm{I}}}{Dt}\right)_{\mathrm{R}} + \boldsymbol{\Omega} \times \boldsymbol{u}_{\mathrm{I}} && \text{(9a)} \\ &= \left(\frac{D\boldsymbol{u}_{\mathrm{R}}}{Dt}\right)_{\mathrm{R}} + 2\boldsymbol{\Omega} \times \boldsymbol{u}_{\mathrm{R}} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{x}) && \text{(9b)} \end{aligned}$$

For a momentum equation in an intertial frame that is

$$\rho \frac{D\boldsymbol{u}}{Dt} = -\nabla P - \rho \nabla \phi + \mu \nabla^2 \boldsymbol{u} \tag{10}$$

derive the corresponding equation in a rotating frame. Hints: Are spatial gradients affected by the choice of reference frame? Showing that the viscous term is invariant can be derived mathematically or argued physically.

(d) The form of the continuity and energy equations is usually unaffected by the transformation to a rotating frame. This invariance occurs because (i) the divergence of velocity, $\nabla \cdot \boldsymbol{u}$ and (ii) the material derivative of any scalar, e.g. $D\rho/Dt$, are both invariant. Argue that both are true physically and then show it mathematically. For (ii) show that (unprimed coordinates are inertial and primed are in the rotating frame)

$$\begin{aligned} \frac{\partial P}{\partial t} &= \frac{\partial P}{\partial t'} - (\boldsymbol{\Omega} \times \boldsymbol{x}') \cdot \nabla' P && \text{(11a)} \\ \boldsymbol{u} \cdot \nabla P &= (\boldsymbol{u}' + \boldsymbol{\Omega} \times \boldsymbol{x}') \cdot \nabla' P && \text{(11b)} \end{aligned}$$

One way (others are possible) to derive Equation (11) is to apply the chain rule with $\boldsymbol{\Omega} = \Omega\hat{z}$ so that $x' = x\cos(\Omega t) + y\sin(\Omega t)$, $y' = \ldots$. Given (i) and (ii) you can (and should) show trivially that the continuity equation is invariant.

(e) When monsoon storms arrive in Tucson from the South, does the Coriolis force deflect them towards the East or West? Not all storms follow this simple expectation. Come up with at least one reason why.

3. Polytropes and the Chanrasekhar Mass:

Consider a polytropic equation of state (EOS) with

$$P = K\rho^{\gamma} \equiv K\rho^{1+1/n}.$$

(a) Give order of magnitude estimates of the central density and pressure in terms of G, M and R; the grav. constant, total mass and radius. Use these estimates to derive a relation between mass and radius, retaining K and G. [Ignore for now the order unity coefficients given by exact solutions to the structure of polytropes, but note that these estimates will fail for $n \geq 5$ because such polytropes have infinite radii.]

(b) Briefly (a few words is enough for your graders) describe the significance of the cases $n = 0, 3/2, 3$. Why does $n = 3/2$ ($\gamma = 5/3$) describe both degenerate (radius decreasing with increasing mass) and more general behavior where radius can increase with mass? I.e. what is different about the polytropic index K.

(c) To order of magnitude, calculate the limiting Chandrasekhar mass in terms of fundamental constants. You need to use K, i.e. $P(\rho)$, for an ultra relativistic sea of degenerate electrons. Express your final result in terms of the Planck mass $m_{\mathrm{Pl}} = \sqrt{\hbar c/G}$ and the mean mass per electron $\mu_e m_p$. (Technically m_p should be the atomic mass unit, but is roughly equal to the proton, or neutron, mass). How close is this dimensional estimate to the accepted value of $M_{\mathrm{ch}} = 1.46(2/\mu_e)^2 M_\odot$?