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Preface

Since the appearance of the first edition of this book, there has been continuing rapid development of our understanding of stellar atmospheres, and it has been clear to me for some time that a new edition was needed. One of the major motivations for producing a new version of the book at this time is the desire to describe the major advances that have been made—in developing methods to solve the transfer equation in moving media, and in the theory of stellar winds. As was true in the first edition, I have not attempted to cover every possible aspect of the subject, but have again treated a limited number of problems in some depth.

It was clear from the outset that, in view of the great demands made upon the student's time in the now-crowded astrophysics curriculum (resulting from the explosion of our knowledge about the Universe), it was pointless to write a book significantly longer than the first edition. Thus, to add new material, it has been necessary to economize the presentation of the old material, and to omit topics that are of specialized interest or that lie outside the mainstream of the developments of primary importance to the book. In particular, given that today's student is most likely to learn what he knows about radiative transfer in a stellar-atmospheres course, but will be interested in applying it to *other* physical situations, I have purposely shifted the emphasis away from strictly stellar applications, and have developed the transfer theory more generally and completely. I believe that a thorough understanding of the radiative transfer theory presented in this book will equip the student to attack a wide variety of transfer problems, whether in the laboratory, the atmospheres of stars and planets, the interstellar medium, X-ray sources, or quasars. Further, I have added exercises in which the student is asked to fill in missing steps of derivations, or to apply the theory himself to simple examples. In most cases the exercises are quite straightforward and should require only a few minutes work; but some of the exercises in Chapter 7 require substantial effort and would make good class projects.

Ideally the material in this book should be taught in a course lasting two quarters, covering Chapters 1–7 in the first quarter, and Chapters 8–15 in the second. If an entire year (two semesters) is available, the book should be supplemented with extra material on subjects of interest to the instructor and students, perhaps drawn from problems of solar physics, stellar spectroscopy, pulsating atmospheres, peculiar stars, abundance analyses, or many others. If only one semester is available, I recommend omitting, first, Chapters 4 and 9 (which are more physics than astrophysics); next, Chapters 3 and 10 (which are fairly elementary and may well have been covered in an earlier course); and, finally, if necessary, Chapter 13 (which is not absolutely essential for a basic understanding of line-formation).

In any case, many fascinating subjects will inevitably be omitted, and teacher and student alike may feel frustrated, as I have been in writing the book, that a more complete coverage is not possible. Again and again I have felt like the traveler in Frost's "The Road Not Taken" (**388**, 105)*, in choosing one of two equally fair paths, knowing full well that way would lead on to way, and that I should not return to the other. I only hope that the students will discover for themselves these other charming paths and will spend a pleasant lifetime in their exploration.

It is no longer possible for me to acknowledge fairly the many people who have helped me learn about stellar atmospheres and line-formation, and I shall not try here, beyond offering a sincere thanks to all in whose debt I am. But I would be remiss if I did not specifically thank Lawrence Auer, David Hummer, and George Rybicki, who (as colleagues, critics, teachers, collaborators, and friends) have greatly deepened and enlarged my understanding of the material in this book. Further, I wish to record my great debt to Professor W. W. Morgan of Yerkes Observatory. His encouragement has stimulated much of the work I have done in the past several years, and his wise counsel has greatly enhanced its value. I also thank him for sharing with me a few glimpses of his perception of the nature of scientific method from the lofty point at which he can view it.

I wish in addition to thank the people who have helped with the writing of this book: Barbara Mihalas, for reading and correcting the manuscript and the typescript; Tom Holzer and Richard Klein for reading and commenting upon Chapter 15; and David Hummer and Paul Kunasz for reading the typescript and offering many corrections and suggestions. Thanks also are due to Gordon Newkirk for helping to provide, through his labors as Director of H.A.O., the scientific environment in which this book could be written. I also thank Paulina Franz for converting hundreds of pages of my spidery handwriting into smooth typed copy, Kathlyn Auer for preparing the index, and Pat Brewer of W. H. Freeman and Company for her effective and careful supervision of the production process.

Finally, I thank my father, M. D. Mihalas, for his unintentional (but priceless) contribution in teaching me, through the example of his life, the meaning of *αὐτοπεποίθησισ* and *φιλοτιμία*.

Oxford, England
October, 1977

Dimitri Mihalas

* A NOTE ABOUT REFERENCES: References are listed serially at the end of the text, and are denoted in the text with boldface numbers—e.g., (**105**). Additional information, such as a page or chapter citation, will be indicated following the reference number—e.g., (**105**, 27) or (**105**, Chap. 4). Citations to two or more references are separated by semicolons—e.g., (**105**, 27; **388**, 105).

Preface

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It has been tempting to include a wider range of subjects, but I have avoided doing so in the belief that it is more worthwhile for the student to consider a smaller number of topics in depth than attempt to survey the entire field superficially. In this vein, I have purposely limited the comparison of theory with observation to a few of the more crucial and illustrative examples. Moreover, I have restricted most of the theoretical discussion to what may be called the *classical stellar-atmospheres* problem—i.e., atmospheres in hydrostatic, radiative, and steady-state statistical equilibrium. This is ample material for a two-quarter course and is understood well enough to require little speculation. Even within this problem, I have limited the variety of techniques treated. For example, I personally favor using differential equations over using integral equations to solve transfer problems. Thus, although the latter method has enjoyed wide application and good success, particularly in the hands of the Harvard–Smithsonian Astrophysical Observatory group, there is little discussion of it in this book. This omission is not arbitrary, however, but is based upon the view that, since the two methods are mathematically equivalent, discussion of one suffices and, in addition, that the one I have chosen seems to offer more promise in future applications—for example, to situations involving hydrodynamics (wherein lies the real frontier of the subject). On the other hand, in my experience, the physics background of astronomy students is often uneven; I have, therefore, not hesitated to develop those aspects of physical theory that are of special interest to the atmospheres problem. In any case, I hope that users of this book will find it a helpful outline, which they can edit, alter, and enlarge upon as their needs dictate.

Williams Bay, Wisconsin
November 1969

Dimitri Mihalas

Stellar Atmospheres

1

The Radiation Field

From quantitative examination of the spectrum of a star, information can be obtained about the frequency distribution of the emergent radiation field. We observe both broad, smooth expanses of *continuum* and *spectrum lines*, where the frequency variation is quite abrupt. The entire spectrum contains an enormous wealth of information, and the primary goal of the theory of stellar atmospheres is to develop methods that can recover this information. To this end we must be able to describe the flow of energy through the outermost layers of a star, and to predict the observable characteristics of the emergent radiation. We apply known physical laws that specify the interaction of radiation with stellar material, and derive mathematical models from which we compute theoretical estimates of observables. We then compare theory and observation, and attempt to infer the physical conditions in stellar atmospheres. Such analyses can provide information about the structure of the envelope (important as a boundary condition for studies of stellar structure), modes of energy transport in the atmosphere, chemical abundances, rates of mass loss, and calibrations for converting observational parameters (e.g., M_v and $B - V$) into theoretically interpretable numbers (luminosity and temperature). By studying large numbers of stars we can establish relations of, say, chemical composition to stellar distributions, kinematics, and dynamics; this information provides clues in developing an understanding of the structure and dynamics of the Galaxy as a whole.

The program outlined above is ambitious, and it is not an easy one to carry out successfully. The observational data are often difficult to acquire, have limited precision, and are the results of very complicated physical structures. Often our physical theories are only primitive, and yet even these may lead to extremely complicated mathematical systems. But the key issue is that the information we deduce from stellar spectra will be a close approximation to reality only if the underlying physical theory is sound and comprehensive. We must, therefore, devote considerable attention to the development of an approach that correctly includes the essential physics.

In this chapter we introduce the basic definitions required to characterize the radiation field itself. The radiation field is treated from three points of view—using macroscopic, electromagnetic, and quantum descriptions. Each of these approaches yields useful information and, taken together, they provide a full picture of the nature of the field. We ignore polarization, but carry along an assumed time-dependence so that in later work we can derive equations of radiation hydrodynamics. In subsequent chapters we shall consider how the radiation interacts with the material and is transported through the atmosphere (Chapter 2), and shall write down detailed descriptions of the atomic parameters that specify the absorptivity of the material (Chapter 4) and the mechanisms that determine the distribution of atoms over available bound and free states (Chapter 5). After consideration of the grey problem, which supplies an ideal testing ground of methods and shows clearly the overall approach used (Chapter 3) and development of general mathematical techniques for solving transfer equations (Chapter 6), we discuss the central problem of the book: the construction of model atmospheres (Chapter 7). We then examine the physics of line formation for a given (static) model (Chapters 8–13), and the methods used to infer chemical abundances in and physical characteristics of stellar atmospheres. Radiative transfer in moving atmospheres is then analyzed (Chapter 14) and, finally, all of the preceding developments are applied in a discussion of stellar winds (Chapter 15).

1-1 The Specific Intensity

MACROSCOPIC DEFINITION

The *specific intensity* $I(\mathbf{r}, \mathbf{n}, \nu, t)$ of radiation at position $\mathbf{r}$, traveling in direction $\mathbf{n}$, with frequency ν, at time t is defined such that the amount of

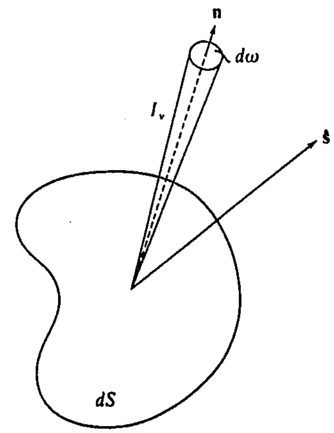

FIGURE 1-1
Pencil of radiation used to define specific intensity. The vector **n** is the direction of propagation of the radiation, while $\hat{\mathbf{s}}$ is the unit vector perpendicular to the element of area dS.

energy transported by radiation of frequencies $(\nu, \nu + d\nu)$ across an element of area dS into a solid angle $d\omega$ in a time interval dt is

$$\delta\mathscr{E} = I(\mathbf{r}, \mathbf{n}, \nu, t)\, dS \cos\theta\, d\omega\, d\nu\, dt \tag{1-1}$$

where θ is the angle between the direction of the beam and the normal to the surface (i.e., $dS \cos\theta = \mathbf{n} \cdot \mathbf{dS}$); see Figure 1-1. The dimensions of I are ergs $\text{cm}^{-2}\ \text{sec}^{-1}\ \text{hz}^{-1}\ \text{sr}^{-1}$. As it has just been defined, the specific intensity provides a complete description of the radiation field from a macroscopic point of view.

In this book consideration will be given only to one-dimensional problems in planar or spherical geometry; that is, the atmosphere will be regarded as composed of either homogeneous plane layers or homogeneous spherical shells. In planar geometries we employ Cartesian (x, y, z) coordinates with planes of constant z being the homogeneous layers; we can then ignore the (x, y) dependence of all variables, as well as derivatives with respect to x and y. It is convenient to introduce polar and azimuthal angles (θ, ϕ) to specify $\mathbf{n}$; we then have $\mathbf{n} \cdot \mathbf{k} = \cos\theta$, $\mathbf{n} \cdot \mathbf{i} = \sin\theta \cos\phi$, $\mathbf{n} \cdot \mathbf{j} = \sin\theta \sin\phi$. For one-dimensional planar geometry I will clearly be independent of ϕ; hence we can write $I = I(z, \theta, \nu, t)$; z is measured as positive upward in the atmosphere (opposite to the direction of gravity). In spherical geometry spatial location is specified by (r, Θ, Φ); but for spherical *symmetry*, I will depend upon r only. The direction of the radiation can be specified in terms of the polar and azimuthal angles (θ, ϕ), now measured with respect to a unit vector $\hat{\mathbf{r}}$ in the radial direction. Spherical symmetry again implies azimuthal invariance, and we can now write $I = I(r, \theta, \nu, t)$. We shall often replace the variable θ with $\mu \equiv \cos\theta$.

Exercise 1-1: By use of Snell's law, $n_1(\nu)\sin\theta_1 = n_2(\nu)\sin\theta_2$, in the calculation of the energy passing through a unit area on the interface between two dispersive media with differing indices of refraction, show that $I_\nu n_\nu^{-2}$ is a constant.

PHOTON DISTRIBUTION FUNCTION

The radiation field can also be described in terms of a *photon distribution function* f_R which is defined such that $f_R(\mathbf{r}, \mathbf{n}, \nu, t)\, d\omega\, d\nu$ is the number of photons per unit volume at location $\mathbf{r}$ and time t, with frequencies on the range $(\nu, \nu + d\nu)$, propagating with velocity c in direction $\mathbf{n}$ into a solid angle $d\omega$. Each photon has an energy $h\nu$. The number of photons crossing an element $\mathbf{dS}$ in time dt is $f_R(c\,dt)(\mathbf{n}\cdot\mathbf{dS})(d\omega\,d\nu)$, so that the energy transported is $\delta\mathscr{E} = (ch\nu)f_R\, dS\cos\theta\, d\omega\, d\nu\, dt$; comparison of this expression with equation (1-1) shows that

$$I(\mathbf{r}, \mathbf{n}, \nu, t) \equiv (ch\nu)f_R(\mathbf{r}, \mathbf{n}, \nu, t) \tag{1-2}$$

INVARIANCE PROPERTIES

An important property of the specific intensity is that it has been defined in such a way as to be independent of the distance between the source and the observer if there are no sources or sinks of radiation along the line of sight. Thus, consider that pencil of rays which passes through both the element of area dS at point P *and* the element dS' at P' (see Figure 1-2). Then the amount of energy $\delta\mathscr{E}$ passing through both areas is

$$\delta\mathscr{E} = I_\nu\, dS\cos\theta\, d\omega\, d\nu\, dt = \delta\mathscr{E}' = I'_\nu\, dS'\cos\theta'\, d\omega'\, d\nu\, dt \tag{1-3}$$

where $d\omega$ is the solid angle subtended by dS' as seen from P, and $d\omega'$ is the solid angle subtended by dS as seen from P'. From Figure 1-2 we see that $d\omega = r^{-2}\, dS'\cos\theta'$ while $d\omega' = r^{-2}\, dS\cos\theta$, where r is the distance from

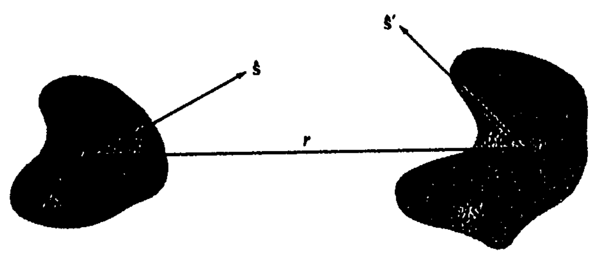

FIGURE 1-2
Geometry used in proof of invariance of specific intensity. The points P and P' are separated by a distance r. Area dS subtends a solid angle $d\omega'$ at P', and the area dS' subtends $d\omega$ at P; $\hat{s}$ and $\hat{s}'$ are unit vectors normal to dS and dS'.

P to P'. Thus it immediately follows from equation (1-3) that $I_\nu \equiv I'_\nu$. Note also that equation (1-3) implies that the energy received per unit area falls off as the inverse square of the distance between P and P'.

OBSERVATIONAL SIGNIFICANCE

The spatial invariance of the specific intensity implies that the actual value of I at the source can be obtained from measurements of the amount of energy falling, in a given time, within a specified frequency band, onto a receiver of known collecting area (and detection efficiency) from a source subtending a definite solid angle. The requirement that $d\omega$ must be specified limits the determination of I to sources that are *spatially resolved*—e.g., nebulae, galaxies, the sun, planets, etc.

In particular, for the sun, the radiation at a given point emerges at a known angle relative to the local normal (in a one-dimensional model); hence measurement of the center-to-limb variation of the radiation allows us to determine the angular variation of I. Note that we do not, in general, see to the same depth in the atmosphere along all rays; hence we do not obtain the angular variation of I at some definite position (z) inside the atmosphere, but rather at some point $\mathbf{r}_{\text{obs}}$ *outside* the atmosphere.

Exercise 1-2: The angular diameter of the sun is 30′. Suppose that atmospheric seeing effects limit resolution to 1″; show that this sets a lower bound on the μ for which we can infer $I(\mu)$ accurately, and determine this $\mu_{\min}$.

1-2 Mean Intensity and Energy Density

MACROSCOPIC DESCRIPTION

In both the physical and the mathematical description of a radiation field it is useful to employ various angular averages, or *moments*. Thus we define the *mean intensity* to be the straight average (zero-order moment) of the specific intensity over all solid angles, i.e.,

$$J(\mathbf{r}, \nu, t) = (4\pi)^{-1}\oint I(\mathbf{r}, \mathbf{n}, \nu, t)\, d\omega \tag{1-4}$$

The mean intensity has dimensions ergs cm^{-2} sec^{-1} hz^{-1}. The element of solid angle $d\omega$ is given by $d\omega = \sin\theta\, d\theta\, d\phi = -d\mu\, d\phi$. If we consider one-dimensional atmospheres, I is independent of ϕ, hence

$$J(z, \nu, t) = (4\pi)^{-1}\int_0^{2\pi} d\phi \int_{-1}^{1} d\mu\, I(z, \mu, \nu, t) = \frac{1}{2}\int_{-1}^{1} I(z, \mu, \nu, t)\, d\mu \tag{1-5}$$

The same result applies in spherical geometry with z replaced by r.

To calculate the *energy density* in the radiation field on the frequency range $(\nu, \nu + d\nu)$, consider a small volume V through which energy flows from all solid angles. The amount flowing from a particular solid angle $d\omega$ through an element of surface area dS of this volume is

$$\delta\mathscr{E} = I(\mathbf{r}, \mathbf{n}, \nu, t)(dS \cos\theta)\, d\omega\, d\nu\, dt$$

Now consider only those photons in flight across V; if the path length across V is l, then the time they will be contained within V is $dt = l/c$. Further, $l\, dS \cos\theta = dV$, the differential element of V through which they sweep. Hence the energy in dV coming from $d\omega$ is $\delta\mathscr{E} = c^{-1} I(\mathbf{r}, \mathbf{n}, \nu, t)\, d\omega\, d\nu\, dV$; by integrating over all solid angles and over the entire volume, we find the total energy contained in V, namely:

$$\mathscr{E}(\mathbf{r}, \nu, t)\, d\nu = c^{-1}\left[\int_V dV \oint d\omega\, I(\mathbf{r}, \mathbf{n}, \nu, t)\right] d\nu \tag{1-6}$$

But if we pass to the limit of *infinitesimal* V, I becomes independent of position in V, and the integrations can be carried out separately. The *monochromatic energy density*, $E_R(\mathbf{r}, \nu, t) \equiv \mathscr{E}(\mathbf{r}, \nu, t)/V$ is thus

$$E_R(\mathbf{r}, \nu, t) = c^{-1} \oint I(\mathbf{r}, \mathbf{n}, \nu, t)\, d\omega = (4\pi/c) J(\mathbf{r}, \nu, t) \tag{1-7}$$

E_R has dimensions of ergs cm^{-3} hz^{-1}. The *total energy density* (dimensions: ergs cm^{-3}) is found by integrating over all frequencies:

$$E_R(\mathbf{r}, t) = \int_0^\infty E_R(\mathbf{r}, \nu, t)\, d\nu = (4\pi/c) \int_0^\infty J(\mathbf{r}, \nu, t)\, d\nu \equiv (4\pi/c) J(\mathbf{r}, t) \tag{1-8}$$

PHOTON PICTURE

It is easy to show that the results derived above are consistent with the photon picture of the radiation field. By definition, $f_R(\mathbf{r}, \mathbf{n}, \nu, t)$ is the number of photons, per unit volume, of energy $h\nu$ propagating in direction $\mathbf{n}$ into intervals $d\nu\, d\omega$. The energy density clearly is just this number, multiplied by the energy per photon, summed over all solid angles: i.e.,

$$E_R(\mathbf{r}, \nu, t) = h\nu \oint f_R(\mathbf{r}, \mathbf{n}, \nu, t)\, d\omega \tag{1-9}$$

But from equation (1-2), $h\nu f_R = c^{-1} I$, hence equation (1-9) is seen to be identical with equation (1-7).

EQUILIBRIUM VALUE

In *thermal equilibrium* the radiation field inside an adiabatic enclosure is uniform, isotropic, time-independent, and has a frequency distribution

given by the *Planck function* $B_\nu(T) = (2h\nu^3/c^2)(e^{h\nu/kT} - 1)^{-1}$ [see (520), (392, 365)]. Thus, in thermal equilibrium the monochromatic energy density is $E_R^*(\nu) = (4\pi/c) B_\nu(T)$, and the total energy density is given by *Stefan's law*:

$$E_R^* = (8\pi h/c^3) \int_0^\infty (e^{h\nu/kT} - 1)^{-1} \nu^3\, d\nu = a_R T^4 \tag{1-10}$$

where $a_R = 8\pi^5 k^4/(15 c^3 h^3)$. Here, as elsewhere in this book, we denote a quantity computed from thermodynamic equilibrium relations with an asterisk.

Exercise 1-3: Derive Stefan's law by substituting $x \equiv h\nu/kT$, and expanding $(e^x - 1)^{-1} = e^{-x}(1 - e^{-x})^{-1}$ as a power series in e^{-x}. The sum obtained from the term-by-term integration is related to the Riemann zeta-function [see (4, 807)].

Stefan's law is valid in the interior of a star, and in the deeper layers of stellar atmospheres, where thermal gradients over a photon mean-free-path are extremely small, and the radiation becomes isotropic and thermalizes to its equilibrium value. At the surface, the radiation field becomes very anisotropic and has a markedly non-Planckian frequency distribution, as a result of steep temperature gradients and the existence of an open boundary through which photons escape into interstellar space; here Stefan's law becomes invalid.

ELECTROMAGNETIC DESCRIPTION

Electromagnetic theory provides an alternative description of the radiation field; we shall show how a one-to-one correspondence can be made between the macroscopic and electromagnetic descriptions of the radiation field. The electromagnetic field is specified by *Maxwell's equations* [see, e.g., (331, Chap. 6)] which, in Gaussian units, are

$$\nabla \cdot \mathbf{D} = 4\pi\rho \tag{1-11a}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1-11b}$$

$$(\nabla \times \mathbf{E}) + c^{-1}(\partial\mathbf{B}/\partial t) = 0 \tag{1-11c}$$

and

$$(\nabla \times \mathbf{H}) - c^{-1}(\partial\mathbf{D}/\partial t) = (4\pi/c)\mathbf{j} \tag{1-11d}$$

The *electric field* $\mathbf{E}$ is related to the *electric displacement* $\mathbf{D}$ in terms of the *permittivity* ε, namely $\mathbf{D} = \varepsilon\mathbf{E}$. Similarly, the *magnetic induction* $\mathbf{B}$ can be expressed in terms of the *magnetic field* $\mathbf{H}$ and the *permeability* μ by the relation $\mathbf{B} = \mu\mathbf{H}$. For vacuum, $\varepsilon = \mu = 1$. In equations (1-11), ρ is the *charge density* and $\mathbf{j}$ is the *current density* $\mathbf{j} = \rho\mathbf{v}$ associated with charges moving with velocity $\mathbf{v}$. The electric field and magnetic induction can be derived from a scalar potential ϕ and a vector potential $\mathbf{A}$, which are defined such that

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{1-12a}$$

and

$$\mathbf{E} = -\nabla\phi - c^{-1}(\partial\mathbf{A}/\partial t) \tag{1-12b}$$

Equation (1-12a) satisfies equation (1-11b), while (1-12b) satisfies (1-11c). Because **B** is defined as the *curl* of **A**, the *divergence* of **A** may be specified arbitrarily; one of the most convenient choices is to impose the *Lorentz condition*

$$\nabla \cdot \mathbf{A} = -c^{-1}(\partial\phi/\partial t) \tag{1-13}$$

With this choice, Maxwell's equations can be reduced to

$$\nabla^2\phi - c^{-2}(\partial^2\phi/\partial t^2) = -4\pi\rho \tag{1-14a}$$

and

$$\nabla^2\mathbf{A} - c^{-2}(\partial^2\mathbf{A}/\partial t^2) = -(4\pi/c)\mathbf{j} \tag{1-14b}$$

The solutions of these equations can be written as [cf. (331, Chap. 6), (494, Chap. 19)],

$$\phi(\mathbf{r}, t) = \int \frac{\rho(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|}\, d^3r' \tag{1-15a}$$

and

$$\mathbf{A}(\mathbf{r}, t) = \frac{1}{c}\int \frac{\rho(\mathbf{r}', t')\mathbf{v}(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|}\, d^3r' \tag{1-15b}$$

where, as indicated, ρ and $\mathbf{v}$ at $\mathbf{r}'$ are evaluated at the *retarded time* $t' = t - c^{-1}|\mathbf{r}' - \mathbf{r}|$ which takes into account the finite speed of propagation of electromagnetic waves.

One of the most important solutions of Maxwell's equations is that for *monochromatic plane waves in vacuum*, propagating in direction $\mathbf{n}_0$ with velocity c:

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 \cos[2\pi(k\mathbf{n}_0 \cdot \mathbf{r} - \nu t)] \tag{1-16a}$$

and

$$\mathbf{H}(\mathbf{r}, t) = \mathbf{H}_0 \cos[2\pi(k\mathbf{n}_0 \cdot \mathbf{r} - \nu t)] \tag{1-16b}$$

where $k = \lambda^{-1} = c^{-1}\nu$. The vectors $(\mathbf{E}_0, \mathbf{H}_0, \mathbf{n}_0)$ form an orthogonal triad with $\mathbf{H}_0 = \mathbf{n}_0 \times \mathbf{E}_0$, so it follows that $|\mathbf{H}_0| = |\mathbf{E}_0|$. The result obtained from electromagnetic theory for the instantaneous energy density $W(t)$ in the field is

$$W(t) = (\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H})/8\pi \tag{1-17}$$

Averaging in time over a cycle introduces a factor of $\langle \cos^2 \omega t \rangle_T = \frac{1}{2}$, and using the relations $|E_0| = |H_0|$ and $\mu = \varepsilon = 1$ (for vacuum), equation (1-17) reduces to $W = \langle W(t) \rangle_T = E_0{}^2/8\pi$. In terms of the macroscopic picture, a monochromatic plane wave propagating in direction $\mathbf{n}_0$ [specified by angles (θ_0, ϕ_0)] has a specific intensity $I(\mu, \phi) = I_0\, \delta(\mu - \mu_0)\, \delta(\phi - \phi_0)$ where δ

denotes the usual Dirac function. Substitution into equation (1-7) yields the energy density $E_R = c^{-1}I_0$, a result that is intuitively obvious for a plane wave propagating with velocity c. Therefore we obtain a correspondence between the two descriptions by making the identification

$$I_0 = cE_0{}^2/8\pi \tag{1-18}$$

It will be shown below that this choice yields consistent relations between the Poynting vector and Maxwell stress tensor and their macroscopic counterparts. The results derived here apply, strictly, only to a monochromatic plane wave, but are easily generalized to fields having arbitrary angle and frequency distributions by summing over suitably chosen elementary plane waves.

1-3 The Flux

MACROSCOPIC DESCRIPTION

We define the *flux* of radiation $\mathscr{F}(\mathbf{r}, \nu, t)$ as a vector quantity such that $\mathscr{F} \cdot d\mathbf{S}$ gives the *net rate of radiant energy flow* across the arbitrarily oriented surface $d\mathbf{S}$ per unit time and frequency interval. Noting that $\mathbf{n} \cdot d\mathbf{S} = dS \cos\theta$, where θ is the angle between the direction of propagation $\mathbf{n}$ and the normal to dS, we immediately recognize that the flux can be derived from the specific intensity via equation (1-1), for $\delta\mathscr{E}$ as written there is, in fact, nothing more than the contribution of the pencil of radiation moving in direction $\mathbf{n}$ to the net energy flux. Thus we merely sum over all solid angles and obtain

$$\mathscr{F}(\mathbf{r}, \nu, t) = \oint I(\mathbf{r}, \mathbf{n}, \nu, t)\mathbf{n}\, d\omega \tag{1-19}$$

The flux has dimensions: ergs cm^{-2} sec^{-1} hz^{-1}. Note that $\mathscr{F}$ is the first moment of the radiation field with respect to angle.

In cartesian coordinates we have

$$(\mathscr{F}_x, \mathscr{F}_y, \mathscr{F}_z) = \left(\oint In_x\, d\omega, \oint In_y\, d\omega, \oint In_z\, d\omega\right) \tag{1-20}$$

where $d\omega = -d\mu\, d\phi$, $n_x = (1 - \mu^2)^{\frac{1}{2}} \cos\phi$, $n_y = (1 - \mu^2)^{\frac{1}{2}} \sin\phi$, $n_z = \mu$. If the radiation field is symmetric with respect to an axis, then there will be a ray-by-ray cancellation in the net energy transport across any surface whose normal is perpendicular to that axis, and the net flux will be identically zero across this surface. In particular, for a planar atmosphere homogeneous in x and y, only $\mathscr{F}_z$ can be nonzero; we shall therefore require only this component of the flux, and shall refer to it as "the" flux, as if it were a scalar, and write

$$\mathscr{F}(z, \nu, t) = 2\pi \int_{-1}^{1} I(z, \mu, \nu, t)\mu \, d\mu \tag{1-21}$$

Exercise 1-4: (a) Show that $\mathscr{F}_x$ and $\mathscr{F}_y$ vanish in an atmosphere with azimuthal (ϕ) independence of I. (b) Show that in a spherically symmetric atmosphere only $\mathscr{F}_r$ is nonzero and is given by equation (1-21) with z replaced by r. (c) Evaluate $\mathscr{F}$ for $I(\mu) = \sum I_n \mu^n$; show that only the odd-order terms contribute to $\mathscr{F}$.

In astrophysical work it is customary to absorb the factor of π appearing in equation (1-21), and to write the *astrophysical flux* as $F(z, \nu, t) \equiv \pi^{-1}\mathscr{F}(z, \nu, t)$. Further, regarding the flux as one of a sequence of moments with respect to μ, one may define the *Eddington flux*

$$H(z, \nu, t) \equiv (4\pi)^{-1}\mathscr{F}(z, \nu, t) = \frac{1}{2}\int_{-1}^{1} I(z, \mu, t)\mu \, d\mu \tag{1-22}$$

which is in a form similar to equation (1-5) for the mean intensity.

PHOTON ENERGY FLUX

The same results for the energy flux may be obtained from the description of the radiation field in terms of photons. The *net number* of photons passing, with velocity c, through a unit surface oriented at angle θ to the beam, per unit time, is clearly

$$N(\mathbf{r}, \nu, t) = c \oint f_R(\mathbf{r}, \mathbf{n}, \nu, t) \cos\theta \, d\omega \tag{1-23}$$

Each photon has energy $h\nu$, so the net energy transport must be

$$\mathscr{F}(\mathbf{r}, \nu, t) = (ch\nu) \oint f_R(\mathbf{r}, \mathbf{n}, \nu, t)\mathbf{n} \, d\omega \tag{1-24}$$

In view of equation (1-2), equation (1-24) is obviously identical to equation (1-19).

Furthermore, photons of energy $h\nu$ propagating in direction $\mathbf{n}$ have momentum $h\nu\mathbf{n}/c$. Thus it is clear that $c^{-1}\mathscr{F} \cdot d\mathbf{S}\, dt$ gives the net momentum transport across the surface dS in time dt, by particles moving with velocity c. It therefore follows that the *momentum density* associated with the radiation field is $\mathbf{G}_R = c^{-2}\mathscr{F}$; we shall find further significance of this result in §2-3 and shall use it in §14-3.

Exercise 1-5: Verify the assertion that $c^{-2}\mathscr{F}$ represents a momentum density; check units for consistency.

THE POYNTING VECTOR

In electromagnetic theory, the energy flux in the field is given by the *Poynting vector*

$$\mathbf{S} = (c/4\pi)(\mathbf{E} \times \mathbf{H}) \tag{1-25}$$

Considering a plane wave as in §1-2, the average power over a cycle is $\langle \mathbf{S} \rangle_T = c\langle \mathbf{E} \times \mathbf{H} \rangle_T/4\pi = (c\langle E^2 \rangle_T \mathbf{n}_0)/4\pi = (cE_0^{\,2})\mathbf{n}_0/8\pi$. On the other hand, in terms of macroscopic quantities, the flux associated with a plane wave is

$$\mathscr{F} = \oint I\mathbf{n} \, d\omega = \oint I_0 \, \delta(\mathbf{n} - \mathbf{n}_0)\mathbf{n} \, d\omega = I_0 \mathbf{n}_0 \tag{1-26}$$

Now using equation (1-18), it is clear that $\mathscr{F}$ defined by equation (1-26) is identical to $\langle \mathbf{S} \rangle_T$. Again, this result can be generalized to arbitrary angle and frequency distributions of the radiation field.

OBSERVATIONAL SIGNIFICANCE

The energy received from a star by a distant observer can be related directly to the flux $\mathscr{F}_\nu$ emitted at the stellar surface. Assume that the distance D between star and observer is very much larger than the stellar radius r_*, so that all rays from star to observer may be considered to be parallel. The energy received, per unit area normal to the line of sight, from a differential area on the star is $df_\nu = I_\nu \, d\omega$ where $d\omega$ is the solid angle subtended by the area, and I_ν is the specific intensity emergent at the stellar surface. Considering the geometry shown in Figure 1-3 we see that $r = r_* \sin\theta$ so that the area of

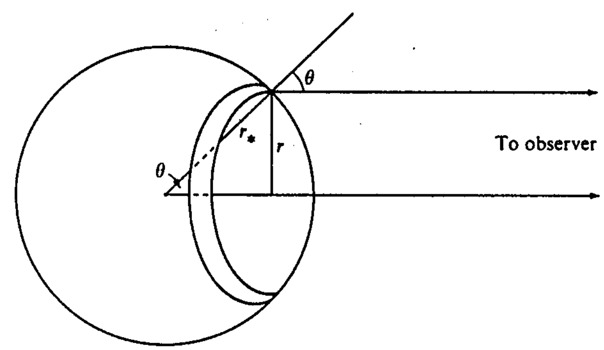

FIGURE 1-3
Geometry of measurement of stellar flux. The annulus on the surface of the star has an area $dS = 2\pi r \, dr = 2\pi r_*^2 \sin\theta \cos\theta \, d\theta$ normal to the line of sight; this area subtends a solid angle $d\omega = dS/D^2$ as seen by the observer.

a differential annulus on the disk is $dS = 2\pi r\,dr = 2\pi r_*^{\,2}\mu\,d\mu$, and $d\omega = 2\pi(r_*/D)^2\mu\,d\mu$. The radiation emitted from this annulus in the direction of the observer emerged at angle θ relative to the normal; hence the appropriate value of the specific intensity is $I(r_*, \mu, \nu)$. Integrating over the disk, we find

$$f_\nu = 2\pi(r_*/D)^2 \int_0^1 I(r_*, \mu, \nu)\mu\,d\mu = (r_*/D)^2 \mathscr{F}(r_*, \nu) = \frac{1}{4}\alpha_*^{\,2}\mathscr{F}(r_*, \nu) \quad (1\text{-}27)$$

where α_* is the *angular diameter* of the star. [In the above calculation we have assumed there is no radiation incident upon the surface of the star; i.e., $I(r_*, -\mu, \nu) \equiv 0$.] For *unresolved* objects (e.g., stars), we can measure only the flux. The energy received falls off as the inverse square of the distance (because the solid angle subtended by the disk varies as D^{-2}). If the angular diameter is known, then the absolute energy flux measured at the earth can be converted to the absolute flux at the star.

Exercise 1-6: Show that the flux emergent from a small aperture in an adiabatic enclosure (blackbody) is $\mathscr{F}_{BB}(\nu) = \pi B_\nu(T)$. Show that the *integrated* flux is $\mathscr{F}_{BB} = \sigma_R T^4$ where $\sigma_R = (c/4)a_R = 2\pi^5 k^4/(15h^3c^2) = 5.67 \times 10^{-5}$ erg cm^{-2} sec^{-1} deg^{-4} is the *Stefan-Boltzmann constant.*

1-4 The Radiation Pressure Tensor

MACROSCOPIC DESCRIPTION AND THE PHOTON MOMENTUM FLUX

The mean intensity and flux are the scalar and vector quantities given by the zero and first angular moments of the specific intensity against the direction cosines between the direction of propagation and an orthogonal triad. The *second* moment yields a tensor quantity that we shall identify as the *radiation pressure tensor* (or *radiation stress tensor*), namely

$$\mathsf{P}(r, \nu, t) = c^{-1} \oint I(\mathbf{r}, \mathbf{n}, \nu, t)\mathbf{nn}\,d\omega \quad (1\text{-}28)$$

or, in component form,

$$P_{ij}(r, \nu, t) = c^{-1} \oint I(\mathbf{r}, \mathbf{n}, \nu, t)n_i n_j\,d\omega \quad (1\text{-}29)$$

The dimensions of P are ergs cm^{-3} hz^{-1}. It is obvious that P is *symmetric*; i.e., $P_{ij} = P_{ji}$.

The physical interpretation of P follows directly from the description of the radiation field in terms of photons. Thus, using equation (1-2) to replace the specific intensity with the photon distribution function f_R, we see that

$$P_{ij}(\mathbf{r}, \nu, t) = \oint [f_R(\mathbf{r}, \mathbf{n}, \nu, t)cn_i](h\nu n_j/c)\,d\omega \quad (1\text{-}30)$$

The above expression clearly gives the net flux of momentum, in the j-direction, per unit time, from radiation of frequency ν, through a unit area oriented perpendicular to the i-direction; this is precisely the definition of *pressure* in any fluid, and hence justifies the term "radiation pressure".

The average of the diagonal components of P may be used to define a *mean radiation pressure*:

$$\bar{P} = \frac{1}{3}(P_{xx} + P_{yy} + P_{zz}) \quad (1\text{-}31)$$

But $(n_x^{\,2} + n_y^{\,2} + n_z^{\,2}) \equiv 1$ for any unit vector $\mathbf{n}$, hence in general

$$\bar{P}(\mathbf{r}, \nu, t) = (3c)^{-1} \oint I(\mathbf{r}, \mathbf{n}, \nu, t)\,d\omega = \frac{1}{3}E_R(\mathbf{r}, \nu, t) \quad (1\text{-}32)$$

However, it must be emphasized that despite the generality of this result, $\bar{P}$ does *not* give the actual radiation pressure unless the radiation field happens to be *isotropic.* In general the radiation field in stellar atmospheres is far from isotropic, and ordinarily the numerical factor relating p_R (a scalar parameter that can be used to calculate radiation forces) and the energy density E_R *exceeds* $\frac{1}{3}$ (see below).

RELATION OF THE PRESSURE TENSOR TO VOLUME FORCES

Let us now examine the relation of the radiation pressure tensor to volume forces exerted by the radiation field. Consider an element of area dS; the flow, per unit time, of the i-component of momentum in the radiation field across this element is $\sum_j P_{ij}n_j\,dS$, where the n_j's are the direction cosines of the normal to dS. Now integrating over a closed surface S, and applying the divergence theorem, we find

$$\oint_S \sum_j P_{ij}n_j\,dS = \int_V \sum_j (\partial P_{ij}/\partial x_j)\,dV = \int_V (\nabla \cdot \mathsf{P})_i\,dV \quad (1\text{-}33)$$

where V is the volume enclosed by S. The integral on the left gives the net flow, per unit time, of the i-component of momentum out of the volume through the surface S; thus from the integral on the right we see that $(\nabla \cdot \mathsf{P})_i$ must be the rate at which the i-component of the momentum density in the field decreases; i.e., $c^{-2}(\partial\mathscr{F}/\partial t)_i$.

Hence *for the radiation field alone* (i.e., in the absence of absorbing or emitting material) we have

$$(\partial \mathbf{G}_R/\partial t) = c^{-2}[\partial \mathscr{F}(\mathbf{r}, \nu, t)/\partial t] = -\nabla \cdot \mathsf{P}(\mathbf{r}, \nu, t) \tag{1-34}$$

Equation (1-34) is essentially identical to the usual momentum equation of hydrodynamics for an ideal fluid with no applied forces (cf. §15-1). We shall generalize this result to include interactions with material in §2-3.

THE MAXWELL STRESS TENSOR

In electromagnetic theory, the stress in the field is described by the *Maxwell stress tensor*, which is defined such that

$$(\partial \mathbf{G}_{\text{em}}/\partial t) = \nabla \cdot \mathsf{T}^M \tag{1-35}$$

Here $\mathbf{G}_{\text{em}}$ is the momentum density associated with the electromagnetic field. The components of T^M are:

$$T_{ij}{}^M = \left[E_i E_j + H_i H_j - \frac{1}{2}\delta_{ij}(E^2 + H^2)\right]\Big/4\pi \tag{1-36}$$

where δ_{ij} denotes the usual Kronecker δ-symbol. By comparison of equations (1-34) and (1-35) we see that the Maxwell stress tensor should be equal to the negative of the radiation pressure tensor; it is instructive to verify this conclusion by direct calculation.

Consider a plane wave propagating in direction $\mathbf{n}_0$; from the macroscopic definition of radiation pressure we have

$$\begin{aligned}\mathsf{P} &= c^{-1}\oint I\mathbf{nn}\,d\omega = c^{-1}I_0\oint \delta(\mathbf{n}-\mathbf{n}_0)\mathbf{nn}\,d\omega = c^{-1}I_0\mathbf{n}_0\mathbf{n}_0\\ &= (E_0{}^2/8\pi)\mathbf{n}_0\mathbf{n}_0\end{aligned} \tag{1-37}$$

which should equal T^M for a plane wave. Choose the electromagnetic field to yield a Poynting vector $\mathbf{S}$ along $\mathbf{n}_0$; in addition to (θ_0, ϕ_0) we must also specify the polarization of the wave via the angle ψ_0. Here ψ_0 measures the angle of rotation of $\mathbf{E}$ around $\mathbf{S}$ from the plane through $\mathbf{n}_0$ and $\mathbf{k}$ (the unit vector in the z-direction); see Figure 1-4. It is easy to see that

$$E_x = E_0(\sin\psi_0 \sin\phi_0 - \cos\psi_0 \cos\phi_0 \cos\theta_0) \tag{1-38a}$$

$$E_y = -E_0(\sin\psi_0 \cos\phi_0 + \cos\psi_0 \sin\phi_0 \cos\theta_0) \tag{1-38b}$$

$$E_z = E_0 \cos\psi_0 \sin\theta_0 \tag{1-38c}$$

Exercise 1-7: Derive expressions analogous to equation (1-38) for (H_x, H_y, H_z).

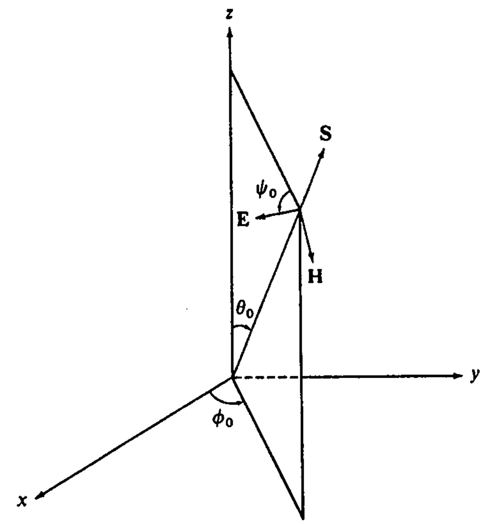

FIGURE 1-4
The plane electromagnetic wave generated by **E** and **H** is propagating along the Poynting vector **S** in direction $\mathbf{n}_0$. The angle ψ_0 measures the rotation of **E** around **S** out of the plane defined by $\mathbf{n}_0$ and **k**, the unit vector in the z-direction.

Substitution of equations (1-38) and the corresponding equations for **H** into equation (1-36) yields components of T^M; for example, for $T_{zz}{}^M$ we find

$$\begin{aligned}T_{zz}{}^M &= \left[E_z{}^2 + H_z{}^2 - \frac{1}{2}(E^2 + H^2)\right]\Big/4\pi = E_0{}^2(\sin^2\theta_0 - 1)/4\pi\\ &= -E_0{}^2\cos^2\theta_0/4\pi\end{aligned} \tag{1-39}$$

Averaging over time yields $\langle T_{zz}{}^M\rangle_T = -(E_0{}^2/8\pi)\cos^2\theta_0$ which is indeed $-P_{zz}$; note that the final result is independent of ψ_0.

Exercise 1-8: Calculate the remaining components of T^M and show that $\mathsf{T}^M = -\mathsf{P}$, independent of ψ_0.

The above results demonstrate that a complete correspondence exists between electromagnetic theory and the macroscopic or photon descriptions of the radiation field; we shall exploit this correspondence in a useful way in §§14-3 and 15-3 where we will be able to use the known Lorentz-transformation properties of electromagnetic field quantities to establish those of their radiation-field-description counterparts.