Motivation:

The aim of this project is to study oscillations in thin disks, in particular the dynamics of the oscillations and the what are the causes of the excitations. A deep understanding of the physics behind these oscillations would allow to study effect as the quasi-periodic oscillations. Also variations in the X-ray flux from black holes and AGNs might be explained. To this aim we follow mainly the review article by **Kato 2001**.

In accretion disks the main forces are the gravitational force and the centrifugal force. When these two forces are in equilibrium the disk is stable. The gravitational force is the one from the central object of the accretion disk.

The excitation mechanisms of disk oscillations are: XXX, XXX, XXX, and viscous processes. Viscosity is the major source of heating in the disk, also the azimuthal force caused by viscosity produces angular momentum transport in the radial direction. The first one can be seen as a thermal process while the second one is a dynamical process.

Accretion disks are collisional and non-selfgravitating.

The dispersion relation of disks:

To study the oscillations generated in the disk we use the dispersion relation derived in the appendix. Which is given by the expression:

$$(\tilde{\omega}^2 - \kappa^2)(\tilde{\omega}^2 - n\Omega_k^2) = \tilde{\omega}^2 c_s^2 k_r^2 \tag{1}$$

Where $om\tilde{e}ga$ is related with number of arms of the perturbation as:

$$\tilde{\omega} = \omega - m\Omega \tag{2}$$

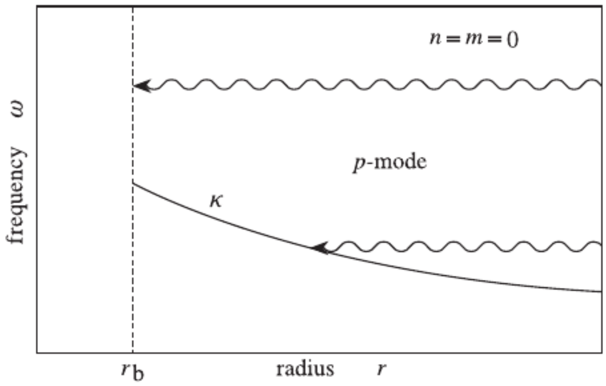
And the epicyclic frequency is:

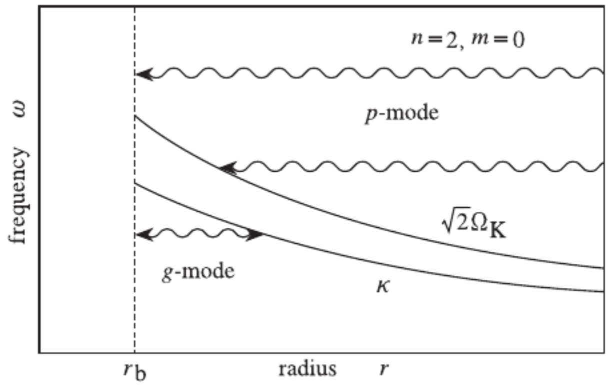
$$\kappa^2 = 2\Omega\left(2\Omega + r\frac{d\Omega}{dr}\right) \tag{3}$$

To understand the physical meaning of this dispersion relation lets study some limiting cases. First if the oscillations take place in the plane of the disk $n = 0$ the relation Eq.3 reduces to:

$$\tilde{\omega}^2 = \kappa^2 + k_r^2 c_s^2 \tag{4}$$

This condition is known as the **inertial-acoustic** waves, and corresponds to the oscillations of a fluid element that is displaced in the radial direction. The oscillations arise due to the resorting force due to rotation that brings back the fluid to the initial position. The oscillation frequency is the epicyclic frequency $\kappa(r)$ first term in the right part of equation 4. While the second term corresponds to acoustic oscillations due to the restoring force from compressible fluids.





If we consider oscillations in the vertical direction ($k_r = 0$) the dispersion relation Eq.4 reduces to the following expressions:

$$\tilde{\omega}^2 = \kappa^2 \tag{5}$$

$$\tilde{\omega}^2 = n\Omega_K^2 \tag{6}$$

Which corresponds to vertical oscillations in the disk, due to a perturbation of a fluid element in the vertical direction. The vertical component of the gravitational force in the restoring force that returns the fluid element to the plane of the disk. The frequency of this oscillations is represented by Ω_K.

The above two limiting cases are present at the same time in disks. The two oscillations are coupled in the form $(\tilde{\omega}^2 - \kappa^2)(\tilde{\omega}^2 - n\Omega_k^2)$ in the dispersion relation Eq.3. Vertical oscillations induce perturbations in the radial direction and radial oscillations induce vertical oscillations due to inhomogeneities in the disk. The coupling is stronger when the radial wavelength is shorter and the acoustic speed is faster.

The dispersion relation Eq.3 is quadratic for $\tilde{\omega}^2$ and the solutions are:

$$\tilde{\omega}^2 = \frac{(n\Omega_k^2 + \kappa^2 + c_s^2 K_r^2) \pm \sqrt{(-4\kappa^2 n\Omega_k^2)}}{2} \tag{7}$$

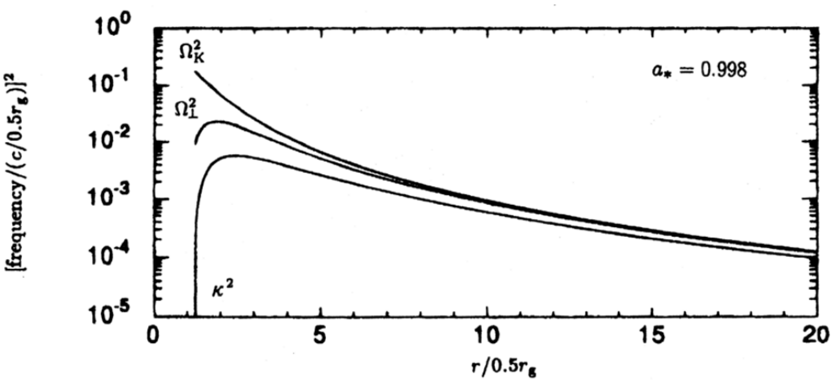
The modes with the sign in Eq.7 are called **p-modes** while the solutions with $-$ are called **g-modes**

0.1 Relativistic effects on the Dispersion Relation

When general relativistic effects are taking into account

$$\kappa^2 = \frac{GM}{r^3}\left(1 + \frac{a}{\hat{r}^{3/2}}\right)^{-2}\left(1 - \frac{6}{\hat{r}} + \frac{8a}{\hat{r}^{3/2}} - \frac{3a^2}{\hat{r}^2}\right) \tag{8}$$

Where a is a dimensionless parameter specifying the amount of angular momentum of the central object if $a = 0$ the object is rotating and if $a = 1$ the central object have the maximum rotation (the case of extreme Kerr).



$$\hat{r} = \frac{r}{GM/c^2} \tag{9}$$

$$\Omega^2 = \Omega_K^2 \left(1 - \frac{4a}{\hat{r}^{3/2} + \frac{3a^2}{\hat{r}^2}}\right) \tag{10}$$

$$\Omega_K^2 = \frac{GM}{r^3}\left[1 + \frac{a}{(8\hat{r}^3)^{1/2}}\right]^{-1} \tag{11}$$

$$(\tilde{\omega}^2 + \kappa^2)(\tilde{\omega}^2 - n\Omega_\perp^2) = \tilde{\omega}^2 c_s^2 k_r^2 \tag{12}$$

$$(\tilde{\omega}^2 - \kappa^2)(\tilde{\omega}^2 - n\Omega_\perp^2) = \tilde{\omega}^2 c_s^2 k_r^2 \tag{13}$$

In Keplerian disks $\kappa \sim \Omega \sim \Omega_K$ and $\Omega_\perp$

1 Low-Frequency Corrugation Waves:

$$\omega \sim -\frac{1}{2}\Omega\frac{\Omega^2}{\Omega^2 - \kappa^2}\frac{k_r^2 c_s^2}{\Omega^2} + \frac{1}{2}\Omega\frac{\Omega^2 - \Omega_\perp^2}{\Omega^2} \tag{14}$$

2 Trapped Oscillations in Relativistic Disks

2.1 Fundamental Mode

For the fundamental mode $n = 0$ the dispersion relation reduces to:

$$\omega^2 = \kappa^2 + c_s^2 k_r^2 \tag{15}$$

This is the same dispersion relation for the non-relativistic case but κ is different see `\ref{fig:X}`. Because $k_r > 0$ the dispersion relation implies $\omega^2 > \kappa^2$, The region in which this is satisfied is plotted in `\ref{fig:X2}`. The propagation regions are three which are explained as follows, In 1 the

3 Types of Oscillations

Oscillations with $m = 0$ that satisfy the dispersion relation 3 would lead to short wavelengths oscillations, which are not interesting, since they can't be observed. Therefore oscillations of interest would be either **global** or **trapped**. Global oscillations are low frequency and one armed $m = 1$. If $n = 0$ they are an eccentric deformation of the disk plane, while the $n = 1$ are a corrugation wave.

3.1 Eccentric deformations of the disk

The dispersion relation for $m = 1$ and $n = 0$ is:

$$(\omega - \Omega)^2 - \kappa^2 = k_r^2 c_s^2 \tag{16}$$

and it reduces to:

$$\omega \sim -\frac{1}{2}\Omega \left(\frac{k_r c_s}{\Omega}\right)^2 \tag{17}$$

The $m = 1$ perturbation is

3.2 tidally deformed disks

Hydrodynamics equations

The basic equations are those derived from Boltzmann equations:

The continuity equation is:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u_i) = 0 \tag{18}$$

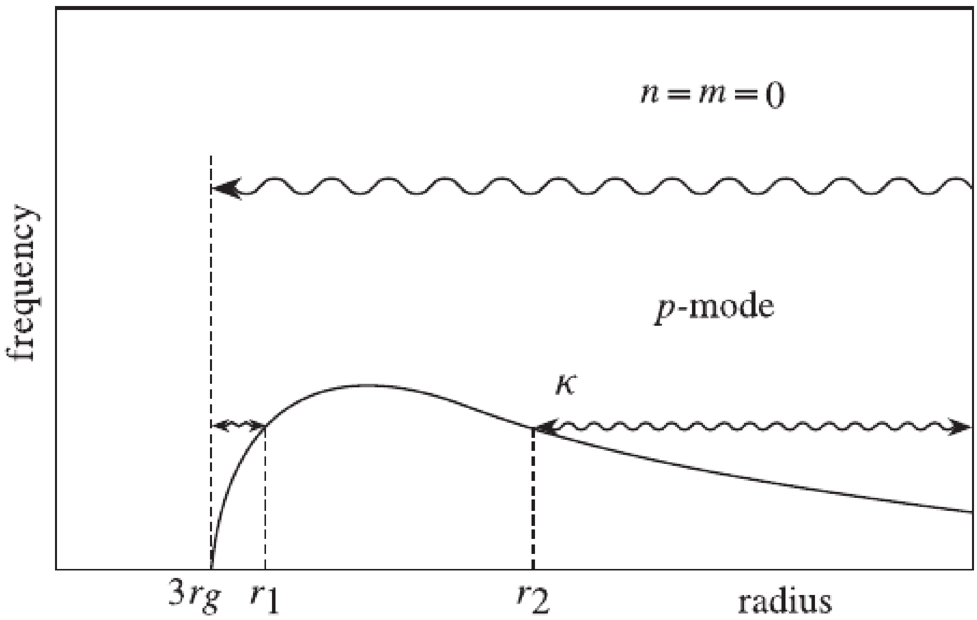
Which could be expressed in cylindrical coordinates as:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{r\partial r}(r\rho v_r) + \frac{\partial}{r\partial \phi}(\rho v_\phi) + \frac{\partial}{\partial z}(\rho v_z) = 0 \tag{19}$$

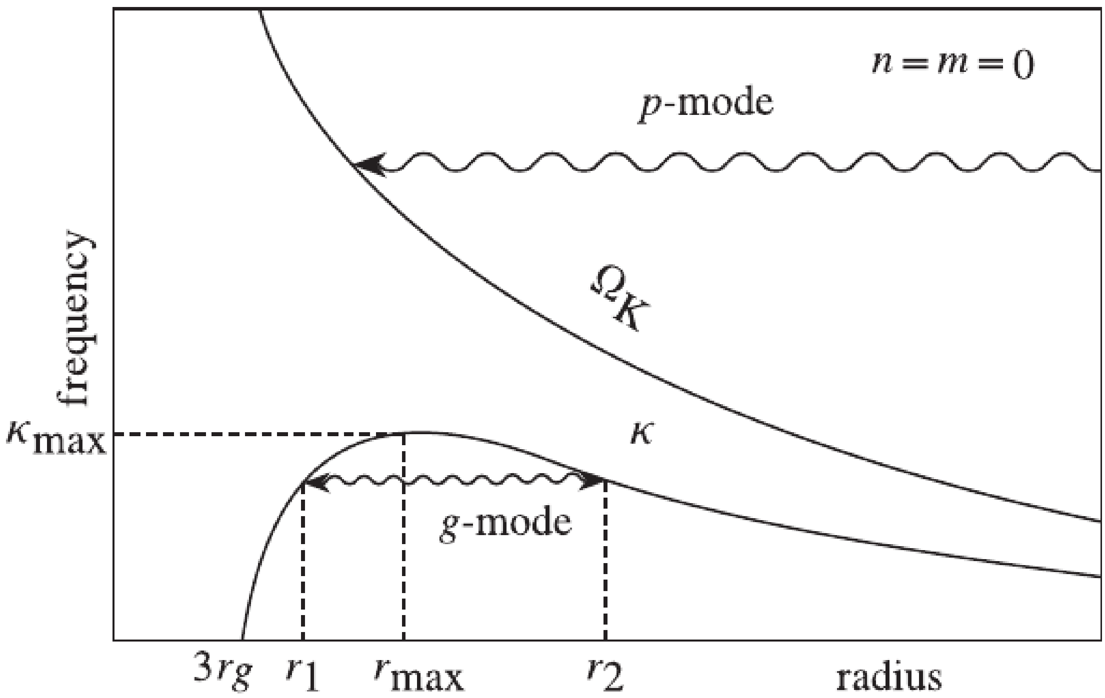
Where v_r, v_ϕ and v_z are the components of the velocity in cylindrical coordinates.

The momentum equation is:

$$\frac{\partial u_j}{\partial t} + u_j \frac{\partial u_i}{\partial x_i} = -a_j - \frac{1}{\rho}\frac{\partial \psi_{ij}}{\partial x_i} \tag{20}$$



$n = m = 0$
frequency
p-mode
κ
$3r_g$
r_1
r_2
radius



$n = m = 0$
p-mode
frequency
Ω_K
κ_{max}
κ
g-mode
$3r_g$
r_1
r_{max}
r_2
radius