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Relation of Kinetic Theory to Fluid Mechanics

Reference: Huang, *Statistical Mechanics*, Chapters 3 and 4.

In this chapter, we consider the microscopic basis for a continuum treatment of a collection of particles. The kinetic theory for electrically neutral gases forms the starting point of our discussion. We begin with a derivation of Boltzmann's equation. In what follows, we shall freely interchange between vector notation and Cartesian tensors (with Einstein's summation convention applied to repeated Latin indices). Chapter 3 of Jeffreys and Jeffreys, *Mathematical Physics*, provides a good primer for the technique.

BOLTZMANN'S EQUATION

To describe the motion of gas particles, we may heuristically divide the Hamiltonian H of a particle of mass m with phase-space coordinates $(\mathbf{x}, \mathbf{p})$ at time t into two parts:

$$H = H_{\text{sm}} + H_{\text{irr}}. \tag{2.1}$$

In H_{sm}, we include the effects of smoothly varying external forces, e.g., macroscopic gravitational fields whose properties we have, in principle, complete knowledge about and whose influence we can try to calculate by the conventional techniques of classical mechanics. In H_{irr}, we put the effects of molecular collisions mediated by irregular and unpredictable microscopic forces that we attempt to treat only by statistical means.

We define the single-particle distribution function $f(\mathbf{x}, \mathbf{p}, t)$ so that

$$f(\mathbf{x}, \mathbf{p}, t)\, d^3x\, d^3p$$

equals the number of particles within a configuration-space and velocity-space volume $d^3x\, d^3p$ centered about $(\mathbf{x}, \mathbf{p})$ at time t. Consider now the time rate of change of the number of particles within any small fixed volume V_6

of phase space with surface S_5. The flux of particles associated with the smooth part of the flow across a face of S_5 oriented in the x_i direction equals $f\dot{x}_i = f\partial H_{\text{sm}}/\partial p_i$; the flux across a face oriented in the p_i direction, $f\dot{p}_i = -f\partial H_{\text{sm}}/\partial x_i$. Violent encounters, however, can lead to sudden appearances and disappearances of particles from the interior of the momentum-space part of the volume V_6 without the particles having to cross the surface S_5 smoothly. In other words, on the mean-free-flight time scale t_c *between collisions* in a dilute gas, we assume that the *duration of a collision* t_d appears negligible because of the short-range nature of the intermolecular forces between neutral atoms and molecules. When $t_d \ll t_c$, we take the presence of molecular collisions into account as volume *source* or *sink* terms in the conservation equation for particles:

$$\int_{V_6} \left[\frac{\partial f}{\partial t} + \frac{\partial}{\partial \mathbf{x}} \cdot \left(f \frac{\partial H_{\text{sm}}}{\partial \mathbf{p}} \right) + \frac{\partial}{\partial \mathbf{p}} \cdot \left(-f \frac{\partial H_{\text{sm}}}{\partial \mathbf{x}} \right) \right] d^3x\, d^3p = \text{sources} - \text{sinks}. \tag{2.2}$$

In equation (2.2), we have used the divergence theorem to convert surface integrals over S_5 into volume integrals over V_6.

Consider, now, the removal of particles of momentum $\mathbf{p}$ by *elastic collisions* with particles of momentum $\mathbf{p}_2$, with the post-encounter pair possessing momenta $\mathbf{p}'$ and $\mathbf{p}'_2$:

$$(\mathbf{p}, \mathbf{p}_2) \rightarrow (\mathbf{p}', \mathbf{p}'_2).$$

Given $\mathbf{p}$ and $\mathbf{p}_2$, the conservation of momentum and energy (we specialize to the nonrelativistic regime),

$$\mathbf{p}' + \mathbf{p}'_2 = \mathbf{p} + \mathbf{p}_2, \tag{2.3}$$

$$\frac{1}{2m}[|\mathbf{p}'|^2 + |\mathbf{p}'_2|] = \frac{1}{2m}[|\mathbf{p}|^2 + |\mathbf{p}_2|^2], \tag{2.4}$$

serve to specify four of the six unknowns in $(\mathbf{p}', \mathbf{p}'_2)$. (The short-range character of the interaction forces allows us to assume that the collision occurs essentially at one value of $\mathbf{x}$; so we need not account for changes of the smooth part of the potential energy associated with the external field.) The other two unknowns in $(\mathbf{p}', \mathbf{p}'_2)$ are fixed (for featureless molecules) once we specify the impact parameter b and the azimuthal orientation ϕ of the collision (see Figure 2.1).

For an elastic collision, the magnitude of the relative velocity constitutes a collisional invariant:

$$|\mathbf{v}' - \mathbf{v}'_2| = |\mathbf{v} - \mathbf{v}_2|. \tag{2.5}$$

Thus we may specify the remaining two pieces of information concerning the collision in terms of the *change* in the orientation of the relative velocity, i.e., in terms of two angles θ and ϕ. In any case, if $b\,db\,d\phi = \sigma(\mathbf{p}, \mathbf{p}_2 | \mathbf{p}', \mathbf{p}'_2)\, d\Omega$
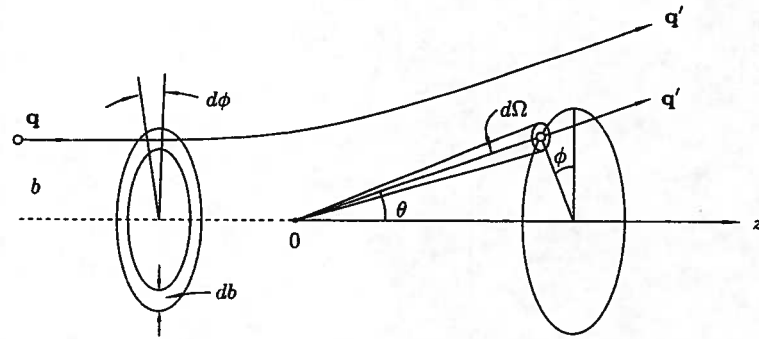
FIGURE 2.1
The geometry of a collision between two molecules of identical mass m and pre-encounter momenta $\mathbf{p} = m\mathbf{v}$ and $\mathbf{p}_2 = m\mathbf{v}_2$, when we reduce the problem to a (classical) description in terms of the relative motion before and after the collision: $\mathbf{q} = \mathbf{v} - \mathbf{v}_2$ and $\mathbf{q}' = \mathbf{v}' - \mathbf{v}_2'$. In an elastic encounter, the collision occurs in a single plane $\phi = $ constant, turning $\mathbf{q}$ through an angle θ into $\mathbf{q}'$ without changing the magnitude of the relative velocity, $|\mathbf{q}'| = |\mathbf{q}|$. For given intermolecular forces, the angle θ will depend on the impact parameter b. The differential cross section for the encounter $\sigma(\mathbf{p}, \mathbf{p}_2 | \mathbf{p}', \mathbf{p}_2')$ is defined so that $\sigma \, d\Omega = b \, db d\phi$, where $d\Omega \equiv \sin\theta \, d\theta d\phi$. A quantum-mechanical calculation yields the probability distributions, or $\sigma(\mathbf{p}, \mathbf{p}_2 | \mathbf{p}', \mathbf{p}_2') = \sigma(\mathbf{p}', \mathbf{p}_2', \mathbf{p}, \mathbf{p}_2)$, without relying on the notion of well-defined trajectories. Similarly, generalization to incorporate relativistic effects presents no difficulties of principle.

represents the differential collisional cross-section associated with scattering to within solid angle $d\Omega$ of (θ, ϕ), we may now write the collisional sink term as

$$\text{sink} = \int |\mathbf{v} - \mathbf{v}_2| f(\mathbf{p}_2) \, d^3 p_2 \, \sigma(\mathbf{p}, \mathbf{p}_2 | \mathbf{p}', \mathbf{p}_2') \, d\Omega f(\mathbf{p}) \, d^3 x d^3 p. \qquad (2.6)$$

In the above, we have suppressed the dependence of f on $\mathbf{x}$ and t for simplicity of notation. The integrations over $d^3 p_2$ and $d\Omega$ take place over all possible values, whereas the integration over $d^3 x d^3 p$ occurs over the small phase-space volume V_6.

By a completely analogous line of reasoning, we may write the source of particles scattered into V_6 centered on $(\mathbf{x}, \mathbf{p})$ due to all encounters of the type

$$(\mathbf{p}', \mathbf{p}_2') \to (\mathbf{p}, \mathbf{p}_2)$$

as

$$\text{source} = \int |\mathbf{v}' - \mathbf{v}_2'| f(\mathbf{p}_2') \, d^3 p_2' \sigma(\mathbf{p}', \mathbf{p}_2' | \mathbf{p}, \mathbf{p}_2) \, d\Omega f(\mathbf{p}') \, d^3 x d^3 p'. \qquad (2.7)$$

Time-reversibility of the process requires equality of the forward-going and backward-going cross sections:

$$\sigma(\mathbf{p}', \mathbf{p}_2' | \mathbf{p}, \mathbf{p}_2) = \sigma(\mathbf{p}, \mathbf{p}_2 | \mathbf{p}', \mathbf{p}_2') \equiv \sigma(\Omega), \qquad (2.8)$$

where $\sigma(\Omega)$ represents a shorthand notation for the angle-dependent elastic cross section. Moreover, it is possible to show that the Jacobian of the transformation from primed momenta to unprimed momenta equals unity:

$$d^3 p' d^3 p_2' = d^3 p d^3 p_2. \qquad (2.9)$$

If we substitute equations (2.5) and (2.9) into equation (2.7), we may now write

$$\text{source} - \text{sink} = \int_{V_6} \left(\frac{\delta f}{\delta t} \right)_c d^3 x d^3 p, \qquad (2.10)$$

where we have defined

$$\left(\frac{\delta f}{\delta t} \right)_c \equiv \int |\mathbf{v} - \mathbf{v}_2| \sigma(\Omega) [f(\mathbf{p}_2') f(\mathbf{p}') - f(\mathbf{p}_2) f(\mathbf{p})] \, d\Omega d^3 p_2, \qquad (2.11)$$

with the integrals with respect to $d\Omega$ and $d^3 p_2$ taken over all solid angles (for the collision outcome) and momenta (for the collisional partner 2).

In equations (2.2) and (2.10), the phase-space volume V_6 is completely arbitrary; consequently, we may set the integrands of the two sides equal to each other, thereby obtaining *Boltzmann's equation*:

$$\frac{\partial f}{\partial t} + \frac{\partial H_{sm}}{\partial \mathbf{p}} \cdot \frac{\partial f}{\partial \mathbf{x}} - \frac{\partial H_{sm}}{\partial \mathbf{x}} \cdot \frac{\partial f}{\partial \mathbf{p}} = \left(\frac{\delta f}{\delta t} \right)_c, \qquad (2.12)$$

where the right-hand side is given by equation (2.11). Our derivation of Boltzmann's equation here has followed the standard (heuristic) route; a more rigorous derivation from first principles is possible by the so-called BBGKY hierarchy (see Uhlenbeck and Ford, *Statistical Mechanics*, for an exposition). We forego such a discussion here, apart from mentioning that such a treatment essentially uses the ratio of the time during a collision and the time between collisions, t_d/t_c, as a small expansion parameter to truncate the hierarchy of equations that result from integrating over successive coordinates of the N-particle distribution function. The latter describes the state of the system in $6N$-dimensional phase space. (When dealing with *long-range* forces, such as electrostatic interactions in a plasma, we need a different expansion procedure—see Montgomery and Tidman, *Plasma Kinetic Theory*, for a derivation of the so-called *Lenard-Balescu equation*.)

COLLISIONAL INVARIANTS

Boltzmann's equation *predicts* the irreversible increase of the entropy of a thermally isolated gas:

$$S \equiv -k \int f \ln f \, d^3x \, d^3p, \qquad (2.13)$$

where k is Boltzmann's constant and where the integration is taken over all of the six-dimensional phase space available to a gas particle (see Problem Set 1). Apart from this result, it is difficult to extract very general statements about the time evolution of f in six-dimensional phase space by a direct attack on equation (2.12). A more fruitful approach considers a *moment equation* approach.

We may motivate such an approach as follows. The kinetic description contains more information than we often really need. We do not usually care to know the details of the distribution in momentum space (or, as we shall see more accurately in Chapter 3, its small departures from a Maxwellian). As a more modest goal, we may be content with knowledge of the behavior, in space x and time t, of the first few velocity moments of the distribution function. Of particular interest are the moments associated with the mass, momentum, and kinetic energy about the mean motion:

$$\begin{pmatrix} \rho \\ \rho \mathbf{u} \\ \rho \mathcal{E} \end{pmatrix} \equiv \int \begin{pmatrix} m \\ m\mathbf{v} \\ m|\mathbf{v} - \mathbf{u}|^2/2 \end{pmatrix} f(\mathbf{x}, \mathbf{v}, t) \, d^3v. \qquad (2.14)$$

In equations (2.14) we have modified conventions slightly so as to let f denote the distribution function in *velocity* space rather than in momentum space. Moreover, we suppose that the function f satisfies Boltzmann's equation in the form:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} - \frac{\partial \mathcal{V}}{\partial \mathbf{x}} \cdot \frac{\partial f}{\partial \mathbf{v}} = \left(\frac{\delta f}{\delta t}\right)_c, \qquad (2.15)$$

where $(\delta f/\delta t)_c$ is given by

$$\left(\frac{\delta f}{\delta t}\right)_c = \int |\mathbf{v} - \mathbf{v}_2| \sigma(\Omega) [f(\mathbf{v}')f(\mathbf{v}_2') - f(\mathbf{v})f(\mathbf{v}_2)] \, d\Omega \, d^3v_2. \qquad (2.16)$$

We have also specialized the consideration of smooth accelerations to those derivable from the negative gradient of a gravitational potential $-\nabla\mathcal{V}$. In $\mathcal{V}$, we allow the possibility of self-gravitation; thus the potential $\mathcal{V}$ satisfies Poisson's equation:

$$\nabla^2 \mathcal{V} = 4\pi G(\rho + \rho_{ext}), \qquad (2.17)$$

with G being the universal gravitational constant, and with ρ_{ext} being the mass density associated with a distribution of matter (if any) external to

our system. In equations (2.14), ρ, $\mathbf{u}$, and $\mathcal{E}$ represent the mass density, the mean (or bulk or fluid) velocity, and specific internal energy (internal energy per unit mass) associated with the gas at $(\mathbf{x}, t)$. In terms of the number density n that we get by integrating f over all $\mathbf{v}$, $\rho = mn$.

We wish to derive equations that govern the spacetime evolution of ρ, $\mathbf{u}$, and $\mathcal{E}$. To do this, consider multiplying equation (2.15) by $\chi(\mathbf{v})$, where $\chi(\mathbf{v})$ represents any constant or power of $\mathbf{v}$, e.g., $\chi(\mathbf{v}) = m$, $m\mathbf{v}$, $m\mathbf{v}\mathbf{v}$, etc. Integrate the result over *all* $\mathbf{v}$ to obtain

$$\int \left(\chi \frac{\partial f}{\partial t} + \chi v_k \frac{\partial f}{\partial x_k} - \chi \frac{\partial \mathcal{V}}{\partial x_k} \frac{\partial f}{\partial v_k}\right) d^3v = \int \chi \left(\frac{\delta f}{\delta t}\right)_c d^3v. \qquad (2.18)$$

If we substitute equation (2.16) into the right-hand side of equation (2.18), we get an integrand in which $\mathbf{v}$ and $\mathbf{v}_2$ are dummy variables, with the two entering symmetrically everywhere except in $\chi(\mathbf{v})$. The integral therefore will have the same value if we make the substitution,

$$\chi(\mathbf{v}) \rightarrow \frac{1}{2}[\chi(\mathbf{v}) + \chi(\mathbf{v}_2)]. \qquad (2.19)$$

Equations (2.5), (2.8), and (2.9) allow us to exchange the roles of the integrations over $d^3v' d^3v_2'$ and $d^3v d^3v_2$. If we subsequently rename primed variables as unprimed variables, and vice versa, we would recover the original expression, except that the term $[f(\mathbf{v}')f(\mathbf{v}_2') - f(\mathbf{v})f(\mathbf{v}_2)]$ changes sign and the right-hand side of equation (2.19) acquires primes on its arguments. In other words, the integral on the right-hand side of equation (2.18) can be evaluated with the substitution,

$$\chi(\mathbf{v}) \rightarrow \frac{1}{4}[\chi(\mathbf{v}) + \chi(\mathbf{v}_2) - \chi(\mathbf{v}') - \chi(\mathbf{v}_2')]. \qquad (2.20)$$

If χ is a conserved quantity in a collision, the right-hand side of equation (2.20) equals zero. For elastic collisions involving short-range forces in the nonrelativistic regime, there exist exactly five such independent χ's in general; they are the mass, momentum, and (kinetic) energy of a particle:

$$\chi = m; \qquad \chi = mv_i; \qquad \text{and } \chi = \frac{m}{2}|\mathbf{v}|^2. \qquad (2.21)$$

If we restrict our attention to such values of χ [or to linear combinations of them, such as $m|\mathbf{v} - \mathbf{u}|^2/2 = m|\mathbf{v}|^2/2 - \mathbf{u} \cdot m\mathbf{v} + m|\mathbf{u}|^2/2$, where $\mathbf{u}(\mathbf{x}, t)$ may be regarded as a constant in any integration over $\mathbf{v}$], we have

$$\int \chi \left(\frac{\delta f}{\delta t}\right)_c d^3v = 0.$$

The above result expresses mathematically the simple notion that collisions can not contribute to the time rate of change of any quantity whose total is conserved in the collisional process.

MOMENT EQUATIONS FOR CONSERVED QUANTITIES

Define the average of any quantity Q by the symbol

$$\langle Q \rangle \equiv n^{-1} \int Q f \, d^3v,$$

where $n \equiv \int f \, d^3v$. Since derivatives with respect to t and $\mathbf{x}$ commute with operations in $\mathbf{v}$, equation (2.18) now yields, if χ represents one of the conserved quantities m, mv_i, or $m|\mathbf{v}|^2/2$:

$$\frac{\partial}{\partial t}(n\langle\chi\rangle) + \frac{\partial}{\partial x_k}(n\langle v_k \chi\rangle) + n\frac{\partial \mathcal{V}}{\partial x_k}\left\langle\frac{\partial \chi}{\partial v_k}\right\rangle = 0. \qquad (2.22)$$

To derive the last term of the above equation, we have used the divergence theorem to convert a volume integral (in $\mathbf{v}$) to a surface integral, and assumed that the distribution function f vanishes faster than any power of $\mathbf{v}$ as $\mathbf{v} \to \infty$ on the latter surface.

MASS CONSERVATION

Substitution of $\chi = m$ into equation (2.22) yields the equation of mass conservation,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_k}(\rho u_k) = 0, \qquad (2.23)$$

also known as the *continuity equation*. This equation implies more than simple mass conservation, for it states that changes in the local matter content due to fluid flow occur in a *continuous* fashion. When mass disappears from any volume element, it does so by flowing in a well-defined manner across the surface of the volume; it does not do so by suddenly reappearing in some completely disconnected region.

Written in vector notation, the equation of continuity reads

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \qquad (2.24)$$

which has the usual expression that we associate with a conservation relation (see Chapter 1 of Volume I). Another form in which the continuity equation often appears can be derived by differentiating out the second term, dividing by ρ, and moving the term not involving derivatives of ρ to the right-hand side:

$$\rho^{-1}\frac{D\rho}{Dt} = -\nabla \cdot \mathbf{u},$$

where D/Dt is the time derivative that follows the motion of a *fluid element* of the substance (not equivalent to the motion of a gas particle):

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla, \qquad (2.25)$$

and is known as the *substantial* (or *Lagrangian*) derivative. Since the reciprocal of ρ equals the volume per unit mass, we may also write the desired relation as

$$\rho\frac{D}{Dt}(\rho^{-1}) = \nabla \cdot \mathbf{u},$$

which demonstrates that the divergence of the velocity field yields the fractional time rate of change of the specific volume ρ^{-1}.

MOMENTUM CONSERVATION

Substitution of $\chi = mv_i$ into equation (2.22) yields the equation of momentum conservation:

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_k}(\rho\langle v_i v_k\rangle) + \rho\frac{\partial \mathcal{V}}{\partial x_i} = 0.$$

We decompose the particle velocity v_i as the sum $u_i + w_i$, and write

$$\langle v_i v_k\rangle = u_i u_k + \langle w_i w_k\rangle,$$

where we have made use of the identity $\langle w_i\rangle = 0$ that follows trivially from the definition of u_i as the average of v_i. Moreover, we follow the usual procedure of separating out the trace of the symmetric dyadic $w_i w_k$; i.e., we write

$$\rho\langle w_i w_k\rangle = P\delta_{ik} - \pi_{ik},$$

where P is the "gas pressure,"

$$P \equiv \frac{1}{3}\rho\langle|\mathbf{w}|^2\rangle, \qquad (2.26)$$

and π_{ik} is the "viscous stress tensor,"

$$\pi_{ik} \equiv \rho\langle\frac{1}{3}|\mathbf{w}|^2\delta_{ik} - w_i w_k\rangle. \qquad (2.27)$$

The *momentum equation*, in its conservation form, now reads

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_k}(\rho u_i u_k + P\delta_{ik} - \pi_{ik}) = -\rho\frac{\partial \mathcal{V}}{\partial x_i}. \qquad (2.28)$$

Notice that the i-th component of the momentum density associated with fluid equals ρu_i. The flux of the i-th component of momentum in the k-th

direction consists of the sum of a mean part, $\rho u_i u_k$, and a random part, with the latter composed of an isotropic component $P\delta_{ik}$ and a nonisotropic (traceless) component $-\pi_{ik}$.

By making use of the equation of continuity, we may also manipulate the momentum equation so that it becomes the *force equation*:

$$\rho \frac{D\mathbf{u}}{Dt} = -\rho\nabla\mathcal{V} - \nabla P + \nabla \cdot \overset{\leftrightarrow}{\pi} . \qquad (2.29)$$

In this form, the equation most closely resembles Newton's second law of dynamics, $m\mathbf{a} = \mathbf{f}$.

ENERGY CONSERVATION

Substitution of $\chi = m|\mathbf{v}|^2/2 = m|\mathbf{u}|^2/2 + m\mathbf{w} \cdot \mathbf{u} + m|\mathbf{w}|^2/2$ into equation (2.22) yields the equation of energy conservation:

$$\frac{\partial}{\partial t}\left[\frac{\rho}{2}(|\mathbf{u}|^2 + \langle|\mathbf{w}|^2\rangle)\right] + \frac{\partial}{\partial x_k}\left[\frac{\rho}{2}\langle(u_k + w_k)(u_i + w_i)^2\rangle\right] + \rho\frac{\partial\mathcal{V}}{\partial x_k}u_k = 0.$$

Expanding the term inside the spatial divergence, we get

$$\langle(u_k + w_k)(u_i + w_i)^2\rangle = |\mathbf{u}|^2 u_k + 2u_i\langle w_i w_k\rangle + u_k\langle|\mathbf{w}|^2\rangle + \langle w_k|\mathbf{w}|^2\rangle.$$

In accordance with equation (2.14), we define the *specific internal energy* $\mathcal{E}$ through

$$\rho\mathcal{E} \equiv \rho\langle\frac{1}{2}|\mathbf{w}|^2\rangle = \frac{3}{2}P, \qquad (2.30)$$

where the second relation follows from equation (2.26). We also define the *conduction heat flux* by

$$F_k \equiv \rho\langle w_k\frac{1}{2}|\mathbf{w}|^2\rangle. \qquad (2.31)$$

With these definitions, we may write the *total energy equation* in its conservation form as

$$\frac{\partial}{\partial t}\left(\frac{\rho}{2}|\mathbf{u}|^2 + \rho\mathcal{E}\right) + \frac{\partial}{\partial x_k}\left[\frac{\rho}{2}|\mathbf{u}|^2 u_k + u_i(P\delta_{ik} - \pi_{ik}) + \rho\mathcal{E}u_k + F_k\right]$$
$$= -\rho u_k\frac{\partial\mathcal{V}}{\partial x_k}. \qquad (2.32)$$

In equation (2.32) we have used the manipulations of the previous section to rewrite $\rho\langle w_i w_k\rangle$ as $P\delta_{ik} - \pi_{ik}$.

Equation (2.32) states that the total fluid energy density is the sum of a part due to bulk motion $\rho|\mathbf{u}|^2$ and a part due to random motions $\rho\mathcal{E}$. The flux of fluid energy in the k-th direction consists of the translation

of the bulk kinetic energy at the k-th component of the mean velocity, $(\rho|\mathbf{u}|^2/2)u_k$, plus the enthalpy (sum of internal energy and pressure) flux $(\rho\mathcal{E} + P)u_k$, plus a viscous contribution, $-u_i\pi_{ik}$, plus the conductive flux, F_k.

We find it convenient for some purposes to express energy conservation in a form that involves only the internal energy plus the PdV work. To do this, first multiply equation (2.28) by u_i and use equation (2.23) to derive the *work equation*:

$$\frac{\partial}{\partial t}\left(\frac{\rho}{2}|\mathbf{u}|^2\right) + \frac{\partial}{\partial x_k}\left(\frac{\rho}{2}|\mathbf{u}|^2 u_k\right) = -\rho u_i\frac{\partial\mathcal{V}}{\partial x_i} - u_i\frac{\partial P}{\partial x_i} + u_i\frac{\partial\pi_{ik}}{\partial x_k}. \qquad (2.33)$$

The subtraction of equation (2.33) from equation (2.32) results in the *internal energy equation*:

$$\frac{\partial}{\partial t}(\rho\mathcal{E}) + \frac{\partial}{\partial x_k}(\rho\mathcal{E}u_k) = -P\frac{\partial u_k}{\partial x_k} - \frac{\partial F_k}{\partial x_k} + \Psi, \qquad (2.34)$$

where Ψ is the *rate of viscous dissipation*:

$$\Psi \equiv \pi_{ik}\frac{\partial u_i}{\partial x_k}. \qquad (2.35)$$

If we use the equation of continuity, we may also write the internal energy equation in the form of the first law of thermodynamics:

$$\rho\frac{D\mathcal{E}}{Dt} = -P\nabla \cdot \mathbf{u} - \nabla \cdot \mathbf{F}_{\text{cond}} + \Psi. \qquad (2.36)$$

We recognize $-P\nabla \cdot \mathbf{u} = -P[\rho D(\rho^{-1})/Dt]$ as the rate of doing PdV work, and $-\nabla \cdot \mathbf{F}_{\text{cond}} + \Psi$ as the time rate of adding heat (through heat conduction and the viscous conversion of ordered energy in differential fluid motions to disordered energy in random particle motions).

THE NEED FOR A CLOSURE RELATION

If we include one each of the equivalent forms of the equations for mass conservation (scalar relation), momentum conservation (vector relation), and energy conservation (scalar relation), we have five linearly independent equations. On the other hand, the same equations have thirteen variables, ρ (1), u_i (3), $P = 2\rho\mathcal{E}/3$ (1), π_{ik} (5 for a symmetric traceless tensor), and F_i (3). Hence we have thirteen variables, but only five equations, a common theme of the moment method before we use any physical arguments to derive *closure relations* (see the comments of Chapter 2 in Volume I). Until we introduce a way to obtain a closed set of moment equations, everything that we have done so far concerning the velocity moments, although

mathematically exact, has no real physical content. This includes our introduction of suggestive names for various quantities, such as viscous stress tensor, conductive heat flux, etc.

When can we find useful closure conditions? If we follow the example of radiative transfer (Volume I), we might guess: at the extremes. And indeed, at the extreme when the mean free path for collisions is much smaller than macroscopic length scale, $\ell \ll L$ (the material analog of the optically thick regime), we expect the concept of *local thermodynamic equilibrium* (LTE) for the translational degrees of freedom to hold, so that, as we shall see in Chapter 3, the *Chapman-Enskog* procedure allows us to derive useful closure relations. To zeroth order in a systematic expansion for small ℓ/L, we shall find that the eight needed relations take the form $\pi_{ik} \approx 0$ and $F_i \approx 0$. The neglect of diffusive effects in this manner leads to a complete set of fluid relations called the *Euler equations*. To next order, the diffusive terms π_{ik} and F_i are not zero, and we get the so-called *Navier-Stokes equations*.

What about the other extreme, $\ell \gg L$? For radiative transfer, the analogous situation corresponds to optically thin conditions. Under these circumstances, photons travel basically in straight lines unimpeded by the presence of matter, and simplifications become possible for the problem of radiative transfer. With material bodies, however, when the mean free path becomes much larger than the macroscopic length scale (say, the size of the system), we can not assume that particles will travel in straight lines, because they will usually be subject to large-scale macroscopic forces that bend the trajectories into curved orbits. Consequently, the dynamics of material bodies remains quite complicated in the regime $\ell \gg L$ (the state of affairs, e.g., in stellar systems), although the kinetic treatment required will generally be simpler than that which applies in the really awkward regime $\ell \sim L$ (the state of affairs, e.g., in planetary rings).