Equation (2-91) [and also equations (2-90)] is referred to as the *diffusion approximation*, primarily because of its formal similarity to other diffusion equations, which are of the form

$$(\text{flux}) = (\text{diffusion coefficient}) \times (\text{gradient of relevant physical variable})$$

e.g., $\Phi = -\kappa \nabla T$ for heat conduction. The coefficient $\frac{1}{3}\chi_\nu^{-1}(\partial B_\nu/\partial T)$ is, in fact, sometimes called the *radiative conductivity*, a designation that is quite appropriate in view of the fact that $\chi_\nu^{-1} = l_\nu$ is the photon mean-free-path. Note that equation (2-91) exhibits the essential physical content of our earlier result that the flux computed by application of the Φ-operator to a linear source function depends only on the gradient of S [see discussion following equation (2-65)]. Also it shows that *the mere fact that energy emerges from the star implies that the temperature must increase inward.* Indeed, replacing H with $(L/4\pi R^2)$ and (dB/dz) with $(\sigma_R T_c^4/\pi R)$, and taking $\langle\chi\rangle \approx 1$, it is easy to show that the central temperature T_c of the sun must be of the order of 6×10^6 °K, a result consistent with our earlier statement that the ultimate energy source in a star is thermonuclear energy-release at the center.

In an intuitive picture of diffusion, one usually conceives of a slow leakage from a reservoir of large capacity by means of a seeping action. These ideas apply in the radiative diffusion limit as well. The diffusion approximation becomes valid at great optical depth (i.e., many photon mean-free-paths from the surface) whence many individual photon flights, with successive absorptions and emissions, are required before the photon finally trickles to the surface, and issues forth into interstellar space.

If we integrate equations (2-90a–c) over all frequencies, we obtain $I(\tau, \mu) \approx B(\tau) + 3\mu H$. The ratio of the anisotropic to isotropic terms gives a measure of the "drift" in the radiation flow; this ratio is

$$\frac{\text{Anisotropic term}}{\text{Isotropic term}} \sim \frac{3H}{B} = \frac{3}{4}\left(\frac{\sigma_R T_{\text{eff}}^4}{\pi}\right) \Big/ \left(\frac{\sigma_R T^4}{\pi}\right) \sim \left(\frac{T_{\text{eff}}}{T}\right)^4 \qquad (2\text{-}92)$$

Clearly at great depth, where T becomes $\gg T_{\text{eff}}$, the "leak" becomes ever smaller. The same result is found by a physical argument from a slightly different point of view. If πF is the energy flux carried from an element of material by photons of velocity c, the rate of *energy flow* per unit volume is $(\pi F/c)$; the *energy content* per unit volume is $(4\pi J/c) \approx (4\pi B/c)$, so that (Rate of energy flow)/(Energy content) $= (F/4B) = \frac{1}{4}(T_{\text{eff}}/T)^4$. Again, we see that diffusion, in the intuitive sense described above, occurs at great depth where $T \gg T_{\text{eff}}$, while *free flow* of radiant energy occurs at the surface where $T \approx T_{\text{eff}}$.

3

The Grey Atmosphere

The grey atmosphere problem provides an excellent introduction to the study of radiative transfer in stellar atmospheres. The nature of the defining assumptions is such that the problem becomes independent of the physical state of the material, and requires the solution of a relatively simple transfer equation. At the same time, the grey problem demonstrates how the constraint of radiative equilibrium can be satisfied, and the solution can be related to more general and more realistic physical situations. Furthermore, an exact solution of the problem can be obtained, and this provides a comparison standard against which we can evaluate the worth of various approximate numerical methods that can be applied in more complex cases.

3-1 Statement of the Problem

The problem is posed by making the simplifying assumption that the opacity of the material is independent of frequency; i.e., $\chi_\nu \equiv \chi$. This assumption is of course unrealistic in many cases. Yet as we shall see in later chapters, the opacity in some stars (e.g., the sun) is not too far from being grey and, in addition, it is possible partially to reduce the nongrey problem to the grey case by suitable choices of *mean opacities.* Thus the solution also provides a valuable starting approximation in the analysis of nongrey atmospheres.

If we assume $\chi_\nu \equiv \chi$, then the standard planar transfer equation (2-36) becomes

$$\mu(\partial I_\nu/\partial\tau) = I_\nu - S_\nu \tag{3-1}$$

Then by integrating over frequency, and writing

$$I \equiv \int_0^\infty I_\nu \, d\nu \tag{3-2}$$

and similarly for J, S, B, etc., we have

$$\mu(\partial I/\partial\tau) = I - S \tag{3-3}$$

If we impose the constraint of radiative equilibrium [equation 2-83b)], we require

$$\int_0^\infty \chi J_\nu \, d\nu = \int_0^\infty \chi S_\nu \, d\nu \tag{3-4}$$

which, for grey material, reduces to $J \equiv S$. Thus equation (3-3) becomes

$$\mu(\partial I/\partial\tau) = I - J \tag{3-5}$$

which has the formal solution [equation (2-57)]

$$J(\tau) = \Lambda_\tau[S(t)] = \Lambda_\tau[J(t)] = \frac{1}{2}\int_0^\infty J(t)E_1|t - \tau| \, dt \tag{3-6}$$

Equation (3-6) is a linear integral equation for J known as *Milne's equation*; the grey problem itself is sometimes called *Milne's problem.* It is important to recognize that, when a solution of equation (3-6) is obtained, it satisfies *simultaneously* the transfer equation and the constraint of radiative equilibrium. The determination of such solutions in the nongrey case will occupy most of Chapter 7.

If we now introduce the additional hypothesis of LTE, then $S_\nu \equiv B_\nu(T)$—which, from the condition of radiative equilibrium, implies that

$$J(\tau) = S(\tau) = B[T(\tau)] = (\sigma_R T^4)/\pi \tag{3-7}$$

Thus, if we are given $J(\tau)$, the solution of the integral equation (3-6), then the additional premise of LTE allows us to associate a temperature with the radiative equilibrium radiation field via equation (3-7).

Several important results may be obtained from moments of equation (3-5). Taking the zero-order moment and imposing radiative equilibrium we have

$$(dH/d\tau) = J - S = J - J \equiv 0 \tag{3-8}$$

which implies *the flux is constant*, while the first moment gives

$$(dK/d\tau) = H \tag{3-9}$$

which, because H is constant, yields the *exact integral*

$$K(\tau) = H\tau + c = \frac{1}{4}F\tau + c \tag{3-10}$$

To make further progress, we must relate $J(\tau)$ to $K(\tau)$. This is easily done on the basis of the discussion in §2-5, where we showed that at great depth the specific intensity is quite accurately represented by $I(\mu) = I_0 + I_1\mu$, which produces a nonzero flux and also implies that, for $\tau \gg 1$, $K(\tau) = \frac{1}{3}J(\tau)$. Thus the fact that $K(\tau) \to \frac{1}{4}F\tau$ for $\tau \gg 1$ implies that at great depth

$$J(\tau) \to \frac{3}{4}F\tau \quad (\tau \gg 1) \tag{3-11}$$

That is, asymptotically the mean intensity varies linearly with optical depth. On general grounds we expect the behavior of $J(\tau)$ to depart most from linearity at the surface [note equation (2-63)], which suggests that a reasonable general expression for $J(\tau)$ is

$$J(\tau) = \frac{3}{4}F[\tau + q(\tau)] = \frac{3}{4}(\sigma_R T_{\text{eff}}^4/\pi)[\tau + q(\tau)] \tag{3-12}$$

The function $q(\tau)$, known as the *Hopf function*, remains to be determined; from equation (3-6) it is clear that $q(\tau)$ is a solution of the equation

$$\tau + q(\tau) = \frac{1}{2}\int_0^\infty [t + q(t)]E_1|t - \tau| \, dt \tag{3-13}$$

Finally, we notice that because

$$\lim_{\tau\to\infty}\left[\frac{1}{3}J(\tau) - K(\tau)\right] = \frac{1}{4}F \lim_{\tau\to\infty}[\tau + q(\tau) - \tau - c] \equiv 0 \tag{3-14}$$

we have $c = q(\infty)$ and hence can write equation (3-10) as

$$K(\tau) = \frac{1}{4}F[\tau + q(\infty)] \tag{3-15}$$

The solution of the grey problem consists of the specification of $q(\tau)$. Given $q(\tau)$, the temperature distribution is obtained by combining equations (3-7) and (3-12) into the relation

$$T^4 = \frac{3}{4}T_{\text{eff}}^4[\tau + q(\tau)] \tag{3-16}$$

We shall derive approximate expressions for $q(\tau)$ in §3-3 and describe the exact solution in §3-4. First, however, it is useful to delineate the nature, and extent, of the correspondence between the grey and nongrey problems.

3-2 Relation to the Nongrey Problem: Mean Opacities

The opacity in real stellar atmospheres usually exhibits strong frequency variations, at least when spectral lines are present. Although numerical methods now exist that allow a refined solution of nongrey transfer equations and an accurate determination of the temperature structure in a nongrey atmosphere, the calculations are, at best, laborious, and it is important to ask whether a significant connection exists between the grey and nongrey cases. We shall show in this section that such a connection, though limited in scope, does exist, and that, among other things, it permits the temperature distribution of the deepest atmospheric layers to be determined quite accurately from the grey solution.

Let us first compare side-by-side the grey and nongrey transfer equations. Starting with the transfer equation and calculating the zero and first-order moments we have, in the nongrey and grey cases respectively:

$$\mu(\partial I_\nu/\partial z) = \chi_\nu(S_\nu - I_\nu) \quad (3\text{-}17a) \qquad \mu(\partial I/\partial z) = \chi(J - I) \quad (3\text{-}17b)$$

$$(\partial H_\nu/\partial z) = \chi_\nu(S_\nu - J_\nu) \quad (3\text{-}18a) \qquad (dH/dz) = 0 \quad (3\text{-}18b)$$

$$(\partial K_\nu/\partial z) = -\chi_\nu H_\nu \quad (3\text{-}19a) \qquad (dK/dz) = -\chi H \quad (3\text{-}19b)$$

Here variables without frequency subscripts denote *integrated* quantities, as in equation (3-2). We now ask whether it is possible to define a mean opacity $\bar{\chi}$, formed as a weighted average of the monochromatic opacity, in such a way that the monochromatic transfer equation, or one of its moments, when integrated over frequency, has exactly the same form as the grey equation. Several possible definitions have been suggested.

FLUX-WEIGHTED MEAN

Suppose we wish to define a mean opacity in such a way as to guarantee an exact correspondence between the integrated form of equation (3-19a) and the grey equation (3-19b). If such a mean can be constructed, then the relation $K(\bar{\tau}) = H\bar{\tau} + c$ will again be an exact integral, as it was in the grey case. Integrating equation (3-19a) over all frequencies we have

$$-(dK/dz) = \int_0^\infty \chi_\nu H_\nu \, d\nu = \bar{\chi}_F H \tag{3-20}$$

where the second equality produces the desired identification with equation (3-19b) if we define

$$\bar{\chi}_F \equiv H^{-1} \int_0^\infty \chi_\nu H_\nu \, d\nu \tag{3-21}$$

The opacity $\bar{\chi}_F$ is called the *flux-weighted* mean. Note that this choice does *not* reduce the nongrey problem completely to the grey problem, for the monochromatic equation (3-18a) does not transform into equation (3-18b) with this choice of $\bar{\chi}$. Further, there is the practical problem that H_ν is not known a priori, and therefore $\bar{\chi}_F$ cannot actually be calculated until after the transfer equation is solved. This latter difficulty can be overcome by an iteration between construction of models and calculation of $\bar{\chi}_F$. Although the desired goal has not been fully attained, the fact that the flux-weighted mean preserves the K-integral is important, for it implies that the correct value is recovered for the *radiation pressure* [cf. equation (1-41)]. It also follows that the correct value of the *radiation force*, which is the gradient of the radiation pressure, is likewise obtained. Thus from equation (2-77) we have

$$(dp_R/d\bar{\tau}) = -\bar{\chi}_F^{-1}(dp_R/dz) = (4\pi/c\bar{\chi}_F) \int_0^\infty \chi_\nu H_\nu \, d\nu = (4\pi H/c) = (\sigma T_{\text{eff}}^4/c) \tag{3-22}$$

so that use of the flux-mean opacity produces a simple expression for the radiation pressure gradient. This is a result of practical value in the computation of model atmospheres for early-type stars, because in these objects radiation forces strongly affect the pressure (or density) structure of the atmosphere via the condition of hydrostatic equilibrium (or momentum balance in steady flow).

ROSSELAND MEAN

Alternatively, suppose we wish to obtain the correct value for the integrated energy flux. From equations (3-19) it follows that this may be done if $\bar{\chi}$ is chosen such that

$$-\int_0^\infty \chi_\nu^{-1}(\partial K_\nu/\partial z) \, d\nu = \int_0^\infty H_\nu \, d\nu = H \equiv -\bar{\chi}^{-1}(dK/dz) \tag{3-23}$$

or, equivalently,

$$\bar{\chi}^{-1} \equiv \int_0^\infty \chi_\nu^{-1}(\partial K_\nu/\partial z) \, d\nu \Big/ \int_0^\infty (\partial K_\nu/\partial z) \, d\nu \tag{3-24}$$

Again we face the practical difficulty that K_ν is not known a priori, and hence the indicated calculation cannot be performed until the transfer equation is solved. But the mean defined in equation (3-24) can be approximated in the following way: at great depth in the atmosphere, $K_\nu \to \frac{1}{3}J_\nu$ while $J_\nu \to B_\nu$. Thus may write $(\partial K_\nu/\partial z) \approx \frac{1}{3}(\partial B_\nu/\partial T)(dT/dz)$. We then define the mean opacity $\bar{\chi}_R$ as

$$\frac{1}{\bar{\chi}_R} \equiv \frac{\frac{1}{3}\left(\frac{dT}{dz}\right)\int_0^\infty \left(\frac{1}{\chi_\nu}\right)\frac{\partial B_\nu}{\partial T}\,d\nu}{\frac{1}{3}\left(\frac{dT}{dz}\right)\int_0^\infty \frac{\partial B_\nu}{\partial T}\,d\nu} = \frac{\int_0^\infty \frac{1}{\chi_\nu}\frac{\partial B_\nu}{\partial T}\,d\nu}{\int_0^\infty \frac{\partial B_\nu}{\partial T}\,d\nu} \tag{3-25}$$

or

$$\bar{\chi}_R^{-1} \equiv (\pi/4\sigma_R T^3)\int_0^\infty \chi_\nu^{-1}(\partial B_\nu/\partial T)\,d\nu \tag{3-26}$$

The opacity $\bar{\chi}_R$ is called the *Rosseland mean* in honor of its originator. Note that the harmonic nature of the averaging process gives highest weight to those regions where the opacity is lowest, and, where as a result, the greatest amount of radiation is transported—a very desirable feature. Again the use of $\bar{\chi}_R$ or the mean defined by equation (3-24) does not permit a correspondence between equations (3-18a) and (3-18b), and hence does not allow the nongrey problem to be replaced by the grey problem. On the other hand, it is obvious that the approximations made to obtain equation (3-26) are precisely those introduced in the derivation of the diffusion approximation to the transfer equation (2-91); i.e.,

$$H_\nu = -\frac{1}{3}\left(\frac{1}{\chi_\nu}\frac{\partial B_\nu}{\partial T}\right)\left(\frac{dT}{dz}\right)$$

Hence use of $\bar{\chi}_R$ is consistent with the diffusion approximation. Therefore on the Rosseland-mean optical-depth scale $\bar{\tau}_R$ we must recover the correct asymptotic solution of the transfer equation and the correct flux transport at great depth. This implies that at great depth ($\bar{\tau}_R \gg 1$) the temperature structure is quite accurately given by the relation $T^4 = \frac{3}{4}T_{\rm eff}^4[\bar{\tau}_R + q(\bar{\tau}_R)]$; see equation (3-16). It is therefore clear why Rosseland mean opacities are employed in studies of stellar interiors. Note also that so long as the diffusion approximation is valid, a simple expression can be written for the radiation force, namely

$$(dp_R/d\bar{\tau}_R) = (16\pi\sigma_R T^3/3c\bar{\chi}_R)(-dT/dz) \tag{3-27}$$

Exercise 3-1: Derive equation (3-27).

Although the diffusion approximation is nearly exact at great depth, and provides the very useful results just discussed, it *must* of course break down at the surface, and exact flux conservation is *not* guaranteed in the upper layers by use of the Rosseland mean, nor will it give the temperature structure or the radiation force correctly in the outermost regions of the atmosphere. This point must be recognized clearly, for it is precisely these layers in which spectrum-formation occurs, and hence which are of primary interest in the analysis of stellar spectra.

PLANCK AND ABSORPTION MEANS

Several other expressions for mean opacities may be chosen. For example, if we demand that the mean be defined to yield the correct integrated value of the thermal emission, then we require

$$\bar{\kappa}_P \equiv \left[\int_0^\infty \kappa_\nu B_\nu(T)\,d\nu\right]\Big/B(T) = \pi\int_0^\infty \kappa_\nu B_\nu(T)\,d\nu\Big/\sigma_R T^4 \tag{3-28}$$

Note that only the true absorption is used, and scattering is omitted. The opacity $\bar{\kappa}_P$ is known as the *Planck mean*; it has the advantage of being calculable without having to solve the transfer equation. On the other hand, $\bar{\kappa}_P$ does not permit a reduction of equation (3-18a) to (3-18b) nor of (3-19a) to (3-19b), and therefore it lacks the desirable features possessed by $\bar{\chi}_F$ or $\bar{\chi}_R$. Nevertheless this mean does have additional significance.

In particular, near the surface of the star, the physical content of the condition of radiative equilibrium is expressed most directly by equation (3-4). In view of this constraint, a correspondence between equations (3-18a) and (3-18b) can be made near the surface if $\bar{\kappa}$ satisfies the relation

$$\int_0^\infty (\kappa_\nu - \bar{\kappa})(J_\nu - B_\nu)\,d\nu = 0 \tag{3-29}$$

Once the material becomes optically thin (i.e., $\tau_\nu < 1$ at all frequencies), J_ν becomes essentially fixed, and the integral above will be dominated by those frequencies where $\kappa_\nu \gg \bar{\kappa}$. If we represent B_ν by a linear expansion on a $\bar{\tau}$-scale, i.e.,

$$B_\nu(t) = B_\nu(\bar{\tau}) + (dB_\nu/d\bar{\tau})(t - \bar{\tau}) \approx B_\nu(\bar{\tau}) + (dB_\nu/d\bar{\tau})(\bar{\kappa}/\kappa_\nu)(t_\nu - \tau_\nu),$$

then by application of the Λ-operator we find [cf. equations (2-57), (2-58), and (2-63)],

$$J_\nu(\bar{\tau}) - B_\nu(\bar{\tau}) \approx -\frac{1}{2}B_\nu(\bar{\tau})E_2(\tau_\nu) + (\bar{\kappa}/\kappa_\nu)(dB_\nu/d\bar{\tau})\left[\frac{1}{2}E_3(\tau_\nu) + \frac{1}{2}\tau_\nu E_2(\tau_\nu)\right] \tag{3-30}$$

In the limit $\tau \to 0$, $E_2(\tau) \to 1$, and $E_3(\tau) \to \frac{1}{2}$, so the first term yields $-\frac{1}{2}B_\nu(\bar{\tau})$ while the second becomes *least* important when $\kappa_\nu \gg \bar{\kappa}$ [precisely the region of highest weight in equation (3-29)]. Thus to satisfy equation (3-29), $\bar{\kappa}$

should essentially fulfill the requirement

$$\int_0^\infty \kappa_\nu B_\nu \, d\nu = \bar{\kappa} \int_0^\infty B_\nu \, d\nu \tag{3-31}$$

which shows that the Planck mean is the choice most nearly consistent with the requirement of radiative equilibrium near the surface.

Alternatively, we might demand that the mean opacity yield the correct total for the amount of energy absorbed. We then obtain the *absorption mean*

$$\bar{\kappa}_J \equiv \int_0^\infty \kappa_\nu J_\nu \, d\nu \bigg/ J \tag{3-32}$$

Again only the true absorption is included, and scattering is omitted. As was true for $\bar{\chi}_F$, we cannot calculate $\bar{\kappa}_J$ until a solution of the transfer equation has been obtained. Further, $\bar{\kappa}_J$ does not permit a strict correspondence between the grey and nongrey forms of the transfer equation or any of its moments (as was also true for the Planck mean).

SUMMARY

We have seen that no one of the mean opacities described above allows, in itself, a complete correspondence of the nongrey problem to the grey problem. Yet mean opacities provide a useful first estimate of the temperature structure in a stellar atmosphere if we assume, as a starting approximation, $T(\bar{\tau}_R) = T_{\text{grey}}(\bar{\tau}_R)$, and then improve this estimate with a *correction procedure* that is designed to enforce radiative equilibrium for the nongrey radiation field. Indeed, the mean opacities $\bar{\chi}_F$, $\bar{\kappa}_P$, and $\bar{\kappa}_J$ appear explicitly in some temperature-correction procedures.

From an historical point of view, it should be recognized that, before the advent of high-speed computers, the nongrey atmosphere problem required far too much calculation to permit a direct attack, and the use of $\bar{\chi}_R$ and $\bar{\kappa}_P$ provided a practical method of approaching an otherwise intractable problem. In fact, the answers obtained in this way often do not compare too unfavorably with more recent results despite the apparent crudeness of the approximation. Only some of the more basic properties of mean opacities have been mentioned here; further information may be found in (419) and (361, §§34–35).

3-3 Approximate Solutions

THE EDDINGTON APPROXIMATION

In §2-5 it was shown that, at great depth in a stellar atmosphere, the relation $J \equiv 3K$ holds; further, in §1-4 (cf. Exercise 1-13), it was demonstrated

that this relation is also valid for a variety of other situations, including the two-stream approximation, which provides a rough representation of the radiation field near the boundary. In view of these results, Eddington made the simplifying assumption that $J \equiv 3K$ *everywhere* in the atmosphere. Then the exact integral $K = \frac{1}{4}F\tau + c$ implies that in the Eddington approximation $J_E(\tau) = \frac{3}{4}F\tau + c'$. To evaluate the constant c' we calculate the emergent flux and fit it to the desired value. Thus from equation (2-59) we have

$$F(0) = 2\int_0^\infty \left(\frac{3}{4}F\tau + c'\right) E_2(\tau)\, d\tau = 2c'E_3(0) + \frac{3}{4}F\left[\frac{4}{3} - 2E_4(0)\right] \tag{3-33}$$

so that, using the relation $E_n(0) = 1/(n-1)$ and demanding $F(0) \equiv F$, we find $c' = \frac{1}{2}F$. Thus

$$J_E(\tau) = \frac{3}{4}F\left(\tau + \frac{2}{3}\right) \tag{3-34}$$

In Eddington's approximation $q(\tau) \equiv \frac{2}{3}$. Imposing the constraint of radiative equilibrium and the assumption of LTE we have from equation (3-16)

$$T^4 \approx \frac{3}{4}T_{\text{eff}}^4\left(\tau + \frac{2}{3}\right) \tag{3-35}$$

Equation (3-35) predicts that the ratio of the boundary temperature to the effective temperature is $(T_0/T_{\text{eff}}) = (\frac{1}{2})^{1/4} = 0.841$, which agrees fairly well with the *exact* value $(T_0/T_{\text{eff}}) = (3^{1/2}/4)^{1/4} = 0.8114$. Assuming $S(\tau) = J_E(\tau)$ we may calculate the angular dependence of the emergent radiation field in the Eddington approximation by substituting equation (3-34) into equation (2-52) to obtain

$$I_E(0,\mu) = \frac{3}{4}F\int_0^\infty \left(\tau + \frac{2}{3}\right)\mu^{-1}\exp(-\tau/\mu)\, d\tau = \frac{3}{4}F\left(\mu + \frac{2}{3}\right) \tag{3-36}$$

which yields a very specific form for the Eddington–Barbier relation [cf. equation (2-53)]. The *center* of a star's disk, as seen by an external observer, corresponds to $\theta = 0°$, or $\mu = 1$; the *limb* is at $\theta = 90°$, $\mu = 0$. The ratio $I(0,\mu)/I(0,1)$, which gives the intensity at angle $\theta = \cos^{-1}\mu$ relative to disk-center, is referred to as the *limb-darkening law*. In the Eddington approximation the limb-darkening is

$$I_E(0,\mu)/I_E(0,1) = \frac{3}{5}\left(\mu + \frac{2}{3}\right) \tag{3-37}$$

This result predicts the limb intensity to be 40 percent of the central intensity. Observations of the sun in the visual regions of the spectrum are actually in good agreement with this value and, in fact, it was precisely this agreement that led K. Schwarzschild (**416**, 25) to propose the validity of radiative equilibrium in the outermost layers of the solar photosphere.

Equation (3-35) predicts that $T = T_{\text{eff}}$ when $\tau = \frac{2}{3}$. This result has given rise to the useful conceptual notion that the "effective depth" of continuum formation is $\tau \approx \frac{2}{3}$; in fact, this is often a rather good estimate. In support of this idea we may note that a photon emitted outward from $\tau = \frac{2}{3}$ has a chance of the order of $e^{-0.67} \approx 0.5$ of emerging from the surface; this corresponds in a reasonable way with the place we would intuitively identify with the region of continuum formation.

Exercise 3-2: The Eddington–Barbier relation shows that the intensity $I(0, \mu)$ is characteristic of $S(\tau)$ at $\tau(\mu) \approx \mu$. Show then that the average depth that characterizes the *flux* is $\langle\tau\rangle = \frac{2}{3}$.

Anticipating the exact solution given in Table 3-2, we can evaluate the accuracy of $J_E(\tau)$; one finds that the worst error occurs at the surface, where $\Delta J/J = (J_E - J_{\text{exact}})/J_{\text{exact}} = 0.155$. Both the size of the error and the fact that it occurs at $\tau = 0$ are unsurprising when we recognize that the basic assumption upon which the derivation is based, namely $J \equiv 3K$, is known to be inaccurate at the surface. We know that $J(\tau)$ must satisfy the integral equation (3-6), and we know further that the Λ-operator has its largest effect at $\tau = 0$ [see equation (2-63) and associated discussion]. This suggests that an improved estimate of $J(\tau)$ can be obtained from

$$J_E^{(2)}(\tau) = \Lambda_\tau[J_E(t)] = \Lambda_\tau\left[\frac{3}{4}F\left(t + \frac{2}{3}\right)\right] = \frac{3}{4}F\left[\tau + \frac{2}{3} - \frac{1}{3}E_2(\tau) + \frac{1}{2}E_3(\tau)\right] \tag{3-38}$$

Recalling the properties of $E_n(\tau)$, it is clear that the largest difference between $J_E^{(2)}(\tau)$ and $J_E(\tau)$ occurs at the surface, where we find $J_E^{(2)}(0)/J_E(0) = \frac{7}{8}$. The new estimate of T_0/T_{eff} is thus $(\frac{7}{16})^{1/4} = 0.813$ (exact value is 0.8114) and $q(0)$ drops from $\frac{2}{3}$ to $\frac{7}{12} = 0.583$ (exact value is $1/\sqrt{3} = 0.577$).

It is thus clear that an application of the Λ-operator has produced a dramatic improvement in the solution near the surface. Note, however, that there is *no* improvement in the solution at $\tau \to \infty$, where q remains at its original value $\frac{2}{3}$. In *principle*, successive applications of the Λ-operator should improve the solution, and, eventually, produce the exact solution. In fact, one can show that $\lim_{n\to\infty} \Lambda^n(1) = 0$ [see (**684**, 31)] so that an initial error ε at any depth can ultimately be reduced to zero by repeated application of the Λ-operator. In practice, however, the convergence is too slow to be of value, for the effective range of the Λ-operator is of order $\Delta\tau = 1$, so errors at large depth are removed only "infinitely slowly." (We shall encounter this problem with Λ-iteration repeatedly in a wide variety of contexts! See, e.g.,

§6-1, §7-2.) Further, even a second application of the Λ-operator to $J_E^{(2)}(\tau)$ introduces the functions $\Lambda_\tau[E_n(t)]$, which are cumbersome to compute [see (**361**), equations (14-50), (14-53), and (37-36) through (37-44)]. Therefore alternative methods for obtaining a solution must be developed.

Exercise 3-3: Using the results of Table 3-2, evaluate the percentage errors of $J_E(\tau)$ and $J_E^{(2)}(\tau)$ and display them in a plot. The required values of $E_n(\tau)$ can be found in (**4**, 245).

Exercise 3-4: Show that, although $J_E(\tau)$ was derived assuming F = constant, the flux computed from $J_E(\tau)$ via equation (2-59) is not constant; make a plot of the error $\Delta F(\tau)/F$.

Exercise 3-5: Apply the X-operator [cf. equations (2-62) and (2-65)] to $J_E(\tau)$ to obtain $K_E^{(2)}(\tau) = \frac{3}{16}F[\frac{4}{3}\tau + \frac{8}{9} - \frac{4}{3}E_4(\tau) + 2E_5(\tau)]$. Use this result to write an analytical expression for the variable Eddington factor $f(\tau) \equiv K(\tau)/J(\tau)$. Show that $f(\tau = \infty) = \frac{1}{3}$ and $f(\tau = 0) = \frac{17}{42} = 0.405$. Using the results of Table 3-2, evaluate the fractional error in $f(\tau)$ [recall equation (3-15)] and plot it.

Exercise 3-6: Show that the improved estimate of the emergent intensity obtained by using $J_E^{(2)}(\tau)$ is $I_E^{(2)}(0, \mu) = \frac{3}{4}F\{\frac{7}{12} + \frac{1}{2}\mu + (\frac{1}{3}\mu + \frac{1}{2}\mu^2)\ln[(1 + \mu)/\mu]\}$. Compare this result and $I_E(0, \mu)$ given by equation (3-36) with the exact result shown in Table 3-1, and plot a graph of their fractional errors.

ITERATION: THE UNSÖLD PROCEDURE

The primary shortcoming of the Λ-iteration procedure is its failure to yield an improvement in the solution at great depth. Unsöld (**638**, 141) proposed an ingenious alternative method that overcomes this inadequacy and can be generalized to the nongrey case. The basic idea is to start from an initial estimate for the source function $B(\tau)$, and to derive a perturbation equation for a change $\Delta B(\tau)$ that more nearly satisfies the requirement of radiative equilibrium.

If we calculate the flux from the initial guess $B(\tau)$, we will find that it is a function of depth, $H(\tau)$, and not exactly constant unless $B(\tau)$ happened to be the exact solution of the problem. From the first-order moment equation (3-9) we then have

$$K(\tau) = \int_0^\tau H(\tau')\,d\tau' + C \tag{3-39}$$

If we make the Eddington approximation $J(\tau) = 3K(\tau)$ and evaluate C by writing $J(0) = 2H(0)$ [cf. equation (3-34)] we obtain

$$J(\tau) \approx 3\int_0^\tau H(\tau')\,d\tau' + 2H(0) \tag{3-40}$$

But from the zero-order moment equation (3-8) we have

$$B(\tau) = J(\tau) - [dH(\tau)/d\tau] \tag{3-41}$$