Chapter 4

Conservation Laws

1. *Introduction*

All fluid mechanics is based on the conservation laws for mass, momentum, and energy. These laws can be stated in the *differential* form, applicable at a point. They can also be stated in the *integral* form, applicable to an extended region. In the integral form, the expressions of the laws depend on whether they relate to a *volume fixed in space*, or to a *material volume*, which consists of the same fluid particles and whose bounding surface moves with the fluid. Both types of volumes will be considered in this chapter; *a fixed region will be denoted by V and a material volume will be denoted by* $\mathcal{V}$. In engineering literature a fixed region is called a *control volume*, whose surfaces are called *control surfaces*.

The integral and differential forms can be derived from each other. As we shall see, during the derivation surface integrals frequently need to be converted to volume integrals (or vice versa) by means of the divergence theorem of Gauss

$$\int_V \frac{\partial F}{\partial x_i} dV = \int_A dA_i F, \tag{4.1}$$

where $F(\mathbf{x}, t)$ is a tensor of *any* rank (including vectors and scalars), V is either a fixed volume or a material volume, and A is its boundary surface. Gauss' theorem was presented in Section 2.13.

2. *Time Derivatives of Volume Integrals*

In deriving the conservation laws, one frequently faces the problem of finding the time derivative of integrals such as

$$\frac{d}{dt} \int_{V(t)} F\, dV,$$

where $F(\mathbf{x}, t)$ is a tensor of any order, and $V(t)$ is any region, which may be fixed or move with the fluid. The d/dt sign (in contrast to $\partial/\partial t$) has been written because only a function of time remains after performing the integration in space. The different possibilities are discussed in what follows.

General Case

Consider the general case in which $V(t)$ is neither a fixed volume nor a material volume. The surfaces of the volume are moving, but not with the local fluid velocity. The rule for differentiating an integral becomes clear at once if we consider a one-dimensional (1D) analogy. In books on calculus,

$$\frac{d}{dt} \int_{x=a(t)}^{b(t)} F(x, t)\, dx = \int_a^b \frac{\partial F}{\partial t} dx + \frac{db}{dt} F(b, t) - \frac{da}{dt} F(a, t). \tag{4.2}$$

This is called the *Leibniz theorem*, and shows how to differentiate an integral whose integrand F as well as the limits of integration are functions of the variable with respect to which we are differentiating. A graphical illustration of the three terms on the right-hand side of the Leibniz theorem is shown in Figure 4.1. The continuous line shows the integral $\int F\, dx$ at time t, and the dashed line shows the integral at time $t + dt$. The first term on the right-hand side in Eq. (4.2) is the integral of $\partial F/\partial t$ over the region, the second term is due to the gain of F at the outer boundary moving at a rate db/dt, and the third term is due to the loss of F at the inner boundary moving at da/dt.

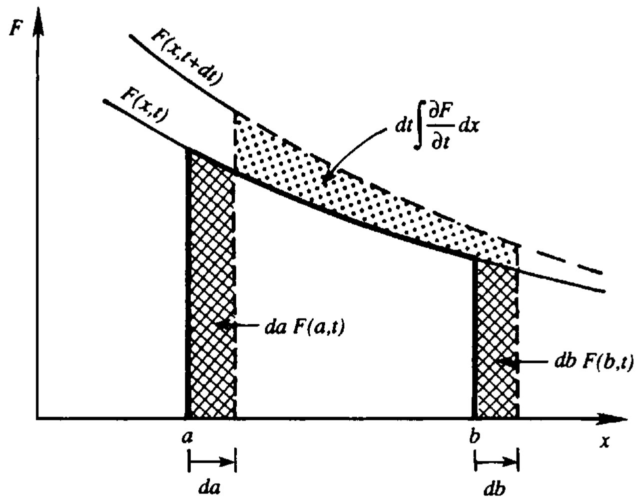


Figure 4.1 Graphical illustration of Leibniz's theorem.

Generalizing the Leibniz theorem, we write

$$\frac{d}{dt}\int_{V(t)} F(\mathbf{x},t)\,dV = \int_{V(t)} \frac{\partial F}{\partial t} dV + \int_{A(t)} \mathbf{dA}\cdot\mathbf{u}_A F, \tag{4.3}$$

where $\mathbf{u}_A$ is the velocity of the boundary and $A(t)$ is the surface of $V(t)$. The surface integral in Eq. (4.3) accounts for both "inlets" and "outlets," so that separate terms as in Eq. (4.2) are not necessary.

Fixed Volume

For a fixed volume we have $\mathbf{u}_A = 0$, for which Eq. (4.3) becomes

$$\boxed{\frac{d}{dt}\int_V F(\mathbf{x},t)\,dV = \int_V \frac{\partial F}{\partial t}\,dV,} \tag{4.4}$$

which shows that the time derivative can be simply taken inside the integral sign if the boundary is fixed. This merely reflects the fact that the "limit of integration" V is not a function of time in this case.

Material Volume

For a material volume $\mathcal{V}(t)$ the surfaces move with the fluid, so that $\mathbf{u}_A = \mathbf{u}$, where $\mathbf{u}$ is the fluid velocity. Then Eq. (4.3) becomes

$$\boxed{\frac{D}{Dt}\int_{\mathcal{V}} F(\mathbf{x},t)\,d\mathcal{V} = \int_{\mathcal{V}} \frac{\partial F}{\partial t}\,d\mathcal{V} + \int_A \mathbf{dA}\cdot\mathbf{u}F.} \tag{4.5}$$

This is sometimes called the *Reynolds transport theorem*. Although not necessary, we have used the D/Dt symbol here to emphasize that we are following a material volume.

Another form of the transport theorem is derived by using the mass conservation relation Eq. (3.32) derived in the last chapter. Using Gauss' theorem, the transport theorem Eq. (4.5) becomes

$$\frac{D}{Dt}\int_{\mathcal{V}} F\, d\mathcal{V} = \int_{\mathcal{V}} \left[\frac{\partial F}{\partial t} + \frac{\partial}{\partial x_j}(F u_j)\right] d\mathcal{V}.$$

Now define a new function f such that $F \equiv \rho f$, where ρ is the fluid density. Then the preceding becomes

$$\frac{D}{Dt}\int \rho f\, d\mathcal{V} = \int \left[\frac{\partial(\rho f)}{\partial t} + \frac{\partial}{\partial x_j}(\rho f u_j)\right] d\mathcal{V}$$
$$= \int \left[\rho\frac{\partial f}{\partial t} + f\frac{\partial \rho}{\partial t} + f\frac{\partial}{\partial x_j}(\rho u_j) + \rho u_j \frac{\partial f}{\partial x_j}\right] d\mathcal{V}.$$

Using the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0.$$

we finally obtain

$$\boxed{\frac{D}{Dt}\int_{\mathcal{V}} \rho f\, d\mathcal{V} = \int_{\mathcal{V}} \rho\,\frac{Df}{Dt}\, d\mathcal{V}.} \tag{4.6}$$

Notice that the D/Dt operates only on f on the right-hand side, although ρ is variable. Applications of this rule can be found in Sections 7 and 14.

3. Conservation of Mass

The differential form of the law of conservation of mass was derived in Chapter 3, Section 13 from a consideration of the volumetric rate of strain of a particle. In this chapter we shall adopt an alternative approach. We shall first state the principle in an integral form for a fixed region and then deduce the differential form. Consider a volume fixed in space (Figure 4.2). The rate of increase of mass inside it is the volume integral

$$\frac{d}{dt}\int_V \rho\, dV = \int_V \frac{\partial \rho}{\partial t}\, dV.$$

The time derivative has been taken inside the integral on the right-hand side because the volume is fixed and Eq. (4.4) applies. Now the rate of mass flow out of the volume is the surface integral

$$\int_A \rho \mathbf{u} \cdot d\mathbf{A},$$

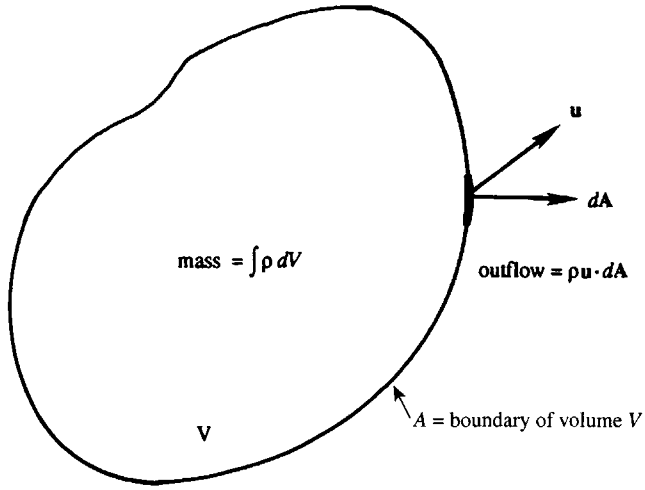


Figure 4.2 Mass conservation of a volume fixed in space.

because $\rho \mathbf{u} \cdot d\mathbf{A}$ is the outward flux through an area element $d\mathbf{A}$. (Throughout the book, we shall write $d\mathbf{A}$ for $\mathbf{n}\, dA$, where $\mathbf{n}$ is the unit outward normal to the surface. Vector $d\mathbf{A}$ therefore has a magnitude dA and a direction along the outward normal.) The law of conservation of mass states that the rate of increase of mass within a fixed volume must equal the rate of inflow through the boundaries. Therefore,

$$\int_V \frac{\partial \rho}{\partial t}\, dV = -\int_A \rho \mathbf{u} \cdot d\mathbf{A}, \tag{4.7}$$

which is the integral form of the law for a volume fixed in space.

The differential form can be obtained by transforming the surface integral on the right-hand side of Eq. (4.7) to a volume integral by means of the divergence theorem, which gives

$$\int_A \rho \mathbf{u} \cdot d\mathbf{A} = \int_V \nabla \cdot (\rho \mathbf{u})\, dV.$$

Equation (4.7) then becomes

$$\int_V \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) \right] dV = 0.$$

The forementioned relation holds for *any* volume, which can be possible only if the integrand vanishes at every point. (If the integrand did not vanish at every point, then we could choose a small volume around that point and obtain a nonzero integral.) This requires

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0,} \tag{4.8}$$

which is called the *continuity equation* and expresses the differential form of the principle of conservation of mass.

The equation can be written in several other forms. Rewriting the divergence term in Eq. (4.8) as

$$\frac{\partial}{\partial x_i}(\rho u_i) = \rho \frac{\partial u_i}{\partial x_i} + u_i \frac{\partial \rho}{\partial x_i},$$

the equation of continuity becomes

$$\boxed{\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla \cdot \mathbf{u} = 0.} \tag{4.9}$$

The derivative $D\rho/Dt$ is the rate of change of density following a fluid particle; it can be nonzero because of changes in pressure, temperature, or composition (such as salinity in sea water). A fluid is usually called *incompressible* if its density does not change with *pressure*. Liquids are almost incompressible. Although gases are compressible, for speeds $\lesssim 100\,\mathrm{m/s}$ (that is, for Mach numbers <0.3) the fractional change of absolute pressure in the flow is small. In this and several other cases the density changes in the flow are also small. The neglect of $\rho^{-1}D\rho/Dt$ in the continuity equation is part of a series of simplifications grouped under the Boussinesq approximation, discussed in Section 18. In such a case the continuity equation (4.9) reduces to the incompressible form

$$\boxed{\nabla \cdot \mathbf{u} = 0,} \tag{4.10}$$

whether or not the flow is steady.

4. Streamfunctions: Revisited and Generalized

Consider the steady-state form of mass conservation from Eq. (4.8),

$$\nabla \cdot (\rho \mathbf{u}) = 0. \tag{4.11}$$

In Exercise 10 of Chapter 2 we showed that the divergence of the curl of any vector field is identically zero. Thus we can represent the mass flow vector as the curl of a vector potential

$$\rho \mathbf{u} = \nabla \times \mathbf{\Omega}, \tag{4.12}$$

where we can write $\mathbf{\Omega} = \chi \nabla \psi + \nabla \phi$ in terms of three scalar functions. We are concerned with the mass flux field $\rho \mathbf{u} = \nabla \chi \times \nabla \psi$ because the curl of any gradient is identically zero (Chapter 2, Exercise 11). The gradients of the surfaces $\chi =$ const. and $\psi =$ const. are in the directions of the surface normals. Thus the cross product is perpendicular to both normals and must lie simultaneously in both surfaces $\chi =$ const. and $\psi =$ const. Thus streamlines are the intersections of the two surfaces, called streamsurfaces or streamfunctions in a three-dimensional (3D) flow. Consider an edge view of two members of each of the families of the two streamfunctions $\chi = a$,

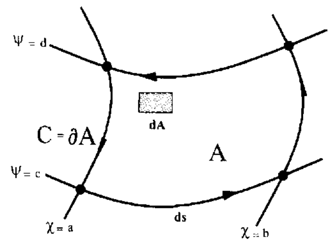


Figure 4.3 Edge view of two members of each of two families of streamfunctions. Contour C is the boundary of surface area A : $C = \partial A$.

$\chi = b$, $\psi = c$, $\psi = d$. The intersections shown as darkened dots in Figure 4.3 are the streamlines coming out of the paper. We calculate the mass per time through a surface A bounded by the four streamfunctions with element **dA** having **n** out of the paper. By Stokes' theorem,

$$\dot{m} = \int_A \rho \mathbf{u} \cdot \mathbf{dA} = \int_A (\nabla \times \mathbf{\Omega}) \cdot \mathbf{dA} = \int_C \mathbf{\Omega} \cdot \mathbf{ds} = \int_C (\chi \nabla \psi + \nabla \phi) \cdot \mathbf{ds}$$
$$= \int_C (\chi d\psi + d\phi) = \int_C \chi d\psi = b(d-c) + a(c-d) = (b-a)(d-c).$$

Here we have used the vector identity $\nabla \phi \cdot \mathbf{ds} = d\phi$ and recognized that integration around a closed path of a single-valued function results in zero. The mass per time through a surface bounded by adjacent members of the two families of streamfunctions is just the product of the differences of the numerical values of the respective streamfunctions. As a very simple special case, consider flow in a z = constant plane (described by x and y coordinates). Because all the streamlines lie in z = constant planes, z is a streamfunction. Define $\chi = -z$, where the sign is chosen to obey the usual convention. Then $\nabla \chi = -\mathbf{k}$ (unit vector in the z direction), and

$$\rho \mathbf{u} = -\mathbf{k} \times \nabla \psi; \quad \rho u = \partial \psi / \partial y, \quad \rho v = -\partial \psi / \partial x,$$

in conformity with Chapter 3, Exercise 14.

Similarly, in cyclindrical polar coordinates as shown in Figure 3.1, flows, symmetric with respect to rotation about the x-axis, that is, those for which $\partial/\partial\phi = 0$, have streamlines in ϕ = constant planes (through the x-axis). For those axisymmetric flows, $\chi = -\phi$ is one streamfunction:

$$\rho \mathbf{u} = -\frac{1}{R} \mathbf{i}_\phi \times \nabla \psi,$$

then gives $\rho R u_x = \partial \psi / \partial R$, $\rho R u_R = -\partial \psi / \partial x$. We note here that if the density may be taken as a constant, mass conservation reduces to $\nabla \cdot \mathbf{u} = 0$ (steady or not) and the entire preceding discussion follows for **u** rather than $\rho \mathbf{u}$ with the interpretation of streamfunction in terms of volumetric rather than mass flux.

5. *Origin of Forces in Fluid*

Before we can proceed further with the conservation laws, it is necessary to classify the various types of forces on a fluid mass. The forces acting on a fluid element can be divided conveniently into three classes, namely, body forces, surface forces, and line forces. These are described as follows:

(1) *Body forces*: Body forces are those that arise from "action at a distance," without physical contact. They result from the medium being placed in a certain *force field*, which can be gravitational, magnetic, electrostatic, or electromagnetic in origin. They are distributed throughout the mass of the fluid and are proportional to the mass. Body forces are expressed either per unit mass or per unit volume. In this book, the body force per unit mass will be denoted by **g**.

Body forces can be conservative or nonconservative. *Conservative body forces* are those that can be expressed as the gradient of a potential function:

$$\mathbf{g} = -\nabla\Pi, \tag{4.13}$$

where Π is called the *force potential*. All forces directed *centrally* from a source are conservative. Gravity, electrostatic and magnetic forces are conservative. For example, the gravity force can be written as the gradient of the potential function

$$\Pi = gz,$$

where g is the acceleration due to gravity and z points vertically upward. To verify this, Eq. (4.13) gives

$$\mathbf{g} = -\nabla(gz) = -\left[\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right](gz) = -\mathbf{k}g,$$

which is the gravity force per unit mass. The negative sign in front of $\mathbf{k}g$ ensures that **g** is downward, along the negative z direction. The expression $\Pi = gz$ also shows that the *force potential equals the potential energy per unit mass*. Forces satisfying Eq. (4.13) are called "conservative" because the resulting motion conserves the sum of kinetic and potential energies, if there are no dissipative processes.

(2) *Surface forces*: Surface forces are those that are exerted on an area element by the surroundings through direct contact. They are proportional to the extent of the area and are conveniently expressed per unit of area. Surface forces can be resolved into components normal and tangential to the area. Consider an element of area dA in a fluid (Figure 4.4). The force $d\mathbf{F}$ on the element can be resolved into a component dF_n normal to the area and a component dF_s tangential to the area. The normal and shear stress on the element are defined, respectively as,

$$\tau_n \equiv \frac{dF_n}{dA} \qquad \tau_s \equiv \frac{dF_s}{dA}.$$

These are scalar definitions of stress components. Note that the component of force tangential to the surface is a two-dimensional (2D) vector in the surface. The state of stress at a point is, in fact, specified by a stress tensor, which has nine components. This was explained in Section 2.4 and is again discussed in the following section.

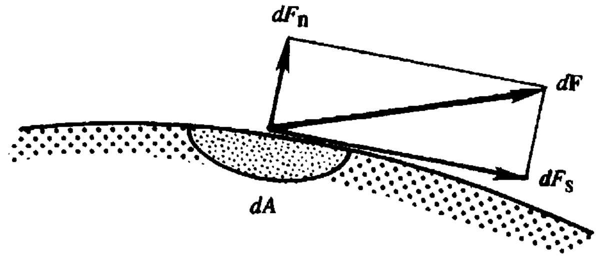


Figure 4.4 Normal and shear forces on an area element.

(3) *Line forces*: Surface tension forces are called *line forces* because they act along a line (Figure 1.4) and have a magnitude proportional to the extent of the line. They appear at the interface between a liquid and a gas, or at the interface between two immiscible liquids. Surface tension forces do not appear directly in the equations of motion, but *enter only in the boundary conditions.*

6. Stress at a Point

It was explained in Chapter 2, Section 4 that the stress at a point can be completely specified by the nine components of the stress tensor $\boldsymbol{\tau}$. Consider an infinitesimal rectangular parallelepiped with faces perpendicular to the coordinate axes (Figure 4.5). On each face there is a normal stress and a shear stress, which can be further resolved into two components in the directions of the axes. The figure shows the directions of *positive* stresses on four of the six faces; those on the remaining two faces are omitted for clarity. The first index of τ_{ij} indicates the direction of the normal to the surface on which the stress is considered, and the second index indicates the direction in which the stress acts. The diagonal elements τ_{11}, τ_{22}, and τ_{33} of the stress matrix are the normal stresses, and the off-diagonal elements are the tangential or shear stresses. Although a cube is shown, the figure really shows the stresses on four of the six orthogonal planes passing through a point; the cube may be imagined to shrink to a point.

We shall now prove that the *stress tensor is symmetric*. Consider the torque on an element about a centroid axis parallel to x_3 (Figure 4.6). This torque is generated only by the shear stresses in the $x_1 x_2$-plane and is (assuming $dx_3 = 1$)

$$T = \left[\tau_{12} + \frac{1}{2}\frac{\partial \tau_{12}}{\partial x_1}dx_1\right] dx_2 \frac{dx_1}{2} + \left[\tau_{12} - \frac{1}{2}\frac{\partial \tau_{12}}{\partial x_1}dx_1\right] dx_2 \frac{dx_1}{2}$$

$$- \left[\tau_{21} + \frac{1}{2}\frac{\partial \tau_{21}}{\partial x_2}dx_2\right] dx_1 \frac{dx_2}{2} - \left[\tau_{21} - \frac{1}{2}\frac{\partial \tau_{21}}{\partial x_2}dx_2\right] dx_1 \frac{dx_2}{2}.$$

After canceling terms, this gives

$$T = (\tau_{12} - \tau_{21})\, dx_1\, dx_2.$$

The rotational equilibrium of the element requires that $T = I\dot{\omega}_3$, where $\dot{\omega}_3$ is the angular acceleration of the element and I is its moment of inertia. For the rectangular element considered, it is easy to show that $I = dx_1\, dx_2(dx_1^2 + dx_2^2)\rho/12$. The

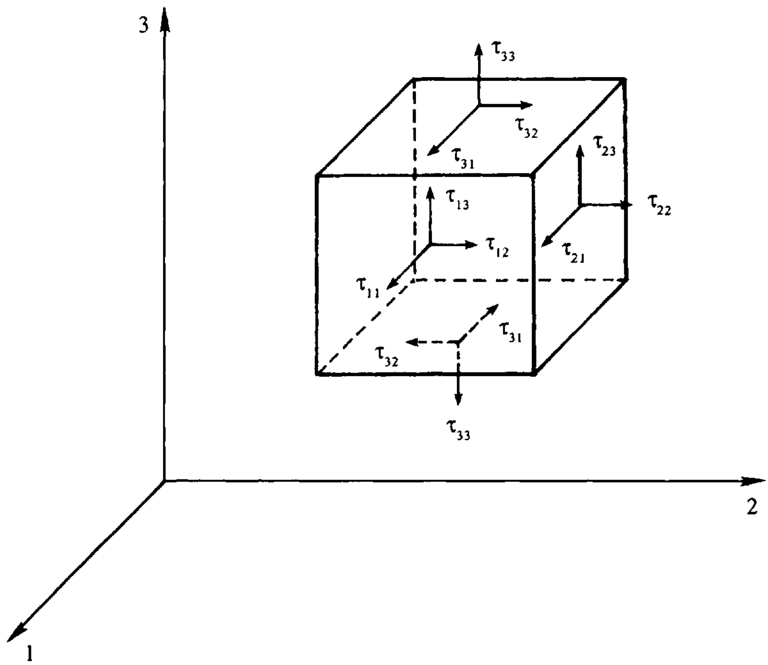


Figure 4.5 Stress at a point. For clarity, components on only four of the six faces are shown.

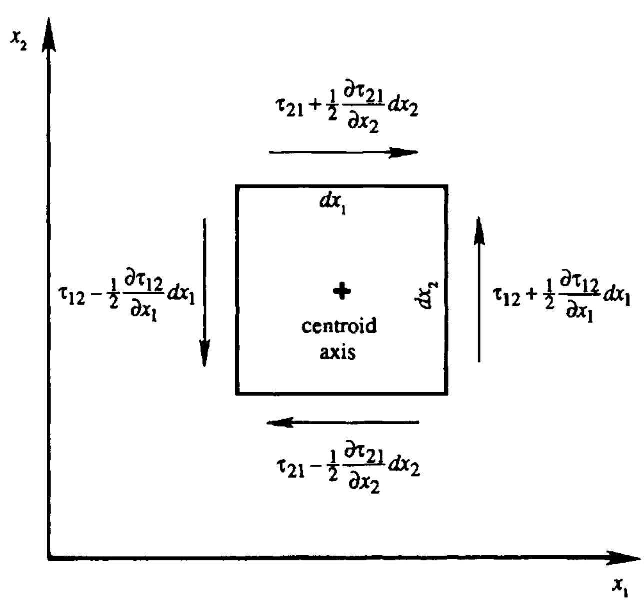


Figure 4.6 Torque on an element.

rotational equilibrium then requires

$$(\tau_{12} - \tau_{21})\, dx_1\, dx_2 = \frac{\rho}{12}\, dx_1\, dx_2 (dx_1^2 + dx_2^2)\, \dot{\omega}_3,$$

that is,

$$\tau_{12} - \tau_{21} = \frac{\rho}{12}(dx_1^2 + dx_2^2)\, \dot{\omega}_3.$$

As dx_1 and dx_2 go to zero, the preceding condition can be satisfied only if $\tau_{12} = \tau_{21}$. In general,

$$\boxed{\tau_{ij} = \tau_{ji}.} \tag{4.14}$$

See Exercise 3 at the end of the chapter.

The stress tensor is therefore symmetric and has only six independent components. The symmetry, however, is violated if there are "body couples" proportional to the mass of the fluid element, such as those exerted by an electric field on polarized fluid molecules. Antisymmetric stresses must be included in such fluids.

7. *Conservation of Momentum*

In this section the law of conservation of momentum will be expressed in the differential form directly by applying Newton's law of motion to an infinitesimal fluid element. We shall then show how the differential form could be derived by starting from an integral form of Newton's law.

Consider the motion of the infinitesimal fluid element shown in Figure 4.7. Newton's law requires that the net force on the element must equal mass times the acceleration of the element. The sum of the surface forces in the x_1 direction equals

$$\left(\tau_{11} + \frac{\partial \tau_{11}}{\partial x_1}\frac{dx_1}{2} - \tau_{11} + \frac{\partial \tau_{11}}{\partial x_1}\frac{dx_1}{2}\right) dx_2\, dx_3$$
$$+ \left(\tau_{21} + \frac{\partial \tau_{21}}{\partial x_2}\frac{dx_2}{2} - \tau_{21} + \frac{\partial \tau_{21}}{\partial x_2}\frac{dx_2}{2}\right) dx_1\, dx_3$$
$$+ \left(\tau_{31} + \frac{\partial \tau_{31}}{\partial x_3}\frac{dx_3}{2} - \tau_{31} + \frac{\partial \tau_{31}}{\partial x_3}\frac{dx_3}{2}\right) dx_1\, dx_2,$$

which simplifies to

$$\left(\frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + \frac{\partial \tau_{31}}{\partial x_3}\right) dx_1\, dx_2\, dx_3 = \frac{\partial \tau_{j1}}{\partial x_j} d\mathcal{V},$$

where $d\mathcal{V}$ is the volume of the element. Generalizing, the i-component of the *surface force per unit volume* of the element is

$$\frac{\partial \tau_{ij}}{\partial x_j},$$

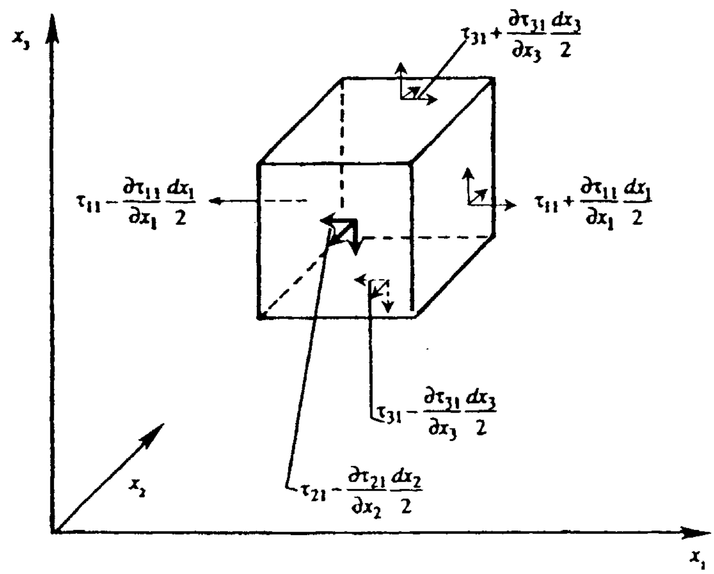


Figure 4.7 Surface stresses on an element moving with the flow. Only stresses in the x_1 direction are labeled.

where we have used the symmetry property $\tau_{ij} = \tau_{ji}$. Let **g** be the body force per unit mass, so that $\rho\mathbf{g}$ is the body force per unit volume. Then Newton's law gives

$$\boxed{\rho \frac{Du_i}{Dt} = \rho g_i + \frac{\partial \tau_{ij}}{\partial x_j}.} \tag{4.15}$$

This is the equation of motion relating acceleration to the net force at a point and holds for any continuum, solid or fluid, no matter how the stress tensor τ_{ij} is related to the deformation field. Equation (4.15) is sometimes called *Cauchy's equation of motion.*

We shall now deduce Cauchy's equation starting from an *integral* statement of Newton's law for a material volume $\mathcal{V}$. In this case we do not have to consider the internal stresses within the fluid, but only the surface forces at the boundary of the volume (along with body forces). It was shown in Chapter 2, Section 6 that the surface force per unit area is $\mathbf{n} \cdot \boldsymbol{\tau}$, where $\mathbf{n}$ is the unit outward normal. The surface force on an area element $d\mathbf{A}$ is therefore $\mathbf{dA} \cdot \boldsymbol{\tau}$. Newton's law for a material volume $\mathcal{V}$ requires that the rate of change of its momentum equals the sum of body forces throughout the volume, plus the surface forces at the boundary. Therefore

$$\frac{D}{Dt}\int_{\mathcal{V}} \rho u_i \, d\mathcal{V} = \int_{\mathcal{V}} \rho \frac{Du_i}{Dt} d\mathcal{V} = \int_{\mathcal{V}} \rho g_i \, d\mathcal{V} + \int_A \tau_{ij} \, dA_j, \tag{4.16}$$

where Eqs. (4.6) and (4.14) have been used. Transforming the surface integral to a volume integral, Eq. (4.16) becomes

$$\int \left[\rho \frac{Du_i}{Dt} - \rho g_i - \frac{\partial \tau_{ij}}{\partial x_j} \right] d\mathcal{V} = 0.$$

As this holds for any volume, the integrand must vanish at every point and therefore Eq. (4.15) must hold. We have therefore derived the differential form of the equation of motion, starting from an integral form.

8. *Momentum Principle for a Fixed Volume*

In the preceding section the momentum principle was applied to a *material* volume of finite size and this led to Eq. (4.16). In this section the form of the law will be derived for a fixed region in space. It is easy to do this by starting from the differential form (4.15) and integrating over a fixed volume V. Adding u_i times the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0,$$

to the left-hand side of Eq. (4.15), we obtain

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = \rho g_i + \frac{\partial \tau_{ij}}{\partial x_j}. \tag{4.17}$$

Each term of Eq. (4.17) is now integrated over a fixed region V. The time derivative term gives

$$\int_V \frac{\partial(\rho u_i)}{\partial t} dV = \frac{d}{dt} \int_V \rho u_i \, dV = \frac{dM_i}{dt}, \tag{4.18}$$

where

$$M_i \equiv \int_V \rho u_i \, dV,$$

is the momentum of the fluid inside the volume. The volume integral of the second term in Eq. (4.17) becomes, after applying Gauss' theorem,

$$\int_V \frac{\partial}{\partial x_j}(\rho u_i u_j) \, dV = \int_A \rho u_i u_j \, dA_j \equiv \dot{M}_i^{\text{out}}, \tag{4.19}$$

where $\dot{M}_i^{\text{out}}$ is the net rate of outflux of i-momentum. (Here $\rho u_j \, dA_j$ is the mass outflux through an area element $d\mathbf{A}$ on the boundary. Outflux of momentum is defined as the outflux of mass times the velocity.) The volume integral of the third term in Eq. (4.17) is simply

$$\int \rho g_i \, dV = F_{bi}, \tag{4.20}$$

where $\mathbf{F}_b$ is the net body force acting over the entire volume. The volume integral of the fourth term in Eq. (4.17) gives, after applying Gauss' theorem,

$$\int_V \frac{\partial \tau_{ij}}{\partial x_j} dV = \int_A \tau_{ij} \, dA_j \equiv F_{si}, \tag{4.21}$$

where $\mathbf{F}_s$ is the net surface force at the boundary of V. If we define $\mathbf{F} = \mathbf{F}_b + \mathbf{F}_s$ as the sum of all forces, then the volume integral of Eq. (4.17) finally gives

$$\boxed{\mathbf{F} = \frac{d\mathbf{M}}{dt} + \dot{\mathbf{M}}^{\text{out}},} \tag{4.22}$$

where Eqs. (4.18)–(4.21) have been used.

Equation (4.22) is the law of conservation of momentum for a fixed volume. It states that the net force on a fixed volume equals the rate of change of momentum within the volume, plus the net outflux of momentum through the surfaces. The equation has three independent components, where the x-component is

$$F_x = \frac{dM_x}{dt} + \dot{M}_x^{\text{out}}.$$

The momentum principle (frequently called the *momentum theorem*) has wide application, especially in engineering. An example is given in what follows. More illustrations can be found throughout the book, for example, in Chapter 9, Section 4, Chapter 10, Section 11, Chapter 13, Section 10, and Chapter 16, Sections 2 and 3.

Example 4.1. Consider an experiment in which the drag on a 2D body immersed in a steady incompressible flow can be determined from measurement of the velocity distributions far upstream and downstream of the body (Figure 4.8). Velocity far upstream is the uniform flow U_∞, and that in the wake of the body is measured to be $u(y)$, which is less than U_∞ due to the drag of the body. Find the drag force D per unit length of the body.

Solution: The wake velocity $u(y)$ is less than U_∞ due to the drag forces exerted by the body on the fluid. To analyze the flow, take a fixed volume shown by the dashed lines in Figure 4.8. It consists of the rectangular region PQRS and has a hole in the center coinciding with the surface of the body. The sides PQ and SR are chosen far enough from the body so that the pressure nearly equals the undisturbed pressure p_∞. The side QR at which the velocity profile is measured is also at a far enough distance for the streamlines to be nearly parallel; the pressure variation across the wake is

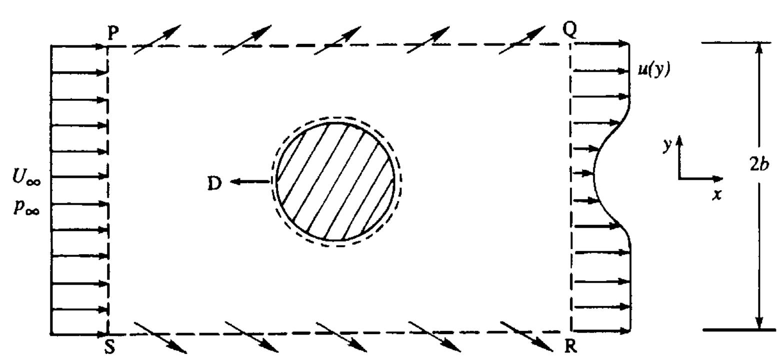


Figure 4.8 Momentum balance of flow over a body (Example 4.1).

therefore small, so that it is nearly equal to the undisturbed pressure p_∞. The surface forces on PQRS therefore cancel out, and the only force acting at the boundary of the chosen fixed volume is D, the force exerted by the body at the central hole.

For steady flow, the x-component of the momentum principle (4.22) reduces to

$$D = \dot{M}^{\text{out}}, \tag{4.23}$$

where $\dot{M}^{\text{out}}$ is the net outflow rate of x-momentum through the boundaries of the region. There is no flow of momentum through the central hole in Figure 4.8. Outflow rates of x-momentum through PS and QR are

$$\dot{M}^{\text{PS}} = -\int_{-b}^{b} U_\infty(\rho U_\infty\, dy) = -2b\rho U_\infty^2, \tag{4.24}$$

$$\dot{M}^{\text{QR}} = \int_{-b}^{b} u(\rho u\, dy) = \rho \int_{-b}^{b} u^2\, dy. \tag{4.25}$$

An important point is that there is an outflow of mass and x-momentum through PQ and SR. A mass flux through PQ and SR is required because the velocity across QR is less than that across PS. Conservation of mass requires that the inflow through PS, equal to $2b\rho U_\infty$, must balance the outflows through PQ, SR, and QR. This gives

$$2b\rho U_\infty = \dot{m}^{\text{PQ}} + \dot{m}^{\text{SR}} + \rho \int_{-b}^{b} u\, dy,$$

where $\dot{m}^{\text{PQ}}$ and $\dot{m}^{\text{SR}}$ are the outflow rates of mass through the sides. The mass balance can be written as

$$\dot{m}^{\text{PQ}} + \dot{m}^{\text{SR}} = \rho \int_{-b}^{b} (U_\infty - u)\, dy.$$

Outflow rate of x-momentum through PQ and SR is therefore

$$\dot{M}^{\text{PQ}} + \dot{M}^{\text{SR}} = \rho U_\infty \int_{-b}^{b} (U_\infty - u)\, dy, \tag{4.26}$$

because the x-directional velocity at these surfaces is nearly U_∞. Combining Eqs. (4.22)–(4.26) gives a net outflow of x-momentum of:

$$\dot{M}^{\text{out}} = \dot{M}^{\text{PS}} + \dot{M}^{\text{QR}} + \dot{M}^{\text{PQ}} + \dot{M}^{\text{SR}} = -\rho \int_{-b}^{b} u(U_\infty - u)\, dy.$$

The momentum balance (4.23) now shows that the body exerts a force on the fluid in the negative x direction of magnitude

$$D = \rho \int_{-b}^{b} u(U_\infty - u)\, dy,$$

which can be evaluated from the measured velocity profile. □

A more general way of obtaining the force on a body immersed in a flow is by using the Euler momentum integral, which we derive in what follows. We must assume that the flow is steady and body forces are absent. Then integrating Eq. (4.19) over a fixed volume gives

$$\int_V \nabla \cdot (\rho \mathbf{uu} - \boldsymbol{\tau}) dV = \int_A (\rho \mathbf{uu} - \boldsymbol{\tau}) \cdot \mathbf{dA}, \tag{4.27}$$

where A is the closed surface bounding V. This volume V contains *only* fluid particles. Imagine a body immersed in a flow and surround that body with a closed surface. We seek to calculate the force on the body by an integral over a possibly distant surface. In order to apply (4.27), A must bound a volume containing only fluid particles. This is accomplished by considering A to be composed of three parts (see Figure 4.9),

$$A = A_1 + A_2 + A_3.$$

Here A_1 is the outer surface, A_2 is wrapped around the body like a tight-fitting rubber glove with dA_2 pointing outwards from the fluid volume and, therefore, into the body, and A_3 is the connection surface between the outer A_1 and the inner A_2. Now

$$\int_{A_3} (\rho \mathbf{uu} - \boldsymbol{\tau}) \cdot \mathbf{dA}_3 \to 0 \qquad \text{as } A_3 \to 0,$$

because it may be taken as the bounding surface of an evanescent thread. On the surface of a solid body, $\mathbf{u} \cdot \mathbf{dA}_2 = 0$ because no mass enters or leaves the surface. Here $\int_{A_2} \boldsymbol{\tau} \cdot \mathbf{dA}_2$ is the force the body exerts on the fluid from our definition of $\boldsymbol{\tau}$. Then the force the fluid exerts on the body is

$$F_B = -\int_{A_2} \boldsymbol{\tau} \cdot \mathbf{dA}_2 = -\int_{A_1} (\rho \mathbf{uu} - \boldsymbol{\tau}) \cdot \mathbf{dA}_1. \tag{4.28}$$

Using similar arguments, mass conservation can be written in the form

$$\int_{A_1} \rho \mathbf{u} \cdot \mathbf{dA}_1 = 0. \tag{4.29}$$

Equations (4.28) and (4.29) can be used to solve Example 4.1. Of course, the same final result is obtained when $\boldsymbol{\tau} \approx$ constant pressure on all of A_1, ρ = constant, and the x component of $\mathbf{u} = U_\infty \mathbf{i}$ on segments PQ and SR of A_1.

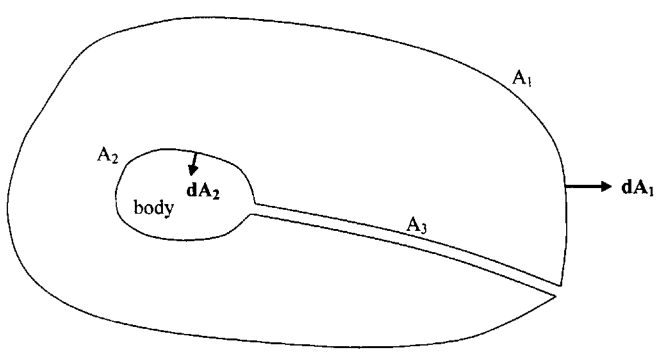


Figure 4.9 Surfaces of integration for the Euler momentum integral.

9. Angular Momentum Principle for a Fixed Volume

In mechanics of solids it is shown that

$$\mathbf{T} = \frac{d\mathbf{H}}{dt}, \tag{4.30}$$

where $\mathbf{T}$ is the torque of all external forces on the body about any chosen axis, and $d\mathbf{H}/dt$ is the rate of change of angular momentum of the body about the same axis. The angular momentum is defined as the "moment of momentum," that is

$$\mathbf{H} \equiv \int \mathbf{r} \times \mathbf{u}\, dm,$$

where dm is an element of mass, and $\mathbf{r}$ is the position vector from the chosen axis (Figure 4.10). The angular momentum principle is *not* a separate law, but can be derived from Newton's law by performing a cross product with $\mathbf{r}$. It can be shown that Eq. (4.30) also holds for a material volume in a fluid. When Eq. (4.30) is transformed to apply to a *fixed volume*, the result is

$$\mathbf{T} = \frac{d\mathbf{H}}{dt} + \dot{\mathbf{H}}^{\text{out}}, \tag{4.31}$$

where

$$\mathbf{T} = \int_A \mathbf{r} \times (\boldsymbol{\tau} \cdot d\mathbf{A}) + \int_V \mathbf{r} \times (\rho \mathbf{g}\, dV),$$

$$\mathbf{H} = \int_V \mathbf{r} \times (\rho \mathbf{u}\, dV),$$

$$\dot{\mathbf{H}}^{\text{out}} = \int_A \mathbf{r} \times [(\rho \mathbf{u} \cdot d\mathbf{A})\mathbf{u}].$$

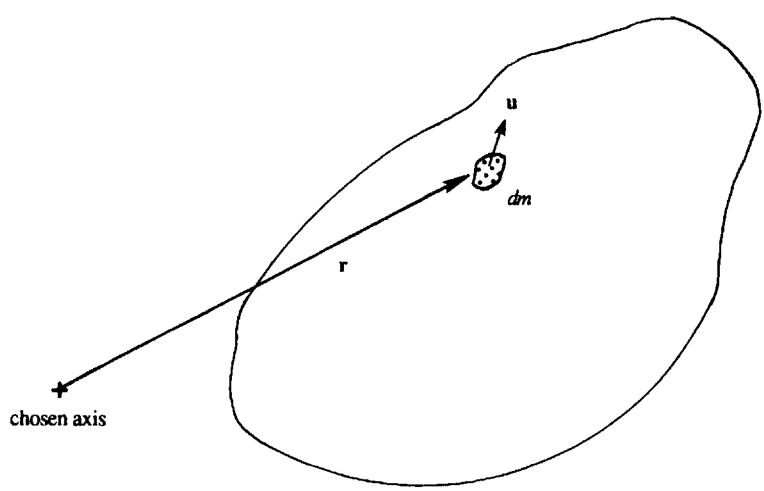


Figure 4.10 Definition sketch for angular momentum theorem.

Here $\mathbf{T}$ represents the sum of torques due to surface and body forces, $\boldsymbol{\tau} \cdot d\mathbf{A}$ is the surface force on a boundary element, and $\rho \mathbf{g} dV$ is the body force acting on an interior element. Vector $\mathbf{H}$ represents the angular momentum of fluid inside the fixed volume because $\rho \mathbf{u} dV$ is the momentum of a volume element. Finally, $\dot{\mathbf{H}}^{\text{out}}$ is the rate of outflow of angular momentum through the boundary, $\rho \mathbf{u} \cdot d\mathbf{A}$ is the mass flow rate, and $(\rho \mathbf{u} \cdot d\mathbf{A})\mathbf{u}$ is the momentum outflow rate through a boundary element $d\mathbf{A}$.

The angular momentum principle (4.31) is analogous to the linear momentum principle (4.22), and is very useful in investigating rotating fluid systems such as turbomachines, fluid couplings, and even lawn sprinklers.

Example 4.2. Consider a lawn sprinkler as shown in Figure 4.11. The area of the nozzle exit is A, and the jet velocity is U. Find the torque required to hold the rotor stationary.

Solution: Select a stationary volume V shown by the dashed lines. Pressure everywhere on the control surface is atmospheric, and there is no net moment due to the pressure forces. The control surface cuts through the vertical support and the torque T exerted by the support on the sprinkler arm is the only torque acting on V. Apply the angular momentum balance

$$T = \dot{H}_z^{\text{out}}.$$

Let $\dot{m} = \rho A U$ be the mass flux through each nozzle. As the angular momentum is the moment of momentum, we obtain

$$\dot{H}_z^{\text{out}} = (\dot{m} U \cos\alpha)a + (\dot{m} U \cos\alpha)a = 2a\rho A U^2 \cos\alpha.$$

Therefore, the torque required to hold the rotor stationary is

$$T = 2a\rho A U^2 \cos\alpha.$$

When the sprinkler is rotating at a steady state, this torque is balanced by both air resistance and mechanical friction.

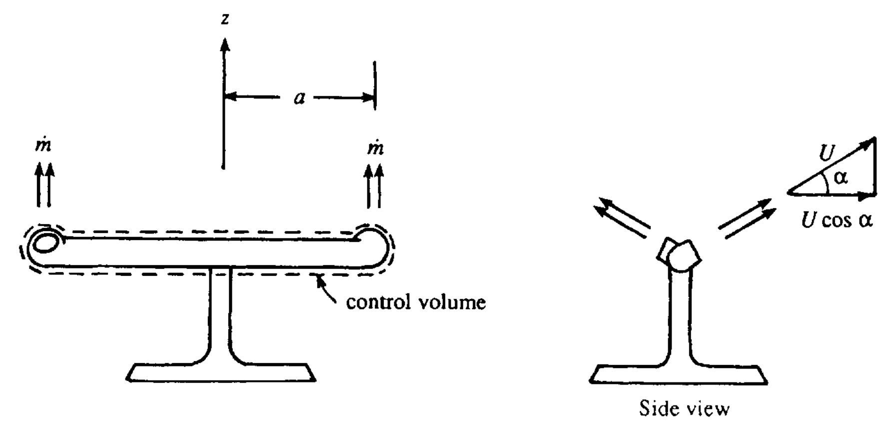


Figure 4.11 Lawn sprinkler (Example 4.2).

10. Constitutive Equation for Newtonian Fluid

The relation between the stress and deformation in a continuum is called a *constitutive equation*. An equation that linearly relates the stress to the rate of strain in a fluid medium is examined in this section.

In a fluid at rest there are only normal components of stress on a surface, and the stress does not depend on the orientation of the surface. In other words, the stress tensor is *isotropic* or spherically symmetric. An isotropic tensor is defined as one whose components do not change under a rotation of the coordinate system (see Chapter 2, Section 7). The only second-order isotropic tensor is the Kronecker delta

$$\boldsymbol{\delta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Any isotropic second-order tensor must be proportional to $\boldsymbol{\delta}$. Therefore, because the stress in a static fluid is isotropic, it must be of the form

$$\tau_{ij} = -p\delta_{ij}, \tag{4.32}$$

where p is the *thermodynamic pressure* related to ρ and T by an equation of state (e.g., the thermodynamic pressure for a perfect gas is $p = \rho RT$). A negative sign is introduced in Eq. (4.32) because the normal components of $\boldsymbol{\tau}$ are regarded as positive if they indicate tension rather than compression.

A moving fluid develops additional components of stress due to viscosity. The diagonal terms of $\boldsymbol{\tau}$ now become unequal, and shear stresses develop. For a moving fluid we can split the stress into a part $-p\delta_{ij}$ that would exist if it were at rest and a part σ_{ij} due to the fluid motion alone:

$$\tau_{ij} = -p\delta_{ij} + \sigma_{ij}. \tag{4.33}$$

We shall assume that p appearing in Eq. (4.33) is still the thermodynamic pressure. The assumption, however, is not on a very firm footing because thermodynamic quantities are defined for equilibrium states, whereas a moving fluid undergoing diffusive fluxes is generally not in equilibrium. Such departures from thermodynamic equilibrium are, however, expected to be unimportant if the relaxation (or adjustment) time of the molecules is small compared to the time scale of the flow, as discussed in Chapter 1, Section 8.

The nonisotropic part $\boldsymbol{\sigma}$, called the *deviatoric stress tensor*, is related to the velocity gradients $\partial u_i/\partial x_j$. The velocity gradient tensor can be decomposed into symmetric and antisymmetric parts:

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right).$$

The antisymmetric part represents fluid rotation without deformation, and cannot by itself generate stress. The stresses must be generated by the strain rate tensor

$$e_{ij} \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

alone. We shall assume a linear relation of the type

$$\sigma_{ij} = K_{ijmn} e_{mn}, \tag{4.34}$$

where K_{ijmn} is a fourth-order tensor having 81 components that depend on the thermodynamic state of the medium. Equation (4.34) simply means that *each* stress component is linearly related to *all* nine components of e_{ij}; altogether 81 constants are therefore needed to completely describe the relationship.

It will now be shown that only two of the 81 elements of K_{ijmn} survive if it is assumed that the medium is isotropic and that the stress tensor is symmetric. An isotropic medium has no directional preference, which means that the stress–strain relationship is independent of rotation of the coordinate system. This is only possible if K_{ijmn} is an isotropic tensor. It is shown in books on tensor analysis (e.g., see Aris (1962), page 30) that all isotropic tensors of even order are made up of products of δ_{ij}, and that a fourth-order isotropic tensor must have the form

$$K_{ijmn} = \lambda \delta_{ij} \delta_{mn} + \mu \delta_{im} \delta_{jn} + \gamma \delta_{in} \delta_{jm}, \tag{4.35}$$

where λ, μ, and γ are scalars that depend on the local thermodynamic state. As σ_{ij} is a symmetric tensor, Eq. (4.34) requires that K_{ijmn} also must be symmetric in i and j. This is consistent with Eq. (4.35) only if

$$\gamma = \mu. \tag{4.36}$$

Only two constants μ and λ, of the original 81, have therefore survived under the restrictions of material isotropy and stress symmetry. Substitution of Eq. (4.35) into the constitutive equation (4.34) gives

$$\sigma_{ij} = 2\mu e_{ij} + \lambda e_{mm}\, \delta_{ij},$$

where $e_{mm} = \nabla \cdot \mathbf{u}$ is the volumetric strain rate (explained in Chapter 3, Section 6). The complete stress tensor (4.33) then becomes

$$\tau_{ij} = -p\delta_{ij} + 2\mu e_{ij} + \lambda e_{mm}\, \delta_{ij}. \tag{4.37}$$

The two scalar constants μ and λ can be further related as follows. Setting $i = j$, summing over the repeated index, and noting that $\delta_{ii} = 3$, we obtain

$$\tau_{ii} = -3p + (2\mu + 3\lambda)\, e_{mm},$$

from which the pressure is found to be

$$p = -\tfrac{1}{3}\tau_{ii} + \left(\tfrac{2}{3}\mu + \lambda\right) \nabla \cdot \mathbf{u}. \tag{4.38}$$

Now the diagonal terms of e_{ij} in a flow may be unequal. In such a case the stress tensor τ_{ij} can have unequal diagonal terms because of the presence of the term proportional to μ in Eq. (4.37). We can therefore take the average of the diagonal terms of $\boldsymbol{\tau}$ and define a *mean pressure* (as opposed to thermodynamic pressure p) as

$$\bar{p} \equiv -\tfrac{1}{3}\tau_{ii}. \tag{4.39}$$

Substitution into Eq. (4.38) gives

$$p - \bar{p} = \left(\tfrac{2}{3}\mu + \lambda\right) \nabla \cdot \mathbf{u}. \tag{4.40}$$

For a completely incompressible fluid we can only define a mechanical or mean pressure, because there is no equation of state to determine a thermodynamic pressure. (In fact, the *absolute pressure in an incompressible fluid is indeterminate,* and only its *gradients* can be determined from the equations of motion.) The λ-term in the constitutive equation (4.37) drops out because $e_{mm} = \nabla \cdot \mathbf{u} = 0$, and no consideration of Eq. (4.40) is necessary. For *incompressible fluids*, the constitutive equation (4.37) takes the simple form

$$\boxed{\tau_{ij} = -p\delta_{ij} + 2\mu e_{ij}} \quad \text{(incompressible)}, \tag{4.41}$$

where p can only be interpreted as the mean pressure. For a compressible fluid, on the other hand, a thermodynamic pressure can be defined, and it seems that p and $\bar{p}$ can be different. In fact, Eq. (4.40) relates this difference to the rate of expansion through the proportionality constant $\kappa = \lambda + 2\mu/3$, which is called the *coefficient of bulk viscosity*. In principle, κ is a measurable quantity; however, extremely large values of $D\rho/Dt$ are necessary in order to make any measurement, such as within shock waves. Moreover, measurements are inconclusive about the nature of κ. For many applications the *Stokes assumption*

$$\lambda + \tfrac{2}{3}\mu = 0, \tag{4.42}$$

is found to be sufficiently accurate, and can also be supported from the kinetic theory of monatomic gases. Interesting historical aspects of the Stokes assumption $3\lambda + 2\mu = 0$ can be found in Truesdell (1952).

To gain additional insight into the distinction between thermodynamic pressure and the mean of the normal stresses, consider a system inside a cylinder in which a piston may be moved in or out to do work. The first law of thermodynamics may be written in general terms as $de = dw + dQ = -\bar{p}dv + dQ = -pdv + TdS$, where the last equality is written in terms of state functions. Then $TdS - dQ = (p - \bar{p})dv$. The Clausius-Duhem inequality (see under Eq. 1.16) tells us $TdS - dQ \geq 0$ for any process and, consequently, $(p - \bar{p})dv \geq 0$. Thus, for an expansion, $dv > 0$, so $p > \bar{p}$, and conversely for a compression. Equation (4.40) is:

$$p - \bar{p} = \left(\frac{2}{3}\mu + \lambda\right) \nabla \cdot \mathbf{u} = -\left(\frac{2}{3}\mu + \lambda\right) \frac{1}{\rho}\frac{D\rho}{Dt} = \left(\frac{2}{3}\mu + \lambda\right) \frac{1}{v}\frac{Dv}{Dt}, \qquad v = \frac{1}{\rho}.$$

Further, we require $(2/3)\mu + \lambda > 0$ to satisfy the Clausius-Duhem inequality statement of the second law.

With the assumption $\kappa = 0$, the constitutive equation (4.37) reduces to

$$\boxed{\tau_{ij} = -\left(p + \tfrac{2}{3}\mu\nabla \cdot \mathbf{u}\right) \delta_{ij} + 2\mu e_{ij}} \tag{4.43}$$

This linear relation between $\boldsymbol{\tau}$ and $\mathbf{e}$ is consistent with Newton's definition of viscosity coefficient in a simple parallel flow $u(y)$, for which Eq. (4.43) gives a shear stress of $\tau = \mu(du/dy)$. Consequently, a fluid obeying Eq. (4.43) is called a *Newtonian fluid.* The fluid property μ in Eq. (4.43) can depend on the local thermodynamic state alone.

The nondiagonal terms of Eq. (4.43) are easy to understand. They are of the type

$$\tau_{12} = \mu\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right),$$

which relates the shear stress to the shear strain rate. The diagonal terms are more difficult to understand. For example, Eq. (4.43) gives

$$\tau_{11} = -p + 2\mu\left[-\frac{1}{3}\frac{\partial u_i}{\partial x_i} + \frac{\partial u_1}{\partial x_1}\right],$$

which means that the normal viscous stress on a plane normal to the x_1-axis is proportional to the *difference* between the extension rate in the x_1 direction and the average expansion rate at the point. Therefore, only those extension rates different from the average will generate normal viscous stress.

Non-Newtonian Fluids

The linear Newtonian friction law is expected to hold for small rates of strain because higher powers of $\mathbf{e}$ are neglected. However, for common fluids such as air and water the linear relationship is found to be surprisingly accurate for most applications. Some liquids important in the chemical industry, on the other hand, display non-Newtonian behavior at moderate rates of strain. These include: (1) solutions containing polymer molecules, which have very large molecular weights and form long chains coiled together in spongy ball-like shapes that deform under shear; and (2) emulsions and slurries containing suspended particles, two examples of which are blood and water containing clay. These liquids violate Newtonian behavior in several ways—for example, shear stress is a *nonlinear* function of the local strain rate, it depends not only on the local strain rate, but also on its *history*. Such a "memory" effect gives the fluid an elastic property, in addition to its viscous property. Most non-Newtonian fluids are therefore *viscoelastic*. Only Newtonian fluids will be considered in this book.

11. Navier–Stokes Equation

The equation of motion for a Newtonian fluid is obtained by substituting the constitutive equation (4.43) into Cauchy's equation (4.15) to obtain

$$\rho\frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \rho g_i + \frac{\partial}{\partial x_j}\left[2\mu e_{ij} - \frac{2}{3}\mu(\nabla\cdot\mathbf{u})\delta_{ij}\right], \qquad (4.44)$$

where we have noted that $(\partial p/\partial x_j)\delta_{ij} = \partial p/\partial x_i$. Equation (4.44) is a general form of the *Navier–Stokes equation.* Viscosity μ in this equation can be a function of the thermodynamic state, and indeed μ for most fluids displays a rather strong dependence on temperature, decreasing with T for liquids and increasing with T for gases.

However, if the temperature differences are small within the fluid, then μ can be taken outside the derivative in Eq. (4.44), which then reduces to

$$\rho\frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \rho g_i + 2\mu\frac{\partial e_{ij}}{\partial x_j} - \frac{2\mu}{3}\frac{\partial}{\partial x_i}(\nabla\cdot\mathbf{u})$$
$$= -\frac{\partial p}{\partial x_i} + \rho g_i + \mu\left[\nabla^2 u_i + \frac{1}{3}\frac{\partial}{\partial x_i}(\nabla\cdot\mathbf{u})\right],$$

where

$$\nabla^2 u_i \equiv \frac{\partial^2 u_i}{\partial x_j \partial x_j} = \frac{\partial^2 u_i}{\partial x_1^2} + \frac{\partial^2 u_i}{\partial x_2^2} + \frac{\partial^2 u_i}{\partial x_3^2},$$

is the Laplacian of u_i. For incompressible fluids $\nabla\cdot\mathbf{u} = 0$, and using vector notation, the Navier–Stokes equation reduces to

$$\boxed{\rho\frac{D\mathbf{u}}{Dt} = -\nabla p + \rho\mathbf{g} + \mu\,\nabla^2\mathbf{u}.} \quad \text{(incompressible)} \tag{4.45}$$

If viscous effects are negligible, which is generally found to be true far from boundaries of the flow field, we obtain the *Euler equation*

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p + \rho\mathbf{g}. \tag{4.46}$$

Comments on the Viscous Term

For an incompressible fluid, Eq. (4.41) shows that the viscous stress at a point is

$$\sigma_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{4.47}$$

which shows that $\boldsymbol{\sigma}$ depends only on the deformation rate of a fluid element at a point, and not on the rotation rate $(\partial u_i/\partial x_j - \partial u_j/\partial x_i)$. We have built this property into the Newtonian constitutive equation, based on the fact that in a solid-body rotation (that is a flow in which the tangential velocity is proportional to the radius) the particles do not deform or "slide" past each other, and therefore they do not cause viscous stress.

However, consider the net viscous force per unit volume at a point, given by

$$F_i = \frac{\partial \sigma_{ij}}{\partial x_j} = \mu\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \mu\frac{\partial^2 u_i}{\partial x_j\,\partial x_j} = -\mu(\nabla\times\boldsymbol{\omega})_i, \tag{4.48}$$

where we have used the relation

$$(\nabla\times\boldsymbol{\omega})_i = \varepsilon_{ijk}\frac{\partial\omega_k}{\partial x_j} = \varepsilon_{ijk}\frac{\partial}{\partial x_j}\left(\varepsilon_{kmn}\frac{\partial u_n}{\partial x_m}\right)$$
$$= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})\frac{\partial^2 u_n}{\partial x_j\,\partial x_m} = \frac{\partial^2 u_j}{\partial x_j\,\partial x_i} - \frac{\partial^2 u_i}{\partial x_j\,\partial x_j}$$
$$= -\frac{\partial^2 u_i}{\partial x_j\,\partial x_j}.$$

In the preceding derivation the "epsilon delta relation," given by Eq. (2.19), has been used. Relation (4.48) can cause some confusion because it seems to show that the net viscous force depends on *vorticity*, whereas Eq. (4.47) shows that viscous stress depends only on strain rate and is independent of local vorticity. The apparent paradox is explained by realizing that the net viscous force is given by either the spatial *derivative* of vorticity or the spatial *derivative* of deformation rate; both forms are shown in Eq. (4.48). The net viscous force vanishes when ω is uniform everywhere (as in solid-body rotation), in which case the incompressibility condition requires that the deformation is zero everywhere as well.

12. Rotating Frame

The equations of motion given in Section 7 are valid in an inertial or "fixed" frame of reference. Although such a frame of reference cannot be defined precisely, experience shows that these laws are accurate enough in a frame of reference stationary with respect to "distant stars." In geophysical applications, however, we naturally measure positions and velocities with respect to a frame of reference fixed on the surface of the earth, which rotates with respect to an inertial frame. In this section we shall derive the equations of motion in a rotating frame of reference. Similar derivations are also given by Batchelor (1967), Pedlosky (1987), and Holton (1979).

Consider (Figure 4.12) a frame of reference (x_1, x_2, x_3) rotating at a uniform angular velocity $\mathbf{\Omega}$ with respect to a fixed frame (X_1, X_2, X_3). Any vector $\mathbf{P}$ is represented in the rotating frame by

$$\mathbf{P} = P_1\mathbf{i}_1 + P_2\mathbf{i}_2 + P_3\mathbf{i}_3.$$

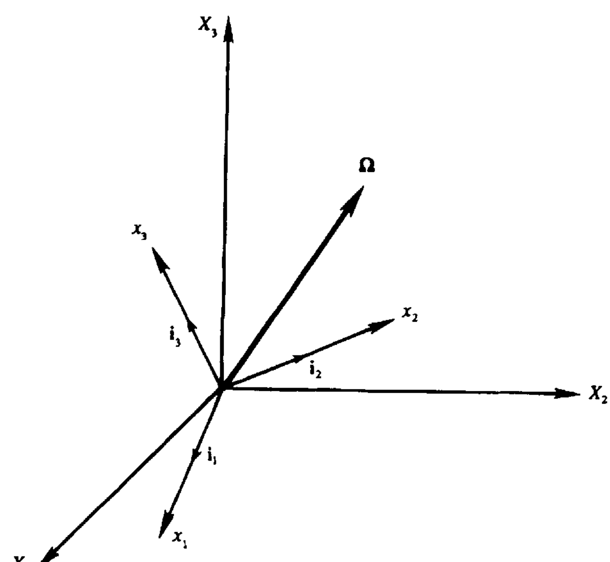


Figure 4.12 Coordinate frame (x_1, x_2, x_3) rotating at angular velocity $\mathbf{\Omega}$ with respect to a fixed frame (X_1, X_2, X_3).

To a fixed observer the directions of the rotating unit vectors $\mathbf{i}_1$, $\mathbf{i}_2$, and $\mathbf{i}_3$ change with time. To this observer the time derivative of $\mathbf{P}$ is

$$\left(\frac{dP}{dt}\right)_F = \frac{d}{dt}(P_1\mathbf{i}_1 + P_2\mathbf{i}_2 + P_3\mathbf{i}_3)$$
$$= \mathbf{i}_1\frac{dP_1}{dt} + \mathbf{i}_2\frac{dP_2}{dt} + \mathbf{i}_3\frac{dP_3}{dt} + P_1\frac{d\mathbf{i}_1}{dt} + P_2\frac{d\mathbf{i}_2}{dt} + P_3\frac{d\mathbf{i}_3}{dt}.$$

To the rotating observer, the rate of change of $\mathbf{P}$ is the sum of the first three terms, so that

$$\left(\frac{d\mathbf{P}}{dt}\right)_F = \left(\frac{d\mathbf{P}}{dt}\right)_R + P_1\frac{d\mathbf{i}_1}{dt} + P_2\frac{d\mathbf{i}_2}{dt} + P_3\frac{d\mathbf{i}_3}{dt}. \tag{4.49}$$

Now each unit vector $\mathbf{i}$ traces a cone with a radius of $\sin\alpha$, where α is a constant angle (Figure 4.13). The magnitude of the change of $\mathbf{i}$ in time dt is $|d\mathbf{i}| = \sin\alpha\, d\theta$, which is the length traveled by the tip of $\mathbf{i}$. The magnitude of the rate of change is therefore $(d\mathbf{i}/dt) = \sin\alpha\,(d\theta/dt) = \Omega\sin\alpha$, and the direction of the rate of change is perpendicular to the $(\boldsymbol{\Omega}, \mathbf{i})$-plane. Thus $d\mathbf{i}/dt = \boldsymbol{\Omega}\times\mathbf{i}$ for any rotating unit vector $\mathbf{i}$. The sum of the last three terms in Eq. (4.49) is then $P_1\boldsymbol{\Omega}\times\mathbf{i}_1 + P_2\boldsymbol{\Omega}\times\mathbf{i}_2 + P_3\boldsymbol{\Omega}\times\mathbf{i}_3 = \boldsymbol{\Omega}\times\mathbf{P}$. Equation (4.49) then becomes

$$\left(\frac{d\mathbf{P}}{dt}\right)_F = \left(\frac{d\mathbf{P}}{dt}\right)_R + \boldsymbol{\Omega}\times\mathbf{P}, \tag{4.50}$$

which relates the rates of change of the vector $\mathbf{P}$ as seen by the two observers.

Application of rule (4.50) to the position vector $\mathbf{r}$ relates the velocities as

$$\mathbf{u}_F = \mathbf{u}_R + \boldsymbol{\Omega}\times\mathbf{r}. \tag{4.51}$$

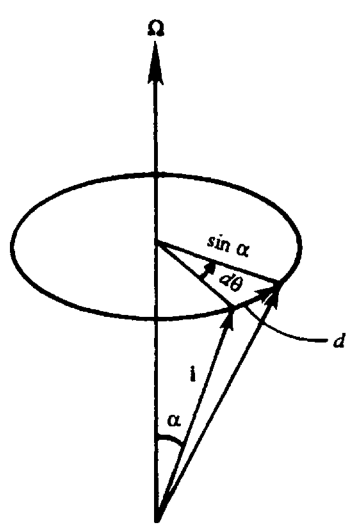


Figure 4.13 Rotation of a unit vector.

Applying rule (4.50) on $\mathbf{u}_F$, we obtain

$$\left(\frac{d\mathbf{u}_F}{dt}\right)_F = \left(\frac{d\mathbf{u}_F}{dt}\right)_R + \boldsymbol{\Omega} \times \mathbf{u}_F,$$

which becomes, upon using Eq. (4.51),

$$\begin{aligned}\frac{d\mathbf{u}_F}{dt} &= \frac{d}{dt}(\mathbf{u}_R + \boldsymbol{\Omega} \times \mathbf{r})_R + \boldsymbol{\Omega} \times (\mathbf{u}_R + \boldsymbol{\Omega} \times \mathbf{r}) \\ &= \left(\frac{d\mathbf{u}_R}{dt}\right)_R + \boldsymbol{\Omega} \times \left(\frac{d\mathbf{r}}{dt}\right)_R + \boldsymbol{\Omega} \times \mathbf{u}_R + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}).\end{aligned}$$

This shows that the accelerations in the two frames are related as

$$\mathbf{a}_F = \mathbf{a}_R + 2\boldsymbol{\Omega} \times \mathbf{u}_R + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}), \qquad \dot{\boldsymbol{\Omega}} = 0, \tag{4.52}$$

The last term in Eq. (4.52) can be written in terms of the vector $\mathbf{R}$ drawn perpendicularly to the axis of rotation (Figure 4.14). Clearly, $\boldsymbol{\Omega} \times \mathbf{r} = \boldsymbol{\Omega} \times \mathbf{R}$. Using the vector identity $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$, the last term of Eq. (4.52) becomes

$$\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{R}) = -(\boldsymbol{\Omega} \cdot \boldsymbol{\Omega})\mathbf{R} = -\Omega^2\mathbf{R},$$

where we have set $\boldsymbol{\Omega} \cdot \mathbf{R} = 0$. Equation (4.52) then becomes

$$\mathbf{a}_F = \mathbf{a} + 2\boldsymbol{\Omega} \times \mathbf{u} - \Omega^2\mathbf{R}, \tag{4.53}$$

where the subscript "R" has been dropped with the understanding that velocity $\mathbf{u}$ and acceleration $\mathbf{a}$ are measured in a rotating frame of reference. Equation (4.53) states

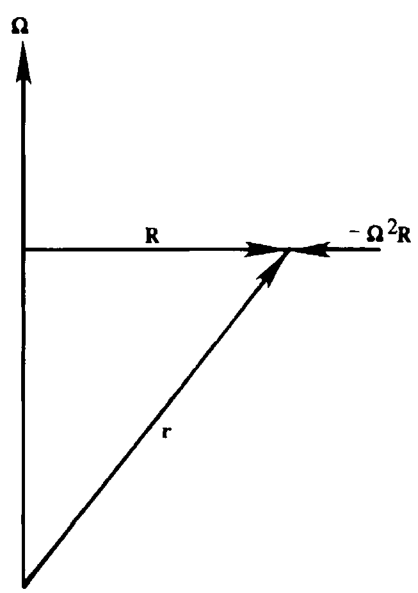


Figure 4.14 Centripetal acceleration.

that the "true" or inertial acceleration equals the acceleration measured in a rotating system, plus the Coriolis acceleration $2\mathbf{\Omega} \times \mathbf{u}$ and the centripetal acceleration $-\Omega^2\mathbf{R}$.

Therefore, Coriolis and centripetal accelerations have to be considered if we are measuring quantities in a rotating frame of reference. Substituting Eq. (4.53) in Eq. (4.45), the equation of motion in a rotating frame of reference becomes

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u} + (\mathbf{g}_\mathrm{n} + \Omega^2\mathbf{R}) - 2\mathbf{\Omega} \times \mathbf{u}, \tag{4.54}$$

where we have taken the Coriolis and centripetal acceleration terms to the right-hand side (now signifying Coriolis and centrifugal *forces*), and added a subscript on $\mathbf{g}$ to mean that it is the body force per unit mass due to (Newtonian) gravitational attractive forces alone.

Effect of Centrifugal Force

The additional apparent force $\Omega^2\mathbf{R}$ can be added to the Newtonian gravity $\mathbf{g}_\mathrm{n}$ to define an *effective gravity force* $\mathbf{g} = \mathbf{g}_\mathrm{n} + \Omega^2\mathbf{R}$ (Figure 4.15). The Newtonian gravity would be uniform over the earth's surface, and be centrally directed, if the earth were spherically symmetric and homogeneous. However, the earth is really an ellipsoid with the equatorial diameter 42 km larger than the polar diameter. In addition, the existence of the centrifugal force makes the effective gravity less at the equator than at the poles, where $\Omega^2\mathbf{R}$ is zero. In terms of the effective gravity, Eq. (4.54) becomes

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u} + \mathbf{g} - 2\mathbf{\Omega} \times \mathbf{u}. \tag{4.55}$$

The Newtonian gravity can be written as the gradient of a scalar potential function. It is easy to see that the centrifugal force can also be written in the same manner.

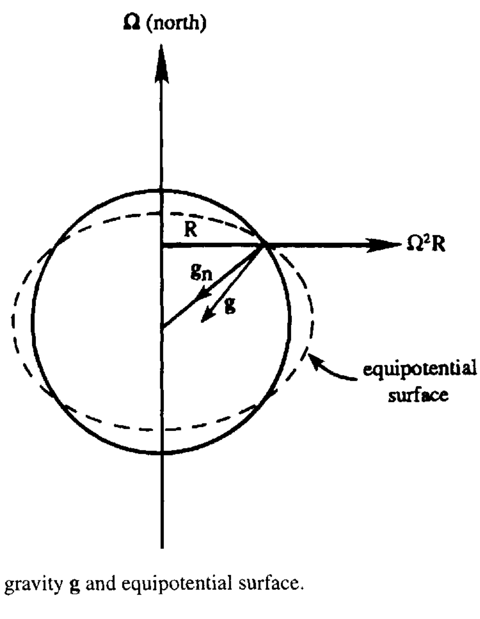


Figure 4.15 Effective gravity g and equipotential surface.

From Definition (2.22), it is clear that the gradient of a spatial direction is the unit vector in that direction (e.g., $\nabla x = \mathbf{i}_x$), so that $\nabla(R^2/2) = R\mathbf{i}_R = \mathbf{R}$. Therefore, $\Omega^2\mathbf{R} = \nabla(\Omega^2 R^2/2)$, and the centrifugal potential is $-\Omega^2 R^2/2$. The *effective gravity* can therefore be written as $\mathbf{g} = -\nabla\Pi$, where Π is now the potential due to the Newtonian gravity, plus the centrifugal potential. The equipotential surfaces (shown by the dashed lines in Figure 4.15) are now perpendicular to the effective gravity. The average sea level is one of these equipotential surfaces. We can then write $\Pi = gz$, where z is measured perpendicular to an equipotential surface, and g is the effective acceleration due to gravity.

Effect of Coriolis Force

The angular velocity vector $\mathbf{\Omega}$ points out of the ground in the northern hemisphere. The Coriolis force $-2\mathbf{\Omega} \times \mathbf{u}$ therefore tends to deflect a particle to the right of its direction of travel in the northern hemisphere (Figure 4.16) and to the left in the southern hemisphere.

Imagine a projectile shot horizontally from the north pole with speed u. The Coriolis force $2\Omega u$ constantly acts perpendicular to its path and therefore does not change the speed u of the projectile. The forward distance traveled in time t is ut, and the deflection is $\Omega u t^2$. The angular deflection is $\Omega u t^2/ut = \Omega t$, which is the earth's rotation in time t. This demonstrates that the projectile in fact travels in a straight line if observed from the inertial outer space; its apparent deflection is merely due to the rotation of the earth underneath it. Observers on earth need an imaginary force to account for the apparent deflection. A clear physical explanation of the Coriolis force, with applications to mechanics, is given by Stommel and Moore (1989).

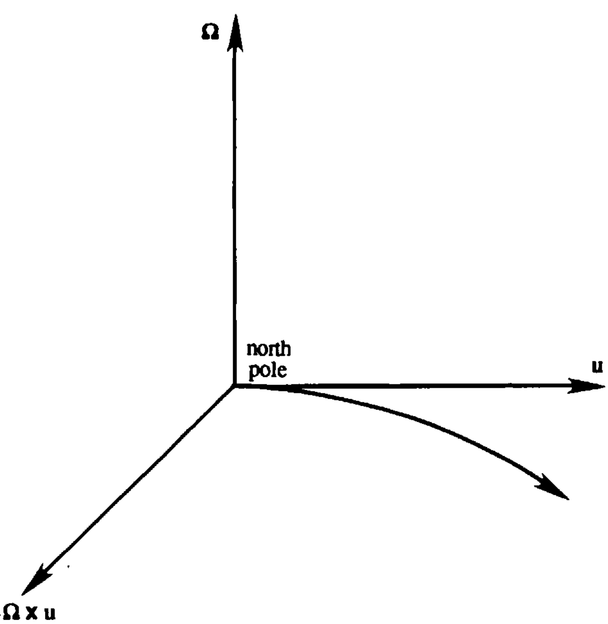


Figure 4.16 Deflection of a particle due to the Coriolis force.

Although the effects of a rotating frame will be commented on occasionally in this and subsequent chapters, most of the discussions involving Coriolis forces are given in Chapter 14, which deals with geophysical fluid dynamics.

13. Mechanical Energy Equation

An equation for kinetic energy of the fluid can be obtained by finding the scalar product of the momentum equation and the velocity vector. The kinetic energy equation is therefore not a separate principle, and is not the same as the first law of thermodynamics. We shall derive several forms of the equation in this section. The Coriolis force, which is perpendicular to the velocity vector, does not contribute to any of the energy equations. The equation of motion is

$$\rho \frac{Du_i}{Dt} = \rho g_i + \frac{\partial \tau_{ij}}{\partial x_j}.$$

Multiplying by u_i (and, of course, summing over i), we obtain

$$\rho \frac{D}{Dt}\left(\frac{1}{2}u_i^2\right) = \rho u_i g_i + u_i \frac{\partial \tau_{ij}}{\partial x_j}, \tag{4.56}$$

where, for the sake of notational simplicity, we have written u_i^2 for $u_i u_i = u_1^2 + u_2^2 + u_3^2$. A summation over i is therefore implied in u_i^2, although no repeated index is explicitly written. Equation (4.56) is the simplest as well as most revealing mechanical energy equation. Recall from Section 7 that the resultant imbalance of the surface forces at a point is $\nabla \cdot \boldsymbol{\tau}$, per unit volume. Equation (4.56) therefore says that the rate of increase of kinetic energy at a point equals the sum of the rate of work done by body force $\mathbf{g}$ and the rate of work done by the net surface force $\nabla \cdot \boldsymbol{\tau}$ per unit volume.

Other forms of the mechanical energy equation are obtained by combining Eq. (4.56) with the continuity equation in various ways. For example, $\rho u_i^2/2$ times the continuity equation is

$$\frac{1}{2}\rho u_i^2 \left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j)\right] = 0,$$

which, when added to Eq. (4.56), gives

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u_i^2\right) + \frac{\partial}{\partial x_j}\left[u_j \frac{1}{2}\rho u_i^2\right] = \rho u_i g_i + u_i \frac{\partial \tau_{ij}}{\partial x_j}.$$

Using vector notation, and defining $E \equiv \rho u_i^2/2$ as the kinetic energy per unit volume, this becomes

$$\frac{\partial E}{\partial t} + \nabla \cdot (\mathbf{u}E) = \rho \mathbf{u} \cdot \mathbf{g} + \mathbf{u} \cdot (\nabla \cdot \boldsymbol{\tau}). \tag{4.57}$$

The second term is in the form of divergence of kinetic energy flux $\mathbf{u}E$. Such *flux divergence* terms frequently arise in energy balances and can be interpreted as the net loss at a point due to divergence of a flux. For example, if the source terms on the right-hand side of Eq. (4.57) are zero, then the local E will increase with time if

$\nabla \cdot (\mathbf{u}E)$ is negative. Flux divergence terms are also called *transport* terms because they transfer quantities from one region to another without making a net contribution over the entire field. When integrated over the entire volume, their contribution vanishes if there are no sources at the boundaries. For example, Gauss' theorem transforms the volume integral of $\nabla \cdot (\mathbf{u}E)$ as

$$\int_V \nabla \cdot (\mathbf{u}E)\, dV = \int_A E\mathbf{u} \cdot d\mathbf{A},$$

which vanishes if the flux $\mathbf{u}E$ is zero at the boundaries.

Concept of Deformation Work and Viscous Dissipation

Another useful form of the kinetic energy equation will now be derived by examining how kinetic energy can be lost to internal energy by deformation of fluid elements. In Eq. (4.56) the term $u_i(\partial\tau_{ij}/\partial x_j)$ is velocity times the net force imbalance at a point due to differences of stress on opposite faces of an element; the net force accelerates the local fluid and increases its kinetic energy. However, this is *not* the total rate of work done by stress on the element, and the remaining part goes into *deforming* the element without accelerating it. The total rate of work done by surface forces on a fluid element must be $\partial(\tau_{ij}u_i)/\partial x_j$, because this can be transformed to a surface integral of $\tau_{ij}u_i$ over the element. (Here $\tau_{ij}\, dA_j$ is the force on an area element, and $\tau_{ij}u_i\, dA_j$ is the scalar product of force and velocity. The total rate of work done by surface forces is therefore the surface integral of $\tau_{ij}u_i$.) The total work rate per volume at a point can be split up into two components:

$$\underbrace{\frac{\partial}{\partial x_j}(u_i\tau_{ij})}_{\substack{\text{total work}\\ \text{(rate/volume)}}} = \underbrace{\tau_{ij}\frac{\partial u_i}{\partial x_j}}_{\substack{\text{deformation}\\ \text{work}\\ \text{(rate/volume)}}} + \underbrace{u_i\frac{\partial \tau_{ij}}{\partial x_j}}_{\substack{\text{increase}\\ \text{of KE}\\ \text{(rate/volume)}}}.$$

We have seen from Eq. (4.56) that the last term in the preceding equation results in an increase of kinetic energy of the element. Therefore, the rest of the work rate per volume represented by $\tau_{ij}(\partial u_i/\partial x_j)$ can only deform the element and increase its *internal* energy.

The *deformation work* rate can be rewritten using the symmetry of the stress tensor. In Chapter 2, Section 11 it was shown that the contracted product of a symmetric tensor and an antisymmetric tensor is zero. The product $\tau_{ij}(\partial u_i/\partial x_j)$ is therefore equal to τ_{ij} times the *symmetric* part of $\partial u_i/\partial x_j$, namely e_{ij}. Thus

$$\text{Deformation work rate per volume} = \tau_{ij}\frac{\partial u_i}{\partial x_j} = \tau_{ij}e_{ij}. \qquad (4.58)$$

On substituting the Newtonian constitutive equation

$$\tau_{ij} = -p\delta_{ij} + 2\mu e_{ij} - \tfrac{2}{3}\mu(\nabla \cdot \mathbf{u})\delta_{ij},$$

relation (4.58) becomes

$$\text{Deformation work} = -p(\nabla \cdot \mathbf{u}) + 2\mu e_{ij}e_{ij} - \tfrac{2}{3}\mu(\nabla \cdot \mathbf{u})^2,$$

where we have used $e_{ij}\delta_{ij} = e_{ii} = \nabla \cdot \mathbf{u}$. Denoting the viscous term by ϕ, we obtain

$$\text{Deformation work (rate per volume)} = -p(\nabla \cdot \mathbf{u}) + \phi, \tag{4.59}$$

where

$$\phi \equiv 2\mu e_{ij}e_{ij} - \tfrac{2}{3}\mu(\nabla \cdot \mathbf{u})^2 = 2\mu\left[e_{ij} - \tfrac{1}{3}(\nabla \cdot \mathbf{u})\delta_{ij}\right]^2. \tag{4.60}$$

The validity of the last term in Eq. (4.60) can easily be verified by completing the square (Exercise 5).

In order to write the energy equation in terms of ϕ, we first rewrite Eq. (4.56) in the form

$$\rho\frac{D}{Dt}\left(\tfrac{1}{2}u_i^2\right) = \rho g_i u_i + \frac{\partial}{\partial x_j}(u_i\tau_{ij}) - \tau_{ij}e_{ij}, \tag{4.61}$$

where we have used $\tau_{ij}(\partial u_i/\partial x_j) = \tau_{ij}e_{ij}$. Using Eq. (4.59) to rewrite the deformation work rate per volume, Eq. (4.61) becomes

$$\rho\frac{D}{Dt}\left(\tfrac{1}{2}u_i^2\right) = \underbrace{\rho\mathbf{g}\cdot\mathbf{u}}_{\substack{\text{rate of work by}\\ \text{body force}}} + \underbrace{\frac{\partial}{\partial x_j}(u_i\tau_{ij})}_{\substack{\text{total rate of}\\ \text{work by } \tau}} + \underbrace{p(\nabla\cdot\mathbf{u})}_{\substack{\text{rate of work}\\ \text{by volume}\\ \text{expansion}}} - \underbrace{\phi}_{\substack{\text{rate of}\\ \text{viscous}\\ \text{dissipation}}} \tag{4.62}$$

It will be shown in Section 14 that the last two terms in the preceding equation (representing pressure and viscous contributions to the rate of deformation work) also appear in the *internal* energy equation but with their signs changed. The term $p(\nabla \cdot \mathbf{u})$ can be of either sign, and converts mechanical to internal energy, or vice versa, by volume changes. The viscous term ϕ is always positive and represents a rate of loss of mechanical energy and a gain of internal energy due to deformation of the element. The term $\tau_{ij}e_{ij} = p(\nabla \cdot \mathbf{u}) - \phi$ represents the total deformation work rate per volume; the part $p(\nabla \cdot \mathbf{u})$ is the reversible conversion to internal energy by volume changes, and the part ϕ is the irreversible conversion to internal energy due to viscous effects.

The quantity ϕ defined in Eq. (4.60) is proportional to μ and represents the rate of *viscous dissipation* of kinetic energy per unit volume. Equation (4.60) shows that it is proportional to the square of velocity gradients and is therefore more important in regions of high shear. The resulting heat could appear as a hot lubricant in a bearing, or as burning of the surface of a spacecraft on reentry into the atmosphere.

Equation in Terms of Potential Energy

So far we have considered kinetic energy as the only form of mechanical energy. In doing so we have found that the effects of gravity appear as work done on a fluid particle, as Eq. (4.62) shows. However, the rate of work done by body forces can be taken to the left-hand side of the mechanical energy equations and be interpreted as changes in the potential energy. Let the body force be represented as the gradient of a scalar potential $\Pi = gz$, so that

$$u_i g_i = -u_i\frac{\partial}{\partial x_i}(gz) = -\frac{D}{Dt}(gz),$$

where we have used $\partial(gz)/\partial t = 0$, because z and t are independent. Equation (4.62) then becomes

$$\rho \frac{D}{Dt}\left(\frac{1}{2}u_i^2 + gz\right) = \frac{\partial}{\partial x_j}(u_i \tau_{ij}) + p(\nabla \cdot \mathbf{u}) - \phi,$$

in which the function $\Pi = gz$ clearly has the significance of *potential energy* per unit mass. (This identification is possible only for conservative body forces for which a potential may be written.)

Equation for a Fixed Region

An integral form of the mechanical energy equation can be derived by integrating the differential form over either a fixed volume or a material volume. The procedure is illustrated here for a fixed volume. We start with Eq. (4.62), but write the left-hand side as given in Eq. (4.57). This gives (in mixed notation)

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_i}(u_i E) = \rho \mathbf{g} \cdot \mathbf{u} + \frac{\partial}{\partial x_j}(u_i \tau_{ij}) + p(\nabla \cdot \mathbf{u}) - \phi,$$

where $E = \rho u_i^2/2$ is the kinetic energy per unit volume. Integrate each term of the foregoing equation over the fixed volume V. The second and fourth terms are in the flux divergence form, so that their volume integrals can be changed to surface integrals by Gauss' theorem. This gives

$$\underbrace{\frac{d}{dt}\int E\, dV}_{\substack{\text{rate of change}\\ \text{of KE}}} + \underbrace{\int E\mathbf{u} \cdot d\mathbf{A}}_{\substack{\text{rate of outflow}\\ \text{across}\\ \text{boundary}}}$$

$$= \underbrace{\int \rho \mathbf{g} \cdot \mathbf{u}\, dV}_{\substack{\text{rate of work}\\ \text{by body}\\ \text{force}}} + \underbrace{\int u_i \tau_{ij}\, dA_j}_{\substack{\text{rate of work}\\ \text{by surface}\\ \text{force}}} + \underbrace{\int p(\nabla \cdot \mathbf{u})\, dV}_{\substack{\text{rate of work}\\ \text{by volume}\\ \text{expansion}}} - \underbrace{\int \phi\, dV}_{\substack{\text{rate of viscous}\\ \text{dissipation}}} \tag{4.63}$$

where each term is a time rate of change. The description of each term in Eq. (4.63) is obvious. The fourth term represents rate of work done by forces at the boundary, because $\tau_{ij}\, dA_j$ is the force in the i direction and $u_i \tau_{ij}\, dA_j$ is the scalar product of the force with the velocity vector.

The energy considerations discussed in this section may at first seem too "theoretical." However, they are very useful in understanding the physics of fluid flows. The concepts presented here will be especially useful in our discussions of turbulent flows (Chapter 13) and wave motions (Chapter 7). It is suggested that the reader work out Exercise 11 at this point in order to acquire a better understanding of the equations in this section.

14. First Law of Thermodynamics: Thermal Energy Equation

The mechanical energy equation presented in the preceding section is derived from the momentum equation and is not a separate principle. In flows with temperature variations we need an independent equation; this is provided by the first law of thermodynamics. Let $\mathbf{q}$ be the heat flux vector per unit area, and e the *internal energy* per unit mass; for a perfect gas $e = C_V T$, where C_V is the specific heat at constant volume (assumed constant). The sum $(e + u_i^2/2)$ can be called the "stored" energy per unit mass. The first law of thermodynamics is most easily stated for a material volume. It says that the rate of *change of stored energy equals the sum of rate of work done and rate of heat addition to a material volume.* That is,

$$\frac{D}{Dt}\int_{\mathcal{V}} \rho\left(e + \tfrac{1}{2}u_i^2\right) d\mathcal{V} = \int_{\mathcal{V}} \rho g_i u_i \, d\mathcal{V} + \int_A \tau_{ij} u_i \, dA_j - \int_A q_i \, dA_i. \tag{4.64}$$

Note that work done by body forces has to be included on the right-hand side if potential energy is *not* included on the left-hand side, as in Eqs. (4.62)–(4.64). (This is clear from the discussion of the preceding section and can also be understood as follows. Imagine a situation where the surface integrals in Eq. (4.64) are zero, and also that e is uniform everywhere. Then a rising fluid particle ($\mathbf{u} \cdot \mathbf{g} < 0$), which is constantly pulled down by gravity, must undergo a decrease of kinetic energy. This is consistent with Eq. (4.64).) The negative sign is needed on the heat transfer term, because the direction of $d\mathbf{A}$ is along the outward normal to the area, and therefore $\mathbf{q} \cdot d\mathbf{A}$ represents the rate of heat *outflow*.

To derive a differential form, all terms need to be expressed in the form of volume integrals. The left-hand side can be written as

$$\frac{D}{Dt}\int_{\mathcal{V}} \rho\left(e + \frac{1}{2}u_i^2\right) d\mathcal{V} = \int_{\mathcal{V}} \rho \frac{D}{Dt}\left(e + \frac{1}{2}u_i^2\right) d\mathcal{V},$$

where Eq. (4.6) has been used. Converting the two surface integral terms into volume integrals, Eq. (4.64) finally gives

$$\rho \frac{D}{Dt}\left(e + \frac{1}{2}u_i^2\right) = \rho g_i u_i + \frac{\partial}{\partial x_j}(\tau_{ij} u_i) - \frac{\partial q_i}{\partial x_i}. \tag{4.65}$$

This is the first law of thermodynamics in the differential form, which has both mechanical and thermal energy terms in it. A thermal energy equation is obtained if the mechanical energy equation (4.62) is subtracted from it. This gives the *thermal energy equation* (commonly called the *heat equation*)

$$\rho \frac{De}{Dt} = -\nabla \cdot \mathbf{q} - p(\nabla \cdot \mathbf{u}) + \phi, \tag{4.66}$$

which says that internal energy increases because of convergence of heat, volume compression, and heating due to viscous dissipation. Note that the last two terms in Eq. (4.66) also appear in mechanical energy equation (4.62) with their signs reversed.

The thermal energy equation can be simplified under the *Boussinesq approximation*, which applies under several restrictions including that in which the flow speeds are small compared to the speed of sound and in which the temperature differences in the flow are small. This is discussed in Section 18. It is shown there that, under these restrictions, heating due to the viscous dissipation term is negligible in Eq. (4.66), and that the term $-p(\nabla \cdot \mathbf{u})$ can be combined with the left-hand side of Eq. (4.66) to give (for a perfect gas)

$$\rho C_{\mathrm{p}} \frac{DT}{Dt} = -\nabla \cdot \mathbf{q}. \tag{4.67}$$

If the heat flux obeys the Fourier law

$$\mathbf{q} = -k\nabla T,$$

then, if $k = \text{const.}$, Eq. (4.67) simplifies to:

$$\boxed{\frac{DT}{Dt} = \kappa \nabla^2 T.} \tag{4.68}$$

where $\kappa \equiv k/\rho C_{\mathrm{p}}$ is the *thermal diffusivity*, stated in m^2/s and which is the same as that of the momentum diffusivity ν.

The viscous heating term ϕ may be negligible in the thermal energy equation (4.66), but not in the mechanical energy equation (4.62). In fact, there must be a sink of mechanical energy so that a steady state can be maintained in the presence of the various types of forcing.

15. Second Law of Thermodynamics: Entropy Production

The second law of thermodynamics essentially says that real phenomena can only proceed in a direction in which the "disorder" of an isolated system increases. Disorder of a system is a measure of the degree of *uniformity* of macroscopic properties in the system, which is the same as the degree of randomness in the molecular arrangements that generate these properties. In this connection, disorder, uniformity, and randomness have essentially the same meaning. For analogy, a tray containing red balls on one side and white balls on the other has more order than in an arrangement in which the balls are mixed together. A real phenomenon must therefore proceed in a direction in which such orderly arrangements decrease because of "mixing." Consider two possible states of an isolated fluid system, one in which there are nonuniformities of temperature and velocity and the other in which these properties are uniform. Both of these states have the same internal energy. Can the system spontaneously go from the state in which its properties are uniform to one in which they are nonuniform? The second law asserts that it cannot, based on experience. Natural processes, therefore, tend to cause mixing due to transport of heat, momentum, and mass.

A consequence of the second law is that there must exist a property called *entropy*, which is related to other thermodynamic properties of the medium. In addition, the second law says that the entropy of an isolated system can only increase; entropy is therefore a measure of disorder or randomness of a system. Let S be the entropy per unit mass. It is shown in Chapter 1, Section 8 that the change of entropy is related to the changes of internal energy e and specific volume v ($= 1/\rho$) by

$$T\,dS = de + p\,dv = de - \frac{p}{\rho^2}\,d\rho.$$

The rate of change of entropy following a fluid particle is therefore

$$T\frac{DS}{Dt} = \frac{De}{Dt} - \frac{p}{\rho^2}\frac{D\rho}{Dt}. \tag{4.69}$$

Inserting the internal energy equation (see Eq. (4.66))

$$\rho\frac{De}{Dt} = -\nabla\cdot\mathbf{q} - p(\nabla\cdot\mathbf{u}) + \phi,$$

and the continuity equation

$$\frac{D\rho}{Dt} = -\rho(\nabla\cdot\mathbf{u}),$$

the entropy production equation (4.69) becomes

$$\begin{aligned}\rho\frac{DS}{Dt} &= -\frac{1}{T}\frac{\partial q_i}{\partial x_i} + \frac{\phi}{T}\\ &= -\frac{\partial}{\partial x_i}\left(\frac{q_i}{T}\right) - \frac{q_i}{T^2}\frac{\partial T}{\partial x_i} + \frac{\phi}{T}.\end{aligned}$$

Using Fourier's law of heat conduction, this becomes

$$\rho\frac{DS}{Dt} = -\frac{\partial}{\partial x_i}\left(\frac{q_i}{T}\right) + \frac{k}{T^2}\left(\frac{\partial T}{\partial x_i}\right)^2 + \frac{\phi}{T}.$$

The first term on the right-hand side, which has the form (heat gain)/T, is the entropy gain due to reversible heat transfer because this term does not involve heat conductivity. The last two terms, which are proportional to the square of temperature and velocity gradients, represent the *entropy production* due to heat conduction and viscous generation of heat. The second law of thermodynamics requires that the entropy production due to irreversible phenomena should be positive, so that

$$\mu, k > 0.$$

An explicit appeal to the second law of thermodynamics is therefore not required in most analyses of fluid flows because it has already been satisfied by taking positive values for the molecular coefficients of viscosity and thermal conductivity.

If the flow is inviscid and nonheat conducting, entropy is preserved along the particle paths.

16. Bernoulli Equation

Various conservation laws for mass, momentum, energy, and entropy were presented in the preceding sections. The well-known Bernoulli (4.46) equation is not a separate law, but is derived from the momentum equation for *inviscid* flows, namely, the Euler equation (4.46):

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial}{\partial x_i}(gz) - \frac{1}{\rho}\frac{\partial p}{\partial x_i},$$

where we have assumed that gravity $\mathbf{g} = -\nabla(gz)$ is the only body force. The advective acceleration can be expressed in terms of vorticity as follows:

$$\begin{aligned} u_j \frac{\partial u_i}{\partial x_j} &= u_j \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) + u_j \frac{\partial u_j}{\partial x_i} = u_j r_{ij} + \frac{\partial}{\partial x_i}\left(\frac{1}{2} u_j u_j \right) \\ &= -u_j \varepsilon_{ijk} \omega_k + \frac{\partial}{\partial x_i}\left(\frac{1}{2} q^2 \right) = -(\mathbf{u} \times \boldsymbol{\omega})_i + \frac{\partial}{\partial x_i}\left(\frac{1}{2} q^2 \right), \end{aligned} \tag{4.70}$$

where we have used $r_{ij} = -\varepsilon_{ijk}\omega_k$ (see Eq. 3.23), and used the customary notation

$$q^2 = u_j^2 = \text{twice kinetic energy}.$$

Then the Euler equation becomes

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_i}\left(\frac{1}{2} q^2 \right) + \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_i}(gz) = (\mathbf{u} \times \boldsymbol{\omega})_i. \tag{4.71}$$

Now assume that ρ is a function of p only. A flow in which $\rho = \rho(p)$ is called a *barotropic flow*, of which isothermal and isentropic (p/ρ^γ = constant) flows are special cases. For such a flow we can write

$$\frac{1}{\rho}\frac{\partial p}{\partial x_i} = \frac{\partial}{\partial x_i} \int \frac{dp}{\rho}, \tag{4.72}$$

where dp/ρ is a perfect differential, and therefore the integral does not depend on the path of integration. To show this, note that

$$\int_{\mathbf{x}_0}^{\mathbf{x}} \frac{dp}{\rho} = \int_{\mathbf{x}_0}^{\mathbf{x}} \frac{1}{\rho}\frac{dp}{d\rho}\, d\rho = \int_{\mathbf{x}_0}^{\mathbf{x}} \frac{dP}{d\rho}\, d\rho = P(\mathbf{x}) - P(\mathbf{x}_0), \tag{4.73}$$

where $\mathbf{x}$ is the "field point," $\mathbf{x}_0$ is any arbitrary reference point in the flow, and we have defined the following function of ρ alone:

$$\frac{dP}{d\rho} \equiv \frac{1}{\rho}\frac{dp}{d\rho}. \tag{4.74}$$

The gradient of Eq. (4.73) gives

$$\frac{\partial}{\partial x_i} \int_{\mathbf{x}_0}^{\mathbf{x}} \frac{dp}{\rho} = \frac{\partial P}{\partial x_i} = \frac{dP}{dp}\frac{\partial p}{\partial x_i} = \frac{1}{\rho}\frac{\partial p}{\partial x_i},$$

where Eq. (4.74) has been used. The preceding equation is identical to Eq. (4.72).

Using Eq. (4.72), the Euler equation (4.71) becomes

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_i}\left[\frac{1}{2} q^2 + \int \frac{dp}{\rho} + gz \right] = (\mathbf{u} \times \boldsymbol{\omega})_i.$$

Defining the Bernoulli function

$$B \equiv \frac{1}{2}q^2 + \int \frac{dp}{\rho} + gz = \frac{1}{2}q^2 + P + gz, \tag{4.75}$$

the Euler equation becomes (using vector notation)

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla B = \mathbf{u} \times \boldsymbol{\omega}. \tag{4.76}$$

Bernouli equations are integrals of the conservation laws and have wide applicability as shown by the examples that follow. Important deductions can be made from the preceding equation by considering two special cases, namely a steady flow (rotational or irrotational) and an unsteady irrotational flow. These are described in what follows.

Steady Flow

In this case Eq. (4.76) reduces to

$$\nabla B = \mathbf{u} \times \boldsymbol{\omega}. \tag{4.77}$$

The left-hand side is a vector normal to the surface B = constant, whereas the right-hand side is a vector perpendicular to both $\mathbf{u}$ and $\boldsymbol{\omega}$ (Figure 4.17). It follows that surfaces of constant B must contain the streamlines and vortex lines. Thus, an inviscid, steady, barotropic flow satisfies

$$\boxed{\frac{1}{2}q^2 + \int \frac{dp}{\rho} + gz = \text{constant along streamlines and vortex lines}} \tag{4.78}$$

which is called *Bernoulli's equation*. If, in addition, the flow is irrotational ($\boldsymbol{\omega} = 0$), then Eq. (4.72) shows that

$$\frac{1}{2}q^2 + \int \frac{dp}{\rho} + gz = \text{constant everywhere.} \tag{4.79}$$

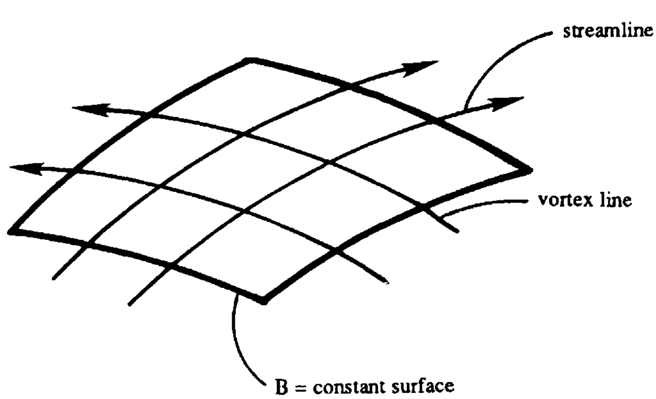


Figure 4.17 Bernoulli's theorem. Note that the streamlines and vortex lines can be at an arbitrary angle.

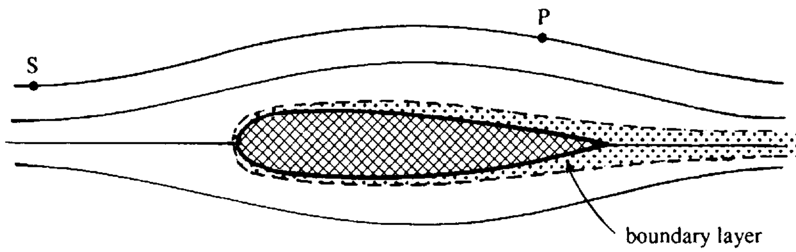


Figure 4.18 Flow over a solid object. Flow outside the boundary layer is irrotational.

It may be shown that a sufficient condition for the existence of the surfaces containing streamlines and vortex lines is that the flow be barotropic. Incidentally, these are called Lamb surfaces in honor of the distinguished English applied mathematician and hydrodynamicist, Horace Lamb. In a general, that is, nonbarotropic flow, a path composed of streamline and vortex line segments can be drawn between any two points in a flow field. Then Eq. (4.78) is valid with the proviso that the integral be evaluated on the specific path chosen. As written, Eq. (4.78) requires the restrictions that the flow be steady, inviscid, and have only gravity (or other conservative) body forces acting upon it. Irrotational flows are studied in Chapter 6. We shall note only the important point here that, in a nonrotating frame of reference, barotropic irrotational flows remain irrotational if viscous effects are negligible. Consider the flow around a solid object, say an airfoil (Figure 4.18). The flow is irrotational at all points outside the thin viscous layer close to the surface of the body. This is because a particle P on a streamline outside the viscous layer started from some point S, where the flow is uniform and consequently irrotational. The Bernoulli equation (4.79) is therefore satisfied everywhere outside the viscous layer in this example.

Unsteady Irrotational Flow

An unsteady form of Bernoulli's equation can be derived only if the flow is irrotational. For irrotational flows the velocity vector can be written as the gradient of a scalar potential ϕ (called velocity potential):

$$\mathbf{u} \equiv \nabla\phi. \tag{4.80}$$

The validity of Eq. (4.80) can be checked by noting that it automatically satisfies the conditions of irrotationality

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial u_j}{\partial x_i} \qquad i \neq j.$$

On inserting Eq. (4.80) into Eq. (4.76), we obtain

$$\nabla\left[\frac{\partial\phi}{\partial t} + \frac{1}{2}q^2 + \int\frac{dp}{\rho} + gz\right] = 0,$$

that is

$$\boxed{\frac{\partial\phi}{\partial t} + \frac{1}{2}q^2 + \int\frac{dp}{\rho} + gz = F(t),} \tag{4.81}$$

where the integrating function $F(t)$ is independent of location. This form of the Bernoulli equation will be used in studying irrotational wave motions in Chapter 7.

Energy Bernoulli Equation

Return to Eq. (4.65) in the steady state with neither heat conduction nor viscous stresses. Then $\tau_{ij} = -p\delta_{ij}$ and Eq. (4.65) becomes

$$\rho u_i \frac{\partial}{\partial x_i}(e + q^2/2) = \rho u_i g_i - \frac{\partial}{\partial x_i}(\rho u_i p/\rho).$$

If the body force per unit mass g_i is conservative, say gravity, then $g_i = -(\partial/\partial x_i)(gz)$, which is the gradient of a scalar potential. In addition, from mass conservation, $\partial(\rho u_i)/\partial x_i = 0$ and thus

$$\rho u_i \frac{\partial}{\partial x_i}\left(e + \frac{p}{\rho} + \frac{q^2}{2} + gz\right) = 0. \tag{4.82}$$

From Eq. (1.13), $h = e + p/\rho$. Eq. (4.82) now states that gradients of $B' = h + q^2/2 + gz$ must be normal to the local streamline direction u_i. Then $B' = h + q^2/2 + gz$ is a constant on streamlines. We showed in the previous section that inviscid, non-heat conducting flows are isentropic (S is conserved along particle paths), and in Eq. (1.18) we had the relation $dp/\rho = dh$ when $S =$ constant. Thus the path integral $\int dp/\rho$ becomes a function h of the endpoints only if, in the momentum Bernoulli equation, both heat conduction and viscous stresses may be neglected. This latter form from the energy equation becomes very useful for high-speed gas flows to show the interplay between kinetic energy and internal energy or enthalpy or temperature along a streamline.

17. *Applications of Bernoulli's Equation*

Application of Bernoulli's equation will now be illustrated for some simple flows.

Pitot Tube

Consider first a simple device to measure the local velocity in a fluid stream by inserting a narrow bent tube (Figure 4.19). This is called a *pitot tube*, after the French mathematician Henry Pitot (1695–1771), who used a bent glass tube to measure the velocity of the river Seine. Consider two points 1 and 2 at the same level, point 1 being away from the tube and point 2 being immediately in front of the open end where the fluid velocity is zero. Friction is negligible along a streamline through 1 and 2, so that Bernoulli's equation (4.78) gives

$$\frac{p_1}{\rho} + \frac{u_1^2}{2} = \frac{p_2}{\rho} + \frac{u_2^2}{2} = \frac{p_2}{\rho},$$

from which the velocity is found to be

$$u_1 = \sqrt{2(p_2 - p_1)/\rho}.$$

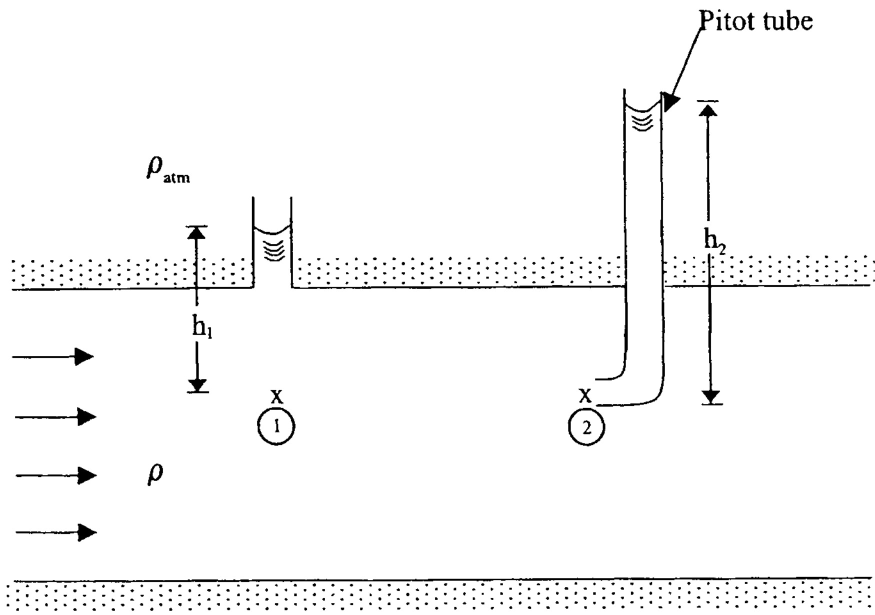


Figure 4.19 Pitot tube for measuring velocity in a duct.

Pressures at the two points are found from the hydrostatic balance

$$p_1 = \rho g h_1 \qquad \text{and} \qquad p_2 = \rho g h_2,$$

so that the velocity can be found from

$$u_1 = \sqrt{2g(h_2 - h_1)}.$$

Because it is assumed that the fluid density is very much greater than that of the atmosphere to which the tubes are exposed, the pressures at the tops of the two fluid columns are assumed to be the same. They will actually differ by $\rho_{atm} g(h_2 - h_1)$. Use of the hydrostatic approximation above station 1 is valid when the streamlines are straight and parallel between station 1 and the upper wall. In working out this problem, the fluid density also has been taken to be a constant.

The pressure p_2 measured by a pitot tube is called "stagnation pressure," which is larger than the local static pressure. Even when there is no pitot tube to measure the stagnation pressure, it is customary to refer to the local value of the quantity $(p + \rho u^2/2)$ as the local *stagnation pressure*, defined as the pressure that would be reached if the local flow is *imagined* to slow down to zero velocity frictionlessly. The quantity $\rho u^2/2$ is sometimes called the *dynamic pressure*; stagnation pressure is the sum of static and dynamic pressures.

Orifice in a Tank

As another application of Bernoulli's equation, consider the flow through an orifice or opening in a tank (Figure 4.20). The flow is slightly unsteady due to lowering of

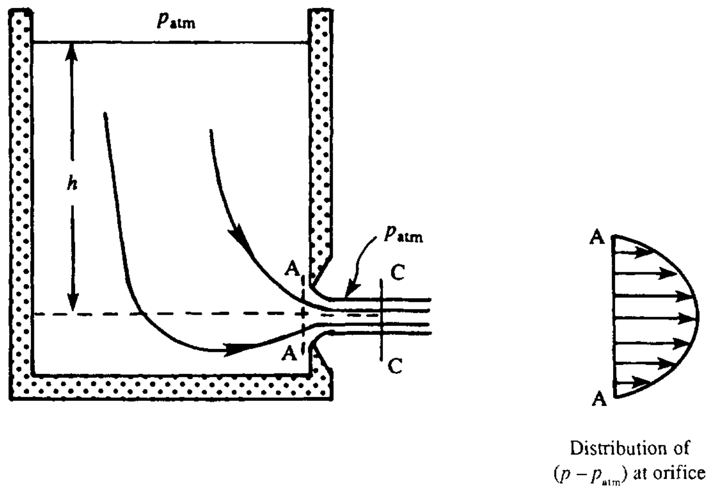


Figure 4.20 Flow through a sharp-edged orifice. Pressure has the atmospheric value everywhere across section CC; its distribution across orifice AA is indicated.

the water level in the tank, but this effect is small if the tank area is large as compared to the orifice area. Viscous effects are negligible everywhere away from the walls of the tank. All streamlines can be traced back to the free surface in the tank, where they have the same value of the Bernoulli constant $B = q^2/2 + p/\rho + gz$. It follows that the flow is irrotational, and B is constant *throughout* the flow.

We want to apply Bernoulli's equation between a point at the free surface in the tank and a point in the jet. However, the conditions right at the opening (section A in Figure 4.20) are not simple because the pressure is *not* uniform across the jet. Although pressure has the atmospheric value everywhere on the free surface of the jet (neglecting small surface tension effects), it is not equal to the atmospheric pressure *inside* the jet at this section. The streamlines at the orifice are curved, which requires that pressure must vary across the width of the jet in order to balance the centrifugal force. The pressure distribution across the orifice (section A) is shown in Figure 4.20. However, the streamlines in the jet become parallel at a short distance away from the orifice (section C in Figure 4.20), where the jet area is smaller than the orifice area. The pressure across section C is uniform and equal to the atmospheric value because it has that value at the surface of the jet.

Application of Bernoulli's equation between a point on the free surface in the tank and a point at C gives

$$\frac{p_{\text{atm}}}{\rho} + gh = \frac{p_{\text{atm}}}{\rho} + \frac{u^2}{2},$$

from which the jet velocity is found as

$$u = \sqrt{2gh},$$

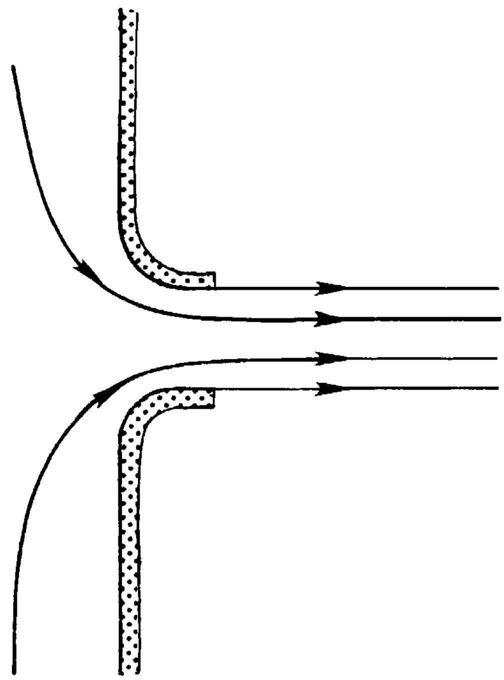


Figure 4.21 Flow through a rounded orifice.

which simply states that the loss of potential energy equals the gain of kinetic energy. The mass flow rate is

$$\dot{m} = \rho A_c u = \rho A_c \sqrt{2gh},$$

where A_c is the area of the jet at C. For orifices having a sharp edge, A_c has been found to be $\approx$62% of the orifice area.

If the orifice happens to have a well-rounded opening (Figure 4.21), then the jet does not contract. The streamlines right at the exit are then parallel, and the pressure at the exit is uniform and equal to the atmospheric pressure. Consequently the mass flow rate is simply $\rho A\sqrt{2gh}$, where A equals the orifice area.

18. Boussinesq Approximation

For flows satisfying certain conditions, Boussinesq in 1903 suggested that the density changes in the fluid can be neglected except in the gravity term where ρ is multiplied by g. This approximation also treats the other properties of the fluid (such as μ, k, C_p) as constants. A formal justification, and the conditions under which the Boussinesq approximation holds, is given in Spiegel and Veronis (1960). Here we shall discuss the basis of the approximation in a somewhat intuitive manner and examine the resulting simplifications of the equations of motion.

Continuity Equation

The Boussinesq approximation replaces the continuity equation

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla \cdot \mathbf{u} = 0, \tag{4.83}$$

by the incompressible form

$$\nabla \cdot \mathbf{u} = 0. \tag{4.84}$$

However, this does not mean that the density is regarded as constant along the direction of motion, but simply that the magnitude of $\rho^{-1}(D\rho/Dt)$ is small in comparison to the magnitudes of the velocity gradients in $\nabla \cdot \mathbf{u}$. We can immediately think of several situations where the density variations cannot be neglected as such. The first situation is a steady flow with large Mach numbers (defined as U/c, where U is a typical measure of the flow speed and c is the speed of sound in the medium). At large Mach numbers the compressibility effects are large, because the large pressure changes cause large density changes. It is shown in Chapter 16 that compressibility effects are negligible in flows in which the Mach number is <0.3. A typical value of c for air at ordinary temperatures is $350\,\mathrm{m/s}$, so that the assumption is good for speeds $<100\,\mathrm{m/s}$. For water $c = 1470\,\mathrm{m/s}$, but the speeds normally achievable in liquids are much smaller than this value and therefore the incompressibility assumption is very good in liquids.

A second situation in which the compressibility effects are important is unsteady flows. The waves would propagate at infinite speed if the density variations are neglected.

A third situation in which the compressibility effects are important occurs when the vertical scale of the flow is so large that the hydrostatic pressure variations cause large changes in density. In a hydrostatic field the vertical scale in which the density changes become important is of order $c^2/g \sim 10\,\mathrm{km}$ for air. (This length agrees with the e-folding height RT/g of an "isothermal atmosphere," because $c^2 = \gamma RT$; see Chapter 1, Section 10.) The Boussinesq approximation therefore requires that the vertical scale of the flow be $L \ll c^2/g$.

In the three situations mentioned the medium is regarded as "compressible," in which the density depends strongly on pressure. Now suppose the compressibility effects are small, so that the density changes are caused by temperature changes alone, as in a thermal convection problem. In this case the Boussinesq approximation applies when the temperature variations in the flow are small. Assume that ρ changes with T according to

$$\frac{\delta\rho}{\rho} = -\alpha\,\delta T,$$

where $\alpha = -\rho^{-1}(\partial\rho/\partial T)_{\mathrm{p}}$ is the thermal expansion coefficient. For a perfect gas $\alpha = 1/T \sim 3 \times 10^{-3}\,\mathrm{K}^{-1}$ and for typical liquids $\alpha \sim 5 \times 10^{-4}\,\mathrm{K}^{-1}$. With a temperature difference in the fluid of $10\,^\circ\mathrm{C}$, the variation of density can be only a few percent at most. It turns out that $\rho^{-1}(D\rho/Dt)$ can also be no larger than a few percent of the velocity gradients in $\nabla \cdot \mathbf{u}$. To see this, assume that the flow field is characterized by a length scale L, a velocity scale U, and a temperature scale δT. By this we mean that the velocity varies by U and the temperature varies by δT, in a distance of order L. The ratio of the magnitudes of the two terms in the continuity equation is

$$\frac{(1/\rho)(D\rho/Dt)}{\nabla \cdot \mathbf{u}} \sim \frac{(1/\rho)u(\partial\rho/\partial x)}{\partial u/\partial x} \sim \frac{(U/\rho)(\delta\rho/L)}{U/L} = \frac{\delta\rho}{\rho} = \alpha\delta T \ll 1,$$

which allows us to replace continuity equation (4.83) by its incompressible form (4.84).

Momentum Equation

Because of the incompressible continuity equation $\nabla \cdot \mathbf{u} = 0$, the stress tensor is given by Eq. (4.41). From Eq. (4.45), the equation of motion is then

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p + \rho\mathbf{g} + \mu\nabla^2\mathbf{u}. \tag{4.85}$$

Consider a hypothetical static reference state in which the density is ρ_0 everywhere and the pressure is $p_0(z)$, so that $\nabla p_0 = \rho_0\mathbf{g}$. Subtracting this state from Eq. (4.85) and writing $p = p_0 + p'$ and $\rho = \rho_0 + \rho'$, we obtain

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p' + \rho'\mathbf{g} + \mu\nabla^2\mathbf{u}. \tag{4.86}$$

Dividing by ρ_0, we obtain

$$\left(1 + \frac{\rho'}{\rho_0}\right)\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p' + \frac{\rho'}{\rho_0}\mathbf{g} + \nu\nabla^2\mathbf{u},$$

where $\nu = \mu/\rho_0$. The ratio ρ'/ρ_0 appears in both the inertia and the buoyancy terms. For small values of ρ'/ρ_0, the density variations generate only a small correction to the inertia term and can be neglected. However, the buoyancy term $\rho' g/\rho_0$ is very important and cannot be neglected. For example, it is these density variations that drive the convective motion when a layer of fluid is heated. The magnitude of $\rho' g/\rho_0$ is therefore of the same order as the vertical acceleration $\partial w/\partial t$ or the viscous term $\nu\nabla^2 w$. We conclude that the density variations are negligible the momentum equation, except when ρ is multiplied by g.

Heat Equation

From Eq. (4.66), the thermal energy equation is

$$\rho\frac{De}{Dt} = -\nabla \cdot \mathbf{q} - p(\nabla \cdot \mathbf{u}) + \phi. \tag{4.87}$$

Although the continuity equation is approximately $\nabla \cdot \mathbf{u} = 0$, an important point is that the volume expansion term $p(\nabla \cdot \mathbf{u})$ is *not* negligible compared to other dominant terms of Eq. (4.87); only for incompressible liquids is $p(\nabla \cdot \mathbf{u})$ negligible in Eq. (4.87). We have

$$-p\nabla \cdot \mathbf{u} = \frac{p}{\rho}\frac{D\rho}{Dt} \simeq \frac{p}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_{\mathrm{p}}\frac{DT}{Dt} = -p\alpha\frac{DT}{Dt}.$$

Assuming a perfect gas, for which $p = \rho RT$, $C_p - C_v = R$ and $\alpha = 1/T$, the foregoing estimate becomes

$$-p\nabla \cdot \mathbf{u} = -\rho RT\alpha \frac{DT}{Dt} = -\rho(C_p - C_v)\frac{DT}{Dt}.$$

Equation (4.87) then becomes

$$\rho C_p \frac{DT}{Dt} = -\nabla \cdot \mathbf{q} + \phi, \tag{4.88}$$

where we used $e = C_v T$ for a perfect gas. Note that we would have gotten C_v (instead of C_p) on the left-hand side of Eq. (4.88) if we had dropped $\nabla \cdot \mathbf{u}$ in Eq. (4.87).

Now we show that the heating due to viscous dissipation of energy is negligible under the restrictions underlying the Boussinesq approximation. Comparing the magnitudes of viscous heating with the left-hand side of Eq. (4.88), we obtain

$$\frac{\phi}{\rho C_p(DT/Dt)} \sim \frac{2\mu e_{ij}e_{ij}}{\rho C_p u_j(\partial T/\partial x_j)} \sim \frac{\mu U^2/L^2}{\rho_0 C_p U \delta T/L} = \frac{\nu}{C_p}\frac{U}{\delta T L}.$$

In typical situations this is extremely small ($\sim 10^{-7}$). Neglecting ϕ, and assuming Fourier's law of heat conduction

$$\mathbf{q} = -k\nabla T,$$

the heat equation (4.88) finally reduces to (if $k =$ const.)

$$\frac{DT}{Dt} = \kappa\nabla^2 T,$$

where $\kappa \equiv k/\rho C_p$ is the *thermal diffusivity.*

Summary: The Boussinesq approximation applies if the Mach number of the flow is small, propagation of sound or shock waves is not considered, the vertical scale of the flow is not too large, and the temperature differences in the fluid are small. Then the density can be treated as a constant in both the continuity and the momentum equations, except in the gravity term. Properties of the fluid such as μ, k, and C_p are also assumed constant in this approximation. Omitting Coriolis forces, the set of equations corresponding to the Boussinesq approximation is

$$\begin{aligned}
&\nabla \cdot \mathbf{u} = 0\\
&\frac{Du}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu\nabla^2 u\\
&\frac{Dv}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \nu\nabla^2 v\\
&\frac{Dw}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - \frac{\rho g}{\rho_0} + \nu\nabla^2 w\\
&\frac{DT}{Dt} = \kappa\nabla^2 T\\
&\rho = \rho_0[1 - \alpha(T - T_0)],
\end{aligned} \tag{4.89}$$

where the z-axis is taken upward. The constant ρ_0 is a reference density corresponding to a reference temperature T_0, which can be taken to be the mean temperature in the flow or the temperature at a boundary. Applications of the Boussinesq set can be found in several places throughout the book, for example, in the problems of wave propagation in a density-stratified medium, thermal instability, turbulence in a stratified medium, and geophysical fluid dynamics.

19. Boundary Conditions

The differential equations we have derived for the conservation laws are subject to boundary conditions in order to properly formulate any problem. Specifically, the Navier-Stokes equations are of a form that requires the velocity vector to be given on all surfaces bounding the flow domain.

If we are solving for an external flow, that is, a flow over some body, we must specify the velocity vector and the thermodynamic state on a closed distant surface. On a solid boundary or at the interface between two immiscible liquids, conditions may be derived from the three basic conservation laws as follows.

In Figure 4.22, a "pillbox" is drawn through the interface surface separating medium 1 (fluid) from medium 2 (solid or liquid immiscible with fluid 1). Here $\mathbf{dA}_1$ and $\mathbf{dA}_2$ are elements of the end face areas in medium 1 and medium 2, respectively, locally tangent to the interface, and separated from each other by a distance l. Now apply the conservation laws to the volume defined by the pillbox. Next, let $l \to 0$, keeping A_1 and A_2 in the different media. As $l \to 0$, all volume integrals $\to 0$ and the integral over the side area, which is proportional to l, tends to zero as well. Define a unit vector $\mathbf{n}$, normal to the interface at the pillbox and pointed into medium 1. Mass conservation gives $\rho_1 \mathbf{u}_1 \cdot \mathbf{n} = \rho_2 \mathbf{u}_2 \cdot \mathbf{n}$ at each point on the interface as the end face area becomes small.

If medium 2 is a solid, then $\mathbf{u}_2 = 0$ there. If medium 1 and medium 2 are immiscible liquids, no mass flows across the boundary surface. In either case, $\mathbf{u}_1 \cdot \mathbf{n} = 0$ on the boundary. The same procedure applied to the integral form of the momentum equation (4.16) gives the result that the force/area on the surface, $n_i \tau_{ij}$ is continuous across the interface if surface tension is neglected. If surface tension is included, a jump in pressure in the direction normal to the interface must be added; see Chapter 1, Section 6.

Applying the integral form of energy conservation (4.64) to a pillbox of infinitesimal height l gives the result $n_i q_i$ as continuous across the interface, or explicity, $k_1(\partial T_1/\partial n) = k_2(\partial T_2/\partial n)$ at the interface surface. The heat flux must be continuous at the interface; it cannot store heat.

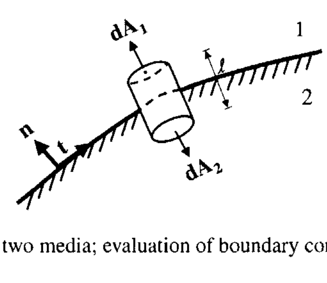


Figure 4.22 Interface between two media; evaluation of boundary conditions.

Two more boundary conditions are required to completely specify a problem and these are not consequences of any conservation law. These boundary conditions are: no slip of a viscous fluid is permitted at a solid boundary $\mathbf{v}_1 \cdot \mathbf{t} = 0$; and no temperature jump is permitted at the boundary $T_1 = T_2$. Here $\mathbf{t}$ is a unit vector tangent to the boundary.

Exercises

1. Let a one-dimensional velocity field be $u = u(x, t)$, with $v = 0$ and $w = 0$. The density varies as $\rho = \rho_0(2 - \cos \omega t)$. Find an expression for $u(x, t)$ if $u(0, t) = U$.

2. In Section 3 we derived the continuity equation (4.8) by starting from the integral form of the law of conservation of mass for a *fixed* region. Derive Eq. (4.8) by starting from an integral form for a *material* volume. [*Hint*: Formulate the principle for a material volume and then use Eq. (4.5).]

3. Consider conservation of angular momentum derived from the angular momentum principle by the word statement: Rate of increase of angular momentum in volume V = net influx of angular momentum across the bounding surface A of V + torques due to surface forces + torques due to body forces. Here, the only torques are due to the same forces that appear in (linear) momentum conservation. The possibilities for body torques and couple stresses have been neglected. The torques due to the surface forces are manipulated as follows. The torque about a point O due to the element of surface force $\tau_{mk} dA_m$ is $\int \epsilon_{ijk} x_j \tau_{mk} dA_m$, where x is the position vector from O to the element dA. Using Gauss' theorem, we write this as a volume integral,

$$\int_V \varepsilon_{ijk} \frac{\partial}{\partial x_m}(x_j \tau_{mk}) dV = \varepsilon_{ijk} \int_V \left(\frac{\partial x_j}{\partial x_m} \tau_{mk} + x_j \frac{\partial \tau_{mk}}{\partial x_m} \right) dV$$
$$= \varepsilon_{ijk} \int_V \left(\tau_{jk} + x_j \frac{\partial \tau_{mk}}{\partial x_m} \right) dV,$$

where we have used $\partial x_j / \partial x_m = \partial_{jm}$. The second term is $\int_V \mathbf{x} \times \nabla \cdot \boldsymbol{\tau}\, dV$ and combines with the remaining terms in the conservation of angular momentum to give $\int_V \mathbf{x} \times (\text{Linear Momentum: Eq. (4.17)})\, dV = \int_V \epsilon_{ijk} \tau_{jk} dV$. Since the left-hand side $= 0$ for any volume V, we conclude that $\varepsilon_{ijk} \tau_{kj} = 0$, which leads to $\tau_{ij} = \tau_{ji}$.

4. Near the end of Section 7 we derived the equation of motion (4.15) by starting from an integral form for a material volume. Derive Eq. (4.15) by starting from the integral statement for a *fixed region*, given by Eq. (4.22).

5. Verify the validity of the second form of the viscous dissipation given in Eq. (4.60). [*Hint*: Complete the square and use $\delta_{ij}\delta_{ij} = \delta_{ii} = 3$.]

6. A rectangular tank is placed on wheels and is given a constant horizontal acceleration a. Show that, at steady state, the angle made by the free surface with the horizontal is given by $\tan \theta = a/g$.

7. A jet of water with a diameter of 8 cm and a speed of 25 m/s impinges normally on a large stationary flat plate. Find the force required to hold the plate stationary.

Compare the average pressure on the plate with the stagnation pressure if the plate is 20 times the area of the jet.

8. Show that the thrust developed by a stationary rocket motor is $F = \rho A U^2 + A(p - p_{\text{atm}})$, where p_{atm} is the atmospheric pressure, and p, ρ, A, and U are, respectively, the pressure, density, area, and velocity of the fluid at the nozzle exit.

9. Consider the propeller of an airplane moving with a velocity U_1. Take a reference frame in which the air is moving and the propeller [disk] is stationary. Then the effect of the propeller is to accelerate the fluid from the upstream value U_1 to the downstream value $U_2 > U_1$. Assuming incompressibility, show that the thrust developed by the propeller is given by

$$F = \frac{\rho A}{2}(U_2^2 - U_1^2),$$

where A is the projected area of the propeller and ρ is the density (assumed constant). Show also that the velocity of the fluid at the plane of the propeller is the average value $U = (U_1 + U_2)/2$. [*Hint*: The flow can be idealized by a pressure jump, of magnitude $\Delta p = F/A$ right at the location of the propeller. Also apply Bernoulli's equation between a section far upstream and a section immediately upstream of the propeller. Also apply the Bernoulli equation between a section immediately downstream of the propeller and a section far downstream. This will show that $\Delta p = \rho(U_2^2 - U_1^2)/2$.]

10. A hemispherical vessel of radius R has a small rounded orifice of area A at the bottom. Show that the time required to lower the level from h_1 to h_2 is given by

$$t = \frac{2\pi}{A\sqrt{2g}}\left[\frac{2}{3}R\left(h_1^{3/2} - h_2^{3/2}\right) - \frac{1}{5}\left(h_1^{5/2} - h_2^{5/2}\right)\right].$$

11. Consider an incompressible planar Couette flow, which is the flow between two parallel plates separated by a distance b. The upper plate is moving parallel to itself at speed U, and the lower plate is stationary. Let the x-axis lie on the lower plate. All flow fields are independent of x. Show that the pressure distribution is hydrostatic and that the solution of the Navier–Stokes equation is

$$u(y) = \frac{Uy}{b}.$$

Write the expressions for the stress and strain rate tensors, and show that the viscous dissipation per unit volume is $\phi = \mu U^2/b^2$.

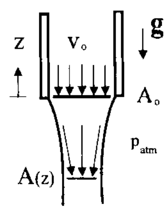
Take a rectangular control volume for which the two horizontal surfaces coincide with the walls and the two vertical surfaces are perpendicular to the flow. Evaluate every term of energy equation (4.63) for this control volume, and show that the balance is between the viscous dissipation and the work done in moving the upper surface.

12. The components of a mass flow vector $\rho\mathbf{u}$ are $\rho u = 4x^2y$, $\rho v = xyz$, $\rho w = yz^2$. Compute the net outflow through the closed surface formed by the planes $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.

(a) Integrate over the closed surface.
(b) Integrate over the volume bounded by that surface.

13. Prove that the velocity field given by $u_r = 0$, $u_\theta = k/(2\pi r)$ can have only two possible values of the circulation. They are (a) $\Gamma = 0$ for any path not enclosing the origin, and (b) $\Gamma = k$ for any path enclosing the origin.

14. Water flows through a pipe in a gravitational field as shown in the accompanying figure. Neglect the effects of viscosity and surface tension. Solve the appropriate conservation equations for the variation of the cross-sectional area of the fluid column $A(z)$ after the water has left the pipe at $z = 0$. The velocity of the fluid at $z = 0$ is uniform at v_0 and the cross-sectional area is A_0.



15. Redo the solution for the "orifice in a tank" problem allowing for the fact that in Fig. 4.20, $h = h(t)$. How long does the tank take to empty?

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Supplemental Reading

Chandrasekhar, S. (1961). *Hydrodynamic and Hydromagnetic Stability*, London: Oxford University Press. (This is a good source to learn the basic equations in a brief and simple way.)