Week 1: (Un)Natural Numbers — Write-Up

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Problem 3 *Higher Induction Practice*

Definition 1 *A binary tree is defined as a tree where each node has 0, 1, or 2 children.*

Theorem 1 *For any binary tree with height h, the number of leaves it has is at most* 2^{h-1}.

Proof. Consider a binary tree with height h. We will prove by induction on h.

Base case Let us assume as a base case where the height of the tree $h = 0$, then the corresponding binary tree has no nodes, i.e. $n_h = 0$ where n_h denotes the number of leaves at a height h. As a result,

$$
\begin{aligned}
2^{h-1} &= 2^{(0)-1} \\
&= 2^{-1} \\
&= \frac{1}{2}
\end{aligned}
$$

Because $0 \leq \frac{1}{2}$, the stated theorem is true for the base case of height $h = 0$.

Inductive Step Now let us assume that for a tree with height h and number of leaves n_h, the tree satisfies the theorem, $n_h \leq 2^{h-1}$. We want to show that the tree with height $h+1$ also satisfies the theorem, $n_{h+1} \leq 2^h$.

Given the definition of a binary tree we know that each node may have up to 2 children. This means $n_{h+1} \leq 2n_h$. But since we assumed $n_h \leq 2^{h-1}$, we would have

$$
\begin{aligned}
n_{h+1} &\leq 2 \cdot 2^{h-1} \\
&= 2^{(h-1)+1} \\
&= 2^h
\end{aligned}
$$

Therefore, we have shown that $n_h \leq 2^{h-1} \implies n_{h+1} \leq 2^h$.

Conclusion By taking the base case where height $h = 0$, and proving the induction step such that $n_h \leq 2^{h-1} \implies n_{h+1} \leq 2^h$, we have shown that $\forall h \in \mathbb{N}, n_h \leq 2^{h-1}$.

□