Week 3: Sweet

Authors: Benjamin (aqn9yv, dlb2ru, ht6xd, iad4de, jmn4fms, lw7jz)

Collaborators and Resources: Definition of Boolean circuits from *Introduction to Theoretical Computer Science* by Boaz Barak

Problem 2 *Maximum number of Inputs (Induction Practice)*

Definition 1 *Let n, m, s be positive integers with $s \geq m$. A Boolean circuit with n inputs, m outputs, and s gates, is a labeled directed acyclic graph (DAG) $G = (V, E)$ with $s + n$ vertices. There are n inputs of the circuit, and each gate s can perform either a unary or binary operation, i.e. it can have either one or two inputs.*

Theorem 1 *The maximum number of inputs for a Boolean circuit that produces one output that depends on all of its inputs with depth d is 2^d for all $d \geq 0$.*

Proof. We will proceed by induction on the depth d of the Boolean circuit.

Base Case Consider a circuit of depth $d = 0$. This implies that the circuit has 1 output, and no gates. Since the output should depend on all the inputs, this implies that the circuit has only 1 input. In this case, $1 \leq 2^d = 2^0 = 1$.

Inductive Hypothesis We now assume that the number of inputs in a circuit with depth d is bounded above by 2^d, in the hopes of concluding that a circuit with depth d+1 takes at most 2^{d+1} inputs.

We begin by noting at a depth of d+1, the longest path from an input to the final output consists of d+1 gates. The $(d+1)^{th}$ gate must take at most 2 inputs, one of which is a circuit of depth d. To achieve a maximal number of inputs, assume the $(d+1)^{th}$ gate takes two inputs from the outputs of two circuits of depth d, which we will call "subcircuits."

We know from our assumption that the number of inputs leading into both subcircuits are at most 2^d. Therefore the circuit with depth d+1 will include at most $2^d + 2^d$ inputs. Therefore, if we denote the number of inputs by i_{d+1} then we have

$$i_{d+1} \leq 2^d + 2^d = 2 \cdot 2^d = 2^{d+1}.$$

Conclusion Thus we have shown in our base case that for a depth of $d = 0$, $i_d \leq 2^d$, and from the induction step, given $i_d \leq 2^d$, we have that $i_{d+1} \leq 2^{d+1}$. So by the principle of induction we have proved that a circuit with a depth of d includes at most 2^d inputs, $\forall$d$\in \mathbb{N}$.

□