Week 11: The Penultimate Problem Party

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Problem 1 *TAUT*

Proof.

Given that 3-SAT is NP-complete, if we can reduce 3-SAT to 3-TAUT, that is, provide an algorithm to solve 3-SAT using 3-TAUT, then we know by the transitive property of polynomial-reduction that 3-TAUT is NP-Hard.

In formal terms, given an arbitrary $F \in NP, F \leq_p$ 3-SAT by definition of NP-Complete.

If 3-$SAT \leq_p$ 3-$TAUT$, then by transitivity $F \leq_p$ 3-$SAT \leq_p$ 3-$TAUT \in NP$ which implies 3-$TAUT$ is NP-Hard by definition.

Thus, it is sufficient to show the reduction of 3-SAT to 3-$TAUT$.

Let x be an arbitrary 3-CNF. That is, x consists of the logical AND of clauses, each of which consists of the logical OR of 3 terms. We can then write x in the form $x = (x_1 \vee x_2 \vee x_3) \wedge (x_4 \vee x_5 \vee x_6) \wedge ...$ such that x_i is a variable whose value is either 0 or 1 for every value i.

The problem 3-SAT consists of determining if there exists a choice of values for each variable in the 3-CNF such that evaluating x returns true. Thus, 3-$SAT(x) = 1$ if and only if there exists such a choice of values, and 0 otherwise.

In the style of reduction, we assume that there exists some implementation of 3-$TAUT$ such that for any 3-DNF y, we can evaluate 3-$TAUT(y)$. If we can solve 3-SAT in terms of this 3-$TAUT$ function, then we have proven that 3-$SAT \leq_p$ 3-$TAUT$.

Define x' to be the complement of an arbitrary 3-CNF x as defined above. Then, $x' = \neg((x_1 \vee x_2 \vee x_3) \wedge ...)$ which by DeMorgan's Law equals $(x_1' \wedge x_2' \wedge x_3') \vee (x_4' \wedge x_5' \wedge x_6') \vee ...$ which is clearly of the form 3-DNF. Thus, given 3-CNF x we know x' is a 3-DNF, and we know 3-$TAUT$ is defined for any 3-DNF, such that we may compute 3-$TAUT(x')$.

If 3-$TAUT(x') = 1$ then all possible assignments to the variables make the complement of the original formula true. Then there does not exist an assignment of variables that would make the original formula true, and 3-$SAT(x) = 0$.

Else if 3-$TAUT(x') = 0$ then there is at least one possible assignment of variables that makes the complement of the original formula false. Then there exists at least one assignment of variables that would make the original formula true, and 3-$SAT(x) = 1$.

Thus we have 3-SAT in terms of 3-$TAUT$, that is 3-$SAT(x) = \neg($3-$TAUT(x'))$ for arbitrary 3-CNF x.

Conclusion: Since the conversion of a 4-CNF to a 4-DNF can certainly be done in polynomial time and this conversion allows the computation of 3-SAT using 3-$TAUT$, then 3-SAT can be reduced to 3-$TAUT$. Because 3-SAT is known to be NP-Hard then, as shown above, by the transitivity of $\leq_p$ 3-$TAUT$ is also NP-Hard.

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