Week 5: Complexity Sells Better

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Problem 1 *Little-o*

(a) Prove that for any function f, $f \notin o(f)$.

Proof.

$f \in o(f) \iff \forall c > 0, \exists n_0$ such that $\forall n \in \mathbb{N}$ where $n > n_0, f(n) < c \cdot f(n)$

Thus, it is sufficient to provide a counterexample where, for some $c > 0$, $f(n) \geq c \cdot f(n)$.

Choose $c = 1$.

In this case, $\forall n \in \mathbb{N}$, $f(n) < 1 \cdot f(n)$ is never true, since $f(n) = f(n)$.

$\implies \exists c > 0$ such that $f(n) < c \cdot f(n)$ is false

$\implies \exists c > 0$ such that $f(n) \geq c \cdot f(n)$

$\implies f(n) \notin o(f(n))$

□

(b) Prove that $n \in o(n \log n)$.

Proof. Assume $log_b n$ has arbitrary base b. Let $c > 0$ an arbitrary constant.

$$\begin{aligned}
& n < c \cdot n \cdot \log(n) \\
\iff & 1 < c \cdot \log(n) \\
\iff & \frac{1}{c} < \log(n) \\
\iff & b^{\frac{1}{c}} < n \\
\iff & n > b^{\frac{1}{c}}
\end{aligned}$$

Thus, choose $n_0 \in \mathbb{N}$ such that $n_0 > b^{\frac{1}{c}}$ for arbitrary b, c.

$$\begin{aligned}
&\implies \forall n \in \mathbb{N}, n \geq n_0, n > b^{\frac{1}{c}} \\
&\implies \forall n \in \mathbb{N}, n \geq n_0, n < c \cdot n \log n, \quad \text{as seen above} \\
&\implies \exists n_0, \quad \text{such that } \forall c > 0, n < c \cdot n \log n \\
&\implies n \in o(n \log n)
\end{aligned}$$

□