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Econometrics I

– Exercise Book –

Exercise 1

Let $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ be a matrix and let $a = (a_1, ..., a_n)'$, $x = (x_1, ..., x_n)'$, $y = (y_1, ..., y_n)'$ be vectors of length n.

a) Write the sum $\sum_{i=1}^{n} a_i \cdot x_i$ as a product of vectors.

b) Write $c_i = \sum_{j=1}^{n} a_{ij} \cdot x_j$ in matrix form.

c) Write $\sum_{i=1}^{n} y_i$ in matrix form. Hint: Use a vector of ones.

d) Write $\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}$ in matrix form.

Exercise 2

Consider the following matrices

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \qquad C = \begin{pmatrix} 2 & 0 & 3 \\ 7 & -1 & 2 \\ 0 & 1 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 2 & 3 \\ -2 & 1 \\ 0 & 5 \end{pmatrix}, \qquad E = \begin{pmatrix} 0 & 2 & 1 \end{pmatrix}.$$

a) Calculate (if possible): AB, CD, EA, BE, ED.

b) Verify that: $(A + C)' = A' + C'$ and $(AC)' = C'A'$.

Exercise 3

Consider the following matrices

$$A = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & -2 \\ 1 & -3 \end{pmatrix}.$$

a) Calculate the determinant of A, B, AB, A', $2A$ and $-B$.

b) Calculate A^{-1} and B^{-1}.

c) Calculate $(AB)^{-1}$.

d) Calculate $B^{-1}A^{-1}$ and compare with (b).

e) Are the matrices A and $C = A'A$ positive definite, negative definite or indefinite?

f) Give an example of a negative definite and indefinite (2×2) matrix.

Exercise 4

In this exercise we take a look at an example to understand why the unbiased variance estimator has a factor $\frac{1}{T-1}$ instead of $\frac{1}{T}$. Imagine a box containing 3 coins with values 0, 2 and 4 denoted as x_1, x_2 and x_3 respectively. Please follow the steps:

- Calculate the *population sample*

$$\mu := \frac{x_1 + x_2 + x_3}{3}$$

  and the *population variance*

$$\sigma^2 = \frac{1}{3} \sum_{t=1}^{3} (x_t - \mu)^2 \ .$$

- We try to estimate the population variance by drawing a coin $T = 2$ times. Write down every combination of two values from the population. Hint: $(0,2) \neq (2,0)$
- For every of these 9 combinations, calculate the sample average and the sample variance with both factors $\frac{1}{T}$ and $\frac{1}{T-1}$.
- Calculate the average of these three key figures over all 9 samples.
- What do you notice?

Exercise 5

An investor wants to invest in two possible, independent stocks A and B. The return of those two stocks is denoted as R_A and R_B. To evaluate the stocks we assume

$$\mathrm{E}(R_A) = 2 \qquad \mathrm{E}(R_B) = 5 \qquad \mathrm{Var}(R_A) = 2 \qquad \mathrm{Var}(R_B) = 6$$

a) Calculate the expected return in case the investor splits his capital evenly.

b) Calculate the variance of the above described portfolio. What do you notice?

c) The investor wants to limit his risk, so that the variance in the portfolio return does not exceed 3.5. What is the maximal return on such a portfolio? How much so you suggest him to invest in A and in B?

Exercise 6

Let a be a column vector of length n and A an $(n \times n)$ matrix. For a scalar valued function $f(x)$ of a vector x,

$$\frac{\partial f(x)}{\partial x} = \begin{bmatrix} \partial f / \partial x_1 \\ \vdots \\ \partial f / \partial x_n \end{bmatrix}$$

is called the gradient vector of f. For the vector $y = Ax$, define

$$\frac{\partial y}{\partial x} = \begin{bmatrix} \partial y_1 / \partial x' \\ \partial y_2 / \partial x' \\ \vdots \\ \partial y_n / \partial x' \end{bmatrix} .$$

a) Show the following equations:
   i) $\frac{\partial a'x}{\partial x} = a$
   ii) $\frac{\partial Ax}{\partial x} = A$
   iii) $\frac{\partial x'Ax}{\partial x} = (A + A')x$

b) What does a) iii) imply for $\frac{\partial x'Ax}{\partial x}$ if A is symmetric?

Exercise 7

Consider the following linear regression:

$$y_t = \alpha + \beta x_t + u_t \ , \qquad \text{for } t = 1, ..., T \ .$$

Show that the OLS estimators $\hat{\alpha}$ and $\hat{\beta}$ are given by

$$\hat{\beta} = \frac{\sum_{t=1}^{T} (x_t - \bar{x})(y_t - \bar{y})}{\sum_{t=1}^{T} (x_t - \bar{x})^2} \ , \qquad \hat{\alpha} = \bar{y} - \hat{\beta}\bar{x}$$

Exercise 8

A supermarket chain plans to investigate the connection between marketing expenditure for a certain product and the number of items sold. $T = 10$ different but comparable markets are included in the study. For market t, marketing expenditure (in USD) and number of items sold are denoted as x_t and y_t respectively.
From this study, the following values are calculated:

$$\bar{x} = 1170 \qquad \sqrt{\frac{1}{10}\sum_{t=1}^{10} (x_t - \bar{x})^2} = 415$$

$$\sum_{t=1}^{10} y_t = 3500 \qquad \sum_{t=1}^{10} y_t^2 = 1'300'000 \qquad \sum_{t=1}^{10} x_t y_t = 4'443'000$$

a) Calculate the OLS-estimates for α and β from the regression model $y_t = \alpha + \beta x_t + u_t$.

b) How is β interpreted from an economic point of view?

c) The manager of a market not included in the study wants to sell 500 products, what would you suggest him to spend on marketing?

d) Calculate the coefficient of determination and interpret it.

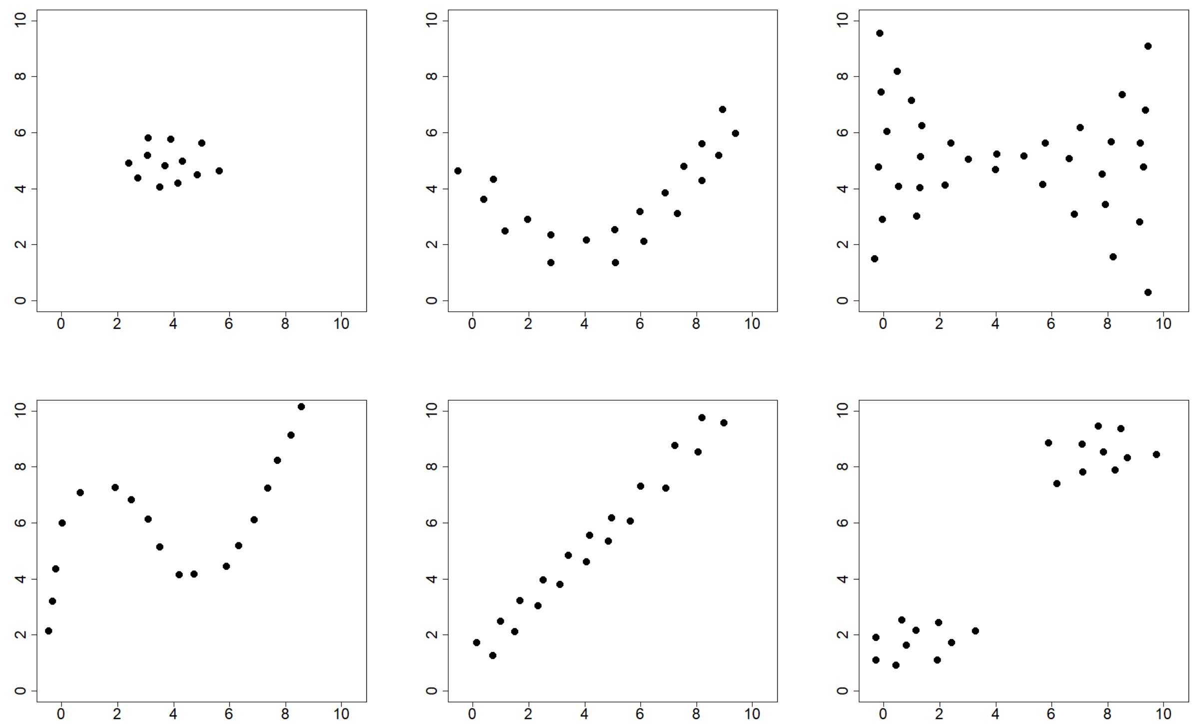
Exercise 9

a) State the A-, B- and C-assumptions in your own words.

b) Give a counterexample of each assumption by drawing a picture if points (x_t, y_t), if possible. Otherwise describe the counterexample in your own words.

Exercise 10

Take a look at the 6 scatterplots below.



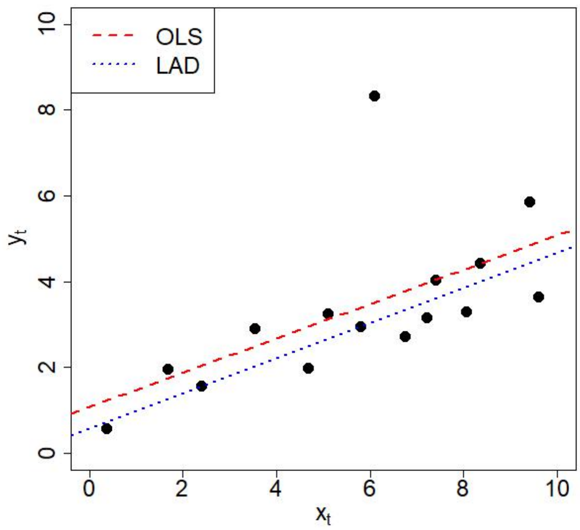
a) Which of the A-, B- and C-assumptions are not satisfied in the individual cases?

b) Which of the assumptions could be violated without being visible in a scatterplot?

Exercise 11

The figure below shows the fitted values of two different regressions $y_t = \alpha + \beta x_t + u_t$. Besides the OLS regression line, the Least Absolute Deviation (LAD) regression line is shown. The LAD-estimators $\tilde{\alpha}$ and $\tilde{\beta}$ are calculated by minimizing

$$\sum_{t=1}^{15} |y_t - (\alpha + \beta x_t)| \; .$$



$$\sum_{t=1}^{15} y_t = 50.67 \qquad \sum_{t=1}^{15} y_t^2 = 219.53$$

$$\sum_{t=1}^{15} \hat{y}_t^2 = 188.77 \qquad \sum_{t=1}^{15} \tilde{y}_t = 44.16$$

$$\sum_{t=1}^{15} \tilde{y}_t^2 = 148.48$$

a) Calculate the coefficient of determination R^2 for both regressions using the above sums.

b) Calculate the correlation between y_t and x_t.

c) Interpret the above results.

Exercise 12

a) Consider the regression model

$$y_t = \alpha + \beta x_t + u_t \; .$$

Is it possible that the coefficient of determination is either negative or larger than 1? Justify your answer briefly.

b) Consider the regression model

$$y_t = \alpha x_t + u_t \; .$$

Is it possible that the coefficient of determination is negative or larger than 1? Justify your answer.

Exercise 13

Consider the following regression:

$$y_t = \alpha x_t + u_t, \qquad \text{where } u_t \sim \mathcal{N}(0, \sigma^2)$$

for $t = 1, ..., T$. Assume validity if all A-, B- and C- assumptions.

a) Verify that the OLS estimator $\hat{\alpha}$ satisfies the following relationships

$$\hat{\alpha} = \frac{\sum_{t=1}^{T} x_t y_t}{\sum_{t=1}^{N} x_t^2}$$

and can be represented as

$$\hat{\alpha} = \alpha + \frac{\sum_{t=1}^{T} x_t u_t}{\sum_{t=1}^{T} x_t^2} \ .$$

b) Calculate the ML-estimator of α.

c) Verify that the OLS estimator $\hat{\alpha}$ is unbiased for α.

d) Find the distribution of $\hat{\alpha}$.

Exercise 14

Consider the simple regression model with binary regressor, i.e.

$$y_t = \alpha + \beta x_t + u_t, \qquad x_t \in \{0, 1\}, \quad y_t \in \mathbb{R}, \quad t = 1, ..., T.$$

Let $\bar{y}_A$ and $\bar{y}_B$ denote the mean of y_t for those observations with $x_t = 1$ and $x_t = 0$ respectively. Further, let T_A denote the number of observations with $x_t = 1$ and correspondingly $T_B = T - T_A$.

a) Show that the OLS estimators can be expressed as

$$\hat{\alpha} = \bar{y}_B \qquad \hat{\beta} = \bar{y}_A - \bar{y}_B \ .$$

b) Sketch the above findings in a plot.

c) What would the OLS estimators be if it wasn't $x_t \in \{0, 1\}$, but $x_t \in \{-1, 1\}$?

d) Show that $S_{xx} = \frac{T_A T_B}{T}$ and $(T-2)\hat{\sigma}^2 = S_{yy}^A + S_{yy}^B$, where

$$S_{yy}^A = \sum_A (y_t - \bar{y}_A)^2 \qquad \text{and} \qquad S_{yy}^B = \sum_A (y_t - \bar{y}_B)^2 \ .$$

Exercise 15

A supermarket chain is interested in the effect of bonus scheme on the profit. The 35 stores in the suburbs offer the bonus scheme, while the 25 stores in the city do not use it. The following table summarizes the results of a study:

	No bonus scheme	Bonus scheme
Number of Stores	25	35
Mean Return	117.58	122.46
Sum of squared Returns	347686	527799

a) What model would be suitable to investigate the effect of the bonus scheme on the return?

b) Estimate the parameters of the above suggested model and interpret them.

c) Calculate 95% confidence intervals for the above estimates. (table of quantiles attached)

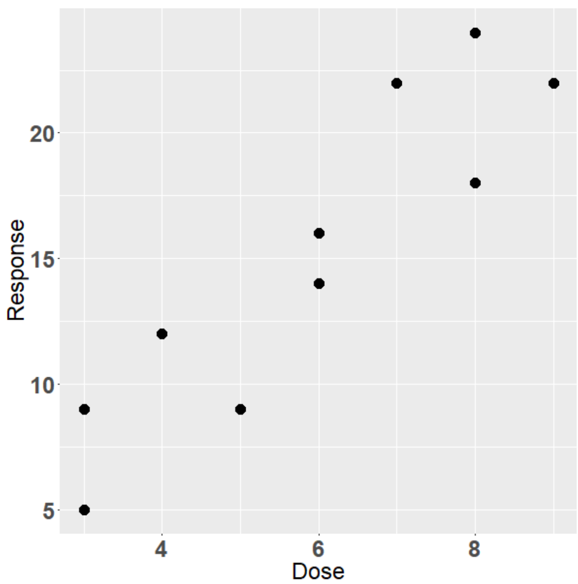
d) The CEO of the chain asks, if you suggest to introduce the bonus scheme to the stores in the city based on the data. What do you reply?

Exercise 16

The following table displays the results of a study with allergy patients. We investigate the effect of medication (Dosage x_t) on the time of relief y_t by assuming the model $y_t = \alpha + \beta x_t + u_t$.

	x_t	y_t	$x_t y_t$	$\hat{y}_t$
	3	9	27	7.15
	3	5	15	7.15
	4	12	48	9.89
	5	9	45	12.63
	6	14	84	15.37
	6	16	96	15.37
	7	22	154	18.11
	8	18	144	20.86
	8	24	192	20.86
	9	22	198	23.60
Σ	59	151	1003	151
SSQ[a]	389	2651	–	2587.35

[a] SSQ: Sum of squared values - $\text{SSQ}(x) = \sum x_t^2$
Not the same as S_{xx}

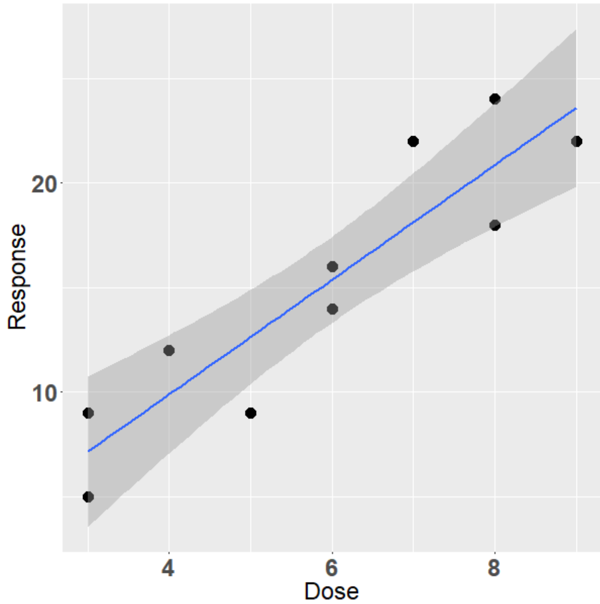


a) Are the assumptions satisfied?

b) Estimate the parameters α and β by OLS.

c) Calculate the 95% confidence intervals for $\hat{\alpha}$ and $\hat{\beta}$ and interpret.

d) The following figure shows the 95% confidence interval around the regression line. Try to think of a reason why the this confidence band is wider at the edges.



Exercise 17

Consider the regression model $y_t = \alpha + \beta x_t + u_t$. Show that $\hat{\sigma}^2 = \frac{1}{T-2}\sum_{t=1}^{T} \hat{u}_t^2$ is an unbiased estimator of the variance of the error terms by following the steps:

Step 1) Show that $y_t - \bar{y} = \beta(x_t - \bar{x}) + (u_t - \bar{u})$

Step 2) Show that $\hat{\beta} = \beta + \frac{S_{xu}}{S_{xx}}$.

Step 3) Show that $\hat{u}_t = (y_t - \bar{y}) - \hat{\beta}(x_t - \bar{x}) = -(\hat{\beta} - \beta)(x_t - \bar{x}) + u_t - \bar{u}$

Step 4) Conclude that $\sum_{t=1}^{T} \hat{u}_t^2 = S_{uu} - (\hat{\beta} - \beta)^2 S_{xx}$

Step 5) Show that $\mathrm{E}(S_{uu}) = (T-1)\sigma^2$

Step 6) Show that $\mathrm{E}\Big((\hat{\beta} - \beta)^2\Big)\, S_{xx} = \sigma^2$ which finishes the proof

Exercise 18

This exercise deals with multicollinearity. As a reminder, consider a regression model $y_t = \beta_0 + \beta_1 x_{1,t} + ... + \beta_k x_{k,t} + u_t$. Multicollinearity exists if there is a vector $\gamma = (\gamma_0, ..., \gamma_k)' \neq (0, ..., 0)'$ so that $\gamma_0 + \gamma_1 x_{1,t} + ... + \gamma_k x_{k,t} = 0$ for every $t = 1, ..., T$.

a) Are the following statements true or false?

   (i) In case of multicollinearity, at least one exogenous variable can be expressed as a linear combination of the remaining ones.

   (ii) If y_t can be expressed as linear combination of the exogenous variables $x_{1,t}, ..., x_{k,t}$, the OLS estimator can not be calculated.

   (iii) To check if multicollinearity is in the data, it suffices to calculate the pairwise correlation coefficients of exogenous variables.

   (iv) In case two exogenous variables are perfectly correlated, i.e. $x_{1,t} = \gamma_0 + \gamma_1 x_{2,t}$, the total impact of those variables on y_t can be estimated.
   If yes, can the impact be separated on the two variables?
   If no, can you think of a way to solve the problem?

   (v) In case of multicollinearity, the determinant of $X'X$ is 0.

b) Decide if there is multicollinearity in the cases below:

   (i) Model: $y_t = \beta_0 + \beta_1 x_t + u_t$
   $S_{xx} = \sum_{t=1}^{T} (x_i - \bar{x})^2 = 0$

   (ii) Model: $y_t = \beta_0 + \beta_1 x_{1,t} + \beta_2 x_{2,t} + \beta_3 x_{3,t} + u_t$

t	$x_{1,t}$	$x_{2,t}$	$x_{3,t}$
1	2	1	12
2	2	1	12
3	3	3	9
4	5	1	9
5	2	3	10
6	5	2	8
7	6	3	6
8	4	6	5
9	3	1	11
10	1	3	11
$\sum_{t=1}^{10} x_{\bullet t}$	33	24	93
$\sum_{t=1}^{10} x_{\bullet t}^2$	133	80	917

$$\sum_{t=1}^{10} x_{1,t} x_{2,t} = 82$$
$$\sum_{t=1}^{10} x_{1,t} x_{3,t} = 280$$
$$\sum_{t=1}^{10} x_{2,t} x_{3,t} = 198$$

Exercise 19

Consider the restricted regression

$$y_t = \alpha + u_t \ , \ t = 1, ..., T \ ,$$

for which the A-, B- and C-assumption are satisfied.

a) Derive the ML estimator for α.

b) Calculate the ML estimator for the error variance given the sample

y_t	4	6	7	4
x_t	2	4	5	3

Exercise 20

Reconsider exercise 8.

a) Test if β is significantly different from 0 at a 5% level and interpret the outcome.

b) Is the intercept α significantly greater than 1? (Choose a 5% level of significance)
Hint: $\text{Var}\,(\hat{\alpha}) = \sigma^2 \left[\frac{1}{T} + \frac{\bar{x}^2}{\sum_{t=1}^{T} (x_t - \bar{x})^2} \right]$

Exercise 21

It is known that the IQ is normally distributed with mean $\mu = 100$ and standard deviation $\sigma = 15$. Four students want to find out if students have a significantly higher IQ. Therefore they do an IQ test, in which they scored 120 on average.

a) Formulate the hypotheses for above question.

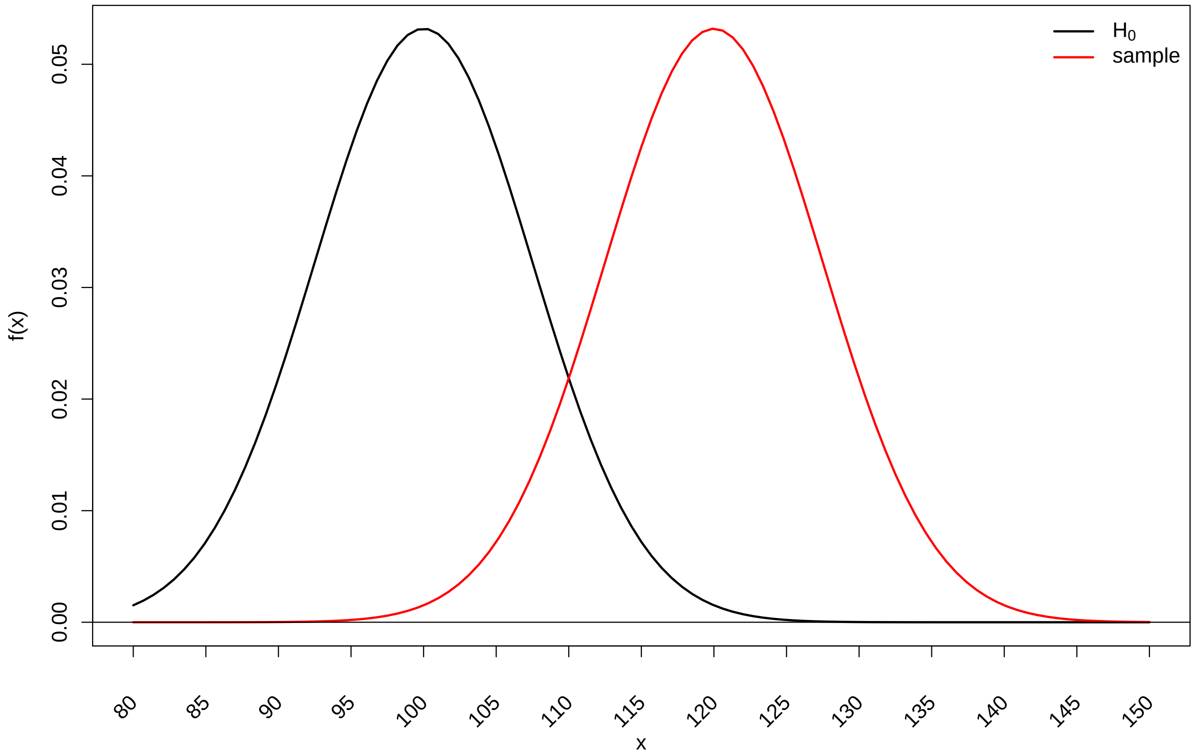
b) We want to use the average score of 120 as a test statistics. Find the critical value for testing with a 5% level of significance.
Hint: Use the 95% quantile of the $\mathcal{N}(0, 1)$ distribution 1.645.

c) Below figure shows the densities of the average score if H_0 was true (black) and the estimated density from the sample (red). Complete the plot by adding

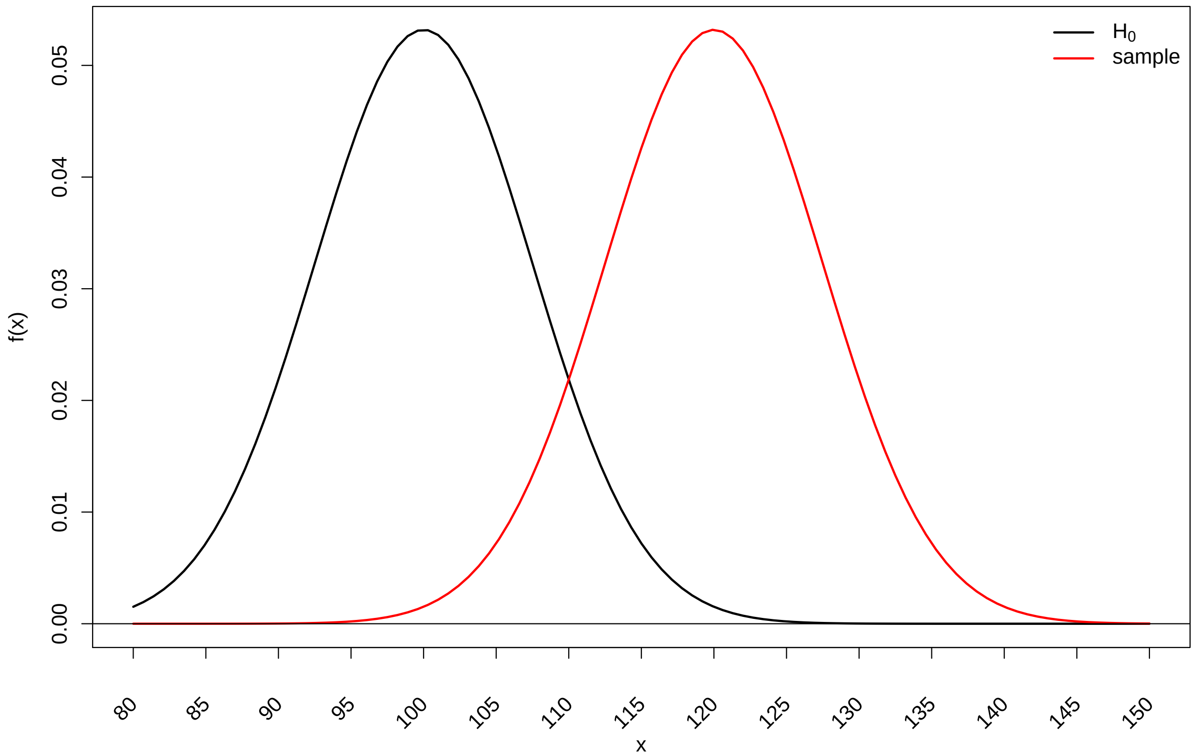
   (i) The critical value

   (ii) The type I error

   (iii) The type II error if the IQ of students is really 120 on average

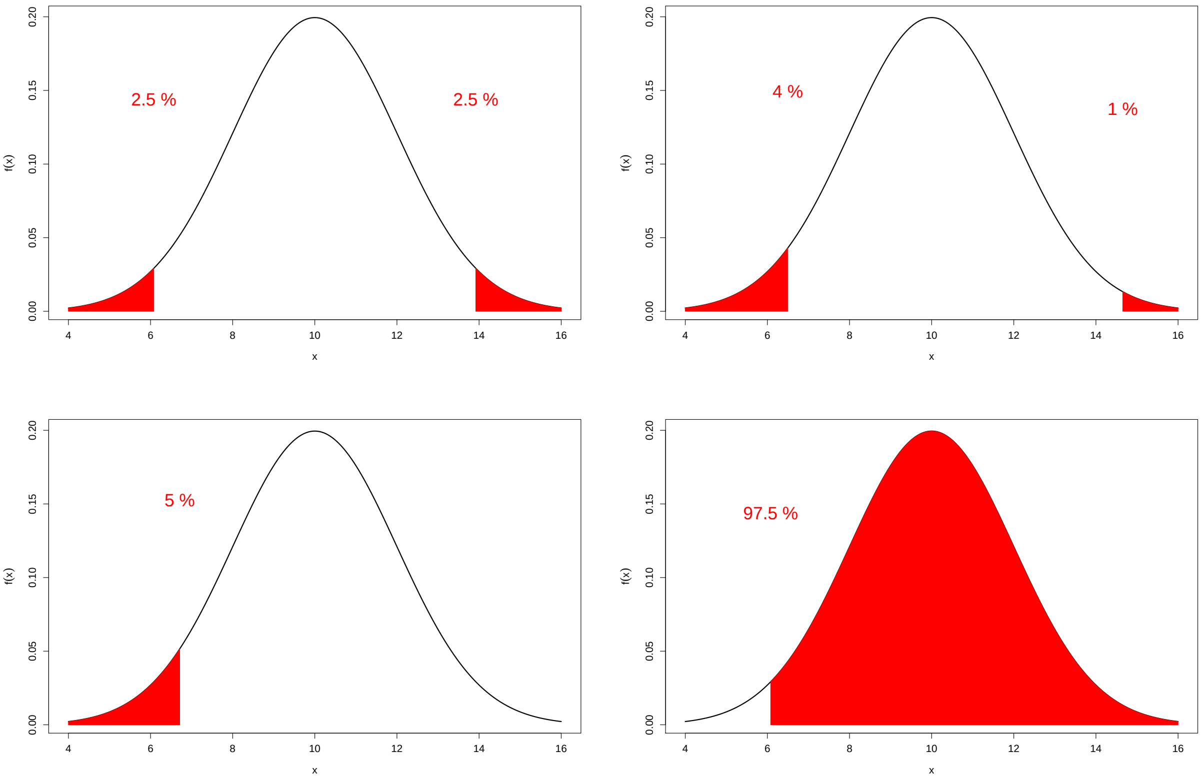


d) Sketch what happens with type I and type II error if the level of significance is increased. Use below figure again.



Exercise 22

The following plots show possible density functions of test statistics and critical values. If it belongs to a valid test, give the hypotheses and the level of significance.



Exercise 23

For the annual turnover of a large store group, the validity of the following linear regression model is assumed:

$$y_t = \alpha + \beta_1 x_{1,t} + \beta_2 x_{2,t} + u_t$$

with y_t: annual turnover (of store t), $x_{1,t}$: sales area (of store t), $x_{2,t}$: average frequency of people passing by (of store t), u_t: error term (of store t). Based on the observation values collected in $\boldsymbol{y}$ and $\boldsymbol{X}$ one obtains

$$\boldsymbol{X}'\boldsymbol{X} = \begin{pmatrix} 20 & 200 & 0 \\ 200 & 2014.4 & 16 \\ 0 & 16 & 40 \end{pmatrix}, \qquad (\boldsymbol{X}'\boldsymbol{X})^{-1} = \begin{pmatrix} 12.55 & -1.25 & 0.5 \\ -1.25 & 0.125 & -0.05 \\ 0.5 & -0.05 & 0.045 \end{pmatrix},$$

$$\boldsymbol{X}'\boldsymbol{y} = \begin{pmatrix} 946 \\ 9532 \\ 85 \end{pmatrix}, \qquad \hat{\boldsymbol{u}}'\hat{\boldsymbol{u}} = 90.28125 \,.$$

a) How many stores were involved in the investigation?

b) Find the OLS estimators of α, β_1 and β_2.

c) Test the null hypothesis that coefficients associated with the regressors $x_{1,t}$ and $x_{2,t}$ are equal at a 5% level of significance.

Exercise 24

Consider a dataset with 500 observations of the variables y, x_1, x_2, x_3 and the specification

$$y_t = \alpha + \beta_1 x_{1,t} + \beta_2 x_{2,t} + \beta_3 x_{3,t} + u_t$$

under the validity of A-, B- and C-assumptions. The following values are given

$$\hat{\boldsymbol{u}}'\hat{\boldsymbol{u}} = 2912.778 \ , \qquad \hat{\boldsymbol{\beta}} = \begin{pmatrix} 10.85 \\ 2.14 \\ -0.28 \\ 9.53 \end{pmatrix}$$

$$(\boldsymbol{X}'\boldsymbol{X})^{-1} = \begin{pmatrix} 0.0229 & -0.0020 & -0.0005 & -0.0060 \\ -0.0020 & 0.0020 & -0.0001 & 0.0001 \\ -0.0005 & -0.0001 & 0.0031 & -0.0019 \\ -0.0060 & 0.0001 & -0.0019 & 0.0033 \end{pmatrix} .$$

a) Test $H_0 : \beta_2 = 3$ vs. $H_1 : \beta_2 \neq 3$.

After your preliminary analysis you find an additional observed variable x_4 and may want to include it in your specification.

$$y_t = \alpha + \beta_1 x_{1,t} + \beta_2 x_{2,t} + \beta_3 x_{3,t} + \beta_4 x_{4,t} + u_t \ .$$

The values for the new specification are given by

$$\hat{\boldsymbol{u}}'\hat{\boldsymbol{u}} = 117.138 \ , \qquad \hat{\boldsymbol{\beta}} = \begin{pmatrix} 1.061 & 1.965 & 2.967 & 4.040 & 4.986 \end{pmatrix}'$$

$$(\boldsymbol{X}'\boldsymbol{X})^{-1} = \begin{pmatrix} 0.0572 & -0.0014 & -0.0119 & 0.0132 & -0.0175 \\ -0.0014 & 0.0020 & -0.0003 & 0.0004 & -0.0003 \\ -0.0119 & -0.0003 & 0.0069 & -0.0083 & 0.0058 \\ 0.0132 & 0.0004 & -0.0083 & 0.0141 & -0.0098 \\ -0.0175 & -0.0003 & 0.0058 & -0.0098 & 0.0089 \end{pmatrix} .$$

b) Test $H_0 : \beta_2 = 3$ vs. $H_1 : \beta_2 \neq 3$.

c) How would you explain the difference in the results from a) and b)?

d) Can you be sure that your result from b) is valid?

Exercise 25

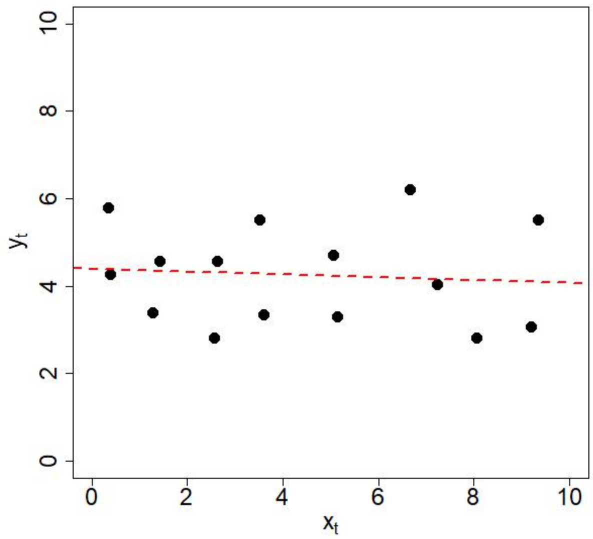
Consider a regression model $y_t = \alpha + \beta x_t + u_t$. The parameters are estimated by OLS and the following sums are obtained:

$$\sum_{t=1}^{15} y_t = 63.8808 \qquad \sum_{t=1}^{15} y_t^2 = 289.9335 \qquad \sum_{t=1}^{15} \hat{y}_t^2 = 272.1864$$

a) Calculate the adjusted coefficient of determination

$$\bar{R}^2 = 1 - \frac{\frac{1}{T-k-1}\sum_{t=1}^{T}(y_t - \hat{y}_t)^2}{\frac{1}{T-1}\sum_{t=1}^{T}(y_t - \bar{y})^2}$$

b) The observations are displayed in the figure below. How do you explain the results in a)?



Exercise 26

Consider the multivariate linear regression model $y = X\beta + u$. The exogenous variables summarized in matrix X were assumed to be non-stochastic. That is not plausible in many economic applications. For this exercise we replace the assumption "X is not stochastic" by the two assumptions $\mathrm{E}(u|X) = 0$ and $\mathrm{Var}(u|X) = \sigma^2 I_T$, where I_T is the identity matrix of dimension $T \times T$.
For the following proofs you can use the *law of iterated expectation* and the *law of total variance*. For two random variables X and Y they are defied as:

$$\begin{aligned}\mathrm{E}(X) &= \mathrm{E}\left[\mathrm{E}\left(X|Y\right)\right]\\ \mathrm{Var}(X) &= \mathrm{Var}\left[\mathrm{E}\left(X|Y\right)\right] + \mathrm{E}\left[\mathrm{Var}\left(X|Y\right)\right]\end{aligned}$$

a) Show that the OLS estimator $\hat{\beta} = (X'X)^{-1}X'y$ is unbiased for β.

b) Derive the variance of the OLS estimator $\hat{\beta}$.

Exercise 27

Consider the multivariate linear regression model $y = X\beta + u$ with $K = 3$ exogenous variables and an intercept. How would you test the following hypotheses? Translate the hypotheses to the notation $R\,\beta(\geq / = / \leq)\,q$ and sketch the steps to find the test decision if possible.

a) $H_0 : \beta_1 = \beta_2 = \beta_3 = 0$

b) $H_0 : \beta_1 = \beta_2 = \beta_3$

c) $H_0 : \beta_1 = 2\beta_2 = 3\beta_3$

d) $H_0 : \frac{\beta_0}{\beta_1} = 2$

e) $H_0 : \beta_1 \geq \beta_2$

Exercise 28

To estimate the consumption function for the years 1962 to 2001, the regression model

$$y_t = \beta_0 + \beta_1 x_{1,t} + \beta_2 x_{2,t} + u_t$$

is considered with

$$\begin{aligned} y_t &: \text{Consumption} \\ x_{1,t} &: \text{Income} \\ X_{2,t} &: \text{Interest rate.} \end{aligned}$$

The following results are obtained:

$$X'X = \begin{pmatrix} 40 & 0 & 10 \\ 0 & 100 & 10 \\ 10 & 10 & 20 \end{pmatrix}$$

The OLS estimators were first calculated without the variable $x_{2,t}$.

a) On what condition are $\hat{\beta}_0$ and $\hat{\beta}_1$ unbiased?

b) Is there an omitted variable bias?

c) Calculate the bias $\mathrm{E}(\hat{\beta}_0) - \beta_0$ and $\mathrm{E}(\hat{\beta}_1) - \beta_1$ by assuming $\beta_2 = -0.1$.

d) Assuming that there is no omitted variable bias. What consequences would if expect when adding $x_{2,t}$ to the model?

Exercise 29

A friend had asked you to do a regression for his master's thesis. For the model

$$y_t = \alpha + \beta x_t + u_t \qquad t = 1, ..., T$$

you estimated the parameters given the following information

$$\sum x_t = 19.3 \qquad \sum x_t^2 = 47.16 \qquad \sum x_t y_t = 654.4$$

$$\sum y_t = 272.48 \qquad \sum y_t^2 = 9223.33 \qquad \bar{x} = 1.93$$

After speaking to his professor, your friends tells you about two problems. First, he omitted a relevant variable and, second, x_t has a quadratic impact on y_t.

a) Your friend asks you why you did not tell him about these problems. Was it possible for you to recognize them without further information about the theoretical background of the data?

b) Which assumptions are violated by the described problems and what are the consequences?

c) Consider the model $y_t = \alpha + \beta_1 x_{1,t} + \beta_2 x_{2,t} + u_t$ where $x_{2,t} = x_t^2$. Assume that the A-, B- and C-assumptions hold. Calculate the OLS esimator.
First calculations provide the following results.

$$X = \begin{pmatrix} 1 & 7.02 & 0.10 \\ 1 & 5.76 & 0.30 \\ 1 & 9.61 & 0.39 \\ 1 & 1.21 & 0.41 \\ 1 & 4.41 & 0.26 \\ 1 & 0.00 & 0.11 \\ 1 & 8.70 & 0.55 \\ 1 & 7.02 & 0.63 \\ 1 & 0.36 & 0.36 \\ 1 & 3.06 & 0.27 \end{pmatrix} \qquad y = \begin{pmatrix} 35.22 \\ 31.12 \\ 47.76 \\ 13.52 \\ 25.73 \\ 8.28 \\ 44.24 \\ 36.99 \\ 9.52 \\ 20.11 \end{pmatrix}$$

$$X'X = \begin{pmatrix} 10.00 & 47.15 & 3.38 \\ 47.15 & 330.19 & 17.98 \\ 3.38 & 17.98 & 1.40 \end{pmatrix} \qquad (X'X)^{-1} = \begin{pmatrix} 0.59 & -0.02 & -1.13 \\ -0.02 & 0.01 & -0.09 \\ -1.13 & -0.09 & 4.54 \end{pmatrix}$$

$$X'y = \begin{pmatrix} 272.49 \\ 1724.82 \\ 101.12 \end{pmatrix} \qquad My = \begin{pmatrix} -0.58 \\ -0.22 \\ 0.67 \\ 0.15 \\ -0.06 \\ 0.68 \\ 0.32 \\ -0.41 \\ -0.28 \\ -0.28 \end{pmatrix}$$

d) Your friend believes that $\beta_1 = \beta_2$ and, in addition, $\alpha = \beta_1 + \beta_2$. Test his hypothesis with one of the tests discussed in the lecture.

Exercise 30

At the football world cup 2010 in South Africa, the atmosphere at match t (y_t) was measured by a rating system. To model it, the regression model

$$y_t = \beta_0 + \beta_1 x_t + u_t$$

is assumed, where x_t is the number of vuvuzelas (in 10 thousand). The following results were obtained from the study

$$X'X = \begin{pmatrix} 64 & 288.128 \\ 288.128 & 1842.3374 \end{pmatrix} \qquad (X'X)^{-1} = \begin{pmatrix} 0.0528 & -0.0083 \\ -0.0083 & 0.0018 \end{pmatrix}$$

$$X'y = \begin{pmatrix} 6080 \\ 27496 \end{pmatrix}$$

a) Calculate the OLS estimator.

b) Calculate the coefficient of determination under the assumption $y'y = 577700$.

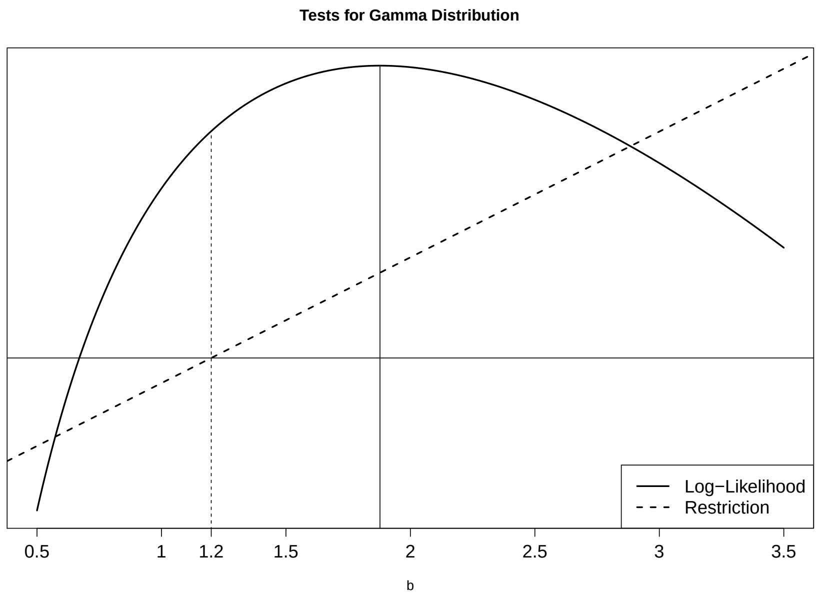
c) Give a prognosis for the atmosphere rating if $x_0 = 3$.

d) Conduct a 95% prognosis interval for y_0 with $x_0 = 3$. Thereby assume an error variance $\hat{\sigma}^2 = 1$.

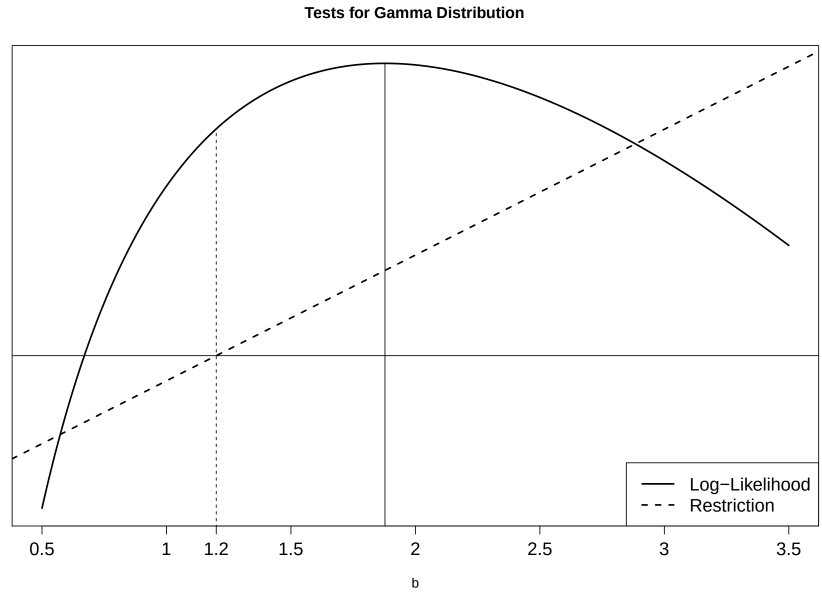
Exercise 31

Explain the idea behind Wald-, LR- and LM-test by summarizing the idea in two sentences and completing the images.

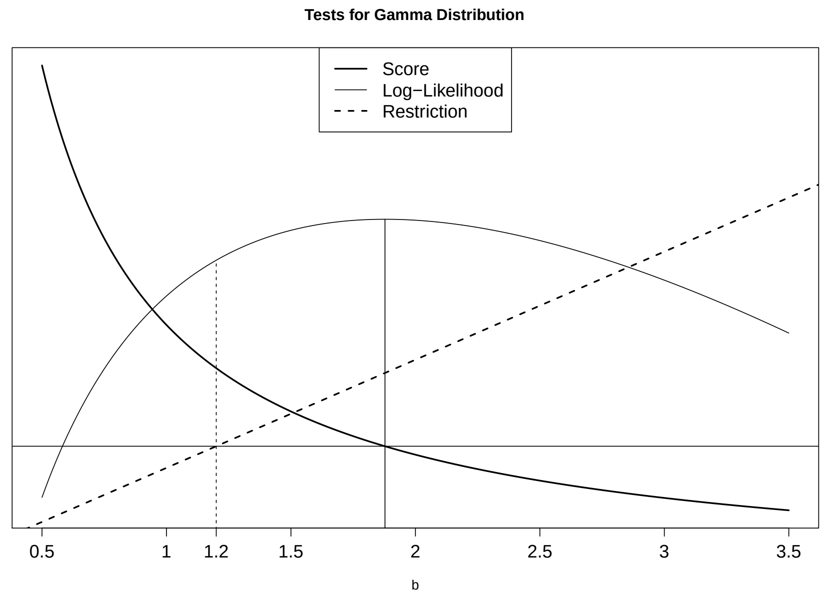
a) Wald-test



b) LR-test



c) LM-test



Exercise 32

The *International Monetary Found* hires you to estimate the credit demand function for the private sector

$$y_t = \alpha + \beta x_t + u_t \qquad t = 1, ..., T$$

based on a sample of $T = 40$ countries. Thereby y_t gives the total of private credits in country t (in billion USD) and x_t the average nominal interest rate in country t (in percent). Further you get the following values:

$$\sum_{t=1}^{T} x_t = 250; \quad \sum_{t=1}^{T} x_t^2 = 1600; \quad \sum_{t=1}^{T} y_t = 200; \quad \sum_{t=1}^{T} y_t^2 = 1015.78; \quad \sum_{t=1}^{T} x_t y_t = 1227.5$$

a) Estimate the coefficients α and β by OLS and interpret.

b) Calculate and interpret the coefficient of determination R^2.

c) Estimate the error variance $\sigma^2 = \text{Var}(u_t)$.

d) Test if the impact of nominal interest rate on the total of private credits is significantly different from zero.

e) Give a prognosis for a country with a nominal interest rate of $x_0 = 3$ (percent) and construct the 95% prognosis interval.

Exercise 33

Consider a multivariate linear regression model $y = X\beta + u$. Show that the estimator $\hat{\beta}^* = (X'BX)^{-1}X'By$ is unbiased. The matrix B is not stochastic, of suitable dimension and chosen so that the inverse $X'BX$ exists.

df	90%	92.5%	95%	97.5%	99%	**df**	90%	92.5%	95%	97.5%	99%
1	3.078	4.165	6.314	12.706	31.821	**51**	1.298	1.462	1.675	2.008	2.402
2	1.886	2.282	2.920	4.303	6.965	**52**	1.298	1.461	1.675	2.007	2.400
3	1.638	1.924	2.353	3.182	4.541	**53**	1.298	1.461	1.674	2.006	2.399
4	1.533	1.778	2.132	2.776	3.747	**54**	1.297	1.460	1.674	2.005	2.397
5	1.476	1.699	2.015	2.571	3.365	**55**	1.297	1.460	1.673	2.004	2.396
6	1.440	1.650	1.943	2.447	3.143	**56**	1.297	1.460	1.673	2.003	2.395
7	1.415	1.617	1.895	2.365	2.998	**57**	1.297	1.459	1.672	2.002	2.394
8	1.397	1.592	1.860	2.306	2.896	**58**	1.296	1.459	1.672	2.002	2.392
9	1.383	1.574	1.833	2.262	2.821	**59**	1.296	1.459	1.671	2.001	2.391
10	1.372	1.559	1.812	2.228	2.764	**60**	1.296	1.458	1.671	2.000	2.390
11	1.363	1.548	1.796	2.201	2.718	**61**	1.296	1.458	1.670	2.000	2.389
12	1.356	1.538	1.782	2.179	2.681	**62**	1.295	1.458	1.670	1.999	2.388
13	1.350	1.530	1.771	2.160	2.650	**63**	1.295	1.457	1.669	1.998	2.387
14	1.345	1.523	1.761	2.145	2.624	**64**	1.295	1.457	1.669	1.998	2.386
15	1.341	1.517	1.753	2.131	2.602	**65**	1.295	1.457	1.669	1.997	2.385
16	1.337	1.512	1.746	2.120	2.583	**66**	1.295	1.456	1.668	1.997	2.384
17	1.333	1.508	1.740	2.110	2.567	**67**	1.294	1.456	1.668	1.996	2.383
18	1.330	1.504	1.734	2.101	2.552	**68**	1.294	1.456	1.668	1.995	2.382
19	1.328	1.500	1.729	2.093	2.539	**69**	1.294	1.456	1.667	1.995	2.382
20	1.325	1.497	1.725	2.086	2.528	**70**	1.294	1.456	1.667	1.994	2.381
21	1.323	1.494	1.721	2.080	2.518	**71**	1.294	1.455	1.667	1.994	2.380
22	1.321	1.492	1.717	2.074	2.508	**72**	1.293	1.455	1.666	1.993	2.379
23	1.319	1.489	1.714	2.069	2.500	**73**	1.293	1.455	1.666	1.993	2.379
24	1.318	1.487	1.711	2.064	2.492	**74**	1.293	1.455	1.666	1.993	2.378
25	1.316	1.485	1.708	2.060	2.485	**75**	1.293	1.454	1.665	1.992	2.377
26	1.315	1.483	1.706	2.056	2.479	**76**	1.293	1.454	1.665	1.992	2.376
27	1.314	1.482	1.703	2.052	2.473	**77**	1.293	1.454	1.665	1.991	2.376
28	1.313	1.480	1.701	2.048	2.467	**78**	1.292	1.454	1.665	1.991	2.375
29	1.311	1.479	1.699	2.045	2.462	**79**	1.292	1.454	1.664	1.990	2.374
30	1.310	1.477	1.697	2.042	2.457	**80**	1.292	1.453	1.664	1.990	2.374
31	1.309	1.476	1.696	2.040	2.453	**81**	1.292	1.453	1.664	1.990	2.373
32	1.309	1.475	1.694	2.037	2.449	**82**	1.292	1.453	1.664	1.989	2.373
33	1.308	1.474	1.692	2.035	2.445	**83**	1.292	1.453	1.663	1.989	2.372
34	1.307	1.473	1.691	2.032	2.441	**84**	1.292	1.453	1.663	1.989	2.372
35	1.306	1.472	1.690	2.030	2.438	**85**	1.292	1.453	1.663	1.988	2.371
36	1.306	1.471	1.688	2.028	2.434	**86**	1.291	1.453	1.663	1.988	2.370
37	1.305	1.470	1.687	2.026	2.431	**87**	1.291	1.452	1.663	1.988	2.370
38	1.304	1.469	1.686	2.024	2.429	**88**	1.291	1.452	1.662	1.987	2.369
39	1.304	1.468	1.685	2.023	2.426	**89**	1.291	1.452	1.662	1.987	2.369
40	1.303	1.468	1.684	2.021	2.423	**90**	1.291	1.452	1.662	1.987	2.368
41	1.303	1.467	1.683	2.020	2.421	**91**	1.291	1.452	1.662	1.986	2.368
42	1.302	1.466	1.682	2.018	2.418	**92**	1.291	1.452	1.662	1.986	2.368
43	1.302	1.466	1.681	2.017	2.416	**93**	1.291	1.452	1.661	1.986	2.367
44	1.301	1.465	1.680	2.015	2.414	**94**	1.291	1.451	1.661	1.986	2.367
45	1.301	1.465	1.679	2.014	2.412	**95**	1.291	1.451	1.661	1.985	2.366
46	1.300	1.464	1.679	2.013	2.410	**96**	1.290	1.451	1.661	1.985	2.366
47	1.300	1.463	1.678	2.012	2.408	**97**	1.290	1.451	1.661	1.985	2.365
48	1.299	1.463	1.677	2.011	2.407	**98**	1.290	1.451	1.661	1.984	2.365
49	1.299	1.462	1.677	2.010	2.405	**99**	1.290	1.451	1.660	1.984	2.365
50	1.299	1.462	1.676	2.009	2.403	**100**	1.290	1.451	1.660	1.984	2.364

Tabelle 1: Quantiles of t-distribution