New York University
Applied Statistics and Econometrics I
Fall 2015

Problem Set 1

Cristian Fuenzalida
cristian.fuenzalida@nyu.edu

(I) The file Data.xls contains prices of 10 highly traded stocks. Write an R program to:

1. Load the data on memory, and create a matrix containing daily returns.
2. Calculate the average return, the standard deviation, and the correlation matrix of the returns. Check whether this matrix is singular or not.
3. Get a histogram of the returns for each stock.

(II) Write an R program to:

1. Generate a sample of 200 replications of a 4-dimensional multivariate normal random vector with mean zero and variance equal to the identity matrix. Each of the rows of this 200×4 matrix are going to be artificial observations in this exercise.
2. Compute the residual maker matrix $M_0 := I - i(i'i)^{-1}i'$, where i is a vector of ones. Check that this matrix is idempotent.
3. Compute the eigenvalues of M_0. How many of them are 1. Compute now the rank of M_0. Is this a coincidence? Prove your statement.
4. Compute the values of $a = x'M_0x$ for each row x. Plot a histogram with these values. What is the distribution of the values of a?

(III) The trace operator $tr(\cdot)$ is defined on the family of squared matrices n-by-n as the sum of the elements of the main diagonal. That is, if $A := [a_{ij}]$, then:

$$tr(A) = \sum_{i=1}^{n} a_{ii}$$

Prove the following properties of the trace operator:

1. The trace is invariant under cyclical permutations (what is a *cyclical permutation*? An ordering that does not change the relative position of the elements is they are arranged in a ring):

$$tr(ABCD) = tr(BCDA) = tr(CDAB) = tr(DABC)$$

Provide an example of matrices such that the trace of the product of a non-cyclical permutation gives different results, i.e. matrices such that $tr(ABC) \neq tr(ACB)$.

2. $tr(X \otimes Y) = tr(X) \cdot tr(Y)$

(IV) The projection matrix P_X is defined as:

$$P_X := X \left(X^\top X\right)^{-1} X^\top$$

Prove the following properties about the projection matrix:

1. P_X is symmetric and idempotent
2. $P_X X = X$
3. The i-th diagonal element of $P_X = X \left(X^\top X\right)^{-1} X^\top$ is:

$$h_{ii} = x_i^\top \left(X^\top X\right)^{-1} x_i$$

which is called the **leverage** of each observation. Prove that:

$$\sum_{i=1}^{n} h_{ii} = tr(P) = k$$

where k is the number of columns in X.

(V) Show the following:

1. The inverse of a diagonal block matrix is given by:

$$\begin{pmatrix} A_{11} & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \quad A_{NN} \end{pmatrix}^{-1} = \begin{pmatrix} A_{11}^{-1} & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \quad A_{NN}^{-1} \end{pmatrix}$$

2. The partitioned inverse 2x2 matrix has the inverse:

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^{-1} = \begin{pmatrix} A_{11}^{-1} + A_{11}^{-1} A_{12} F_2 A_{21} A_{11}^{-1} & -A_{11}^{-1} A_{12} F_2 \\ -F_2 A_{21} A_{11}^{-1} & F_2 \end{pmatrix}$$

where $F_2 = (A_{11} - A_{12} A_{22}^{-1} A_{21})^{-1}$.

(VI) Consider the following matrix:

$$A = \begin{pmatrix} 0.4 & 1-c \\ 0.6 & c \end{pmatrix}$$

1. Find all the eigenvalues of A (as a function of c).
2. Show that A has just one eigenvector dimension when $c = 1.6$. Find a unitary vector with this direction.

3. When $c = 0.4$, this is a Markov matrix. What happens to A^n when $n \to \infty$? (Hint: use the spectral decomposition of a matrix. What is the *spectral decomposition*? Start here, section 2.3: `http://web.stanford.edu/~damle/refresher/notes/matrixanalysis.pdf`)

(VII) A portfolio manager wants to know what is the tradeoff between returns and volatility in his portfolio. There are N assets in the economy. The return of each asset is a random variable whose mean and variance are known. These returns are not independent, so there is a $N \times N$ covariance matrix Ω including all covariances of returns between the assets. Suppose the manager is going to choose a portfolio w, which corresponds to a $N \times 1$ vector including the weights assigned to each asset in the portfolio. For example, if the portfolio has an equal weight for every asset, the vector has $1/N$ as entry in every position. A portfolio that allocates all the money in the first asset has a first entry 1 and the rest zero. The $N \times 1$ vector of expected returns is denoted by e. In order to study this mean-variance trade-off, the portfolio manager sets a reference expected return μ he wants to reach and calculates the minimum variance he can get by choosing the appropriate weights.

1. Express the variance of the portfolio in terms of w and Ω.
2. Write down the constraints expressing that the weights must be such that the expected return of the portfolio is equal to μ, and the one that expresses that all the weights sum up to one.
3. Write down the variance minimization problem: minimize the portfolio variance subject to the two previous constraints. Write the Langrangian of the problem and get the first order conditions of the optimization problem.
4. Express the optimal weight as a function of μ (without the Lagrange multipliers). You have just characterized the *porfolio frontier* in the Markowitz problem. Nobel Prize insight!

(VIII) Give an non-trivial example of the following:

1. Two uncorrelated random variables X and Y that are not independent.
2. Two different discrete random variables X and Y with the same distribution.
3. Two normal random variables X and Y that are not jointly normal.

(IX) The random variables X and Y are defined on $\mathbb{N}$.

1. Suppose that know the joint probability distribution of X and Y:

$$\mathbb{P}(a \leq x \leq b, c \leq y \leq d) = \sum_{a \leq x \leq b} \sum_{c \leq y \leq d} f(x, y)$$

- Explain how to obtain the marginal distribution of X and Y from $f(x, y)$

- Define the covariance between X and Y. Show that:

$$Cov(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$$

- State the condition we need to check that X and Y are independent

2. Let $\mathbb{P}_X(x = 1) = 0.5$, $\mathbb{P}_X(x = 2) = 0.3$, $\mathbb{P}_X(x = 3) = 0.2$. Define $Y = X^2$. Derive the marginal distribution for both random variables, their covariance, and check if whether they are independent or not.