PS5

$$dS_t = \mu S_t dt + \sigma S_t d\bar{W}_t$$

1.

$$dC_t = dV_t = h^* dS_t + h^0 B r\, dt$$
$$= h^*(\mu S_t dt + \sigma S_t d\bar{W}_t) + h^0 B r\, dt$$

$h^* = \frac{\partial C}{\partial S}$

$h^0 = \frac{C - \frac{\partial C}{\partial S} \cdot S}{B}$

$$= \frac{\partial C}{\partial S}\mu S_t dt + \frac{\partial C}{\partial S}\sigma S_t d\bar{W}_t + h^0 B r\, dt$$
$$= \frac{\partial C}{\partial S}\mu S_t dt + \frac{\partial C}{\partial S}\sigma S_t d\bar{W}_t + \frac{C_t - \frac{\partial C}{\partial S} \cdot S}{B} \cdot B \cdot r\, dt$$
$$= \frac{\partial C}{\partial S}\mu S_t dt + \frac{\partial C}{\partial S}\sigma S_t d\bar{W}_t + [C_t r - \frac{\partial C}{\partial S} \cdot S \cdot r] dt$$
$$= \left[\frac{\partial C}{\partial S}\mu S_t - \frac{\partial C}{\partial S} S_t r + C_t \cdot r\right] dt + \frac{\partial C}{\partial S}\sigma S_t d\bar{W}_t$$
$$= \left[\frac{\partial C}{\partial S}(\mu - r) S_t + C_t r\right] dt + \frac{\partial C}{\partial S}\sigma S_t d\bar{W}_t$$
$$= \left[\frac{\frac{\partial C}{\partial S}(\mu - r) S_t}{C_t} + r\right] C_t dt + \frac{\partial C}{\partial S}\frac{\sigma S_t}{C_t} \cdot C_t d\bar{W}_t$$

Given $dC_t = \mu_C C_t dt + \sigma_C C_t d\bar{W}_t$

$$\mu_C = \frac{\frac{\partial C}{\partial S}(\mu - r) S_t}{C_t} + r$$

$$\sigma_C = \frac{\partial C}{\partial S} \cdot \frac{\sigma S_t}{C_t}$$

Call option Formula: $C_t = S_t N[d_1] - e^{-r(T-t)} K N[d_2]$

$$S_t N[d_1] > C_t$$

$N[d_1] = \frac{\partial C}{\partial S}$ $\qquad \frac{\frac{\partial C}{\partial S} \cdot S_t}{C_t} > 1, \quad \frac{\frac{\partial C}{\partial S}\sigma \cdot S_t}{C_t} > \sigma \Rightarrow \boxed{\sigma_C > \sigma}$

$$\frac{\frac{\partial C}{\partial S} \cdot S_t}{C_t} > \frac{\mu - r}{\mu - r} \Rightarrow \frac{\frac{\partial C}{\partial S} S_t (\mu - r)}{C_t} > \mu - r$$

$\mu > r$

$$\Rightarrow \frac{\frac{\partial C}{\partial S} S_t (\mu - r)}{C_t} + r > \mu$$

$$\Rightarrow \boxed{\mu_C > \mu}$$

plug in μ_c & σ_c

$$\frac{\mu_c - r}{\sigma_c} = \frac{\cancel{\frac{\partial c}{\partial S}}(\mu-r)S_t}{C_t} + r - r \Big/ \cancel{\frac{\partial c}{\partial S}} \cdot \frac{\sigma S_t}{C_t}$$

$$= \left(\frac{\mu - r}{\sigma}\right)$$

2. put-call parity formula

$$\cancel{P(t,S_t) + S_t = C_t}$$

$$P + S = C + ke^{-r(T-t)}$$

P – put option value

C – call option value

S – Stock value.

$$P = C + ke^{-r(T-t)} - S$$

$$= SN(d_1) - ke^{-r(T-t)} \cdot N(d_2) - S + ke^{-r(T-t)}$$

$$= S[N(d_1) - 1] - [N(d_2) - 1]\, k \cdot e^{-r(T-t)}.$$

Using Black shole formula, $N(d_1) - 1 = \frac{\partial P}{\partial S}$.

$\Rightarrow$ # of stock to hold in order to replicate is $N(d_1) - 1$.