HW 2
MFE 403: Stochastic Calculus
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Problem 1

Let $Z_s = W_s^3$

Apply Ito's lemma to get:

$$dZ_s = 3W_s^2 dW_s + \frac{1}{2} \cdot 6W_s (dW_s)^2$$

Given:

$$Z(0) = 0$$

Integrate on both sides to get:

$$\int_0^t dZ_s = W_t^3$$

$$W_t^3 = 3\int_0^t W_s^2 dW_s + W_s ds$$

$$\int_0^t W_s^2 dW_s = \frac{1}{3}W_t^3 - \int_0^t W_s ds$$

Problem 2

$$X(t) = W_1(t) \times W_2(t)$$

Fix any $u < t$, we have

$$E[X_t|\mathcal{F}_u] = E[W_1(t) \times W_2(t)|\mathcal{F}_u]$$
$$= E[W_1(t)|\mathcal{F}_u] \times E[W_2(t)|\mathcal{F}_u]$$
$$= W_1(u) \times W_2(u) = X(u)$$

Hence it is a martingale

Problem 3

$$X_t = \int_0^t g(s) dW_s$$

$$dX_t = g(t)W_t$$

$$Z_t = e^{-\frac{\eta^2}{2}\int_0^t g^2(s)ds + \eta\int_0^t g(s)dW_s}$$

Apply Ito's Lemma:

$$dZ_t = -\frac{\eta^2}{2} g^2(t)dt \cdot Z_t + \eta g(t) dW_t \cdot Z_t + \frac{\eta^2}{2} g^2(t) dt Z_t$$

$$dZ_t = \eta g(t) dW_t \cdot Z_t = \eta dX_t \cdot Z_t$$

$$Z(0) = 1$$

$$E[\int_0^t dZ_t] = E[\int_0^t \eta dX_s \cdot Z_s]$$

$$E[Z_t - Z(0)] = E[\int_0^t \eta g(s) dW_s Z_s] = \eta \sum_{k=0}^{n-1} E[g(t_k) E_{t_k}[W(t_{k+1}) - W(t_k)] \cdot Z_k]$$

hence

$$E[Z_t] = 1 = Z(0)$$

It is a martingale

$$E[Z_t] = e^{-\frac{\eta^2}{2}\int_0^t g^2(s)ds} E[e^{\eta \int_0^t g(s)dW_s}] = 1$$

$$E[e^{\eta \int_0^t g(s)dW_s}] = e^{\frac{\eta^2}{2}\int_0^t g^2(s)ds}$$

$$E[e^{\eta X_t}] = e^{\frac{\eta^2}{2}\int_0^t g^2(s)ds}$$

Based on the Moment Generating function, the Standard Deviation should be:

$$\sqrt{\int_0^t g^2(s)ds}$$

Problem 4

$$Z_t = W_t^k$$

Apply Ito's Lemma

$$dZ_t = 0 \cdot dt + kW_t^{k-1} dW_t + \frac{1}{2}[k(k-1)W_t^{k-2}(dW_t)^2]$$

$$= kW_t^{k-1} dW_t + \frac{1}{2}[k(k-1)W_t^{k-2} dt]$$

$Z_0 = 0$

$$\int_0^t dZ_s = Z_t = W_t^k$$

$$W_t^k = \int_0^t kW^{k-1} dW_s + \frac{1}{2}k(k-1)W_s^{k-2} ds$$

$$B_k(t) = E[Z_t] = E[W_t^k]$$

$$= E[\int_0^t kW^{k-1} dW_s + \frac{1}{2}k(k-1)W_s^{k-2} ds]$$

$$= 0 + \frac{1}{2}k(k-1)\int_0^t E[W_s^{k-2}] ds$$

$$= \frac{1}{2}k(k-1)\int_0^t B_{k-2}(s) ds$$