CS 377P Spring 2026
Assignment 3
Due: 11 PM, March 4th, 2026

February 22, 2026

No late submissions will be accepted for this assignment.

1. (Finite-differences, 10 points)

   In lecture, we considered the ode $\frac{du}{dt} = -3u(t) + 2$ with initial condition $u(0) = 1$. Using the forward-Euler discretization scheme, we came up with the following recurrence equation:

   $$u_f(0) = 1$$
   $$u_f(nh + h) = (1 - 3h)u_f(nh) + 2h$$

   where $u_f(nh)$ is the approximation for u at $t = nh$. In this problem, you will study the behavior of the approximate solution for different values of h.

   (a) Consider the following values of $h : 1/6, 1/3, 1/2, 2/3, 1$. On a single graph, plot the points for each value of h, using a different color for each h, in the interval $0 \leq t \leq 1$.

   (b) At what value of h does the approximate solution start to oscillate instead of decreasing monotonically?

   (c) At what value of h does the approximate solution become unstable and blow up?

   (d) Explain these results analytically using the difference equation. Hint: look at the values of $(1 - 3h)$.

2. (Iterative solution of linear systems, 5 points) Consider the linear system

   $4x + 2y = 6$
   $x - 5y = -4$

   (a) (2 points) Write down the recurrence relation that corresponds to solving this system using the Jacobi method, starting with the initial approximation $(x_1 = 0, y_1 = 0)$. Use the first equation to refine the approximation for x and the second equation to refine the approximation for y. Express this recurrence as a computation involving matrices and vectors.

(b) (1 points) Compute the first 10 approximations (x_i, y_i) and plot a 3D plot (x, y, i) in which the z-axis is the iteration number i. Give an intuitive explanation of this 3D plot. You do not need to turn in any code but turn in your plot and explanation.

(c) (2 points) Repeat these two parts for the Gauss-Seidel method. You can find a description of the Gauss-Seidel method online.

3. (ODE's, 15 points) Consider the second-order differential equation

$\frac{d^2y}{dx^2} = -y$

with initial conditions $y(0) = 0, y'(0) = 1$. The exact solution of this equation is $y = sin(x)$.

(a) (3 points) What is the difference equation if we use the forward-Euler method to discretize derivatives? Assume the step size is h.

(b) (2 points) Discretize the initial conditions to find expressions for the first two terms in the solution to the difference equation.

(c) (5 points) Calculate the solutions to the difference equation in the interval $x = [0, 2\pi]$ for $h = 0.01, 0.1, 1.0, 2.0$. Graph each solution together with the exact solution, using a separate graph for each value of h. What trends do you see in your plots? No need to turn in code for the calculations.

(d) (5 points) Repeat these steps with the backward-Euler discretization. For this part, you need to discretize the initial condition $y'(0) = 1$ by considering the value of $y(-h)$ (i.e., one time-step before 0). Using the backward-Euler formula for the first derivative, we get $\frac{y_b(0)-y_b(-h)}{h} = 1$, so $y_b(-h) = y_b(0) - h = -h$ since $y_b(0) = 0$ from the other boundary condition. Use the values of $y_b(-h)$ and $y_b(0)$ to "turn the crank" and compute the remaining values of $y_b(nh)$.

4. (PDE's, 20 points) Consider the 2D heat conduction problem discussed in lecture in which we solved the heat equation for given boundary conditions, using a grid that had 4 interior points (see Slide 23). Repeat this exercise using a grid obtained by dividing the x and y ranges into 6 equal sized intervals, rather than 3 intervals as in the lecture example. You should have 25 interior points so you will have to construct a 25x25 matrix A, and solve a linear system Ax = b to find the solution. You may want to write a program to construct this matrix. Use a linear solver from MATLAB, Octave or any other program of your choice.

   What to turn in: What are the temperature values at the 4 original grid points when you use this finer grid?

5. (PDE's, 40 points) In this problem, we consider a membrane in the unit square that is clamped along its edges. At $t = 0$, the membrane is pulled into some initial shape given by the initial conditions, and then released. Intuitively, we would expect the membrane to keep vibrating as shown in Figure 1. The problem is to solve a pde to find the shape of the membrane over time. Here are the details:

   - The independent variables (x, y) are in the unit square [0,1]x[0,1].
   - $u(x, y, t)$ is the displacement at position (x, y) at time t. u is the dependent variable, and its value tells you how far a given point (x, y) has moved from the (x, y) plane at time t. Figure 1 illustrates this for $t = 0$ and $t = 500$.
   - Wave equation: $\frac{\delta^2 u}{\delta t^2} = \frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2}$
   - $u(0, y, t) = u(1, y, t) = u(x, 0, t) = u(x, 1, t) = 0$ (clamped boundary conditions, see Figure 1)
   - $u(x, y, 0) = 4 * x^2 * y(1 - x)(1 - y)$ (initial condition, see Figure 1(a))
   - $u'(x, y, 0) = 0$ (initial condition, membrane is at rest at t=0)
   - Use centered differences to discretize both space and time.
   - Spatial discretization step Δx, Δy = 0.01
   - Time discretization step = Δt = 0.0025
   - Number of time steps = 500

   Figure 1 shows the function $u(x, y, t)$ at $t = 0$ (initial condition) and at $t = 500$. Use the second diagram to check your answer.

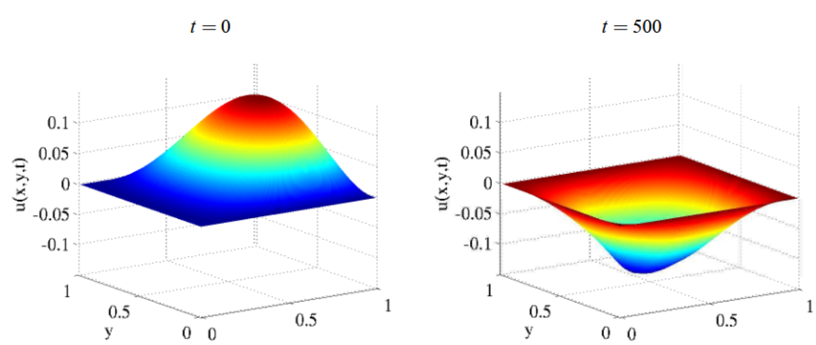

Figure 1: Vibrating membrane at $t = 0$ and at $t = 500$

   Conceptually, you are filling in a series of arrays of size (100x100) that has one such array for each time step.

   What to turn in:

(a) The difference equation obtained by discretizing the pde.

(b) A short paragraph on how you discretized the initial conditions. Note that because you are using centered differences, you will have to compute the values of $u(x, y, -\Delta t)$ using the boundary condition for u' just like you did in Problem 3(d) for the backward-Euler method. Here you would use the centered difference approximation to the first derivative to compute the values of $u(x, y, -\Delta t)$, from which you can turn the crank and compute the remaining values of u.

(c) Plot a graph similar to the ones in Figure 1 for $t = 200$.

(d) Plot a graph of $u(0.5, 0.5, t$ for $0 \leq t \leq 500$. Intuitively, this shows you the displacement of the point $(0.5, 0.5)$ in the membrane over time.

(e) Your code for computing the solution.

6. (Short answers, 10 points) Explain the following terms in a few sentences each.

   (a) (2 points) What is a commutative function? Associative function? Give an example of a function that is commutative but not associative. Give an example of a function that is associative but not commutative.

   (b) (2 points) What is the difference between a problem and an algorithm? Is SSSP a problem or algorithm? If it is a problem, name two algorithms for solving the SSSP problem and write down the asymptotic complexity of each algorithm. What algorithm would you use in a parallel implementation? For the last question, justify your answer briefly.

   (c) (2 points) Explain briefly why the average diameter of a very large power-law graph with billions of vertices may be as small as 5-10. Who was Stanley Milgram and how is he connected to power-law graphs and social networks?

   (d) (2 points) In implementing the Barnes-Hut algorithm, we usually rebuild the spatial decomposition tree from scratch rather than incrementally updating it between time steps. Explain why, using the phrase "Amdahl's Law" in your answer.

   (e) (2 points) How do direct methods for solving linear systems work? Name two direct methods. Why are direct methods not used very often for solving sparse linear systems?

Errata:

2/22: Fixed a typo in the hint for Problem 5(b). You need to compute approximations for $u(x, y, -\Delta t)$ from the boundary conditions, not $u(x, y, -1)$.