Hailstone Numbers and the Collatz Conjecture

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Abstract

Choose any positive integer, call it *n*. Now, if *n* is even, half it. Otherwise, triple it and add one. This is what is known as the hailstone operation. Repeatedly applying it will generate a sequence of hailstone numbers. Like hailstones move up and down sporadically in a cloud, hailstone sequences behave in a similar way. The Collatz conjecture claims that the hailstone sequence will always converge to one for any chosen positive integer *n* as the initial value. In this paper, we explore Hailstone Sequences and the Collatz conjecture.

Section 1: Hailstone Sequences

We begin with a simple idea. Choose any positive integer, n. If n is even, half it. Otherwise, if n is odd, triple it then add one. This algorithm is called the *hailstone operation*. The modular version of this is constructed as follows:

$$f(n) = \begin{cases} \frac{n}{2}, & n \equiv 0 \ (mod\ 2) \\ 3n+1, & n \equiv 1 \ (mod\ 2) \end{cases}$$

Definition 1.1 (Hailstone Sequence). Using the function f from above, let n be any chosen positive integer. We define the sequence a as:

$$a_0 = n, \ \ n > 0$$

$$a_{i+1} = f(a_i)$$

Then a is a sequence of hailstone numbers, also known as a *hailstone sequence* with initial value n.

Example 1.2.

We observe the hailstone sequences of some starting values:

$\boldsymbol{a_0 = 5}$**:**

$5 \to 16 \to 8 \to 4 \to 2 \to 1 \ \to 4 \to 2 \to 1 \to \cdots$

$\boldsymbol{a_0 = 17}$**:**

$17 \to 52 \to 26 \to 13 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1 \to 4 \to 2 \to 1 \to \cdots$

$\boldsymbol{a_0 = 128}$**:**

$128 \to 64 \to 32 \to 16 \to 8 \to 4 \to 2 \to 1 \to 4 \to 2 \to 1 \to \cdots$

Code 1.
The following block in Python 3 can be used to perform the hailstone operation:

```
def hailstone(n):
    if n == 1:
        result = n
    else:
        return n//2 if n % 2 == 0 else 3*n + 1
```

The following block can be used for generating a hailstone sequence:

```
def hailstoneSeq(n):
    result = [n]
    while n != 1:
        n = hailstone(n)
        result.append(n)
    return result
```

From Example 1.2, we begin to see a possible pattern emerging. Each of the sequences eventually reaches the cycle (4, 2, 1). This is known as the trivial cycle, and we will return to this. However, hailstone sequences do behave in a rather hectic way.

Consider the following sequence of hailstone numbers:

$$n \to a_1 \to a_2 \to \cdots \to 4 \to 2 \to 1$$

Then, relative to the above sequence let D be the function from the positive integers on themselves given by $D(n) = k$. That is, $D(n)$ is the amount of times the hailstone operation must be applied to initial value n until its sequence converges to one. Using the sequences from Example 1.2, we see $D(5) < D(128) < D(17)$ even though $5 < 17 < 128$. This is because the numbers fluctuate in a seemingly unpredictable way. In fact, the speed at which a hailstone sequence converges to one is related to the speed at which the odd instances of the sequence converge to one. This will be explored later in the paper, when we discuss the Syracuse Function.

Code 2.

The following block in Python 3 can be used to calculate $D(n)$:

```
def distanceToOne(n):
    result = 0
    while n != 1:
        n = hailstone(n)
        result += 1
    return result
```

This brings us to the inspiration of the name- just as hailstones move up and down in a cloud during formation, hailstone numbers "move up and down" until their sequence converges to one.

Section 2: Extension to the Real Numbers

Let x be a positive integer. We define two binary functions:

$$I_O = \begin{cases} 1, & x \text{ is odd} \\ 0, & x \text{ is even} \end{cases} \quad , \quad I_E = \begin{cases} 1, & x \text{ is even} \\ 0, & x \text{ is odd} \end{cases}$$

Then, we can confirm that the hailstone operation can be given by:

$$f(x) = (3x+1)I_O + (\frac{x}{2})I_E$$

We can even further write f as follows:

$$f(x) = (3x+1)\left(\frac{1-(-1)^x}{2}\right) + (\frac{x}{2})(\frac{1+(-1)^x}{2})$$

Now, whenever x is not an integer, $(-1)^x$ is imaginary. To counter this, we use the following knowledge:

- Whenever x is an odd integer, $cos^2\left(\frac{\pi}{2}x\right) = 0$. If x is even, then it is equal to 1.
- Whenever x is an odd integer, $sin^2\left(\frac{\pi}{2}x\right) = 1$. If x is even, then it is equal to 0.

From here, we can form the function that extends the hailstone operation to the real numbers:

$$f(x) = \frac{x}{2}cos^2\left(\frac{\pi}{2}x\right) + (3x+1)sin^2\left(\frac{\pi}{2}x\right)$$

This can be tested by comparing results to those of the modular form given in Section 1.

Section 3: The Collatz Conjecture

In 1937, Lothar Collatz introduced the following conjecture:

Definition 3.1 (Collatz Conjecture). No matter what positive integer is chosen as an initial value, its hailstone sequence will eventually reach one.

While the Collatz Conjecture has not yet been proven, it is almost unanimously agreed to be true. It has been tested for values up to 87*(2^60). Also, each odd number in a given hailstone sequence is on average ¾ of the previous one.

Code 3.
The following block in Python 3 can be used to test the claim:

```
def listOddHails(n):
    result = hailstoneSeq(n)
    result = [i for i in result if i%2==1]
    return result

def averageOddRatio(n):
    odds = listOddHails(n)
    ratios = []
    for i in range(1,len(odds)-1):
        ratios.append(odds[i]/odds[i-1])
    result = sum(ratios)/len(ratios)
    return result
```

As mentioned before, hailstone sequences converging to one seems to be reliant on their odd instances converging to one. Therefore, this provides evidence that they do not diverge. However, this does not tell us anything about whether there are cycles other than the trivial cycle. If there were, then the Collatz Conjecture would fail. There are, however, approaches to proving the problem.

Section 4: The Syracuse Function

If n is odd, then $3n$+1 is even. (If j is an integer, then 3(2j+1)+1= 2(3j+2), which is even). That is, $3n + 1 = 2^k n'$, where n' is odd.

Definition 4.1 (Syracuse Function). Let O be the set of odd positive integers. Then, the *Syracuse function,* $f: O \to O$, is given by:

$$f(n) = n'$$

Example 4.2. $a_0 = 17$

We have the hailstone sequence:

$17 \to 52 \to 26 \to 13 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$

Now, we have the sequence formed by using the Syracuse function:

$17 \to 13 \to 5 \to 1$

What this does is "speed up" the hailstone operation. More specifically, it skips all the even instances. Therefore, we generate a sequence of the odd instances. Earlier, we claimed that the convergence of a hailstone sequence to one is dependent on its odd instances converging to one. We will now confirm that.

Claim 4.3. *The Collatz Conjecture holds if, for all n in O, there exists a positive integer, k, such that* $f^k(n) = 1$.

Before we confirm this, consider the following example.

Example 4.4.

Consider the hailstone sequence with initial value three:

$3 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$

We rewrite the sequence by changing the even values to the form $2^k j$:

$3 \to (2^1)(5) \to 5 \to (2^4)(1) \to (2^3)(1) \to (2^2)(1) \to (2^1)(1) \to 1$

We now show the odd hailstone sequence being formed with the Syracuse operation:

$3 \to f(3) = 5 \to f^2(3) = f(5) = 1$

We will now prove the claim.

Proof. Assume that for any odd *n*, the sequence formed by the Syracuse operation converges to one. We have two cases for the initial value, *n*:

- Case 1: *n* is odd. Then, by our assumption, we have the following hailstone sequence for *n* by filling in the even instances:

  $n \to 2^{k_1}n_1' \to 2^{k_1-1}n_1' \to \cdots \to n_1' \to \cdots \to 2^{k_j} \to 2^{k_j-1} \to \cdots 4 \to 2 \to 1$

  Therefore, the Collatz Conjecture holds for *n*.

- Case 2: *n* is even. By the hailstone operation, we divide by two until we reach an odd integer. We have the following for some *n* and odd *n'*:

  $$n \to \frac{n}{2} \to \frac{n}{4} \to \cdots \to n'$$

  Since *n'* is odd, by our assumption and previous result, its hailstone sequence converges to one. This sequence terminates the sequence with *n* as initial value. Therefore, the Collatz Conjecture holds for *n*.

  Thus, Claim 4.3 holds is true. ■

We discussed the notion of hailstone sequence's convergence to one being dependent on its odds converging to one. Since this is the exact same as the sequence formed using the Syracuse operation, the proof confirms this.

Section 5: Inverse Collatz

It can be useful to think of operations in reverse. Hence, from the hailstone operation, we derive the inverse function.

Definition 5.1 (Reverse Hailstone Operation). Let *r* be given by:

$$r(n) = \begin{cases} \{2n\}, & n \equiv 0,1,2,3,5 \\ \{2n, \dfrac{n-1}{3}\}, & n \equiv 4 \end{cases} \pmod 6$$

Then *r* is the Inverse Collatz, or *reverse hailstone operation*.

The reasoning for the derivation:

- Let $n \ (mod\ 6) = 0, 2, 3,$ or 5. Then $\frac{n-1}{3}$ is not an integer.
- Let $n \ (mod\ 6) = 1$. Then $\frac{n-1}{3}$ is even, and hence does not lead to *n* using the hailstone operation.
- Let $n \ (mod\ 6) = 4$. Then $\frac{n-1}{3}$ is an odd integer, and hence leads to *n* using the hailstone operation.

Code 4.
The following block in Python 3 can be used to compute the reverse hailstone operation:

```
def reverseHailstone(n):
    return [2*n, (n-1)//3] if n % 6 == 4 else [2*n]
```

Now, we will build a set *R* using the following process:

- $R = \{1\}$
- $R \rightarrow R \cup r(1)$ $\qquad R = \{1,2\}$
- $R \rightarrow R \cup \{\cup\, r(i) \mid i \in r(1)\}$ $\qquad R = \{1,2,4\}$
- …

Claim 5.2. *The Collatz conjecture holds if* $R = \mathbb{N}$.

Proof. Assume $R = \mathbb{N}$. Then for any positive integer, n, repeatedly applying the reverse hailstone sequence to one will return n. That is, the hailstone sequence of n will converge to one. Thus the Collatz conjecture holds. ■

Conclusion

On the surface, hailstone sequences are incredibly simple. One can easily explain it colleagues outside of mathematics. However, when exploring further, the problem becomes far more difficult. When exposed to the Collatz conjecture, Paul Erdős said that "mathematics is not yet ready for such problems," and offered a five-hundred dollar reward for its proof. Though unproved, most (if not all) mathematicians agree that it is likely true.

References

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