Forward and Backward Difference Interpolation Formulae

Given the set of tabular values $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ satisfying the relation $y = f(x)$ where the explicit nature of $f(x)$ is not known, it is required to find a simple function, say $\phi(x)$, such that $f(x)$ and $\phi(x)$ agree at the tabulated points. Such a process is called *interpolation*. i.e, we will be predicting the value of $f(x)$ using $\phi(x)$. Also, if we predict the value of $f(x)$ such that x lies outside the interval $[x_0, x_n]$ then it is called *extrapolation*. The function $\phi(x)$ is called *interpolating function* or *smoothing function*.

If the function $\phi(x)$ is a polynomial then such interpolations are called *polynomial interpolations* and the function $\phi(x)$ is called *interpolating polynomial.*

Forward and Backward Differences

Suppose that we have a table of values $(x_i, y_i), i = 0, 1, 2, \ldots, n$ of any function $y = f(x)$. Assume that the values of x are equally spaced, i.e, $x_i = x_0 + ih, i = 0, 1, 2, \ldots, n$.

Forward Difference

If $y_0, y_1, \ldots, y_n$ are the set of values of y, then $y_1 - y_0, y_2 - y_1, \ldots, y_n - y_{n-1}$ are called *differences* of y.

- $\Delta y_0 = y_1 - y_0, \Delta y_1 = y_2 - y_1, \ldots, \Delta y_{n-1} = y_n - y_{n-1}$, are called *first forward differences* and Δ is called the forward difference operator.
- $\Delta^2 y_0 = \Delta y_1 - \Delta y_0, \Delta^2 y_1 = \Delta y_2 - \Delta y_1, \ldots$ (differences of first forward differences) are called *second forward differences*
- $\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0, \Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1, \ldots$ (differences of second forward differences) are called *third forward differences*.
- Similarly we can define fourth forward differences, fifth forward difference etc.

Backward Difference

- $\nabla y_1 = y_1 - y_0, \nabla y_2 = y_2 - y_1, \ldots, \nabla y_n = y_n - y_{n-1}$, are called *first backward differences* and ∇ is called the backward difference operator.
- $\nabla^2 y_2 = \nabla y_2 - \nabla y_1, \nabla^2 y_3 = \nabla y_3 - \nabla y_2, \ldots$ (differences of first backward differences) are called *second backward differences*
- $\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2, \nabla^3 y_4 = \nabla^2 y_4 - \nabla^2 y_3, \ldots$ (differences of second backward differences) are called *third backward differences*.
- Similarly we can define fourth backward differences, fifth backward difference etc.

Nowton's Formulae for Interpolation

Suppose that $n+1$ points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ of x and y are given. Assume that the values of x are equally spaced, i.e, $x_i = x_0 + ih, i = 0, 1, 2, \ldots, n$. We want to find the n^{th} degree interpolating polynomial.

Newton's forward diffrence interpolating formula

- Let $x = x_0 + ph$.
- Newton's forward diffrence interpolating formula is

$$\begin{aligned} y_n(x) &= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots + \\ &\quad \frac{p(p-1)(p-2)\ldots(p-n+1)}{n!}\Delta^n y_0 \end{aligned}$$

- This formula is useful for interpolation near the beginning of a set of tabular values.

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Newton's backward diffrence interpolating formula

- Let $x = x_n + ph$.
- Newton's backward diffrence interpolating formula is

$$\begin{aligned} y_n(x) &= y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots + \\ &\quad \frac{p(p+1)(p+2)\ldots(p+n-1)}{n!}\nabla^n y_n \end{aligned}$$

- This formula is useful for interpolation near the end of a set of tabular values.