Numerical Analysis

by

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Chapter 2

Interpolation

Module No. 3

Central Difference Interpolation Formulae

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There is a limitation on Newton's forward and Newton's backward formulae. This formulae are useful only when the unknown point is either in the beginning or in the ending of the table. But, when the point is on the middle of the table then these formulae give more error. So some different methods are required and fortunately developed for central point. These methods are known as central difference formulae.

Many central difference methods are available in literature. Among them Gaussian forward and backward, Stirling's and Bessel's interpolation formulae are widely used and these formulae are discussed in this module.

Like Newton's formulae, there are two types of Gaussian formulae, viz. forward and backward difference formulae. Again, if the number of points is odd then there is only one middle point, but for even number of points there are two middle points. Thus two formulae are developed for odd number and even number of points.

3.1 Gauss's forward difference formula

There are two types of Gauss's forward difference formulae are deduced, one for even number of arguments and other for odd number of arguments.

3.1.1 For odd $(2n+1)$ number of arguments

Let $y = f(x)$ be given at $2n+1$ equally spaced points $x_{-n}, x_{-(n-1)}, \ldots, x_{-1}, x_0, x_1, \ldots, x_{n-1}, x_n$. That is, $y_i = f(x_i)$, $i = 0, \pm1, \pm2, \ldots, \pm n$ are known.

Based on these values we construct a polynomial $\phi(x)$ of degree at most $2n$ which pass through the points (x_i, y_i), i.e.

$$\phi(x_i) = y_i, i = 0, \pm1, \pm2, \ldots, \pm n, \tag{3.1}$$

where $x_i = x_0 + ih$, h is the spacing.

The form of Gauss's forward difference interpolation formula is similar to the Newton's forward difference interpolation formula. Let the function $\phi(x)$ be of the form

$$\begin{aligned}
\phi(x) = a_0 &+ a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_{-1})(x - x_0)(x - x_1) \\
&+ a_4(x - x_{-1})(x - x_0)(x - x_1)(x - x_2) + \cdots \\
&+ a_{2n-1}(x - x_{-n+1})(x - x_{-n+2}) \cdots (x - x_{-1})(x - x_0) \cdots (x - x_{n-1}) \\
&+ a_{2n}(x - x_{-n+1})(x - x_{-n+2}) \cdots (x - x_{-1})(x - x_0) \cdots (x - x_n).
\end{aligned} \tag{3.2}$$

The coefficients a_i's are unknown and their values are to be determined by substituting $x = x_0, x_1, x_{-1}, x_2, x_{-2}, \ldots, x_n, x_{-n}$ to (3.2).

Note that the appearance of the arguments to (3.2) follow the order x_0, x_1, x_{-1}, x_2, $x_{-2}, \ldots, x_n, x_{-n}$ and the same order is maintained to calculate the values of a_i's.

By successive substitution of the values of x, we get the following equations.

$$
\begin{aligned}
y_0 &= a_0 \\
y_1 &= a_0 + a_1(x_1 - x_0) \;\; \textit{i.e.}, \;\; y_1 = y_0 + a_1 h, \\
\text{i.e.,} \quad a_1 &= \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y_0}{h}. \\
\text{Again, } y_{-1} &= y_0 + a_1(-h) + a_2(-h)(-2h) \\
&= y_0 - h\frac{\Delta y_0}{h} + a_2 h^2 \cdot 2! \\
\text{i.e.,} \quad a_2 &= \frac{y_{-1} - 2y_0 + y_1}{2!\, h^2} = \frac{\Delta^2 y_{-1}}{2!\, h^2}. \\
y_2 &= a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) \\
&\quad + a_3(x_2 - x_{-1})(x_2 - x_0)(x_2 - x_1) \\
&= y_0 + \frac{y_1 - y_0}{h}(2h) + \frac{y_{-1} - 2y_0 + y_1}{2!h^2}(2h)(h) + a_3(3h)(2h)(h) \\
\text{or,} \quad a_3 &= \frac{y_2 - 3y_1 + 3y_0 - y_{-1}}{3!h^3} = \frac{\Delta^3 y_{-1}}{3!h^3}.
\end{aligned}
$$

In this manner, the values of the remaining a_i's can be determined. That is,

$$
a_4 = \frac{\Delta^4 y_{-2}}{4!h^4}, a_5 = \frac{\Delta^5 y_{-2}}{5!h^5}, \ldots, a_{2n-1} = \frac{\Delta^{2n-1} y_{-(n-1)}}{(2n-1)!h^{2n-1}}, a_{2n} = \frac{\Delta^{2n} y_{-n}}{(2n)!h^{2n}}.
$$

Therefore, the Gauss's forward difference interpolation formula is given by

$$
\begin{aligned}
\phi(x) = y_0 &+ (x - x_0)\frac{\Delta y_0}{h} + (x - x_0)(x - x_1)\frac{\Delta^2 y_{-1}}{2!h^2} \\
&+ (x - x_{-1})(x - x_0)(x - x_1)\frac{\Delta^3 y_{-1}}{3!h^3} + \cdots \\
&+ (x - x_{-(n+1)}) \cdots (x - x_{n-1})\frac{\Delta^{2n-1} y_{-(n-1)}}{(2n-1)!h^{2n-1}} \\
&+ (x - x_{-(n+1)}) \cdots (x - x_{n-1})(x - x_n)\frac{\Delta^{2n} y_{-n}}{(2n)!h^{2n}}. \qquad (3.3)
\end{aligned}
$$

In this formula, we assumed that the arguments are in equispaced, i.e. $x_{\pm i} = x_0 \pm ih, i = 0, 1, 2, \ldots, n$. So, a new variable s is introduced, where $x = x_0 + sh$.

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Thus, $x - x_{\pm i} = (s \mp i)h$.

By these substitution, $x - x_0 = sh, x - x_1 = (s-1)h, x - x_{-1} = (s+1)h$, $x - x_2 = (s-2)h, x - x_{-2} = (s+2)h$ and so on.

Using these values $\phi(x)$ reduces to

$$
\begin{aligned}
\phi(x) &= y_0 + sh\frac{\Delta y_0}{h} + sh(s-1)h\frac{\Delta^2 y_{-1}}{2!h^2} + (s+1)hsh(s-1)h\frac{\Delta^3 y_{-1}}{3!h^3} + \cdots \\
&\quad + (s+\overline{n-1})h\cdots sh(s-1)h\cdots(s-\overline{n-1})h\frac{\Delta^{2n-1} y_{-(n-1)}}{(2n-1)!h^{2n-1}} \\
&\quad + (s+\overline{n-1})h\cdots sh(s-1)h\cdots(s-\overline{n-1})h(s-n)h\frac{\Delta^{2n} y_{-n}}{(2n)!h^{2n}} \\
&= y_0 + s\Delta y_0 + s(s-1)\frac{\Delta^2 y_{-1}}{2!} + (s+1)s(s-1)\frac{\Delta^3 y_{-1}}{3!} + \cdots \\
&\quad + (s+\overline{n-1})\cdots s(s-1)\cdots(s-\overline{n-1})\frac{\Delta^{2n-1} y_{-(n-1)}}{(2n-1)!} \\
&\quad + (s+\overline{n-1})\cdots s(s-1)\cdots(s-\overline{n-1})(s-n)\frac{\Delta^{2n} y_{-n}}{(2n)!} \\
&= y_0 + s\Delta y_0 + s(s-1)\frac{\Delta^2 y_{-1}}{2!} + s(s^2-1^2)\frac{\Delta^3 y_{-1}}{3!} \\
&\quad + s(s^2-1^2)(s-2)\frac{\Delta^4 y_{-2}}{4!} + \cdots \\
&\quad + s(s^2-\overline{n-1}^2)(s^2-\overline{n-2}^2)\cdots(s^2-1^2)\frac{\Delta^{2n-1} y_{-(n-1)}}{(2n-1)!} \\
&\quad + s(s^2-\overline{n-1}^2)(s^2-\overline{n-2}^2)\cdots(s^2-1^2)(s-n)\frac{\Delta^{2n} y_{-n}}{(2n)!}. \qquad (3.4)
\end{aligned}
$$

The formula (3.3) or (3.4) is known as **Gauss's forward central difference interpolation formula** or **the first interpolation formula of Gauss**.

3.1.2 For even ($2n$) number of arguments

In this case, there are two points in the middle position. So, the arguments be taken as $x_0, x_{\pm 1}, \ldots, x_{\pm(n-1)}$ and x_n.

By using the previous process, the Gauss's forward interpolation formula for even number of arguments becomes:

$$\begin{aligned}
\phi(x) = y_0 + s\Delta y_0 + s(s-1)\frac{\Delta^2 y_{-1}}{2!} + (s+1)s(s-1)\frac{\Delta^3 y_{-1}}{3!} \\
+(s+1)s(s-1)(s-2)\frac{\Delta^4 y_{-2}}{4!} \\
+(s+2)(s+1)s(s-1)(s-2)\frac{\Delta^5 y_{-2}}{5!} + \cdots \\
+(s+\overline{n-1})\cdots s\cdots(s-\overline{n-1})\frac{\Delta^{2n-1} y_{-(n-1)}}{(2n-1)!} \\
= y_0 + s\Delta y_0 + s(s-1)\frac{\Delta^2 y_{-1}}{2!} + s(s^2-1^2)\frac{\Delta^3 y_{-1}}{3!} \\
+s(s^2-1^2)(s-2)\frac{\Delta^4 y_{-2}}{4!} + (s^2-2^2)(s^2-1^2)s\frac{\Delta^5 y_{-2}}{5!} + \cdots \\
+(s^2-\overline{n-1}^2)\cdots(s^2-1^2)s\frac{\Delta^{2n-1} y_{-(n-1)}}{(2n-1)!}.
\end{aligned} \tag{3.5}$$

3.1.3 Error in Gauss's forward central difference formula

From general expression of error term (discussed in Module number 1 of Chapter 2), we have for $2n+1$ arguments

$$\begin{aligned}
E(x) &= (x-x_{-n})(x-x_{-(n-1)})\cdots(x-x_{-1})(x-x_0)\cdots(x-x_n)\frac{f^{2n+1}(\xi)}{(2n+1)!} \\
&= (s+n)(s+n-1)\cdots(s+1)s(s-1)\cdots(s-n+1)(s-n) \\
&\qquad \times\ h^{2n+1}\frac{f^{2n+1}(\xi)}{(2n+1)!} \\
&= s(s^2-1^2)\cdots(s^2-n^2).h^{2n+1}\frac{f^{2n+1}(\xi)}{(2n+1)!}
\end{aligned} \tag{3.6}$$

where $x = x_0 + sh$ and ξ lies between $\min\{x_{-n}, x_{-(n-1)}, \ldots, x_0, x_1, \ldots, x_{n-1}, x_n\}$ and $\max\{x_{-n}, x_{-(n-1)}, \ldots, x_0, x_1, \ldots, x_{n-1}, x_n\}$.

For $2n$ arguments, the error term is

$$\begin{aligned}
E(x) &= (s+n-1)\cdots(s+1)s(s-1)\cdots(s-n+1)(s-n)h^{2n}\frac{f^{2n}(\xi)}{(2n)!} \\
&= s(s^2-1^2)\cdots(s^2-\overline{n-1}^2)(s-n).h^{2n}\frac{f^{2n}(\xi)}{(2n)!},
\end{aligned} \tag{3.7}$$

where, $\min\{x_{-n}, x_{-(n-1)}, \dots, x_0, x_1, \dots, x_{n-1}, x_n\} < \xi$
$< \max\{x_{-n}, x_{-(n-1)}, \dots, x_0, x_1, \dots, x_{n-1}, x_n\}$.

3.2 Gauss's backward difference formula

Like previous case, there are two formulae for Gauss's backward difference interpolation one for odd number of arguments and other for even number number of arguments.

3.2.1 For odd $(2n+1)$ number of arguments

Assumed that the function $y = f(x)$ be known for $2n+1$ equispaced arguments $x_{\pm i}, i = 0, 1, 2, \dots, n$, where $x_{\pm i} = x_0 \pm ih, i = 0, 1, 2, \dots, n$.

Let $y_{\pm i} = f(x_{\pm i}), i = 0, 1, 2, \dots, n$.

Suppose $\phi(x)$ be the approximate polynomial which passes through the $2n$ points $x_{\pm i} = x_0 \pm ih, i = 0, 1, 2, \dots, n$ and the degree of it is at most $2n$. That is,

$$\phi(x_{\pm i}) = y_{\pm i}, i = 0, 1, \dots, n. \tag{3.8}$$

Let the polynomial $\phi(x)$ be of the following form.

$$\begin{aligned}
\phi(x) = a_0 &+ a_1(x - x_0) + a_2(x - x_{-1})(x - x_0) + a_3(x - x_{-1})(x - x_0)(x - x_1) \\
&+ a_4(x - x_{-2})(x - x_{-1})(x - x_0)(x - x_1) \\
&+ a_5(x - x_{-2})(x - x_{-1})(x - x_0)(x - x_1)(x - x_2) + \cdots \\
&+ a_{2n-1}(x - x_{-(n-1)}) \cdots (x - x_{-1})(x - x_0) \cdots (x - x_{n-1}) \\
&+ a_{2n}(x - x_{-n})(x - x_{-(n-1)}) \cdots (x - x_{-1})(x - x_0) \cdots (x - x_{n-1}),
\end{aligned} \tag{3.9}$$

where a_i's are unknown constants and their values are to be determined by using the relations (3.8).

Also, $x_i = x_0 + ih, x_{-j} = x_0 - jh$ for $i, j = 0, 1, 2, \dots, n$. Therefore, $x_i - x_{-j} = (i+j)h$ and $(x_{-i} - x_j) = -(i+j)h$.

To find the values of a_i' we substitute $x = x_0, x_{-1}, x_1, x_{-2}, x_2, \dots, x_{-n}, x_n$ to (3.9) in succession.

The values of a_i's are given by

$$\begin{aligned}
y_0 &= a_0\\
\phi(x_{-1}) &= a_0 + a_1(x_{-1} - x_0)\\
\text{i.e.,}\quad y_{-1} &= y_0 + a_1(-h),\\
\text{i.e., } a_1 &= \frac{y_0 - y_{-1}}{h} = \frac{\Delta y_{-1}}{h}
\end{aligned}$$

$$\begin{aligned}
\phi(x_1) &= a_0 + a_1(x_1 - x_0) + a_2(x_1 - x_{-1})(x_1 - x_0)\\
y_1 &= y_0 + h.\frac{\Delta y_{-1}}{h} + a_2(2h)(h)\\
\text{i.e.,}\quad a_2 &= \frac{y_1 - y_0 - (y_0 - y_{-1})}{2!h^2} = \frac{\Delta^2 y_{-1}}{2!h^2}
\end{aligned}$$

$$\begin{aligned}
\phi(x_{-2}) &= a_0 + a_1(x_{-2} - x_0) + a_2(x_{-2} - x_{-1})(x_{-2} - x_0)\\
&\quad + a_3(x_{-2} - x_{-1})(x_{-2} - x_0)(x_{-2} - x_1)\\
\text{i.e.,}\quad y_{-2} &= y_0 + \frac{\Delta y_{-1}}{h}(-2h) + \frac{\Delta^2 y_{-1}}{2!h^2}(-h)(-2h) + a_3(-h)(-2h)(-3h)\\
&= y_0 - 2(y_0 - y_{-1}) + (y_1 - 2y_0 + y_{-1}) + a_3(-1)^3(3!)h^3\\
\text{or,}\quad a_3 &= \frac{y_1 - 3y_0 + 3y_{-1} - y_{-2}}{3!h^3} = \frac{\Delta^3 y_{-2}}{3!h^3}.
\end{aligned}$$

The other values can be obtained in similar way.

$$a_4 = \frac{\Delta^4 y_{-2}}{4!h^4}, a_5 = \frac{\Delta^5 y_{-3}}{5!h^5}, \ldots, a_{2n-1} = \frac{\Delta^{2n-1} y_{-n}}{(2n-1)!h^{2n-1}}, a_{2n} = \frac{\Delta^{2n} y_{-n}}{(2n)!h^{2n}}.$$

Using the values of a_i's, equation (3.9) reduces to

$$\begin{aligned}
\phi(x) &= y_0 + (x - x_0)\frac{\Delta y_{-1}}{1!h} + (x - x_{-1})(x - x_0)\frac{\Delta^2 y_{-1}}{2!h^2}\\
&\quad + (x - x_{-1})(x - x_0)(x - x_1)\frac{\Delta^3 y_{-2}}{3!h^3}\\
&\quad + (x - x_{-2})(x - x_{-1})(x - x_0)(x - x_1)\frac{\Delta^4 y_{-2}}{4!h^4} + \cdots\\
&\quad + (x - x_{-(n-1)})\cdots(x - x_{-1})(x - x_0)\cdots(x - x_{n-1})\frac{\Delta^{2n-1} y_{-n}}{(2n-1)!h^{2n-1}}\\
&\quad + (x - x_{-n})(x - x_{-1})(x - x_0)(x - x_1)\cdots(x - x_{n-1})\frac{\Delta^{2n} y_{-n}}{(2n)!h^{2n}}. \qquad (3.10)
\end{aligned}$$

Like previous case, we introduced a unit less variable s, where $x = x_0 + sh$. The advantage to use such variable is that the formula becomes simple and easy to calculate.

Let us consider two identities

$$\frac{x - x_i}{h} = \frac{x - x_0 - ih}{h} = s - i \text{ and}$$
$$\frac{x - x_{-i}}{h} = \frac{x - x_0 + ih}{h} = s + i, \quad i = 0, 1, 2, \ldots, n.$$

Then the above formula becomes

$$\begin{aligned}\phi(x) = y_0 + s\Delta y_{-1} &+ \frac{(s+1)s}{2!}\Delta^2 y_{-1} + \frac{(s+1)s(s-1)}{3!}\Delta^3 y_{-2} \\ &+ \frac{(s+2)(s+1)s(s-1)}{4!}\Delta^4 y_{-2} + \cdots \\ &+ \frac{(s+n-1)\cdots(s+1)s(s-1)\cdots(s-n+1)}{(2n-1)!}\Delta^{2n-1} y_{-n} \\ &+ \frac{(s+n)(s+n-1)\cdots(s+1)s(s-1)\cdots(s-n+1)}{(2n)!}\Delta^{2n} y_{-n}. \end{aligned} \tag{3.11}$$

The formula (3.11) is known as **Gauss's backward interpolation formula** or **second interpolation formula of Gauss** for odd number of arguments.

3.2.2 For even $(2n)$ number of arguments

Here the number of middle values is two. So we take the arguments as $x_0, x_{\pm 1}, \ldots, x_{\pm(n-1)}$ and x_{-n}, where $x_{\pm i} = x_0 \pm ih$, $i = 0, 1, \ldots, n-1$ and $x_{-n} = x_0 - nh$.

Proceeding as in previous case we obtained the Gauss's backward interpolation formula as

$$\begin{aligned}\phi(x) = y_0 + s\Delta y_{-1} &+ \frac{(s+1)s}{2!}\Delta^2 y_{-1} + \frac{(s+1)s(s-1)}{3!}\Delta^3 y_{-2} \\ &+ \frac{(s+2)(s+1)s(s-1)}{4!}\Delta^4 y_{-2} + \cdots \\ &+ \frac{(s+n-1)\cdots(s+1)s(s-1)\cdots(s-n+1)}{(2n-1)!}\Delta^{2n-1} y_{-n}, \end{aligned} \tag{3.12}$$

where $s = \dfrac{x - x_0}{h}$.

3.2.3 Error term in Gauss's backward central difference formula

The error term for the $(2n+1)$ equispaced arguments is

$$\begin{aligned}
E(x) &= (x-x_{-n})(x-x_{-(n-1)})\cdots(x-x_{-1})(x-x_0)\cdots(x-x_n)\frac{f^{2n+1}(\xi)}{(2n+1)!} \\
&= (s+n)(s+n-1)\cdots(s+1)s(s-1)\cdots(s-n+1)(s-n) \\
&\qquad\qquad\qquad\qquad \times\ h^{2n+1}\frac{f^{2n+1}(\xi)}{(2n+1)!}, \\
&= s(s^2-1^2)(s^2-2^2)\cdots(s^2-n^2)\times h^{2n+1}\frac{f^{2n+1}(\xi)}{(2n+1)!},
\end{aligned}$$

The error term for the $2n$ equispaced arguments is

$$\begin{aligned}
E(x) &= (x-x_{-n})(x-x_{-(n-1)})\cdots(x-x_{-1})(x-x_0)\cdots(x-x_{n-1})\frac{f^{2n}(\xi)}{(2n)!} \\
&= (s+n)(s+n-1)\cdots(s+1)s(s-1)\cdots(s-n+1)h^{2n}\frac{f^{2n}(\xi)}{(2n)!}, \\
&= s(s^2-1^2)(s^2-2^2)\cdots(s^2-\overline{n-1}^2)(s+n)h^{2n}\frac{f^{2n}(\xi)}{(2n)!},
\end{aligned}$$

In both the cases ξ lies between $\min\{x_{-n}, x_{-n}, \ldots, x_0,\, x_1, \ldots, x_{n-1}, x_{n-1}\}$ and $\max\{x_{-n}, x_{-(n-1)}, \ldots, x_0, x_1, \ldots, x_{n-1}\}$.

3.3 Stirling's interpolation formula

The average of Gauss's forward and backward difference formulae for **odd number** of equispaced arguments gives Stirling's interpolation formula.

The Stirling's formula is obtained from the equations (3.4) and (3.11) as

$$\begin{aligned}
\phi(x) &= \frac{\phi(x)_{\text{forward}}+\phi(x)_{\text{backward}}}{2} \\
&= y_0 + \frac{s}{1!}\frac{\Delta y_{-1}+\Delta y_0}{2} + \frac{s^2}{2!}\Delta^2 y_{-1} + \frac{s(s^2-1^2)}{3!}\frac{\Delta^3 y_{-2}+\Delta^3 y_{-1}}{2} \\
&\quad + \frac{s^2(s^2-1^2)}{4!}\Delta^4 y_{-2} + \frac{s(s^2-1^2)(s^2-2^2)}{5!}\frac{\Delta^5 y_{-3}+\Delta^5 y_{-2}}{2} + \cdots \\
&\quad + \frac{s^2(s^2-1^2)(s^2-2^2)\cdots(s^2-\overline{n-1}^2)}{(2n)!}\Delta^{2n} y_{-n}.
\end{aligned} \tag{3.13}$$

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The error term of this formula is given by

$$E(x) = \frac{s(s^2 - 1^2)(s^2 - 2^2)\cdots(s^2 - n^2)}{(2n+1)} h^{2n+1} f^{2n+1}(\xi), \tag{3.14}$$

where $\min\{x_{-n}, \ldots, x_0, \ldots, x_n\} < \xi < \max\{x_{-n}, \ldots, x_0, \ldots, x_n\}$.

The formula (3.13) is known as **Stirling's central difference interpolation formula**.

Note 3.1 *(a) It may be noted that the Stirling's interpolation formula is applicable when the argument x, for which $f(x)$ to be calculated, is at the centre of the table and the number of arguments is odd.*

If the number of arguments in the given tabulated values is even, then discard one end point from the table to make odd number of points such that x_0 be in the middle of the table.

(b) It is verified that the Stirling's interpolation formula gives the better approximate result when $-0.25 < s < 0.25$. So it is suggested that, assign the subscripts to the points in such a way that $s = \frac{x - x_0}{h}$ satisfies this condition.

3.4 Bessel's interpolation formula

This is another useful central difference interpolation formula obtained from Gauss's forward and backward interpolation formulae. It is also obtained by taking average of Gauss's forward and backward interpolation formulae after shifting one step of backward formula. This formula is application for the even number of arguments.

Let us consider $2n$ equispaced arguments $x_{-(n-1)}, \ldots, x_{-1}, x_0, x_1, \ldots, x_{n-1}, x_n$, where $x_{\pm i} = x_0 \pm ih$, h is the spacing. Since the number of points is even, there are two points in the middle position. For the above numbering, the number of arguments to the right of x_0 is n and to the left is $n - 1$.

For this representation the Gauss's backward difference interpolation formula (3.12) is

$$\begin{aligned}\phi(x) = y_0 + s\Delta y_{-1} + \frac{s(s+1)}{2!}\Delta^2 y_{-1} + \frac{(s+1)s(s-1)}{3!}\Delta^3 y_{-2} \\ + \frac{(s+2)(s+1)s(s-1)}{4!}\Delta^4 y_{-2} + \cdots \\ + \frac{(s+n-1)\cdots(s+1)s(s-1)\cdots(s-n+1)}{(2n-1)!}\Delta^{2n-1} y_{-n}.\end{aligned} \tag{3.15}$$

Now we consider x_1 be the middle argument of the data. That is, the number of points to the right of x_1 is $n-1$ and to the left is n. Therefore, for this assumption, we have to shift the points one step to the right.

Then

$$\frac{x-x_1}{h} = \frac{x-(x_0+h)}{h} = \frac{x-x_0}{h} - 1 = s-1.$$

Also, the indices of all the differences of (3.15) will be increased by 1. So we replace s by $s-1$ and increase the indices of (3.15) by 1, and the Gauss's backward difference interpolation formula becomes

$$\begin{aligned}\phi_1(x) = y_1 &+ (s-1)\Delta y_0 + \frac{s(s-1)}{2!}\Delta^2 y_0 + \frac{s(s-1)(s-2)}{3!}\Delta^3 y_{-1}\\ &+\frac{(s+1)s(s-1)(s-2)}{4!}\Delta^4 y_{-1}\\ &+\frac{(s+1)s(s-1)(s-2)(s-3)}{5!}\Delta^5 y_{-2} + \cdots\\ &+\frac{(s+n-2)\cdots(s+1)s(s-1)(s-2)\cdots(s-n)}{(2n-1)!}\Delta^{2n-1} y_{-n+1}. \end{aligned} \quad (3.16)$$

The average of (3.16) and Gauss's forward interpolation formula (3.5) gives,

$$\begin{aligned}\phi(x) &= \frac{\phi_1(x)+\phi(x)_{\text{forward}}}{2}\\ &= \frac{y_0+y_1}{2} + \Big(s-\frac{1}{2}\Big)\Delta y_0 + \frac{s(s-1)}{2!}\frac{\Delta^2 y_0 + \Delta^2 y_{-1}}{2}\\ &\quad+\frac{(s-\frac{1}{2})s(s-1)}{3!}\Delta^3 y_{-1} + \frac{s(s-1)(s+1)(s-2)}{4!}\frac{\Delta^4 y_{-2}+\Delta^4 y_{-1}}{2}\\ &\quad+\frac{(s-\frac{1}{2})s(s-1)(s+1)(s-2)}{5!}\Delta^5 y_{-2} + \cdots\\ &\quad+\frac{(s-\frac{1}{2})s(s-1)(s+1)\cdots(s+n-2)(s-\overline{n-1})}{(2n-1)!}\Delta^{2n-1} y_{-(n-1)}, \end{aligned} \quad (3.17)$$

where $x = x_0 + sh$.

As in previous cases, we introduce the new variable u defined by $u = s - \dfrac{1}{2} = \dfrac{x-x_0}{h} - \dfrac{1}{2}$. By this substitution the above formula becomes

$$\begin{aligned}\phi(x) = \frac{y_0+y_1}{2} &+ u\Delta y_0 + \frac{u^2-\frac{1}{4}}{2!}\cdot\frac{\Delta^2 y_{-1}+\Delta^2 y_0}{2} + \frac{u(u^2-\frac{1}{4})}{3!}\Delta^3 y_{-1}\\ &+\frac{(u^2-\frac{1}{4})(u^2-\frac{9}{4})\cdots(u^2-\frac{(2n-3)^2}{4})}{(2n-1)!}\Delta^{2n-1} y_{-(n-1)}. \end{aligned} \quad (3.18)$$

This formula is known as **Bessel's central difference interpolation formula**.

Note 3.2 *(a) It can be shown that the Bessel's formula gives the best result when u lines between -0.25 and 0.25, i.e. $0.25 < u < 0.75$.*

*(b) Bessel's central difference interpolation formula is used when the number of arguments is **even** and the interpolating point is near the middle of the table.*

Example 3.1 *Use Bessel central difference interpolation formula to find the values of y at $x = 1.55$ from the following table*

x	:	1.0	1.5	2.0	2.5	3.0
y	:	10.2400	12.3452	15.2312	17.5412	19.3499

Solution. The central difference table is given below:

i	x_i	y_i	Δy_i	$\Delta^2 y_i$	$\Delta^3 y_i$
-2	1.0	10.2400			
			2.1052		
-1	1.5	12.3452		0.7808	
			2.8860		-1.3568
0	2.0	15.2312		-0.5760	
			2.3100		0.0747
1	2.5	17.5412		-0.5013	
			1.8087		
2	3.0	19.3499			

Here $x = 1.55$. Let $x_0 = 2.0$. Therefore, $s = (1.55 - 2.0)/0.5 = -0.9$.

By Bessel's formula

$$\begin{aligned}
y(1.55) &= \frac{y_0 + y_1}{2} + \Big(s - \frac{1}{2}\Big)\Delta y_0 + \frac{s(s-1)}{2!}\frac{\Delta^2 y_0 + \Delta^2 y_{-1}}{2} \\
&\quad + \frac{1}{3!}\Big(s - \frac{1}{2}\Big)s(s-1)\Delta^3 y_{-1} \\
&= \frac{15.2312 + 17.5412}{2} + (-0.9 - 0.5) \times 2.3100 \\
&\quad + \frac{-0.9(-0.9-1)}{2!}\frac{-0.5760 - 0.5013}{2} \\
&\quad + \frac{1}{6}(-0.9 - 0.5)(-0.9)(-0.9 - 1) \times 0.0747 \\
&= 13.5829.
\end{aligned}$$

Example 3.2 *Use Stirling central difference interpolation formula to find the values of y at $x = 2.30$ from the table of above example.*

In this case $x = 2.30$. Let $x_0 = 2.0$. Thus $s = (2.30 - 2.0)/0.5 = 0.6$.

Now, by Stirling's formula

$$y(x) = y_0 + s\frac{\Delta y_{-1} + \Delta y_0}{2} + \frac{s^2}{2!}\Delta^2 y_{-1} + \frac{s(s^2 - 1^2)}{3!}\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2}$$

$$\begin{aligned} y(2.30) &= 15.2312 + 0.6\frac{2.3100 + 2.8860}{2} + \frac{(0.6)^2}{2} \times (-0.5760) \\ &\quad + \frac{0.6(0.36 - 1)}{6}\frac{-1.3568 + 0.0747}{2} \\ &= 15.2312 + 1.5588 - 0.10368 + 0.0410272 \simeq 16.2773. \end{aligned}$$

Note 3.3 *In Newton's forward and backward interpolation formulae the first or the last interpolating point is taken as initial point. But, in central difference interpolation formulae, a middle point is taken as the initial point x_0.*