System of Nonlinear Equations (Newtopn-Raphson Method)

Consider a system of nonlinear equations

$$\begin{aligned} f_1(x_1, x_2, \ldots, x_n) &= 0 \\ f_2(x_1, x_2, \ldots, x_n) &= 0 \\ &\vdots \\ f_n(x_1, x_2, \ldots, x_n) &= 0. \end{aligned}$$

It can be represented as $f(\mathbf{x}) = \mathbf{0}$, where $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$,

$$f(\mathbf{x}) = \begin{pmatrix} f_1(x_1, x_2, \ldots, x_n) \\ f_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ f_n(x_1, x_2, \ldots, x_n) \end{pmatrix}, \text{ and } \mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Let $J(\mathbf{x})$ denotes the Jacobian matrix,

$$J(\mathbf{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \dots & \frac{\partial f_2}{\partial x_n} \\ \dots & \dots & \dots & \dots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \dots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}.$$

Let $\mathbf{x}$ be the current approximate and $\mathbf{x} + \Delta\mathbf{x}$, be the next approximate, then

$$J(\mathbf{x})\Delta\mathbf{x} = -f(\mathbf{x})$$

. If determinant of $J(\mathbf{x})$ is **non-zero**, then $\Delta\mathbf{x} = -J(\mathbf{x})^{-1} f(\mathbf{x})$.

For instance, let $n = 2$ and $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be the current iterate. Then the next iterate is $\begin{pmatrix} x_1 + \Delta x_1 \\ x_2 + \Delta x_2 \end{pmatrix}$ where $\begin{pmatrix} \Delta x_1 \\ \Delta x_2 \end{pmatrix}$ is the solution of

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \end{pmatrix} = - \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix}.$$

Algorithm:

1. Estimate solution vector $\mathbf{x}$.
2. Evaluate $f(\mathbf{x})$.
3. Compute the Jacobian matrix $J(\mathbf{x})$.
4. Solve the equation $J(\mathbf{x})\Delta\mathbf{x} = -f(\mathbf{x})$ for $\Delta\mathbf{x}$.
5. Let $\mathbf{x} \leftarrow \mathbf{x} + \Delta\mathbf{x}$ and repeat 2-5 until $|\Delta\mathbf{x}| < \epsilon$.

Q: Find the real root of the equations $x_1^2 - x_2^2 = 3$ and $x_1^2 + x_2^2 = 13$. Take the initial vector as $\begin{pmatrix} 2.54951 \\ 2.54951 \end{pmatrix}$ and do two iterations.

Ans: Given $f(\mathbf{x}) = \begin{pmatrix} x_1^2 - x_2^2 - 3 \\ x_1^2 + x_2^2 - 13 \end{pmatrix}$. Now,

$$
\begin{aligned}
J(\mathbf{x}) &= \begin{pmatrix} \dfrac{\partial x_1^2 - x_2^2 - 3}{\partial x_1} & \dfrac{\partial x_1^2 - x_2^2 - 3}{\partial x_2} \\ \dfrac{\partial x_1^2 + x_2^2 - 13}{\partial x_1} & \dfrac{\partial x_1^2 + x_2^2 - 13}{\partial x_2} \end{pmatrix} \\
&= \begin{pmatrix} 2x_1 & -2x_2 \\ 2x_1 & 2x_2 \end{pmatrix}
\end{aligned}
$$

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First iteration: Let $\mathbf{x} = \begin{pmatrix} 2.54951 \\ 2.54951 \end{pmatrix}$. Hence,

$$J(\mathbf{x}) = \begin{pmatrix} 2 \times 2.54951 & -2 \times 2.54951 \\ 2 \times 2.54951 & 2 \times 2.54951 \end{pmatrix} = 5.09902 \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

. Therefore,

$$\Delta\mathbf{x} = \begin{pmatrix} 0.29417 \\ -0.29417 \end{pmatrix}$$

and $\mathbf{x} + \Delta\mathbf{x} = \begin{pmatrix} 2.54951 \\ 2.54951 \end{pmatrix} + \begin{pmatrix} 0.29417 \\ -0.29417 \end{pmatrix} = \begin{pmatrix} 2.84368 \\ 2.25534 \end{pmatrix}$.

Second iteration: $J(\mathbf{x}) = \begin{pmatrix} 5.68736 & -4.51068 \\ 5.68736 & 4.51068 \end{pmatrix}$,
$f(\mathbf{x}) = \begin{pmatrix} -0.000042573 \\ 0.173074458 \end{pmatrix}$. We get $\Delta\mathbf{x} = \begin{pmatrix} -0.01521 \\ -0.01919 \end{pmatrix}$ and
$\mathbf{x} = \begin{pmatrix} 2.82847 \\ 2.23615 \end{pmatrix}$

Problems:

Solve the following system of nonlinear equations,

- $x^2 + y^2 = 3$ and $xy = 1$. Take $\begin{pmatrix} 0.5 \\ 1.5 \end{pmatrix}$ as the starting value and perform 3 iterations.

  Final answer : $\begin{pmatrix} 0.61803 \\ 1.61803 \end{pmatrix}$
- $x^2 - y^2 = 4$ and $y^2 + x^2 = 16$. Take $x = y = 2.8284$.

  Answer: x=3.162 and y=2.450