Notes

Thus, $\text{at } x = 35, p = \frac{35-30}{10} = 0.5,$

$y_0 = 439$, $\Delta y_0 = -93$, $\Delta y_{-1} = -73$, $\Delta^2 y_{-1} = -20$, $\Delta^3 y_{-1} = -10$, $\Delta^3 y_{-2} = -35$ and $\Delta^4 y_{-2} = 145$

Substituting the above values in Stirling's formula, we get

$$y_{0.5} = 435 + (0.5)\left[\frac{(-93)+(-73)}{2}\right] + \frac{(0.5)^2}{2!}(-20)$$

$$+\frac{(0.5)\left[(0.5)^2-1\right]}{3!}\left[\frac{10+(-35)}{2}\right] + \frac{(0.5)^2\left[(0.5)^2-1\right]}{4!}(145)$$

$$= 435 - 41.5 - 2.5 + 0.78125 - 1.1328125$$

$$= 390.64844$$

$$\therefore\ y_{35} \approx 390.648$$

3.3 Methods Based on Finite Differences

The shifting operator E is defined by $Ey_{(x)} = y_{x+h}$ or $E(f(x)) = f(x+h)$ i.e. the effect of E on y_x is to shift the value of y_x to the next higher value y_{x+h}. Consider the relation

$$Ef(x) = f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}(x) + ...$$

$$= \left(1 + hD + \frac{h^2D^2}{2!} + ...\right) f(x)$$

$$Ef(x) = e^{hD} f(x)$$

$$\Rightarrow \quad E = e^{hD}$$

$$\log E = hD \qquad ....(i)$$

Where $f'(x) = Df(x) = \frac{d}{dx} f(x)$

$f''(x) = D^2 f(x) = \frac{d^2}{dx^2}(f(x))$, D is the differential operator

The forward difference operator

Δ' is defined by $\Delta f(x) = f(x+h) - f(x)$; h is interval of x.

$$\Delta f(x) = f(x+h) - f(x) = E(f(x)) - f(x) = (E-1)f(x)$$

Notes

i.e., $\Delta = E - 1$ or $E = 1 + \Delta$ ....(ii)

i.e., $\log E = \log(1+\Delta)$

$$hD = \log E = \log(1+\Delta) = \Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \ldots\ldots \qquad \text{........(iii)}$$

The backward difference operator "∇" is defined by

$$\nabla f(x) = f(x) - f(x-h)$$

$$= f(x) - E^{-1}f(x); \; \left[E^{-1} \rightarrow \text{inverse shifting operator}\right]$$

$$\nabla f(x) = \left[1 - E^{-1}\right] f(x)$$

$$\nabla = 1 - E^{-1} \;\Rightarrow\; E^{-1} = 1 - \nabla \qquad \text{........(iv)}$$

$$\Rightarrow \log E^{-1} = \log(1-\nabla)$$

$$-\log E = \log(1-\nabla) \qquad \Rightarrow \log E = -\log(1-\nabla)$$

i.e., $$hD = \log E = -\log(1-\nabla) = \nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} + \frac{\nabla^4}{4} + \qquad \text{........(v)}$$

$$\delta f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right) \qquad \text{(or)}$$

$$\delta y_x = y_{x+\frac{h}{2}} - y_{x-\frac{h}{2}}$$

$$\delta f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right) = E^{-1/2}f(x) - E^{-1/2}f(x)$$

$$\delta f(x) = \left(E^{-1/2} - E^{-1/2}\right) f(x)$$

$$\Rightarrow \quad \delta = E^{1/2} - E^{-1/2} = \left(e^{hD}\right)^{1/2} - \left(e^{hD}\right)^{-1/2}$$

$$\Rightarrow \quad \delta = E^{1/2} - E^{-1/2} = \left(e^{hD}\right)^{1/2} - \left(e^{hD}\right)^{-1/2}$$

$$\Rightarrow \quad \delta = E^{1/2} - E^{-1/2} = \left(e^{hD}\right)^{1/2} - \left(e^{hD}\right)^{-1/2} \qquad \text{........(vi)} \; \left(\text{since, } e^{hD} = E\right)$$

$$\delta = e^{hD/2} - e^{-hD/2}$$

$$\Rightarrow \quad \delta = 2\sinh\left(\frac{hD}{2}\right) \qquad \left(\text{since} \quad \sinh x = \frac{e^x - e^{-x}}{2}\right)$$

$$\Rightarrow \quad hD = \log E = 2\sin h^{-1}\left(\frac{\delta}{2}\right) = \delta - \frac{1^2}{2^2.3!}\delta^3 + \qquad \text{...(vii)}$$

Notes

Thus (iii), (v), (vii) gives

$$hD = \log E = \begin{vmatrix} \Delta - \dfrac{\Delta^2}{2} + \dfrac{\Delta^3}{3}.......... \\ \nabla + \dfrac{\nabla^2}{2} + \dfrac{\nabla^3}{3}........ \\ \delta - \dfrac{1^2}{2^2.3!}\delta^3 + ... \end{vmatrix} \qquad ...(viii)$$

In general, we get

$$h^r D^r f(x) = \begin{vmatrix} (\Delta^r - \dfrac{1}{2} r\Delta^{r+1} + \dfrac{r(3r+5)}{24}\Delta^{r+2} - ....)f(x) \\ (\nabla^r - \dfrac{1}{2} r\nabla^{r+1} + \dfrac{r(3r+5)}{24}\nabla^{r+2} + ...)f(x) \\ (\delta^r - \dfrac{r}{24}\delta^{r+2} + r\dfrac{(5r+22)}{5760}\delta^{r+4} - ....)f(x) \end{vmatrix} \qquad ...(ix)$$

In particular, differentiation methods for r=1,2 at $x = x_k$ become

$$f'(x_k) = \begin{vmatrix} \dfrac{1}{h}(\Delta f_k - \dfrac{\Delta^2}{2} f_k + \dfrac{\Delta^3}{3} f_k - ....) \\ \dfrac{1}{h}(\nabla f_k - \dfrac{\nabla^2}{2} f_k + \dfrac{\nabla^3}{3} f_k + ...) \\ \dfrac{1}{h}(\delta f_k - \dfrac{1\delta^3}{24} f_k + ...) \end{vmatrix} \qquad ...(x)$$

$$f'(x_k) = \begin{vmatrix} \dfrac{1}{h}(\Delta^2 f_k - \Delta^3 f_k + \dfrac{11}{12}\Delta^4 f_k - ....) \\ \dfrac{1}{h}(\nabla^2 f_k - \nabla^3 f_k + \dfrac{11}{12}\nabla^4 f_k + ...) \\ \dfrac{1}{h}(\delta^2 f_k - \dfrac{1}{12} f_k + \dfrac{1}{90}\delta^6 f_k...........) \end{vmatrix} \qquad ...(xi)$$

Keeping only the first term in each of the methods, we get

$$f'_k = \begin{vmatrix} \dfrac{1}{h}(f_{k+1} - f_k) \;\; - \text{first order method} & ...(xiia) \\ \dfrac{1}{h}(f_k - f_{k-1}) \;\; - \text{first order method} & ...(xiib) \\ \dfrac{1}{h}(f_{k+\frac{1}{2}} - f_{k-\frac{1}{2}}) - \text{second order method} & ...(xiic) \end{vmatrix}$$

Notes

Note: The averaging operator μ is defined by

$$\mu f(x) = \frac{1}{2}\left[f\left(x + \frac{h}{2}\right) + f\left(x - \frac{h}{2}\right)\right] \qquad \text{…(xiii)}$$

$$\mu = \frac{1}{2}\left(E^{\frac{1}{2}} + E^{\frac{1}{2}}\right) = \sqrt{1 + \frac{\delta^2}{4}} \Rightarrow \frac{\mu}{\sqrt{a + \frac{\delta^2}{4}}} = 1 \qquad \text{…. (xiv)}$$

w.r.t.,
$$hD = 2\sinh^{-1}\left(\frac{\delta}{2}\right) = \frac{\mu}{\sqrt{1 + \frac{\delta^2}{4}}} 2\sinh^{-1}\left(\frac{\delta}{2}\right)$$

$$hD = \mu(\delta - \frac{1^2}{3!}\delta^3 + \frac{1^2.2^2}{5!}\delta^3 .....)$$

$$f'(x_k) = \left[\frac{1}{h}\mu\delta f_k - \frac{1}{6}\mu\delta f_k + \frac{1}{30}\mu\delta^5 f_k - .....\right] \qquad \text{…(xv)}$$

We can use (xv) instead of (xiic)

Finite Difference Methods:

Let the second order ordinary differential equation be of the form

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = f(x)$$

i.e., $$D^2y(x) + a_1 Dy(x) + a_2 y(x) = f(x)$$

$$y''(x) + a_1 y'(x) + a_2 y(x) = f(x)$$

In this method, we can replace the derivatives in (1) by central differences

(We can also use forward or backward differences.)

Then (1) becomes a difference equation whose solution is an approximation to the given differential equation. This method will be very clear from the following examples.

Examples

1. Solve the differential equation $\frac{d^2y}{dx^2} - y = x$ with $y(0) = 0, y(1) = 0$ with $h = \frac{1}{4}$.

Solution:

The given differential equation can be written as

$$y''(x) - y(x) = x \qquad \text{…(1)}$$

Notes

Using the central difference approximation, we have

$$y'' = \frac{y_{k-1} - 2y_k + y_{k+1}}{h^2} \qquad \ldots(A)$$

Substituting (A) in (1), we get

$$\frac{y_{k-1} - 2y_k + y_{k+1}}{h^2} - y_k = x_k$$

i.e., $$y_{k-1} - 2y_k + y_{k+1} - h^2 - y_k = h^2 x_k$$

i.e., $$y_{k-1} - 2y_k + y_{k+1} - \frac{1}{16} y_k = \frac{1}{16} x_k$$

i.e., $$16y_{k-1} - 33y_k + 16y_{k+1} = x_k \qquad \ldots(2)$$

Put k=1, 2 and 3 in (2), we get

$$\left.\begin{aligned} 16y_0 - 33y_1 + 16y_2 &= \frac{1}{4} \\ 16y_1 - 33y_2 + 16y_3 &= \frac{1}{4} \\ 16y_2 - 33y_3 + 16y_4 &= \frac{3}{4} \end{aligned}\right\} \qquad \ldots(3)$$

Given: $y(0) = 0$ i.e., $x_0 = 0,\ y_0 = 0$

$y(1) = 0$ i.e., $x_4 = 1,\ y_4 = 0$ ...(4)

since $h = \frac{1}{4}$, we have

$$x_1 = \frac{1}{4}, x_2 = \frac{1}{2}, x_3 = \frac{3}{4}$$

Substituting (4) in (3), we get

$$0 - 33y_1 + 16y_2 = \frac{1}{4}$$

$$16y_1 - 33y_2 + 16y_3 = \frac{1}{2}$$

$$16y_2 - 33y_3 + 0 = \frac{3}{4}$$

Solving the above equations, we get

$$y_1 = -0.03488 \qquad y_2 = -0.05632 \qquad y_3 = -0.05003$$

Notes

2. Solve the boundary value problem at $x = 0.5$. $\frac{d^2y}{dx^2} + y + 1 = 0$, $y(0) = y(1) = 0$ with $h = \frac{1}{4}$.

Solution:

The given differential equation can be written as

$$y''(x) + y(x) + 1 = 0 \qquad \ldots(1)$$

Using the central difference approximation, we have

$$y''(x) = \frac{y_{k-1} - 2y_k + y_{k+1}}{h^2} \qquad \ldots(2)$$

Substituting (2) in (1), we get

$$\frac{y_{k-1} - 2y_k + y_{k+1}}{h^2} + y_k + 1 = 0$$

i.e., $$y_{k-1} - 2y_k + y_{k+1} + h^2 y_{k+1} = 0$$

i.e., $$y_{k-1} - 2y_k + y_{k+1} + \frac{1}{16} y_{k+1} = 0 \qquad [\because h = 1/4]$$

$$16y_{k-1} - 31y_k + 16y_{k+1} + 16 = 0$$

Putting k =1, 2, 3 (i.e. at $h = 1/4$, $h = 1/2$, $h = 3/4$)

$$\text{We get} \quad \left.\begin{aligned} 16y_0 - 31y_1 + 16y_2 + 16 = 0 \\ 16y_1 - 31y_2 + 16y_3 + 16 = 0 \\ 16y_2 - 31y_3 + 16y_4 + 16 = 0 \end{aligned}\right\} \qquad \ldots(A)$$

Given $y(0) = 0$ and $y(1) = 0$

i.e., $y_0 = 0$ and $y_4 = 0$ $\qquad \ldots(B)$

Substituting (B) in (1), we get

$$\left.\begin{aligned} -31y_1 + 16y_2 = -16 \\ 16y_1 - 31y_2 + 16y_3 = -16 \\ 61y_2 + 31y_3 = -16 \end{aligned}\right\} \qquad \ldots(C)$$

Solving (c), we get

$$y_2 = 0.14031$$

i.e., $$y(0.5) = 0.14031$$

Analytical Method:

A.E. is $m^2 + 1 = 0$

$m = \pm i$

$$C.F = c_1 \cos x + c_2 \sin x$$

Notes

$$P.I = \frac{1}{D^2+1}(-1) = \frac{-1}{D^2+1}e^{0x} = -1$$

$$\because y(x) = c_1 \cos x + c_2 \sin x - 1$$

Given $\quad y(0) = 0 \Rightarrow c_1 = 1$

$$y(x) = \cos x + c_2 \sin x - 1$$

Given $\quad y(1) = 0 \Rightarrow \cos 1 + c_2 \sin 1 - 1 = 0$

$$c_2 = \frac{1-\cos 1}{\sin 1} = \frac{1-0.5406}{0.8414}$$

$$= 0.5459$$

$$\because y(x) = \cos x + 0.5459 \sin x - 1$$

$$= 0.8777 + 0.2617 - 1 \quad = 0.1394$$

Check Your Progress

Apply Bessel's formula to obtain y_{25} given that y_{20} = 2854, y_{24} = 3162, y_{28} = 3544, y_{32} = 3992.

....

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....

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Lesson End Activity

Derive numerical differentiation method for the solution of a differential equations.

Let us Sum up

In this unit, we have seen how numerical differentiation technique may be used to obtain the derivative of continuous as well as tabulate functions. We have used forward, backward and

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