Muller Method

In this method, first we assume three approximate roots x_{i-1}, x_i and x_{i+1} of the equation f(x) = 0. The next better approximation x_{i+2} is obtained as the root of second degree polynomial equation p(x) = 0, where the second degree parabola y = p(x) is assumed to pass through the points $\{x_{i-1}, f(x_{i-1})\}$, $\{x_i, f(x_i)\}$ and $\{x_{i+2}, f(x_{i+2})\}$.

Algorithm

1. Let x_0, x_1 and x_2 be the initial approximate roots of f(x) = 0. Compute $f(x_0)$, $f(x_1)$ and $f(x_2)$.

2. Compute $h_2 = x_2 - x_1$; $h_1 = x_1 - x_0$

   $f(x_2, x_1) = [f(x_2) - f(x_1)] / h_2$; $f(x_1, x_0) = [f(x_1) - f(x_0)] / h_1$

3. Set k =2

Compute $f(x_k, x_{k-1}, x_{k-2}) = [\ f(x_k, x_{k-1}) - f(x_{k-1}, x_{k-2})] / (h_k + h_{k-1})$

$c_k = f(x_k, x_{k-1}) + h_k\, f(x_k, x_{k-1}, x_{k-2})$

$$h_{k+1} = \frac{-2f(x_k)}{c_k \pm \sqrt{{c_k}^2 - 4f(x_k)\, f(x_k, x_{k-1}, x_{k-2})}}$$

Choosing the sign so that the denominator is largest in magnitude

Set $x_{k+1} = x_k + h_{k+1}$

4. Compute $f(x_{k+1})$ and $f(x_{k+1}, x_k) = \dfrac{f(x_{k+1}) - f(x_k)}{h_{k+1}}$

Set k = k + 1 and repeat steps (3) to (4) until we get the root with required degree of accuracy.

NOTE: This method converges for all initial approximations. If no better approximations are known, we choose $x_0 = -1$; $x_1 = 0$ and $x_2 = 1$.

1. Perform five iterations for the Muller method to find the root of the equation
   $f(x) = \cos x - xe^x = 0$

Sol : Let $x_0 = -1$; $x_1 = 0$ and $x_2 = 1$

$h_2 = x_2 - x_1 = 1$; $h_1 = x_1 - x_0 = 1$

$f(x_2, x_1) = -3.1780$: $f(x_1, x_0) = 0.0918$

$f(x_2, x_1, x_0) = -1.6349$

$c_k = c_2 = f(x_2, x_1) + h_2\, f(x_2, x_1, x_0) = -4.8129$

$h_3 = -0.5584$; $\mathbf{x_3 = x_2 + h_3 = 0.4416}$

$c_3 = f(x_3, x_2) + h_3\, f(x_3, x_2, x_1)$

$f(x_3, x_2) = -4.2896$

$h_4 = 0.0710$; $\mathbf{x_4 = x_3 + h_4 = 0.5126}$

$c_4 = -2.8408 + (0.0710)*(-2.9725) = -3.0518$

$h_5 = 0.0051$

$\mathbf{x_5 = x_4 + h_5 = 0.5177.}$

2. Find the root of the equation $x^3 + x^2 - 1 = 0$ that lies between 0 and 1, correct to four places of decimals using Muller method.

Sol : Let $x_0 = 0$; $x_1 = 0.5$ and $x_2 = 1$

$h_2 = x_2 - x_1 = 0.5$; $h_1 = x_1 - x_0 = 0.5$

$f(x_3, x_2, x_1) = -2.5172$; $c_3 = -2.8834$

$f(x_2, x_1) = 3.25$: $f(x_1, x_0) = 0.75$

$f(x_2, x_1, x_0) = 2.5$

$c_k = c_2 = f(x_2, x_1) + h_2\, f(x_2, x_1, x_0) = 4.5$

$h_3 = -0.2597$; $\mathbf{x_3 = x_2 + h_3 = 0.7403}$

$c_3 = f(x_3, x_2) + h_3\, f(x_3, x_2, x_1)$

$f(x_3, x_2) = 4.0286$

$f(x_3, x_2, x_1) = 3.2403$; $c_3 = 3.1872$

$h_4 = 0.0142$; $\mathbf{x_4 = x_3 + h_4 = 0.7546}$

$\mathbf{x_5 = x_6 = 0.7549}$

3. Use Muller method to find the root of $x^3 - 5x - 6 = 0$ that lies between 2 and 3.

Ans : 2.689

Birge-Vieta Method

In this method we seek to determine a real number p such that $(x - p)$ is a factor of the polynomial equation $p_n(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n = 0$. If we divide the given equation by the factor $(x-p)$ then we get a quotient Q and remainder R. The value of R depends upon P. We apply Newton-Raphson method and improve the initial approximation value. For the polynomial equations, the computation are systematized using synthetic division.

	a_0	a_1	a_2	a_3	.	.	.	.	a_n
p	0	b_0p	b_1p	b_2p	.	.	.	.	$b_{n-1}p$
	$a_0=b_0$	b_1	b_2	b_3	.	.	.	.	b_n
	0	c_0p	c_1p	c_2p	.	.	.	$b_{n-1}p$	
	$b_0=c_0$	c_1	c_2	c_3				c_{n-1}	

$P_{k+1} = p_k - (b_n / c_{n-1})$; $k = 0, 1, 2, 3, \ldots$

1. Use synthetic division and perform two iterations by Birge- Vieta method to find the smallest positive root of the equation $x^4 - 3x^3 + 3x^2 - 3x + 2 = 0$

Sol : Let $p_0 = 0.5$

	1	-3	3	-3	2
0.5		0.5	-1.25	0.875	-1.0625
	1	-2.5	1.75	-2.125	$0.9375 = b_4$
		0.5	-1.00	0.375	
	1	-2.0	0.75	$-1.750 = c_3$	

$P_1 = p_0 - (b_4 / c_3) = 0.5 + (0.9375 / 1.750) = 1.0356$

	1	-3	3	-3	2
1.0356		1.0356	-2.0343	1.0001	-2.0711

1	-1.9644	0.9657	-1.9999	-0.0711
	1.0356	-0.9619	0.0039	
1	-0.9288	0.0038	- 1.9960	

$P_2 = p_1 - (b_4 / c_3) = 1.0356 - (- 0.0711 / - 1. 9960)$

$= 0.999979$

The exact root is 1.0

2.Using Birge-Vieta method find a real root correct to three decimal places of the equation $x^3-11x^2+32x-22=0$ with P = 0.5.

Sol : Using synthetic division

	1	-11	32	-22
0.5	0	0.5	-5.25	13.375
	1	-10.5	26.75	$-8.625= b_n$
0.5	0	0.5	-5	
	1	-9.5	$21.75= c_{n-1}$	

$p_1= 0.5-(-8.625)/(21.75)=0.89655$

Now we divide the given equation with x-0.89655.

	1	-11	32	-22
0.89655		0.89655	-9.058248	20.56842776
	1	-10.10345	22.941752	$-1.4315722=b_n$
		0.89655	-8.254446	
	1	-9.2609	$14.687306= c_{n-1}$	

$p_2 = 0.89655-(-1.43115722)/(14.687306)=0.99402$. Now we divide the given equation with $x - 0.99402$.

	1	-11	32	-22
0.99402		0.99402	-9.94614	21.921977
	1	-10.00598	22.05386	$-0.078023=b_n$
		0.99402	-8.958068	
	1	-9.01196	$13.095792 = c_{n-1}$	

$P_3 = 0.99402-(-0.078023)/(13.095792)=0.999978$.

Now we divide the given equation with $x-0.999978$.

	1	-11	32	-22
0.999978		0.999978	-9.999802	21.999714
	1	-10.000022	22.000198	$-0.000286= b_n$
		0.999978	-8.999845999	
	1	-9.000044	$13.000352= c_{n-1}$	

$P_4 = 0.999978 - (-0.000286) / (13.000352)$

$= 0.99999$.

Therefore, the root correct to three decimal places is 0.999.