POINT-TO-ELLIPSE FOURIER SERIES

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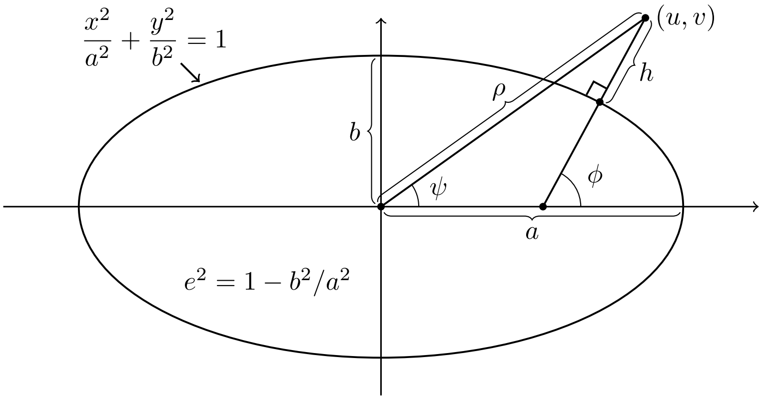
ABSTRACT. Fourier series with power series coefficients for the normal and distance to a point from an ellipse are provided. These expressions are the first of their kind and opens up a range of analysis and computational possibilities.

1. INTRODUCTION



Determining the normal ϕ and the distance h to a point (u, v) from an ellipse with axes a and b, as depicted above, is an extensively studied problem over the centuries: Classical geometry techniques for the normal are known since the antiquity p.180 *Heath 1896*; Quartic equations have been around since at least the early 1900-hundreds p.382 *Gibson and Pinkerton 1911*, even if solid solution methods did not appear until recently *Vermeille 2011*; 100s of exact and approximate methods for computing ϕ and h have been published since the 1960s *Nilsson 2024a*. Despite this, only quartic equation, iterative, and closed-form approximation solutions are available, fundamentally limiting analysis and computations related to ϕ and h. In contrast, provided here are Fourier and power series in ψ and $\sin(\psi)$ for ϕ, h, and $\sin(\phi)$ and $\cos(\phi)$ (vector normal) whose coefficients are, in turn, power series and polynomials in a/ρ and e^2 with rational coefficients. The series are valuable in that:

- They are fundamental results for the ellipse conic and provide the first general series expansions for the point-to-ellipse relation.
- Point-to-ellipse being a fundamental relation means that they enable series expansions of many related quantities.
- They are differentiable, making updates for e.g. ellipse fitting, coordinate transformations, and differential equation solutions conceivable.
- Truncated series give simple algebraic approximations with potentially boundable errors and competitive performance.

Fourier series expansions have been attempted before by *Morrison and Pines 1961* and *Pick 1967*, but these efforts have only resulted in a few initial and partial terms. Series expansion inspiration is also drawn from *Nilsson 2024b*. The main limitation of this work is that no region of convergence is sought. The series are given in the next section. A brief discussion about them follows. Finally, derivations and tabulated coefficients are found in Appendices A and B, respectively. A Python

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implementation of series and coefficients can be found at `https://github.com/jnil02/point2ellipse_series`

2. The series

The series are provided below in Fourier multiple-angle and sin-power series forms. The former is fundamental and useful from a theoretical perspective whilst the latter brings more structure (fewer coefficients) and is more useful from a computational perspective, since truncations directly give polynomials in the ratios $\varrho = a/\rho$, $\sin(\psi) = v/\rho$, and $\cos(\psi) = u/\rho$.

Theorem 1. *For the point-to-ellipse relation and sufficiently small ae^2/u*

$$\begin{aligned}
\phi - \psi &= \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\left(\sum_{l=\max(n,k)}^{\infty} c^{\phi}_{n,k,l} e^{2l}\right)\varrho^k\right)\sin(2n\psi)\\
\frac{\phi-\psi}{\cos(\psi)\sin(\psi)} &= \sum_{n=0}^{\infty}\left(\sum_{k=1}^{\infty}\left(\sum_{l=\max(n+1,k)}^{n+k} d^{\phi}_{n,k,l} e^{2l}\right)\varrho^k\right)\sin^{2n}(\psi)\\
\frac{h+a-\rho}{a} &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\left(\sum_{l=\max(n,k+1)}^{\infty} c^{h}_{n,k,l} e^{2l}\right)\varrho^k\right)\cos(2n\psi)\\
&= \sum_{n=1}^{\infty}\left(\sum_{k=0}^{\infty}\left(\sum_{l=\max(n,k+1)}^{n+k} d^{h}_{n,k,l} e^{2l}\right)\varrho^k\right)\sin^{2n}(\psi)\\
\frac{\sin(\phi)}{\sin(\psi)} - 1 &= \sum_{n=0}^{\infty}\left(\sum_{k=1}^{\infty}\left(\sum_{l=\max(n,k)}^{\infty} c^{\sin}_{n,k,l} e^{2l}\right)\varrho^k\right)\cos(2n\psi)\\
&= \sum_{n=0}^{\infty}\left(\sum_{k=1}^{\infty}\left(\sum_{l=\max(n,k)}^{n+k} d^{\sin}_{n,k,l} e^{2l}\right)\varrho^k\right)\sin^{2n}(\psi)\\
\frac{\cos(\phi)}{\cos(\psi)} - 1 &= \sum_{n=0}^{\infty}\left(\sum_{k=1}^{\infty}\left(\sum_{l=\max(n,k)}^{\infty} c^{\cos}_{n,k,l} e^{2l}\right)\varrho^k\right)\cos(2n\psi)\\
&= \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\left(\sum_{l=\max(n,k)}^{n+k-1} d^{\cos}_{n,k,l} e^{2l}\right)\varrho^k\right)\sin^{2n}(\psi)
\end{aligned}$$

where the coefficients $c^*_{n,k,l}$ and $d^*_{n,k,l}$ are rational and, in order, given by equations (14)*,* (16)*,* (34)*,* (32)*,* (22)*,* (20)*,* (27)*, and* (25)*, respectively.*

Proof. See Appendix A. □

Note, in practice, the region of convergence appear to be of useful size, even including $u = 0$ but not too small ϱ. Potentially it is outside the ellipse evolute. Further note, the $\cos(\psi)\sin(\psi)$ in the sin-power series of $\phi - \psi$ could easily be integrated into the series, but this severely limits the convergence, see A.1. Finally, for computing both $\cos(\phi)$ and $\sin(\phi)$, it may be preferable to compute them jointly with slightly different formulas, see A.3. For more computational details, see *Nilsson 2024b*.

3. Discussion

To my knowledge, the series are the first general series expansions of ϕ, h, $\sin(\phi)$, and $\cos(\phi)$ in terms of ψ and ρ, or, alternatively, u and v. Again, the series are:

- Essentially series solutions to the quartic latitude equation.
- Series of fundamental relations, meaning that series for dependent quantities could be derived, enabling related analysis and computations.
- Trivially differentiable, both with respect to the point and the parameters, providing new paths for data fitting and transformation updates.
- Easily approximated by truncation with granularly tuneable accuracy and potentially boundable errors for a given ρ.

Hence, the series opens up a range of analysis and computational possibilities not obtainable from previous quartic equation, iterative, or closed-form approximate solutions. Contributing to this are the sin-power series variants, with its additional series structure. Depending on the use-case, e^2 may be a constant, e.g. geodesy, or a variable, e.g. ellipse fitting. For the latter case, this is obviously particularly valuable. For the former case, the sin-power series structure is still valuable since $\sin^{2n}(\psi)$ is typically easier to evaluate for different n than $\sin(2n\psi)$ and $\cos(2n\psi)$. Unfortunately, the series come with some disadvantages:

- They appear not to converge for all points and ellipses and the convergence properties are so far not clear.
- The series appear to have geometric convergence with respect to n but, due to the inner series, subgeometric convergence with respect to arithmetic operations, i.e. other methods have better asymptotic convergence.

However, convergence can probably be clarified and complementary series in ϱ^{-1} could possibly be derived. Further, series accelerations may possibly be applied. Finally, for many practical applications, e^2 is small or ρ is large giving fast initial convergence and potentially competitive performance for required accuracies.

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Appendix A. Derivations

Let (u,v) be a Cartesian coordinate of a point. Let a and b designate the major and minor axes of an ellipse aligned with the coordinate axes and centered at the origin. Let h and ϕ designate the distance to the ellipse and the angle to the normal from the ellipse passing through (u,v). Further, let $\rho=(u^2+v^2)^{1/2}$ and $\psi=\text{atan2}(v/u)$ be the polar coordinates of (u,v). Then, from basic geometry,

$$\rho\cos(\psi)=(N+h)\cos(\phi) \tag{1}$$

$$\rho\sin(\psi)=((1-e^2)N+h)\sin(\phi) \tag{2}$$

where the eccentricity squared $e^2=1-b^2/a^2$ and the radii of curvature $N=a(1-e^2\sin(\phi))^{-1/2}$. (1) and (2) are the basis for the following series expansions derivations. Further, in the derivation, the following formulas are used:
Lagrange reversion theorem:

$$\text{with}\quad y=x+dg(y),\quad \text{then for sufficiently small } d$$
$$f(y)=f(x)+\sum_{k=1}^{\infty}\frac{d^k}{k!}\left(\frac{\partial}{\partial x}\right)^{k-1}(f'(x)g^k(x)) \tag{3}$$

The *general Leibniz rule*:

$$\frac{d^n}{dx^n}f(x)g(x)=\sum_{r=0}^{n}\binom{n}{r}\frac{d^{n-r}}{dx^{n-r}}f(x)\frac{d^r}{dx^r}g(x) \tag{4}$$

The *Faá di Bruno's formula*:

$$\frac{d^n}{dx^n}f(g(x))=\sum\frac{n!}{m_1!\dots m_n!}f^{(m_1+\cdots+m_n)}(g(x))\prod_{i=1}^{n}\left(\frac{g^{(i)}(x)}{i!}\right)^{m_i} \tag{5}$$

Finally, for all but the ϕ expansion, the multi-angle Fourier expansions are obtained from the sin-power expansions and the following relation. Let

$$f(\phi)=\sum_{n=n_{\min}}^{\infty}\left(\sum_{k=k_{\min}}^{\infty}\left(\sum_{l=\max(n,k+k')}^{n+k+k''}e^{2l}d_{n,k,l}\right)\varrho^k\right)\sin^{2n}(\phi)$$

From the power reduction formulas

$$\sin^{2n}(\phi)=\frac{1}{2^{2n}}\binom{2n}{n}+\frac{2}{2^{2n}}\sum_{i=0}^{n-1}(-1)^{n-i}\binom{2n}{i}\cos(2(n-i)\phi)$$

Rearranging the sums gives

$$f(\phi)=\sum_{n=0}^{\infty}\left(\sum_{k=k_{\min}}^{\infty}\left(\sum_{l=\max(\max(n,n_{\min}),k+k')}^{\infty}e^{2l}c_{n,k,l}\right)\varrho^k\right)\cos(2n\phi) \tag{6}$$

where

$$c_{n,k,l}=\sum_{i=\max(\max(n,n_{\min}),l-k-k'')}^{l}d_{i,k,l}\frac{2}{2^{2i+\delta_n}}(-1)^n\binom{2i}{i-n}$$

where δ_* is the *Kronecker delta*. In the following subsections, expansions are derived for the desirable quantities and and some auxiliary quantities.

A.1. **Expansions for ϕ.** Let $\varrho = a/\rho$. Dividing (2) by (1) and solving for $\tan(\phi)$

$$\tan(\phi) = \tan(\psi) + \varrho \frac{e^2}{\cos(\psi)} \frac{\sin(\phi)}{(1 - e^2 \sin^2(\phi))^{1/2}} \tag{7}$$

Using the *Lagrange reversion theorem* (3) as in *Morrison and Pines 1961* with

$$y = \tan(\phi), \quad x = \tan(\psi)$$

$$g(y) = \frac{y}{(1 + (1 - e^2)y^2)^{1/2}} = \frac{\sin(\phi)}{(1 - e^2 \sin^2(\phi))^{1/2}}$$

$$d = \varrho e^2 (1 + x^2)^{1/2} = \frac{\varrho e^2}{\cos(\psi)}$$

$$\text{and} \quad f(y) = \tan^{-1}(y)$$

gives

$$\tan^{-1}(y) = \tan^{-1}(x) + \sum_{k=1}^{\infty} \varrho^k \frac{e^{2k}(1 + x^2)^{k/2}}{k!} \left(\frac{\partial}{\partial x}\right)^{k-1} \frac{x^k}{1 + x^2} \frac{1}{(1 + (1 - e^2)x^2)^{k/2}}$$

To proceed beyond *Morrison and Pines 1961*, an expression for the n$^{\text{th}}$ derivative of the latter part is needed. Further, series expansions of h and extra structure is enabled by derivation of alternative sin-power series expansions. Finally, this expansion holds for *sufficiently small* $d = \varrho e^2/\cos(\psi) = ae^2/u$.

The n$^{\text{th}}$ derivative of x^k

$$\frac{d^n}{dx^n} x^k = \begin{cases} \frac{k!}{(k-n)!} x^{k-n} & : n \leq k \\ 0 & : n > k \end{cases} \tag{8}$$

The n$^{\text{th}}$ derivative of $\frac{1}{1+x^2}$

$$\begin{aligned}
\frac{d^n}{dx^n} \frac{1}{1 + x^2} &= \frac{d^n}{dx^n} \frac{1}{2i} \left(\frac{1}{x - i} - \frac{1}{x + i}\right) \\
&= (-1)^n \frac{n!}{2i} \left(\frac{1}{(x - i)^{n+1}} - \frac{1}{(x + i)^{n+1}}\right) \\
&= (-1)^n \frac{n!}{2i(x^2 + 1)^{n+1}} \left((x + i)^{n+1} - (x - i)^{n+1}\right) \\
&= (-1)^n \frac{n!}{2i(x^2 + 1)^{n+1}} \left(\sum_{k=0}^{n+1} \binom{n+1}{k} x^{n+1-k} i^k - \sum_{k=0}^{n+1} \binom{n+1}{k} x^{n+1-k} (-i)^k\right) \\
&= (-1)^n \frac{n!}{2i(x^2 + 1)^{n+1}} \sum_{k=0}^{n+1} \binom{n+1}{k} x^{n+1-k} i^k \left(1 - (-1)^k\right) \\
&= \frac{n!}{(x^2 + 1)^{n+1}} \sum_{k \in \mathbf{N}_o}^{n+1} \binom{n+1}{k} x^{n+1-k} (-1)^{n+(k-1)/2} \\
&= \sum_{k \in \mathbf{N}_o}^{n+1} (-1)^{n+(k-1)/2} n! \binom{n+1}{k} \frac{x^{n+1-k}}{(x^2 + 1)^{n+1}} \\
&= \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^{n+k} n! \binom{n+1}{2k+1} \frac{x^{n-2k}}{(x^2 + 1)^{n+1}} \qquad (9)
\end{aligned}$$

where $\mathbf{N}_o$ designates the odd natural numbers. Using (8) and (9) and the *general Leibniz rule* (4) gives the n$^{\text{th}}$ derivative of $\frac{x^k}{1+x^2}$

$$\begin{aligned}\frac{d^n}{dx^n}\frac{x^k}{1+x^2} &= \sum_{m=0}^{n}\binom{n}{m}\frac{k!}{(k-(n-m))!}x^{k-(n-m)} \\ &\quad \sum_{p=0}^{\lfloor m/2\rfloor}(-1)^{m+p}m!\binom{m+1}{2p+1}\frac{x^{m-2p}}{(1+x^2)^{m+1}} \\ &= \sum_{m=0}^{n}\sum_{p=0}^{\lfloor m/2\rfloor}\binom{m+1}{2p+1}\binom{k}{n-m}n!(-1)^{m+p}\frac{x^{m+1}}{(1+x^2)^{m+1}}x^{k-n+m-2p-1}\end{aligned}$$

Required next is the n$^{\text{th}}$ derivative of $\frac{1}{(1+(1-e^2)x^2)^{k/2}}$. Trivially

$$\frac{d^n}{dx^n}\frac{1}{(1+x)^{k/2}} = (-1)^n(k/2)^{\overline{n}}\frac{1}{(1+x)^{k/2+n}} \tag{10}$$

where $(\cdot)^{\overline{n}}$ indicates the rising factorial. Further

$$\frac{d^n}{dx^n}(1-e^2)x^2 = \begin{cases}0 & \text{if} \quad n>2 \\ \frac{2!}{(2-n)!}(1-e^2)x^{2-n} & \text{otherwise}\end{cases} \tag{11}$$

Plugging this into the *Faá di Bruno's formula* (5) where the sum is over all partitions $m_1+2m_2+\cdots+nm_n=n$, with $f(x)=\frac{1}{(1+x)^{k/2}}$ and $g(x)=(1-e^2)x^2$ and noting that the derivative of $g(x)=(1-e^2)x^2$ equals zero for $n>2$ which means that $m_n=0\forall n>2$, $m_1+2m_2=n$ and $m_2\in[0,\lfloor n/2\rfloor]$, giving

$$\begin{aligned}&\frac{d^n}{dx^n}\frac{1}{(1+(1-e^2)x^2)^{k/2}} \\ &\quad= \sum\frac{n!}{m_1!m_2!}(-1)^{m_1+m_2}(k/2)^{\overline{m_1+m_2}}\frac{(2(1-e^2)x)^{m_1}(1-e^2)^{m_2}}{(1+(1-e^2)x^2)^{k/2+(m_1+m_2)}} \\ &\quad= \sum_{m_2=0}^{\lfloor n/2\rfloor}\frac{n!(k/2)^{\overline{n-m_2}}}{(n-2m_2)!m_2!}(-1)^{n-m_2}\frac{(2(1-e^2)x)^{(n-2m_2)}(1-e^2)^{m_2}}{(1+(1-e^2)x^2)^{k/2+(n-m_2)}} \\ &\quad= \sum_{m=0}^{\lfloor n/2\rfloor}(-1)^{n-m}\frac{n!(k/2)^{\overline{n-m}}2^{(n-2m)}}{(n-2m)!m!}\frac{(1-e^2)^{(n-m)}x^{(n-2m)}}{(1+(1-e^2)x^2)^{k/2+(n-m)}}\end{aligned}$$

Again, applying the *general Leibniz rule* with the factors $\frac{x^k}{1+x^2}$ and $\frac{1}{(1+(1-e^2)x^2)^{k/2}}$ and expanding $(1-e^2)^{r-q}$ in a binomial sum gives the k$^{\text{th}}$ term of the Lagrange reversion expansion

$$\varrho^k \frac{e^{2k}(1+x^2)^{k/2}}{k!}\left(\frac{\partial}{\partial x}\right)^{k-1} \frac{x^k}{1+x^2}\frac{1}{(1+(1-e^2)x^2)^{k/2}}$$

$$= \varrho^k \frac{e^{2k}(1+x^2)^{k/2}}{k!}\sum_{r=0}^{k-1}\binom{k-1}{r}\frac{d^{k-1-r}}{dx^{k-1-r}}\frac{x^k}{1+x^2}\frac{d^r}{dx^r}\frac{1}{(1+(1-e^2)x^2)^{k/2}}$$

$$= \varrho^k \frac{e^{2k}(1+x^2)^{k/2}}{k!}\sum_{r=0}^{k-1}\binom{k-1}{r}\sum_{m=0}^{k-1-r}\sum_{p=0}^{\lfloor m/2\rfloor}\binom{m+1}{2p+1}\binom{k}{k-1-r-m}$$

$$(k-1-r)!(-1)^{m+p}\frac{x^{m+1}}{(1+x^2)^{m+1}}x^{k-(k-1-r)+m-2p-1}$$

$$\sum_{q=0}^{\lfloor r/2\rfloor}(-1)^{r-q}\frac{r!(k/2)^{\overline{r-q}}2^{(r-2q)}}{(r-2q)!q!}\frac{(1-e^2)^{(r-q)}x^{(r-2q)}}{(1+(1-e^2)x^2)^{k/2+(r-q)}}$$

$$= \varrho^k\sum_{r=0}^{k-1}\sum_{m=0}^{k-1-r}\sum_{p=0}^{\lfloor m/2\rfloor}\sum_{q=0}^{\lfloor r/2\rfloor}\sum_{t=0}^{r-q}(-1)^{r-q+m+p+t}\frac{(k/2)^{\overline{r-q}}2^{(r-2q)}e^{2(t+k)}}{(r-2q)!q!(1+r+m)}\binom{r-q}{t}$$
$$\binom{k-1}{r+m}\binom{m+1}{2p+1}x^{-2p-1}\left(\frac{x^2}{1+x^2}\right)^{m+1-k/2}\left(\frac{x}{(1+(1-e^2)x^2)^{1/2}}\right)^{k+2(r-q)} \tag{12}$$

Substitute $\tan(\phi)$ back for x, applying binomial expansion of $\frac{1}{(1-e^2\sin^2(\phi))}$, and collecting sin and cos factors gives the x factors of (12)

$$x^{-(2p+1)}\left(\frac{x^2}{1+x^2}\right)^{m+1-k/2}\left(\frac{x}{(1+(1-e^2)x^2)^{1/2}}\right)^{k+2(r-q)}$$
$$= \tan^{-(2p+1)}(\phi)\sin^{2(m+1-k/2)}(\phi)\frac{\sin^{k+2(r-q)}(\phi)}{(1-e^2\sin^2(\phi))^{(k+2(r-q))/2}}$$
$$= \sum_{s=0}^{\infty}\binom{\frac{k}{2}+r-q+s-1}{s}e^{2s}\cos^{2p+1}(\phi)\sin^{2(m+r-q+s-p)+1}(\phi) \tag{13}$$

From here the derivations of the Fourier multi-angle and the sin-power series forms split. For the Fourier multi-angle, proceed from (13) by applying the trigonometric power-reduction and product-to-sum formulas

$$x^{-(2p+1)}\left(\frac{x^2}{1+x^2}\right)^{m+1-k/2}\left(\frac{x}{(1+(1-e^2)x^2)^{1/2}}\right)^{k+2(r-q)}$$
$$= \sum_{s=0}^{\infty}\binom{\frac{k}{2}+r-q+s-1}{s}e^{2s}\frac{2}{2^{2p+1}}\sum_{i}^{p}\binom{2p+1}{i}\cos((2p+1-2i)\phi)$$
$$\frac{2}{2^{2(m+r-q+s-p)+1}}\sum_{j}^{m+r-q+s-p}(-1)^{m+r-q+s-p-j}$$
$$\binom{2(m+r-q+s-p)+1}{j}\sin((2(m+r-q+s-p)+1-2j)\phi)$$
$$= \sum_{s=0}^{\infty}e^{2s}\sum_{i}^{p}\sum_{j}^{w}\binom{\frac{k}{2}+r-q+s-1}{s}\binom{2p+1}{i}\binom{2w+1}{j}\frac{(-1)^{w-j}}{2^{2(w+p)+1}}$$
$$\big(\sin(2(w-j+p-i+1)\phi)+\sin(2(w-j-(p-i))\phi)\big)$$

where $w = m + r - q + s - p$. Inserting back into (12) gives $\phi - \psi$ in terms of sin-multiples

$$\phi - \psi = \sum_{k=1}^{\infty}\sum_{s=0}^{\infty}\sum_{r=0}^{k-1}\sum_{m=0}^{k-1-r}\sum_{p=0}^{\lfloor m/2\rfloor}\sum_{q=0}^{\lfloor r/2\rfloor}\sum_{t=0}^{r-q}\sum_{i=0}^{p}\sum_{j=0}^{w}\varrho^k \frac{(-1)^{s-j+t}(k/2)^{\overline{r-q}}}{q!(r-2q)!(m+1+r)}\frac{e^{2(k+s+t)}}{2^{2(m+s)+r+1}}$$
$$\binom{r-q}{t}\binom{k-1}{m+r}\binom{m+1}{2p+1}\binom{\frac{k}{2}+r-q+s-1}{s}\binom{2p+1}{i}\binom{2w+1}{j}$$
$$\big(\sin(2(w-j+p-i+1)\phi) + \sin(2(w-j-(p-i))\phi)\big)$$

To collect equal terms of $\sin(2n\psi)$, e^{2l} and ϱ^k, note

- $w - j + p - i + 1 \neq w - j - (p - i)$ so there is no $n = 0$ term.
- $l = s + k + t$ meaning that $s = l - k - t$.
- t is limited by l since obviously $t + k \leq l$.
- $w - j + p - i + 1 \in [1, k + s]$ meaning the first term can only contributes with a positive angle.
- $w - j + p - i + 1 = n$ imply $j = w + p - i + 1 - n$ which, together with $j \in [0, w]$ and $i \in [0, p]$, imply $i \in [\max(0, p - n + 1), \min(p, w + p - n + 1)]$.
- $w - j - (p - i) \in [-\lceil (k-1)/2\rceil, k + s - 1]$ so the second term can contribute with a positive and a negative term.
- $w - j - (p - i) = n$ imply $j = w - p + i - n$ which, together with $j \in [0, w]$ and $i \in [0, p]$, imply $i \in [p - w + n, p]$.
- $w - j - (p - i) = -n$ imply $j = w - p + i + n$ which, together with $j \in [0, w]$ and $i \in [0, p]$, imply $i \in [p - w - n, p - n]$.
- l is obviously below limited by k, but also by n, since for a given n, $l \geq k + s \geq n$.

giving

$$\phi - \psi = \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\left(\sum_{l=\max(n,k)}^{\infty} c^{\phi}_{n,k,l} e^{2l}\right)\varrho^k\right)\sin(2n\phi)$$

where the coefficients are given by

$$c^{\phi}_{n,k,l} = \sum_{r=0}^{k-1}\sum_{m=0}^{k-1-r}\sum_{p=0}^{\lfloor m/2\rfloor}\sum_{q=0}^{\lfloor r/2\rfloor}\sum_{t=0}^{\min(l-k,r-q)} \frac{(-1)^{l-k}(k/2)^{\overline{r-q}}}{q!(r-2q)!(m+1+r)2^{2(m+l-k-t)+r+1}}$$
$$\binom{r-q}{t}\binom{k-1}{m+r}\binom{m+1}{2p+1}\binom{\frac{k}{2}+r-q+l-k-t-1}{l-k-t}$$
$$\Bigg(\sum_{i=\max(0,p-n+1)}^{\min(p,p-n+1+w)}(-1)^{w+p-i+1-n}\binom{2p+1}{i}\binom{2w+1}{w+p-i+1-n}$$
$$+\sum_{i=p-w+n}^{p}(-1)^{w-p+i-n}\binom{2p+1}{i}\binom{2w+1}{w-p+i-n}$$
$$-\sum_{i=p-w-n}^{p-n}(-1)^{w-p+i+n}\binom{2p+1}{i}\binom{2w+1}{w-p+i+n}\Bigg) \tag{14}$$

In turn, the sin-power series form is found by proceeding from (13) as follows.

$$\begin{aligned}
&x^{-(2p+1)}\left(\frac{x^2}{1+x^2}\right)^{m+1-k/2}\left(\frac{x}{(1+(1-e^2)x^2)^{1/2}}\right)^{k+2(r-q)}\\
&=\cos(\phi)\sum_{s=0}^{\infty}\binom{\frac{k}{2}+r-q+s-1}{s}e^{2s}(1-\sin^2(\phi))^p\sin^{2(m+r-q+s-p)+1}(\phi)\\
&=\cos(\phi)\sum_{s=0}^{\infty}\binom{\frac{k}{2}+r-q+s-1}{s}e^{2s}\sum_{i=0}^{p}\binom{p}{i}(-1)^i\sin^{2i}(\phi)\sin^{2(m+r-q+s-p)+1}(\phi)\\
&=\cos(\phi)\sum_{s=0}^{\infty}e^{2s}\sum_{i=0}^{p}(-1)^i\binom{\frac{k}{2}+r-q+s-1}{s}\binom{p}{i}\sin^{2(m+r-q+s-p+i)+1}(\phi)
\end{aligned}$$

Inserting back into (12) gives $\phi-\psi$ in terms of sin-powers

$$\begin{aligned}
\phi-\psi=\cos(\phi)&\sum_{k=1}^{\infty}\sum_{s=0}^{\infty}\sum_{r=0}^{k-1}\sum_{m=0}^{k-1-r}\sum_{p=0}^{\lfloor m/2\rfloor}\sum_{q=0}^{\lfloor r/2\rfloor}\sum_{t=0}^{r-q}\sum_{i=0}^{p}\varrho^k e^{2(k+s+t)}\\
&\binom{r-q}{t}\binom{k-1}{m+r}\binom{m+1}{2p+1}\binom{\frac{k}{2}+r-q+s-1}{s}\binom{p}{i}\\
&\frac{(-1)^{m+r-q+p+t+i}(k/2)^{\overline{r-q}}2^{r-2q}}{q!(r-2q)!(m+1+r)}\sin^{2(m+r-q+s-p+i)+1}(\phi)
\end{aligned}$$

To collect equal terms of $\sin^{2n+1}(\psi)$, e^{2l} and ϱ^k, note

- $l=s+k+t$ meaning that $s=l-k-t$.
- t is limited by l and k since obviously $t+k\le l$.
- l is obviously below limited by k.
- For a given n, from the sin factor $i=n-(m+r-q+s-p)$
- For a given n and k, l is above limited since $n-(m+r-q+s-p)=i\ge 0$ implying $l\le n+k$.
- For a given k, l is below limited since $p\ge i=n-(m+r-q+s-p)$ implying $l\ge n+1$.

giving

$$\frac{\phi-\psi}{\cos(\psi)\sin(\psi)}=\sum_{n=0}^{\infty}\left(\sum_{k=1}^{\infty}\left(\sum_{l=\max(n+1,k)}^{n+k}d^{\phi}_{n,k,l}e^{2l}\right)\varrho^k\right)\sin^{2n}(\psi)\tag{15}$$

where

$$\begin{aligned}
d^{\phi}_{n,k,l}=&\sum_{r=0}^{k-1}\sum_{m=0}^{k-1-r}\sum_{p=0}^{\lfloor m/2\rfloor}\sum_{q=0}^{\lfloor r/2\rfloor}\sum_{t=0}^{\min(l-k,r-q)}\frac{(-1)^{2(p+t)+n-l+k}(k/2)^{\overline{r-q}}2^{r-2q}}{q!(r-2q)!(m+1+r)}\\
&\binom{r-q}{t}\binom{k-1}{m+r}\binom{m+1}{2p+1}\binom{\frac{k}{2}+r-q+l-k-t-1}{l-k-t}\binom{p}{n-m-r+q-l+k+t+p}
\end{aligned}\tag{16}$$

Note, $\cos(\psi)$ in (15) (an obiously $\sin(\psi)$) can easily be integrated into the series, by replacing the lower limit of l with k rather than $\max(n+1,k)$ and replacing p with $p+\frac{1}{2}$ in the last binomial of (16). However, this gives a series whose convergence is limited by the expansion of $\cos(\psi)$. It is typically better to use $\cos(\psi)=(1-\sin^2(\psi))^{1/2}$ and, therefore, this is not done.

A.2. **Expansions for** $(\phi - \psi)^i$**.** For brevity, let

$$a_{n,k} = \sum_{l=\max(n+1,k)}^{n+k} d^{\phi}_{n,k,l} e^{2l} \quad \text{and} \quad a_n = \sum_{k=1}^{\infty} a_{n,k} \varrho^k$$

Then, from (15)

$$\begin{aligned}(\phi - \psi)^i &= \cos^i(\psi) \sin^i(\psi) \left(\sum_{n=0}^{\infty} a_n \sin^{2n}(\psi) \right)^i \\ &= \cos^i(\psi) \sin^i \sum_{n=0}^{\infty} b_{n,i} \sin^{2n}(\psi)\end{aligned}$$

where $b_{n,i}$ is a polynomial in $a_0, \ldots, a_n$. The recurrence formula of powers of power series with non-zero constant term *Gould 1974* implicitly defines a *power of power series polynomials*

$$\begin{aligned}b_{n,i} &= P_{n,i}(a_0, \ldots, a_n) \\ &= \begin{cases} {a_0}^i & n = 0 \\ \frac{1}{n a_0} \sum_{k=1}^{n} (ki - n + k) a_k b_{n-k,i} & n > 0 \end{cases}\end{aligned}$$

Further, since the series of a_l has a zero constant term, its powers *Taghavian 2023*

$$a_l^{i_l} = \begin{cases} 1 & i_l = 0 \\ \sum_{k=i_l}^{\infty} \hat{B}_{k,i_l}(a_{l,1}, \ldots, a_{l,k-i_l+1}) \varrho^k & i_l > 0 \end{cases}$$

where $\hat{B}_{k,i_l}(a_{l,1}, \ldots, a_{l,k-i_l+1})$ are the *partial ordinary Bell polynomials* with a similar recurrence relation

$$\hat{B}_{k,i}(x) = \begin{cases} 1 & k = 1, i = 1 \\ \sum_{j=1}^{k-i+1} x_j \hat{B}_{k-j,i-1}(x) & k > i, i > 1 \end{cases}$$

Finally, to collect equal terms of e^{2l} and ϱ^k in $b_{n,i}$, note

- For a given i, since k starts at 1, the lowest power of ϱ is ϱ^i.
- For a given k, for each $a_{n,k}$ $l \geq k$ meaning that the power e^{2l} is lower bounded by k.
- For a given n, the lowest power of e^2 comes from multiplying the lowest powers in a_l. Writing $b_{n,i}$ as

  $$P_{n,i}(a_0, \ldots, a_n) = b_{n,i} = \sum_{\substack{i_0 + \cdots + i_n = i \\ i_1 + 2i_2 + \cdots + n i_n = n}} \binom{i}{i_0, \ldots, i_n} \prod_{l=0}^{i} a_l^{i_l}$$

  and replacing each a_l with the lowest power $e^{2(n+1)}$ gives

  $$\prod_{l=0}^{i} a_l^{i_l} \sim e^{2(i_0 + 2i_1 + 3i_2 + \cdots + (n+1)i_n)} = e^{2(n+i)}$$

  meaning that the power of e^{2l} is lower bounded by $n + i$.
- For a given n and k, the highest power of e^2 is trivially bounded by $n + k$.

Consequently

$$b_{n,i} = \sum_{k=i}^{\infty} \left(\sum_{l=\max(n+i,k)}^{n+k} c^{\phi}_{n,k,l,i} e^{2l} \right) \varrho^k$$

where

$$d^{\phi}_{n,k,l,i} = [e^{2l}][\varrho^k]\left(P_{n,i}(a_0,\ldots,a_n)\left[a_n^{i_n} \to \sum_{k=i_n}^{\infty} \hat{B}_{k,i_n}(a_{n,1},\ldots,a_{n,k-i_n+1})\varrho^k\right]\right)$$
$$= [e^{2l}][\varrho^k]\left(P_{n,i}(a_0,\ldots,a_n)\left[a_n^{i_n} \to \right.\right.$$
$$\left.\left.\sum_{k=i_n}^{\infty} \hat{B}_{k,i_n}\left(\sum_{l=\max(1,n+1)}^{n+1} d^{\phi}_{n,1,l}e^{2l},\ldots,\sum_{l=\max(n+1,k-i_n+1)}^{n+k-i_n+1} d^{\phi}_{n,k-i_n+1,l}e^{2l}\right)\varrho^k\right]\right)$$

where $[x^k]f(x)$ means the coefficient of x^k in the series or polynomial $f(x)$ and $f(x)[x \to y]$ means the series or polynomial $f(x)$ with x substituted with y. Note, the substitution of $a_n^{i_n}$ is done for all n and i_n. Further, note, in determining $[x^k]f(x)$ in the coefficient expression above, the polynomial of series has to be expanded with Cauchy products. Combining it all gives

$$(\phi-\psi)^i = \cos^i(\psi)\sin^i\sum_{n=0}^{\infty}\left(\sum_{k=i}^{\infty}\left(\sum_{l=\max(n+i,k)}^{n+k} d^{\phi}_{n,k,l,i}e^{2l}\right)\varrho^k\right)\sin^{2n}(\psi) \tag{17}$$

A.3. **Expansions for** $\sin(\phi)$ **and** $\cos(\phi)$**.** Expanding $\sin(\phi)$ in a Taylor series around ψ, using the sin-power series expansion for $(\phi-\psi)^i$ from (17) and expanding $\cos^{2x}(\psi)$ in $\sin^2(\psi)$ gives

$$\frac{\sin(\phi)}{\sin(\psi)} = \sum_{i=0}^{\infty}\frac{\partial}{\partial\varphi^i}\left.\frac{\sin(\varphi)}{i!}\right|_{\psi}\frac{1}{\sin(\psi)}(\phi-\psi)^i$$
$$= \sum_{i\in\mathbf{N}_e}^{\infty}\frac{(-1)^{i/2}}{i!}(\phi-\psi)^i + \sum_{i\in\mathbf{N}_o}^{\infty}\frac{(-1)^{(i-1)/2}}{i!}\frac{\cos(\psi)}{\sin(\psi)}(\phi-\psi)^i \tag{18}$$
$$= 1 + \sum_{i=1}^{\infty}\frac{(-1)^{\lfloor i/2\rfloor}}{i!}\cos^{2\lceil i/2\rceil}(\psi)\sin^{2\lfloor i/2\rfloor}\sum_{n=0}^{\infty}b_{n,i}\sin^{2n}(\psi)$$
$$= 1 + \sum_{i=1}^{\infty}\sum_{j=0}^{\lceil i/2\rceil}\binom{\lceil i/2\rceil}{j}\frac{(-1)^{\lfloor i/2\rfloor+j}}{i!}\sum_{n=0}^{\infty}b_{n,i}\sin^{2(n+\lfloor i/2\rfloor+j)}(\psi)$$
$$= 1+\sum_{i=1}^{\infty}\sum_{j=0}^{\lceil i/2\rceil}\binom{\lceil i/2\rceil}{j}\frac{(-1)^{\lfloor i/2\rfloor+j}}{i!}\sum_{n=0}^{\infty}\left(\sum_{k=i}^{\infty}\left(\sum_{l=\max(n+i,k)}^{n+k}e^{2l}d^{\phi}_{n,k,l,i}\right)\varrho^k\right)\sin^{2(n+\lfloor i/2\rfloor+j)}(\psi)$$
$$= 1 + \sum_{n=0}^{\infty}\left(\sum_{k=1}^{\infty}\left(\sum_{l=\max(n,k)}^{n+k}d^{\sin}_{n,k,l}e^{2l}\right)\varrho^k\right)\sin^{2n}(\psi) \tag{19}$$

where $\mathbf{N}_e$ and $\mathbf{N}_o$ designate the even and odd natural numbers and where

$$d^{\sin}_{n',k',l'}$$
$$= \sum_{i=1}^{\infty}\sum_{j=0}^{\lceil i/2\rceil}\binom{\lceil i/2\rceil}{j}\frac{(-1)^{\lfloor i/2\rfloor+j}}{i!}\sum_{n=0}^{\infty}\left(\sum_{k=i}^{\infty}\left(\sum_{l=\max(n+i,k)}^{n+k}\delta_{l-l'}d^{\phi}_{n,k,l,i}\right)\delta_{k-k'}\right)\delta_{n+\lfloor i/2\rfloor+j-n'}$$
$$= \sum_{i=1}^{\min(k',2n'+1)}\sum_{j=\max(0,\lceil i/2\rceil-l'+n')}^{\min(\lceil i/2\rceil,n'-\lfloor i/2\rfloor)}\binom{\lceil i/2\rceil}{j}\frac{(-1)^{\lfloor i/2\rfloor+j}}{i!}d^{\phi}_{n'-\lfloor i/2\rfloor-j,k',l',i} \tag{20}$$

where the summation limits are due to

- $\delta_{n+\lfloor i/2\rfloor+j-n'}$, $i \geq 1$, $j \geq 0$ and $n \geq 0$ imply

- $n' \geq 0$
- $i \leq 2n' + 1$
- $j \leq n' - \lfloor i/2 \rfloor$
- $n = n' - \lfloor i/2 \rfloor - j$

- $\delta_{k-k'}$, $i \geq 1$ and $k \geq i$ imply
  - $k' = k$
  - $k' \geq 1$
  - $i \leq k'$
- $\delta_{l-l'}$ and $\max(n+i, k) \leq l \leq n+k$, imply
  - $l' = l$
  - $i \leq l' - n = l' - n' + \lfloor i/2 \rfloor + j$ meaning $\lceil i/2 \rceil - l' + n' \leq j$
  - $\max(n', k') \leq l' \leq n' + k'$

Using (6), the Fourier series follows as

$$\frac{\sin(\phi)}{\sin(\psi)} - 1 = \sum_{n=0}^{\infty}\left(\sum_{k=1}^{\infty}\left(\sum_{l=\max(n,k)}^{\infty} c_{n,k,l}^{\sin} e^{2l}\right)\varrho^k\right)\cos(2n\psi) \tag{21}$$

where

$$c_{n,k,l}^{\sin} = \sum_{i=\max(n,l-k)}^{l} d_{i,k,l}^{\sin} \frac{2}{2^{2i+\delta_n}}(-1)^n \binom{2i}{i-n} \tag{22}$$

The derivation for $\cos(\phi)$ is identical with $\lfloor\cdot\rfloor$ changed to $\lceil\cdot\rceil$ and *vice versa*, i.e.

$$\frac{\cos(\phi)}{\cos(\psi)} = \sum_{i=0}^{\infty} \frac{\partial}{\partial\varphi^i} \left.\frac{\cos(\varphi)}{i!}\right|_{\psi} \frac{1}{\cos(\psi)}(\phi-\psi)^i$$

$$= \sum_{i\in\mathbf{N}_e}^{\infty} \frac{(-1)^{i/2}}{i!}(\phi-\psi)^i - \frac{\sin^2(\psi)}{\cos^2(\psi)} \sum_{i\in\mathbf{N}_o}^{\infty} \frac{(-1)^{(i-1)/2}}{i!} \frac{\cos(\psi)}{\sin(\psi)}(\phi-\psi)^i \tag{23}$$

$$= 1 + \sum_{n=0}^{\infty}\left(\sum_{k=1}^{\infty}\left(\sum_{l=\max(n,k)}^{n+k-1} d_{n,k,l}^{\cos} e^{2l}\right)\varrho^k\right)\sin^{2n}(\psi) \tag{24}$$

where

(25)

$$d_{n',k',l'}^{\cos} = \sum_{i=1}^{\min(k',2n'+1)} \sum_{j=\max(0,\lfloor i/2\rfloor - l' + n')}^{\min(\lfloor i/2\rfloor, n'-\lceil i/2\rceil)} \binom{\lfloor i/2\rfloor}{j} \frac{(-1)^{\lceil i/2\rceil + j}}{i!} d_{n'-\lceil i/2\rceil - j,k',l',i}^{\phi}$$

and

$$\frac{\cos(\phi)}{\cos(\psi)} - 1 = \sum_{n=0}^{\infty}\left(\sum_{k=1}^{\infty}\left(\sum_{l=\max(n,k)}^{\infty} c_{n,k,l}^{\cos} e^{2l}\right)\varrho^k\right)\cos(2n\psi) \tag{26}$$

where

$$c_{n,k,l}^{\cos} = \sum_{i=\max(n,l-k+1)}^{l} d_{i,k,l}^{\cos} \frac{2}{2^{2i+\delta_n}}(-1)^n \binom{2i}{i-n} \tag{27}$$

Note, the reason (18) and (23) are spelt out is that if both $\sin(\phi)$ and $\cos(\phi)$ are to be computed, it may be preferable to use them. The reason is that (18) and (23) are essentially univariate polynomials whereas (19) and (24) are bivariate polynomials. In addition, apart from the factor $-\frac{\sin^2(\psi)}{\cos^2(\psi)}$, they are identical. See *Nilsson 2024b* for some more comments about it.

A.4. **Expansions for** $\cos(\phi-\psi)$**.** Trivially

$$\cos(x) = 1 + \sum_{i=1}^{\infty} \frac{(-1)^i}{(2i)!} x^{2i} \tag{28}$$

Combining with (17)

$$\begin{aligned}
&\cos(\phi-\psi)\\
&= 1 + \sum_{i=1}^{\infty} \frac{(-1)^i}{(2i)!}(1-\sin^2(\phi))^i \sin^{2i}(\phi) \sum_{n=0}^{\infty} \left(\sum_{k=2i}^{\infty} \left(\sum_{l=\max(n+2i,k)}^{n+k} d^{\phi}_{n,k,l,2i} e^{2l} \right) \varrho^k \right) \sin^{2n}(\psi)\\
&= 1 + \sum_{i=1}^{\infty} \sum_{j=0}^{i} \frac{(-1)^{i+j}}{(2i)!} \binom{i}{j} \sum_{n=0}^{\infty} \left(\sum_{k=2i}^{\infty} \left(\sum_{l=\max(n+2i,k)}^{n+k} e^{2l} d^{\phi}_{n,k,l,2i} \right) \varrho^k \right) \sin^{2n+2i+2j}(\phi)\\
&= 1 + \sum_{n=1}^{\infty} \left(\sum_{k=2}^{\infty} \left(\sum_{l=\max(n,k)}^{n-1+k} d'_{n,k,l} e^{2l} \right) \varrho^k \right) \sin^{2n}(\phi) \qquad (29)
\end{aligned}$$

where

$$\begin{aligned}
d'_{n',k',l'} &= \sum_{i=1}^{\infty} \sum_{j=0}^{i} \frac{(-1)^{i+j}}{(2i)!} \binom{i}{j} \sum_{n=0}^{\infty} \left(\sum_{k=2i}^{\infty} \left(\sum_{l=\max(n+2i,k)}^{n+k} \delta_{l-l'} d^{\phi}_{n,k,l,2i} \right) \delta_{k-k'} \right) \delta_{n+i+j-n'}\\
&= \sum_{i=1}^{\min(n',\lfloor k'/2 \rfloor)} \sum_{j=0}^{\min(i,n'-i)} \frac{(-1)^{i+j}}{(2i)!} \binom{i}{j} d^{\phi}_{n'-i-j,k',l',2i}
\end{aligned}$$

The last equalities of the two proceeding equations are based on

- $\delta_{n+i+j-n'}$, $n \geq 0$ $i \geq 1$ and $0 \leq j \leq i$ imply that
  - $n' \geq 1$
  - $n = n' - i - j$
  - $0 \leq j \leq n' - i$
  - $1 \leq i \leq n'$
- $\delta_{k-k'}$, $i \geq 1$, $k \geq 2i$ and $i \geq 1$ imply that
  - $k' = k$
  - $k' \geq 2$
  - $i \leq \lfloor k'/2 \rfloor$
- $\delta_{l-l'}$, $\max(n+2i,k) \leq l \leq n+k$ and $j \leq i$ imply that
  - $l' = l$
  - $\max(n',k') \leq l' \leq n' - 1 + k'$

A.5. **Expansions for** $(1-e^2\sin^2(\phi))^{1/2}$**.** Applying binomial expansion twice

$$\begin{aligned}
(1-e^2\sin^2(\phi))^{1/2} &= 1 + \sum_{i=1}^{\infty} \binom{1/2}{i} (-1)^i e^{2i} \sin^{2i}(\phi)\\
&= 1 + \sum_{i=1}^{\infty} \binom{1/2}{i} (-1)^i e^{2i} \sin^{2i}(\psi) \left(1 + \left(\frac{\sin(\phi)}{\sin(\psi)} - 1\right)\right)^{2i}\\
&= 1 + \sum_{i=1}^{\infty} \binom{1/2}{i} (-1)^i e^{2i} \sin^{2i}(\psi) \sum_{j=0}^{2i} \binom{2i}{j} \left(\frac{\sin(\phi)}{\sin(\psi)} - 1\right)^j
\end{aligned}$$

Similar to the series expansion for $(\phi-\psi)^i$ (17), the last factor can be expanded with $\max(n+1,k)$ replaced with $\max(n,k)$, i.e.

$$\begin{aligned}\left(\frac{\sin(\phi)}{\sin(\psi)}-1\right)^j &= \left(\sum_{n=0}^{\infty}\left(\sum_{k=1}^{\infty}\left(\sum_{l=\max(n,k)}^{n+k} d^{\sin}_{n,k,l}e^{2l}\right)\varrho^k\right)\sin^{2n}(\psi)\right)^j \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=j}^{\infty}\left(\sum_{l=\max(n,k)}^{n+k} d^{\sin}_{n,k,l,j}e^{2l}\right)\varrho^k\right)\sin^{2n}(\psi)\end{aligned}$$

where

$$d^{\sin}_{n,k,l,i} = [e^{2l}][\varrho^k]\left(P_{n,i}(a_0,\ldots,a_n)\left[a_n^{i_n}\to \sum_{k=i_n}^{\infty}\hat{B}_{k,i_n}\left(\sum_{l=\max(1,n)}^{n+1} d^{\sin}_{n,1,l}e^{2l},\ldots,\sum_{l=\max(n,k-i_n+1)}^{n+k-i_n+1} d^{\sin}_{n,k-i_n+1,l}e^{2l}\right)\varrho^k\right]\right)$$

This in turn gives

$$\begin{aligned}&(1-e^2\sin^2(\phi))^{1/2} \\ &= \sum_{i=1}^{\infty}\binom{1/2}{i}(-1)^i e^{2i}\sin^{2i}(\psi)\sum_{j=0}^{2i}\binom{2i}{j}\sum_{n=0}^{\infty}\left(\sum_{k=j}^{\infty}\left(\sum_{l=\max(n,k)}^{n+k} d^{\sin}_{n,k,l,j}e^{2l}\right)\varrho^k\right)\sin^{2n}(\psi) \\ &= \sum_{i=1}^{\infty}\binom{1/2}{i}(-1)^i \sum_{j=0}^{2i}\binom{2i}{j}\sum_{n=0}^{\infty}\left(\sum_{k=j}^{\infty}\left(\sum_{l=\max(n,k)}^{n+k} d^{\sin}_{n,k,l,j}e^{2(l+i)}\right)\varrho^k\right)\sin^{2(n+i)}(\psi)\end{aligned}$$

which may be written as a plain triple sum

$$(1-e^2\sin^2(\phi))^{1/2} = \sum_{n=1}^{\infty}\left(\sum_{k=0}^{\infty}\left(\sum_{l=\max(n,k+1)}^{n+k} d^{N}_{n,k,l}e^{2l}\right)\varrho^k\right)\sin^{2n}(\psi) \tag{30}$$

where

$$\begin{aligned}d^N_{n',k',l'} &= \sum_{i=1}^{\infty}\binom{1/2}{i}(-1)^i\sum_{j=0}^{2i}\binom{2i}{j}\sum_{n=0}^{\infty}\left(\sum_{k=j}^{\infty}\left(\sum_{l=\max(n,k)}^{n+k}\delta_{l+i-l'}d^{\sin}_{n,k,l,j}\right)\delta_{k-k'}\right)\delta_{n+i-n'} \\ &= \sum_{i=1}^{\min(n',l')}\binom{1/2}{i}(-1)^i\sum_{j=0}^{\min(2i,k')}\binom{2i}{j}d^{\sin}_{n'-i,k',l'-i,j}\end{aligned}$$

The last equalities of the two proceeding equations are based on

- $\delta_{n+i-n'}$, $n\ge 0$ and $i\ge 1$ imply
  - $n'\ge 1$
  - $1\le i\le n'$
  - $n=n'-i$
- $\delta_{k-k'}$ and $k\ge j$ imply
  - $k'=k$
  - $k'\ge 0$
  - $j\le k'$
- $\delta_{l+i-l'}$, $\max(n,k)\le l\le n+k$ and $j\le 2i$ imply
  - $l=l'-i$
  - $i=l'-l\le l'$
  - $\max(n'-i,k')\le l'-i$, i.e. $\max(n',k'+1)\le l'$
  - $l'-i\le n'-i+k'$, i.e. $l'\le n'+k'$

A.6. **Expansions for h.** Multiplying (1) with $\cos(\phi)$ and (2) with $\sin(\phi)$, add, simplifying and solving for h gives

$$h = \rho\cos(\phi - \psi) - a(1 - e^2\sin^2(\phi))^{1/2}$$

Combining with (29) and (30) directly gives

$$\frac{h + a - \varrho}{a} = \sum_{n=1}^{\infty}\left(\sum_{k=0}^{\infty}\left(\sum_{l=\max(n,k+1)}^{n+k} d^h_{n,k,l}e^{2l}\right)\varrho^k\right)\sin^{2n}(\phi) \tag{31}$$

where

$$d^h_{n,k,l} = \begin{cases} -d^N_{n,k,l} & k = 0 \\ d'_{n,k+1,l} - d^N_{n,k,l} & \text{otherwise} \end{cases} \tag{32}$$

Using (6), the Fourier series follows as

$$\frac{h + a - \varrho}{a} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\left(\sum_{l=\max(n,k+1)}^{\infty} c^h_{n,k,l}e^{2l}\right)\varrho^k\right)\cos(2n\psi) \tag{33}$$

where

$$c^h_{n,k,l} = \sum_{i=\max(n,l-k)}^{l} d^h_{i,k,l}\frac{2}{2^{2i+\delta_n}}(-1)^n\binom{2i}{i-n} \tag{34}$$

Appendix B. Tabulated coefficients

The rational series expansion coefficients, i.e. (14), (16), (34), (32), (22), (20), (27), and (25), are tabulated in Table 1-8.

n	$k \setminus l$	$e^{2\cdot 1}$	$e^{2\cdot 2}$	$e^{2\cdot 3}$	$e^{2\cdot 4}$	$e^{2\cdot 5}$	$e^{2\cdot 6}$	$e^{2\cdot 7}$	$e^{2\cdot 8}$
1	ρ^{-1}	$1/2$	$1/8$	$15/256$	$35/1024$	$735/32768$	$2079/131072$	$99099/8388608$	$306735/33554432$
	ρ^{-2}	$\times$	0	$1/32$	$1/32$	$7/256$	$3/128$	$165/8192$	$143/8192$
	ρ^{-3}	$\times$	$\times$	$-3/64$	$-9/256$	$-105/4096$	$-315/16384$	$-31185/2097152$	$-99099/8388608$
	ρ^{-4}	$\times$	$\times$	$\times$	0	0	0	0	0
	ρ^{-5}	$\times$	$\times$	$\times$	$\times$	$-5/2048$	$-25/8192$	$-1575/524288$	$-5775/2097152$
	ρ^{-6}	$\times$	$\times$	$\times$	$\times$	$\times$	0	0	0
	ρ^{-7}	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$-35/131072$	$-245/524288$
	ρ^{-8}	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	0
2	ρ^{-1}	$\times$	$-1/16$	$-3/64$	$-35/1024$	$-105/4096$	$-10395/524288$	$-33033/2097152$	$-429429/33554432$
	ρ^{-2}	$\times$	$1/4$	$1/8$	$1/16$	$1/32$	$15/1024$	$11/2048$	0
	ρ^{-3}	$\times$	$\times$	0	$15/256$	$75/1024$	$4725/65536$	$17325/262144$	$495495/8388608$
	ρ^{-4}	$\times$	$\times$	$\times$	$-1/16$	$-1/16$	$-51/1024$	$-19/512$	$-55/2048$
	ρ^{-5}	$\times$	$\times$	$\times$	$\times$	0	$-175/32768$	$-1225/131072$	$-24255/2097152$
	ρ^{-6}	$\times$	$\times$	$\times$	$\times$	$\times$	0	0	0
	ρ^{-7}	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	0	$-147/524288$
	ρ^{-8}	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	0
3	ρ^{-1}	$\times$	$\times$	$3/256$	$15/1024$	$945/65536$	$3465/262144$	$99099/8388608$	$351351/33554432$
	ρ^{-2}	$\times$	$\times$	$-3/32$	$-3/32$	$-39/512$	$-15/256$	$-363/8192$	$-273/8192$
	ρ^{-3}	$\times$	$\times$	$35/192$	$35/256$	$525/8192$	$1225/98304$	$-40425/2097152$	$-315315/8388608$
	ρ^{-4}	$\times$	$\times$	$\times$	0	$3/32$	$9/64$	$39/256$	$75/512$
	ρ^{-5}	$\times$	$\times$	$\times$	$\times$	$-315/4096$	$-1575/16384$	$-41895/524288$	$-112455/2097152$
	ρ^{-6}	$\times$	$\times$	$\times$	$\times$	$\times$	0	$-9/512$	$-9/256$
	ρ^{-7}	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$693/131072$	$4851/524288$
	ρ^{-8}	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	0
4	ρ^{-1}	$\times$	$\times$	$\times$	$-5/2048$	$-35/8192$	$-693/131072$	$-3003/524288$	$-195195/33554432$
	ρ^{-2}	$\times$	$\times$	$\times$	$1/32$	$3/64$	$13/256$	$25/512$	$91/2048$
	ρ^{-3}	$\times$	$\times$	$\times$	$-63/512$	$-315/2048$	$-2205/16384$	$-6615/65536$	$-567567/8388608$
	ρ^{-4}	$\times$	$\times$	$\times$	$5/32$	$5/32$	$15/256$	$-5/128$	$-225/2048$
	ρ^{-5}	$\times$	$\times$	$\times$	$\times$	0	$1155/8192$	$8085/32768$	$606375/2097152$
	ρ^{-6}	$\times$	$\times$	$\times$	$\times$	$\times$	$-3/32$	$-9/64$	$-15/128$
	ρ^{-7}	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	0	$-21021/524288$
	ρ^{-8}	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$7/512$
5	ρ^{-1}	$\times$	$\times$	$\times$	$\times$	$35/65536$	$315/262144$	$15015/8388608$	$75075/33554432$
	ρ^{-2}	$\times$	$\times$	$\times$	$\times$	$-5/512$	$-5/256$	$-215/8192$	$-245/8192$
	ρ^{-3}	$\times$	$\times$	$\times$	$\times$	$495/8192$	$3465/32768$	$259875/2097152$	$1029105/8388608$
	ρ^{-4}	$\times$	$\times$	$\times$	$\times$	$-5/32$	$-15/64$	$-55/256$	$-75/512$
	ρ^{-5}	$\times$	$\times$	$\times$	$\times$	$3003/20480$	$3003/16384$	$21021/524288$	$-315315/2097152$
	ρ^{-6}	$\times$	$\times$	$\times$	$\times$	$\times$	0	$105/512$	$105/256$
	ρ^{-7}	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$-15015/131072$	$-105105/524288$
	ρ^{-8}	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	0
6	ρ^{-1}	$\times$	$\times$	$\times$	$\times$	$\times$	$-63/524288$	$-693/2097152$	$-19305/33554432$
	ρ^{-2}	$\times$	$\times$	$\times$	$\times$	$\times$	$3/1024$	$15/2048$	$3/256$
	ρ^{-3}	$\times$	$\times$	$\times$	$\times$	$\times$	$-5005/196608$	$-15015/262144$	$-693693/8388608$
	ρ^{-4}	$\times$	$\times$	$\times$	$\times$	$\times$	$105/1024$	$105/512$	$525/2048$
	ρ^{-5}	$\times$	$\times$	$\times$	$\times$	$\times$	$-6435/32768$	$-45045/131072$	$-675675/2097152$
	ρ^{-6}	$\times$	$\times$	$\times$	$\times$	$\times$	$7/48$	$7/32$	0
	ρ^{-7}	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	0	$153153/524288$
	ρ^{-8}	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$-9/64$
7	ρ^{-1}	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$231/8388608$	$3003/33554432$
	ρ^{-2}	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$-7/8192$	$-21/8192$
	ρ^{-3}	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$20475/2097152$	$225225/8388608$
	ρ^{-4}	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$-7/128$	$-35/256$
	ρ^{-5}	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$85085/524288$	$765765/2097152$
	ρ^{-6}	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$-63/256$	$-63/128$
	ρ^{-7}	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$138567/917504$	$138567/524288$
	ρ^{-8}	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	0
8	ρ^{-1}	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$-429/67108864$
	ρ^{-2}	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$1/4096$
	ρ^{-3}	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$-58905/16777216$
	ρ^{-4}	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$105/4096$
	ρ^{-5}	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$-440895/4194304$
	ρ^{-6}	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$63/256$
	ρ^{-7}	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$-323323/1048576$
	ρ^{-8}	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$165/1024$

Table 1. Coefficients $c^{\phi}_{n,k,l}$ for $\phi - \psi$ in terms of $\sin(2n\psi)$. Note, the coefficients provided in *Morrison and Pines 1961* are contained in the table.

n	$k \setminus l$	$e^{2\cdot 1}$	$e^{2\cdot 2}$	$e^{2\cdot 3}$	$e^{2\cdot 4}$	$e^{2\cdot 5}$	$e^{2\cdot 6}$	$e^{2\cdot 7}$	$e^{2\cdot 8}$	$e^{2\cdot 9}$
0	ρ^{-1}	1	×	×	×	×	×	×	×	×
	ρ^{-2}	×	1	×	×	×	×	×	×	×
	ρ^{-3}	×	×	1	×	×	×	×	×	×
	ρ^{-4}	×	×	×	1	×	×	×	×	×
	ρ^{-5}	×	×	×	×	1	×	×	×	×
	ρ^{-6}	×	×	×	×	×	1	×	×	×
	ρ^{-7}	×	×	×	×	×	×	1	×	×
	ρ^{-8}	×	×	×	×	×	×	×	1	×
1	ρ^{-1}	×	1/2	×	×	×	×	×	×	×
	ρ^{-2}	×	−2	2	×	×	×	×	×	×
	ρ^{-3}	×	×	−35/6	5	×	×	×	×	×
	ρ^{-4}	×	×	×	−12	10	×	×	×	×
	ρ^{-5}	×	×	×	×	−21	35/2	×	×	×
	ρ^{-6}	×	×	×	×	×	−100/3	28	×	×
	ρ^{-7}	×	×	×	×	×	×	−99/2	42	×
	ρ^{-8}	×	×	×	×	×	×	×	−70	60
2	ρ^{-1}	×	×	3/8	×	×	×	×	×	×
	ρ^{-2}	×	×	−3	3	×	×	×	×	×
	ρ^{-3}	×	×	35/6	−77/4	105/8	×	×	×	×
	ρ^{-4}	×	×	×	30	−72	42	×	×	×
	ρ^{-5}	×	×	×	×	3843/40	−819/4	441/4	×	×
	ρ^{-6}	×	×	×	×	×	730/3	−490	252	×
	ρ^{-7}	×	×	×	×	×	×	4257/8	−2079/2	2079/4
	ρ^{-8}	×	×	×	×	×	×	×	1050	−2016
3	ρ^{-1}	×	×	×	5/16	×	×	×	×	×
	ρ^{-2}	×	×	×	−4	4	×	×	×	×
	ρ^{-3}	×	×	×	63/4	−675/16	105/4	×	×	×
	ρ^{-4}	×	×	×	−20	140	−240	120	×	×
	ρ^{-5}	×	×	×	×	−3003/20	11187/16	−3927/4	3465/8	×
	ρ^{-6}	×	×	×	×	×	−660	2580	−3240	1320
	ρ^{-7}	×	×	×	×	×	×	−246675/112	31317/4	−73359/8
	ρ^{-8}	×	×	×	×	×	×	×	−6160	20664
4	ρ^{-1}	×	×	×	×	35/128	×	×	×	×
	ρ^{-2}	×	×	×	×	−5	5	×	×	×
	ρ^{-3}	×	×	×	×	495/16	−7315/96	5775/128	×	×
	ρ^{-4}	×	×	×	×	−80	405	−600	275	×
	ρ^{-5}	×	×	×	×	3003/40	−7293/8	181181/64	−105105/32	165165/128
	ρ^{-6}	×	×	×	×	×	2240/3	−5705	14070	−42350/3
	ρ^{-7}	×	×	×	×	×	×	469755/112	−828685/32	3571425/64
	ρ^{-8}	×	×	×	×	×	×	×	17430	−95200
5	ρ^{-1}	×	×	×	×	×	63/256	×	×	×
	ρ^{-2}	×	×	×	×	×	−6	6	×	×
	ρ^{-3}	×	×	×	×	×	5005/96	−31395/256	9009/128	×
	ρ^{-4}	×	×	×	×	×	−210	924	−1260	546
	ρ^{-5}	×	×	×	×	×	6435/16	−105105/32	1085175/128	−1126125/128
	ρ^{-6}	×	×	×	×	×	−896/3	5600	−27216	161840/3
	ρ^{-7}	×	×	×	×	×	×	−415701/112	1349205/32	−20357415/128
	ρ^{-8}	×	×	×	×	×	×	×	−25452	227304
6	ρ^{-1}	×	×	×	×	×	×	231/1024	×	×
	ρ^{-2}	×	×	×	×	×	×	−7	7	×
	ρ^{-3}	×	×	×	×	×	×	20475/256	−93555/512	105105/1024
	ρ^{-4}	×	×	×	×	×	×	−448	1820	−2352
	ρ^{-5}	×	×	×	×	×	×	85085/64	−580125/64	21500325/1024
	ρ^{-6}	×	×	×	×	×	×	−2016	24192	−94920
	ρ^{-7}	×	×	×	×	×	×	138567/112	−1062347/32	29665935/128
	ρ^{-8}	×	×	×	×	×	×	×	18480	−292320
7	ρ^{-1}	×	×	×	×	×	×	×	429/2048	×
	ρ^{-2}	×	×	×	×	×	×	×	−8	8
	ρ^{-3}	×	×	×	×	×	×	×	58905/512	−527527/2048
	ρ^{-4}	×	×	×	×	×	×	×	−840	3240
	ρ^{-5}	×	×	×	×	×	×	×	440895/128	−10818885/512
	ρ^{-6}	×	×	×	×	×	×	×	−8064	78400
	ρ^{-7}	×	×	×	×	×	×	×	323323/32	−21427497/128
	ρ^{-8}	×	×	×	×	×	×	×	−5280	192192
8	ρ^{-1}	×	×	×	×	×	×	×	×	6435/32768
	ρ^{-2}	×	×	×	×	×	×	×	×	−9
	ρ^{-3}	×	×	×	×	×	×	×	×	323323/2048
	ρ^{-4}	×	×	×	×	×	×	×	×	−1440
	ρ^{-5}	×	×	×	×	×	×	×	×	7834365/1024
	ρ^{-6}	×	×	×	×	×	×	×	×	−24640
	ρ^{-7}	×	×	×	×	×	×	×	×	6084351/128
	ρ^{-8}	×	×	×	×	×	×	×	×	−50688

TABLE 2. Coefficients $d^{\phi}_{n,k,l}$ for $(\phi - \psi)/(\cos(\psi)\sin(\psi))$ in terms of $\sin^{2n}(\psi)$.

n	$k \setminus l$	$e^{2\cdot 1}$	$e^{2\cdot 2}$	$e^{2\cdot 3}$	$e^{2\cdot 4}$	$e^{2\cdot 5}$	$e^{2\cdot 6}$	$e^{2\cdot 7}$	$e^{2\cdot 8}$
0	ρ^{-0}	1/4	3/64	5/256	175/16384	441/65536	4851/1048576	14157/4194304	2760615/1073741824
	ρ^{-1}	×	1/16	1/32	5/256	7/512	21/2048	33/4096	429/65536
	ρ^{-2}	×	×	0	0	0	0	0	0
	ρ^{-3}	×	×	×	1/1024	1/1024	7/8192	3/4096	165/262144
	ρ^{-4}	×	×	×	×	0	0	0	0
	ρ^{-5}	×	×	×	×	×	1/16384	3/32768	27/262144
	ρ^{-6}	×	×	×	×	×	×	0	0
	ρ^{-7}	×	×	×	×	×	×	×	25/4194304
1	ρ^{-0}	−1/4	−1/16	−15/512	−35/2048	−735/65536	−2079/262144	−99099/16777216	−306735/67108864
	ρ^{-1}	×	0	−1/64	−1/64	−7/512	−3/256	−165/16384	−143/16384
	ρ^{-2}	×	×	1/32	3/128	35/2048	105/8192	10395/1048576	33033/4194304
	ρ^{-3}	×	×	×	0	1/1024	3/2048	27/16384	55/32768
	ρ^{-4}	×	×	×	×	0	0	0	0
	ρ^{-5}	×	×	×	×	×	0	5/131072	5/65536
	ρ^{-6}	×	×	×	×	×	×	0	0
	ρ^{-7}	×	×	×	×	×	×	×	0
2	ρ^{-0}	×	1/64	3/256	35/4096	105/16384	10395/2097152	33033/8388608	429429/134217728
	ρ^{-1}	×	−1/16	−1/32	−1/64	−1/128	−15/4096	−11/8192	0
	ρ^{-2}	×	×	0	−1/64	−5/256	−315/16384	−1155/65536	−33033/2097152
	ρ^{-3}	×	×	×	5/256	5/256	255/16384	95/8192	275/32768
	ρ^{-4}	×	×	×	×	0	5/2048	35/8192	693/131072
	ρ^{-5}	×	×	×	×	×	−35/32768	−105/65536	−7/4096
	ρ^{-6}	×	×	×	×	×	×	0	0
	ρ^{-7}	×	×	×	×	×	×	×	−21/524288
3	ρ^{-0}	×	×	−1/512	−5/2048	−315/131072	−1155/524288	−33033/16777216	−117117/67108864
	ρ^{-1}	×	×	1/64	1/64	13/1024	5/512	121/16384	91/16384
	ρ^{-2}	×	×	−1/32	−3/128	−45/4096	−35/16384	3465/1048576	27027/4194304
	ρ^{-3}	×	×	×	0	−35/2048	−105/4096	−455/16384	−875/32768
	ρ^{-4}	×	×	×	×	1/64	5/256	133/8192	357/32768
	ρ^{-5}	×	×	×	×	×	0	567/131072	567/65536
	ρ^{-6}	×	×	×	×	×	×	−1/512	−7/2048
	ρ^{-7}	×	×	×	×	×	×	×	0
4	ρ^{-0}	×	×	×	5/16384	35/65536	693/1048576	3003/4194304	195195/268435456
	ρ^{-1}	×	×	×	−1/256	−3/512	−13/2048	−25/4096	−91/16384
	ρ^{-2}	×	×	×	1/64	5/256	35/2048	105/8192	9009/1048576
	ρ^{-3}	×	×	×	−21/1024	−21/1024	−63/8192	21/4096	945/65536
	ρ^{-4}	×	×	×	×	0	−5/256	−35/1024	−2625/65536
	ρ^{-5}	×	×	×	×	×	231/16384	693/32768	1155/65536
	ρ^{-6}	×	×	×	×	×	×	0	7/1024
	ρ^{-7}	×	×	×	×	×	×	×	−3003/1048576
5	ρ^{-0}	×	×	×	×	−7/131072	−63/524288	−3003/16777216	−15015/67108864
	ρ^{-1}	×	×	×	×	1/1024	1/512	43/16384	49/16384
	ρ^{-2}	×	×	×	×	−25/4096	−175/16384	−13125/1048576	−51975/4194304
	ρ^{-3}	×	×	×	×	33/2048	99/4096	363/16384	495/32768
	ρ^{-4}	×	×	×	×	−1/64	−5/256	−35/8192	525/32768
	ρ^{-5}	×	×	×	×	×	0	−3003/131072	−3003/65536
	ρ^{-6}	×	×	×	×	×	×	7/512	49/2048
	ρ^{-7}	×	×	×	×	×	×	×	0
6	ρ^{-0}	×	×	×	×	×	21/2097152	231/8388608	6435/134217728
	ρ^{-1}	×	×	×	×	×	−1/4096	−5/8192	−1/1024
	ρ^{-2}	×	×	×	×	×	35/16384	315/65536	14553/2097152
	ρ^{-3}	×	×	×	×	×	−143/16384	−143/8192	−715/32768
	ρ^{-4}	×	×	×	×	×	35/2048	245/8192	3675/131072
	ρ^{-5}	×	×	×	×	×	−429/32768	−1287/65536	0
	ρ^{-6}	×	×	×	×	×	×	0	−7/256
	ρ^{-7}	×	×	×	×	×	×	×	7293/524288
7	ρ^{-0}	×	×	×	×	×	×	−33/16777216	−429/67108864
	ρ^{-1}	×	×	×	×	×	×	1/16384	3/16384
	ρ^{-2}	×	×	×	×	×	×	−735/1048576	−8085/4194304
	ρ^{-3}	×	×	×	×	×	×	65/16384	325/32768
	ρ^{-4}	×	×	×	×	×	×	−49/4096	−441/16384
	ρ^{-5}	×	×	×	×	×	×	2431/131072	2431/65536
	ρ^{-6}	×	×	×	×	×	×	−3/256	−21/1024
	ρ^{-7}	×	×	×	×	×	×	×	0
8	ρ^{-0}	×	×	×	×	×	×	×	429/1073741824
	ρ^{-1}	×	×	×	×	×	×	×	−1/65536
	ρ^{-2}	×	×	×	×	×	×	×	231/1048576
	ρ^{-3}	×	×	×	×	×	×	×	−425/262144
	ρ^{-4}	×	×	×	×	×	×	×	441/65536
	ρ^{-5}	×	×	×	×	×	×	×	−4199/262144
	ρ^{-6}	×	×	×	×	×	×	×	21/1024
	ρ^{-7}	×	×	×	×	×	×	×	−46189/4194304

TABLE 3. Coefficients $c^h_{n,k,l}$ for $(h + a - \rho)/a$ in terms of $\cos(2n\psi)$.

n	$k \setminus l$	$e^{2\cdot 1}$	$e^{2\cdot 2}$	$e^{2\cdot 3}$	$e^{2\cdot 4}$	$e^{2\cdot 5}$	$e^{2\cdot 6}$	$e^{2\cdot 7}$	$e^{2\cdot 8}$
1	ρ^{-0}	$1/2$	×	×	×	×	×	×	×
	ρ^{-1}	×	$1/2$	×	×	×	×	×	×
	ρ^{-2}	×	×	$1/2$	×	×	×	×	×
	ρ^{-3}	×	×	×	$1/2$	×	×	×	×
	ρ^{-4}	×	×	×	×	$1/2$	×	×	×
	ρ^{-5}	×	×	×	×	×	$1/2$	×	×
	ρ^{-6}	×	×	×	×	×	×	$1/2$	×
	ρ^{-7}	×	×	×	×	×	×	×	$1/2$
2	ρ^{-0}	×	$1/8$	×	×	×	×	×	×
	ρ^{-1}	×	$-1/2$	$1/2$	×	×	×	×	×
	ρ^{-2}	×	×	$-3/2$	$5/4$	×	×	×	×
	ρ^{-3}	×	×	×	$-25/8$	$5/2$	×	×	×
	ρ^{-4}	×	×	×	×	$-11/2$	$35/8$	×	×
	ρ^{-5}	×	×	×	×	×	$-35/4$	7	×
	ρ^{-6}	×	×	×	×	×	×	-13	$21/2$
	ρ^{-7}	×	×	×	×	×	×	×	$-147/8$
3	ρ^{-0}	×	×	$1/16$	×	×	×	×	×
	ρ^{-1}	×	×	$-1/2$	$1/2$	×	×	×	×
	ρ^{-2}	×	×	1	$-13/4$	$35/16$	×	×	×
	ρ^{-3}	×	×	×	$21/4$	$-49/4$	7	×	×
	ρ^{-4}	×	×	×	×	17	-35	$147/8$	×
	ρ^{-5}	×	×	×	×	×	$693/16$	-84	42
	ρ^{-6}	×	×	×	×	×	×	95	$-357/2$
	ρ^{-7}	×	×	×	×	×	×	×	$3003/16$
4	ρ^{-0}	×	×	×	$5/128$	×	×	×	×
	ρ^{-1}	×	×	×	$-1/2$	$1/2$	×	×	×
	ρ^{-2}	×	×	×	2	$-85/16$	$105/32$	×	×
	ρ^{-3}	×	×	×	$-21/8$	18	$-243/8$	15	×
	ρ^{-4}	×	×	×	×	-20	$725/8$	$-1995/16$	$3465/64$
	ρ^{-5}	×	×	×	×	×	$-1419/16$	$5379/16$	$-825/2$
	ρ^{-6}	×	×	×	×	×	×	$-595/2$	$4095/4$
	ρ^{-7}	×	×	×	×	×	×	×	$-106821/128$
5	ρ^{-0}	×	×	×	×	$7/256$	×	×	×
	ρ^{-1}	×	×	×	×	$-1/2$	$1/2$	×	×
	ρ^{-2}	×	×	×	×	$25/8$	$-245/32$	$1155/256$	×
	ρ^{-3}	×	×	×	×	$-33/4$	$165/4$	$-121/2$	$55/2$
	ρ^{-4}	×	×	×	×	8	-95	$4641/16$	$-5313/16$
	ρ^{-5}	×	×	×	×	×	$1287/16$	$-9581/16$	$11583/8$
	ρ^{-6}	×	×	×	×	×	×	455	$-10927/4$
	ρ^{-7}	×	×	×	×	×	×	×	$60775/32$
6	ρ^{-0}	×	×	×	×	×	$21/1024$	×	×
	ρ^{-1}	×	×	×	×	×	$-1/2$	$1/2$	×
	ρ^{-2}	×	×	×	×	×	$35/8$	$-2625/256$	$3003/512$
	ρ^{-3}	×	×	×	×	×	$-143/8$	78	$-845/8$
	ρ^{-4}	×	×	×	×	×	35	$-1127/4$	$46011/64$
	ρ^{-5}	×	×	×	×	×	$-429/16$	$7865/16$	$-18785/8$
	ρ^{-6}	×	×	×	×	×	×	-336	3724
	ρ^{-7}	×	×	×	×	×	×	×	$-148291/64$
7	ρ^{-0}	×	×	×	×	×	×	$33/2048$	×
	ρ^{-1}	×	×	×	×	×	×	$-1/2$	$1/2$
	ρ^{-2}	×	×	×	×	×	×	$735/128$	$-6699/512$
	ρ^{-3}	×	×	×	×	×	×	$-65/2$	$525/4$
	ρ^{-4}	×	×	×	×	×	×	98	$-1323/2$
	ρ^{-5}	×	×	×	×	×	×	$-2431/16$	$14365/8$
	ρ^{-6}	×	×	×	×	×	×	96	-2520
	ρ^{-7}	×	×	×	×	×	×	×	$46189/32$
8	ρ^{-0}	×	×	×	×	×	×	×	$429/32768$
	ρ^{-1}	×	×	×	×	×	×	×	$-1/2$
	ρ^{-2}	×	×	×	×	×	×	×	$231/32$
	ρ^{-3}	×	×	×	×	×	×	×	$-425/8$
	ρ^{-4}	×	×	×	×	×	×	×	$441/2$
	ρ^{-5}	×	×	×	×	×	×	×	$-4199/8$
	ρ^{-6}	×	×	×	×	×	×	×	672
	ρ^{-7}	×	×	×	×	×	×	×	$-46189/128$

TABLE 4. Coefficients $d^h_{n,k,l}$ for $(h + a - \rho)/a$ in terms of $\sin^{2n}(\psi)$.

n	$k \setminus l$	$e^{2\cdot 1}$	$e^{2\cdot 2}$	$e^{2\cdot 3}$	$e^{2\cdot 4}$	$e^{2\cdot 5}$	$e^{2\cdot 6}$	$e^{2\cdot 7}$	$e^{2\cdot 8}$
0	ρ^{-1}	1/2	1/16	3/128	25/2048	245/32768	1323/262144	7623/2097152	184041/67108864
	ρ^{-2}	×	3/16	1/32	3/256	3/512	7/2048	9/4096	99/65536
	ρ^{-3}	×	×	1/8	1/32	15/1024	35/4096	735/131072	2079/524288
	ρ^{-4}	×	×	×	85/1024	23/1024	87/8192	25/4096	1025/262144
	ρ^{-5}	×	×	×	×	1/16	5/256	21/2048	105/16384
	ρ^{-6}	×	×	×	×	×	791/16384	519/32768	2217/262144
	ρ^{-7}	×	×	×	×	×	×	5/128	7/512
	ρ^{-8}	×	×	×	×	×	×	×	135489/4194304
1	ρ^{-1}	1/2	0	−3/256	−5/512	−245/32768	−189/32768	−38115/8388608	−61347/16777216
	ρ^{-2}	×	1/2	7/64	3/64	13/512	1/64	171/16384	121/16384
	ρ^{-3}	×	×	1/4	1/16	15/512	35/2048	735/65536	2079/262144
	ρ^{-4}	×	×	×	11/64	51/1024	51/2048	245/16384	325/32768
	ρ^{-5}	×	×	×	×	1/8	5/128	21/1024	105/8192
	ρ^{-6}	×	×	×	×	×	199/2048	4257/131072	1161/65536
	ρ^{-7}	×	×	×	×	×	×	5/64	7/256
	ρ^{-8}	×	×	×	×	×	×	×	8479/131072
2	ρ^{-1}	×	−1/16	−3/128	−5/512	−35/8192	−945/524288	−2541/4194304	0
	ρ^{-2}	×	5/16	−1/32	−3/64	−5/128	−125/4096	−195/8192	−77/4096
	ρ^{-3}	×	×	3/8	1/8	15/256	245/8192	4095/262144	2079/262144
	ρ^{-4}	×	×	×	49/256	19/256	783/16384	285/8192	875/32768
	ρ^{-5}	×	×	×	×	1/8	5/128	21/1024	105/8192
	ρ^{-6}	×	×	×	×	×	3219/32768	2251/65536	327/16384
	ρ^{-7}	×	×	×	×	×	×	5/64	7/256
	ρ^{-8}	×	×	×	×	×	×	×	33979/524288
3	ρ^{-1}	×	×	3/256	5/512	455/65536	315/65536	27951/8388608	39039/16777216
	ρ^{-2}	×	×	−7/64	−3/64	−15/1024	0	105/16384	147/16384
	ρ^{-3}	×	×	1/4	−1/16	−105/1024	−385/4096	−315/4096	−15939/262144
	ρ^{-4}	×	×	×	21/64	327/2048	327/4096	535/16384	175/32768
	ρ^{-5}	×	×	×	×	5/32	25/256	357/4096	315/4096
	ρ^{-6}	×	×	×	×	×	363/4096	3179/131072	867/65536
	ρ^{-7}	×	×	×	×	×	×	5/64	7/256
	ρ^{-8}	×	×	×	×	×	×	×	8437/131072
4	ρ^{-1}	×	×	×	−5/2048	−105/32768	−819/262144	−5775/2097152	−39039/16777216
	ρ^{-2}	×	×	×	9/256	17/512	49/2048	63/4096	147/16384
	ρ^{-3}	×	×	×	−5/32	−75/1024	−35/4096	3465/131072	693/16384
	ρ^{-4}	×	×	×	231/1024	−99/1024	−1551/8192	−785/4096	−10675/65536
	ρ^{-5}	×	×	×	×	5/16	55/256	231/2048	105/4096
	ρ^{-6}	×	×	×	×	×	2145/16384	4433/32768	10569/65536
	ρ^{-7}	×	×	×	×	×	×	7/128	0
	ρ^{-8}	×	×	×	×	×	×	×	64493/1048576
5	ρ^{-1}	×	×	×	×	35/65536	63/65536	9933/8388608	21021/16777216
	ρ^{-2}	×	×	×	×	−11/1024	−1/64	−261/16384	−231/16384
	ρ^{-3}	×	×	×	×	75/1024	315/4096	1785/32768	7623/262144
	ρ^{-4}	×	×	×	×	−429/2048	−429/4096	325/16384	3325/32768
	ρ^{-5}	×	×	×	×	7/32	−35/256	−1323/4096	−735/2048
	ρ^{-6}	×	×	×	×	×	1287/4096	38753/131072	10569/65536
	ρ^{-7}	×	×	×	×	×	×	7/64	49/256
	ρ^{-8}	×	×	×	×	×	×	×	2431/131072
6	ρ^{-1}	×	×	×	×	×	−63/524288	−1155/4194304	−429/1048576
	ρ^{-2}	×	×	×	×	×	13/4096	51/8192	33/4096
	ρ^{-3}	×	×	×	×	×	−245/8192	−12495/262144	−13167/262144
	ρ^{-4}	×	×	×	×	×	2145/16384	1235/8192	3325/32768
	ρ^{-5}	×	×	×	×	×	−35/128	−147/1024	735/8192
	ρ^{-6}	×	×	×	×	×	7293/32768	−12155/65536	−8619/16384
	ρ^{-7}	×	×	×	×	×	×	21/64	105/256
	ρ^{-8}	×	×	×	×	×	×	×	46189/524288
7	ρ^{-1}	×	×	×	×	×	×	231/8388608	1287/16777216
	ρ^{-2}	×	×	×	×	×	×	−15/16384	−37/16384
	ρ^{-3}	×	×	×	×	×	×	735/65536	6237/262144
	ρ^{-4}	×	×	×	×	×	×	−1105/16384	−3825/32768
	ρ^{-5}	×	×	×	×	×	×	441/2048	2205/8192
	ρ^{-6}	×	×	×	×	×	×	−46189/131072	−12597/65536
	ρ^{-7}	×	×	×	×	×	×	15/64	−63/256
	ρ^{-8}	×	×	×	×	×	×	×	46189/131072
8	ρ^{-1}	×	×	×	×	×	×	×	−429/67108864
	ρ^{-2}	×	×	×	×	×	×	×	17/65536
	ρ^{-3}	×	×	×	×	×	×	×	−2079/524288
	ρ^{-4}	×	×	×	×	×	×	×	8075/262144
	ρ^{-5}	×	×	×	×	×	×	×	−2205/16384
	ρ^{-6}	×	×	×	×	×	×	×	88179/262144
	ρ^{-7}	×	×	×	×	×	×	×	−231/512
	ρ^{-8}	×	×	×	×	×	×	×	1062347/4194304

TABLE 5. Coefficients $c^{\sin}_{n,k,l}$ for $\sin(\phi)/\sin(\psi) - 1$ in terms of $\cos(2n\psi)$.

n	$k \setminus l$	$e^{2\cdot 1}$	$e^{2\cdot 2}$	$e^{2\cdot 3}$	$e^{2\cdot 4}$	$e^{2\cdot 5}$	$e^{2\cdot 6}$	$e^{2\cdot 7}$	$e^{2\cdot 8}$
0	ρ^{-1}	1	×	×	×	×	×	×	×
	ρ^{-2}	×	1	×	×	×	×	×	×
	ρ^{-3}	×	×	1	×	×	×	×	×
	ρ^{-4}	×	×	×	1	×	×	×	×
	ρ^{-5}	×	×	×	×	1	×	×	×
	ρ^{-6}	×	×	×	×	×	1	×	×
	ρ^{-7}	×	×	×	×	×	×	1	×
	ρ^{-8}	×	×	×	×	×	×	×	1
1	ρ^{-1}	-1	$1/2$	×	×	×	×	×	×
	ρ^{-2}	×	$-7/2$	2	×	×	×	×	×
	ρ^{-3}	×	×	-8	5	×	×	×	×
	ρ^{-4}	×	×	×	-15	10	×	×	×
	ρ^{-5}	×	×	×	×	-25	$35/2$	×	×
	ρ^{-6}	×	×	×	×	×	$-77/2$	28	×
	ρ^{-7}	×	×	×	×	×	×	-56	42
	ρ^{-8}	×	×	×	×	×	×	×	-78
2	ρ^{-1}	×	$-1/2$	$3/8$	×	×	×	×	×
	ρ^{-2}	×	$5/2$	$-11/2$	3	×	×	×	×
	ρ^{-3}	×	×	15	-27	$105/8$	×	×	×
	ρ^{-4}	×	×	×	$427/8$	-91	42	×	×
	ρ^{-5}	×	×	×	×	146	-245	$441/4$	×
	ρ^{-6}	×	×	×	×	×	$2709/8$	-567	252
	ρ^{-7}	×	×	×	×	×	×	700	-1176
	ρ^{-8}	×	×	×	×	×	×	×	$5313/4$
3	ρ^{-1}	×	×	$-3/8$	$5/16$	×	×	×	×
	ρ^{-2}	×	×	$7/2$	$-15/2$	4	×	×	×
	ρ^{-3}	×	×	-8	42	-60	$105/4$	×	×
	ρ^{-4}	×	×	×	$-273/4$	$1017/4$	-306	120	×
	ρ^{-5}	×	×	×	×	-330	1075	$-4725/4$	$3465/8$
	ρ^{-6}	×	×	×	×	×	$-18975/16$	$7227/2$	-3762
	ρ^{-7}	×	×	×	×	×	×	-3520	10332
	ρ^{-8}	×	×	×	×	×	×	×	$-36465/4$
4	ρ^{-1}	×	×	×	$-5/16$	$35/128$	×	×	×
	ρ^{-2}	×	×	×	$9/2$	$-19/2$	5	×	×
	ρ^{-3}	×	×	×	-20	$675/8$	$-875/8$	$5775/128$	×
	ρ^{-4}	×	×	×	$231/8$	$-561/2$	$5973/8$	-770	275
	ρ^{-5}	×	×	×	×	320	$-4075/2$	$35175/8$	$-63525/16$
	ρ^{-6}	×	×	×	×	×	$31317/16$	$-165737/16$	$79365/4$
	ρ^{-7}	×	×	×	×	×	×	8715	-41650
	ρ^{-8}	×	×	×	×	×	×	×	$4033315/128$
5	ρ^{-1}	×	×	×	×	$-35/128$	$63/256$	×	×
	ρ^{-2}	×	×	×	×	$11/2$	$-23/2$	6	×
	ρ^{-3}	×	×	×	×	$-75/2$	$1155/8$	$-2835/16$	$9009/128$
	ρ^{-4}	×	×	×	×	$429/4$	$-3003/4$	$3445/2$	-1625
	ρ^{-5}	×	×	×	×	-112	1750	$-59535/8$	$212415/16$
	ρ^{-6}	×	×	×	×	×	$-24453/16$	$238095/16$	$-399165/8$
	ρ^{-7}	×	×	×	×	×	×	-11312	88396
	ρ^{-8}	×	×	×	×	×	×	×	$-1922921/32$
6	ρ^{-1}	×	×	×	×	×	$-63/256$	$231/1024$	×
	ρ^{-2}	×	×	×	×	×	$13/2$	$-27/2$	7
	ρ^{-3}	×	×	×	×	×	$-245/4$	$28665/128$	$-33957/128$
	ρ^{-4}	×	×	×	×	×	$2145/8$	-1625	$27375/8$
	ρ^{-5}	×	×	×	×	×	-560	5880	$-83055/4$
	ρ^{-6}	×	×	×	×	×	$7293/16$	$-167739/16$	$520455/8$
	ρ^{-7}	×	×	×	×	×	×	7392	-102312
	ρ^{-8}	×	×	×	×	×	×	×	$4110821/64$
7	ρ^{-1}	×	×	×	×	×	×	$-231/1024$	$429/2048$
	ρ^{-2}	×	×	×	×	×	×	$15/2$	$-31/2$
	ρ^{-3}	×	×	×	×	×	×	$-735/8$	$10395/32$
	ρ^{-4}	×	×	×	×	×	×	$1105/2$	$-12325/4$
	ρ^{-5}	×	×	×	×	×	×	-1764	15435
	ρ^{-6}	×	×	×	×	×	×	$46189/16$	$-340119/8$
	ρ^{-7}	×	×	×	×	×	×	-1920	61152
	ρ^{-8}	×	×	×	×	×	×	×	$-1154725/32$
8	ρ^{-1}	×	×	×	×	×	×	×	$-429/2048$
	ρ^{-2}	×	×	×	×	×	×	×	$17/2$
	ρ^{-3}	×	×	×	×	×	×	×	$-2079/16$
	ρ^{-4}	×	×	×	×	×	×	×	$8075/8$
	ρ^{-5}	×	×	×	×	×	×	×	-4410
	ρ^{-6}	×	×	×	×	×	×	×	$88179/8$
	ρ^{-7}	×	×	×	×	×	×	×	-14784
	ρ^{-8}	×	×	×	×	×	×	×	$1062347/128$

TABLE 6. Coefficients $d^{\sin}_{n,k,l}$ for $\sin(\phi)/\sin(\psi) - 1$ in terms of $\sin^{2n}(\psi)$.

n	$k \setminus l$	$e^{2\cdot 1}$	$e^{2\cdot 2}$	$e^{2\cdot 3}$	$e^{2\cdot 4}$	$e^{2\cdot 5}$	$e^{2\cdot 6}$	$e^{2\cdot 7}$	$e^{2\cdot 8}$
0	ρ^{-1}	−1/2	−3/16	−15/128	−175/2048	−2205/32768	−14553/262144	−99099/2097152	−2760615/67108864
	ρ^{-2}	×	3/16	5/32	35/256	63/512	231/2048	429/4096	6435/65536
	ρ^{-3}	×	×	−1/8	−5/32	−175/1024	−735/4096	−24255/131072	−99099/524288
	ρ^{-4}	×	×	×	85/1024	147/1024	1575/8192	957/4096	70785/262144
	ρ^{-5}	×	×	×	×	−1/16	−35/256	−441/2048	−4851/16384
	ρ^{-6}	×	×	×	×	×	791/16384	4227/32768	61545/262144
	ρ^{-7}	×	×	×	×	×	×	−5/128	−63/512
	ρ^{-8}	×	×	×	×	×	×	×	135489/4194304
1	ρ^{-1}	1/2	1/4	45/256	35/256	3675/32768	6237/65536	693693/8388608	306735/4194304
	ρ^{-2}	×	−1/2	−25/64	−21/64	−147/512	−33/128	−3861/16384	−3575/16384
	ρ^{-3}	×	×	1/4	5/16	175/512	735/2048	24255/65536	99099/262144
	ρ^{-4}	×	×	×	−11/64	−301/1024	−801/2048	−7755/16384	−17875/32768
	ρ^{-5}	×	×	×	×	1/8	35/128	441/1024	4851/8192
	ρ^{-6}	×	×	×	×	×	−199/2048	−33951/131072	−30855/65536
	ρ^{-7}	×	×	×	×	×	×	5/64	63/256
	ρ^{-8}	×	×	×	×	×	×	×	−8479/131072
2	ρ^{-1}	×	−1/16	−9/128	−35/512	−525/8192	−31185/524288	−231231/4194304	−429429/8388608
	ρ^{-2}	×	5/16	11/32	21/64	39/128	1155/4096	2145/8192	1001/4096
	ρ^{-3}	×	×	−3/8	−7/16	−115/256	−3675/8192	−116655/262144	−231231/524288
	ρ^{-4}	×	×	×	49/256	79/256	6543/16384	3905/8192	17875/32768
	ρ^{-5}	×	×	×	×	−1/8	−35/128	−441/1024	−4851/8192
	ρ^{-6}	×	×	×	×	×	3219/32768	17063/65536	7733/16384
	ρ^{-7}	×	×	×	×	×	×	−5/64	−63/256
	ρ^{-8}	×	×	×	×	×	×	×	33979/524288
3	ρ^{-1}	×	×	3/256	5/256	1575/65536	3465/131072	231231/8388608	117117/4194304
	ρ^{-2}	×	×	−7/64	−11/64	−207/1024	−55/256	−3575/16384	−3549/16384
	ρ^{-3}	×	×	1/4	7/16	535/1024	2275/4096	1155/2048	147147/262144
	ρ^{-4}	×	×	×	−21/64	−1017/2048	−2397/4096	−10505/16384	−22425/32768
	ρ^{-5}	×	×	×	×	5/32	75/256	1757/4096	4683/8192
	ρ^{-6}	×	×	×	×	×	−363/4096	−31669/131072	−29357/65536
	ρ^{-7}	×	×	×	×	×	×	5/64	63/256
	ρ^{-8}	×	×	×	×	×	×	×	−8437/131072
4	ρ^{-1}	×	×	×	−5/2048	−175/32768	−2079/262144	−21021/2097152	−195195/16777216
	ρ^{-2}	×	×	×	9/256	37/512	209/2048	507/4096	2275/16384
	ρ^{-3}	×	×	×	−5/32	−325/1024	−1785/4096	−67305/131072	−147147/262144
	ρ^{-4}	×	×	×	231/1024	561/1024	6369/8192	3755/4096	65325/65536
	ρ^{-5}	×	×	×	×	−5/16	−145/256	−1491/2048	−861/1024
	ρ^{-6}	×	×	×	×	×	2145/16384	8437/32768	26585/65536
	ρ^{-7}	×	×	×	×	×	×	−7/128	−49/256
	ρ^{-8}	×	×	×	×	×	×	×	64493/1048576
5	ρ^{-1}	×	×	×	×	35/65536	189/131072	21021/8388608	15015/4194304
	ρ^{-2}	×	×	×	×	−11/1024	−7/256	−741/16384	−1015/16384
	ρ^{-3}	×	×	×	×	75/1024	735/4096	9345/32768	98637/262144
	ρ^{-4}	×	×	×	×	−429/2048	−2145/4096	−13403/16384	−34395/32768
	ρ^{-5}	×	×	×	×	7/32	175/256	4557/4096	11613/8192
	ρ^{-6}	×	×	×	×	×	−1287/4096	−84799/131072	−56615/65536
	ρ^{-7}	×	×	×	×	×	×	7/64	49/256
	ρ^{-8}	×	×	×	×	×	×	×	−2431/131072
6	ρ^{-1}	×	×	×	×	×	−63/524288	−1617/4194304	−6435/8388608
	ρ^{-2}	×	×	×	×	×	13/4096	79/8192	75/4096
	ρ^{-3}	×	×	×	×	×	−245/8192	−22785/262144	−82929/524288
	ρ^{-4}	×	×	×	×	×	2145/16384	3055/8192	21525/32768
	ρ^{-5}	×	×	×	×	×	−35/128	−833/1024	−11613/8192
	ρ^{-6}	×	×	×	×	×	7293/32768	55913/65536	25415/16384
	ρ^{-7}	×	×	×	×	×	×	−21/64	−189/256
	ρ^{-8}	×	×	×	×	×	×	×	46189/524288
7	ρ^{-1}	×	×	×	×	×	×	231/8388608	429/4194304
	ρ^{-2}	×	×	×	×	×	×	−15/16384	−53/16384
	ρ^{-3}	×	×	×	×	×	×	735/65536	9933/262144
	ρ^{-4}	×	×	×	×	×	×	−1105/16384	−7225/32768
	ρ^{-5}	×	×	×	×	×	×	441/2048	5733/8192
	ρ^{-6}	×	×	×	×	×	×	−46189/131072	−79781/65536
	ρ^{-7}	×	×	×	×	×	×	15/64	273/256
	ρ^{-8}	×	×	×	×	×	×	×	−46189/131072
8	ρ^{-1}	×	×	×	×	×	×	×	−429/67108864
	ρ^{-2}	×	×	×	×	×	×	×	17/65536
	ρ^{-3}	×	×	×	×	×	×	×	−2079/524288
	ρ^{-4}	×	×	×	×	×	×	×	8075/262144
	ρ^{-5}	×	×	×	×	×	×	×	−2205/16384
	ρ^{-6}	×	×	×	×	×	×	×	88179/262144
	ρ^{-7}	×	×	×	×	×	×	×	−231/512
	ρ^{-8}	×	×	×	×	×	×	×	1062347/4194304

TABLE 7. Coefficients $c^{\cos}_{n,k,l}$ for $\cos(\phi)/\cos(\psi) - 1$ in terms of $\cos(2n\psi)$.

n	$k \setminus l$	$e^{2\cdot 1}$	$e^{2\cdot 2}$	$e^{2\cdot 3}$	$e^{2\cdot 4}$	$e^{2\cdot 5}$	$e^{2\cdot 6}$	$e^{2\cdot 7}$	$e^{2\cdot 8}$
1	ρ^{-1}	-1	×	×	×	×	×	×	×
	ρ^{-2}	×	$-3/2$	×	×	×	×	×	×
	ρ^{-3}	×	×	-2	×	×	×	×	×
	ρ^{-4}	×	×	×	$-5/2$	×	×	×	×
	ρ^{-5}	×	×	×	×	-3	×	×	×
	ρ^{-6}	×	×	×	×	×	$-7/2$	×	×
	ρ^{-7}	×	×	×	×	×	×	-4	×
	ρ^{-8}	×	×	×	×	×	×	×	$-9/2$
2	ρ^{-1}	×	$-1/2$	×	×	×	×	×	×
	ρ^{-2}	×	$5/2$	$-5/2$	×	×	×	×	×
	ρ^{-3}	×	×	9	$-15/2$	×	×	×	×
	ρ^{-4}	×	×	×	$175/8$	$-35/2$	×	×	×
	ρ^{-5}	×	×	×	×	44	-35	×	×
	ρ^{-6}	×	×	×	×	×	$315/4$	-63	×
	ρ^{-7}	×	×	×	×	×	×	130	-105
	ρ^{-8}	×	×	×	×	×	×	×	$1617/8$
3	ρ^{-1}	×	×	$-3/8$	×	×	×	×	×
	ρ^{-2}	×	×	$7/2$	$-7/2$	×	×	×	×
	ρ^{-3}	×	×	-8	26	$-35/2$	×	×	×
	ρ^{-4}	×	×	×	$-189/4$	$441/4$	-63	×	×
	ρ^{-5}	×	×	×	×	-170	350	$-735/4$	×
	ρ^{-6}	×	×	×	×	×	$-7623/16$	924	-462
	ρ^{-7}	×	×	×	×	×	×	-1140	2142
	ρ^{-8}	×	×	×	×	×	×	×	$-39039/16$
4	ρ^{-1}	×	×	×	$-5/16$	×	×	×	×
	ρ^{-2}	×	×	×	$9/2$	$-9/2$	×	×	×
	ρ^{-3}	×	×	×	-20	$425/8$	$-525/16$	×	×
	ρ^{-4}	×	×	×	$231/8$	-198	$2673/8$	-165	×
	ρ^{-5}	×	×	×	×	240	$-2175/2$	$5985/4$	$-10395/16$
	ρ^{-6}	×	×	×	×	×	$18447/16$	$-69927/16$	$10725/2$
	ρ^{-7}	×	×	×	×	×	×	4165	$-28665/2$
	ρ^{-8}	×	×	×	×	×	×	×	$1602315/128$
5	ρ^{-1}	×	×	×	×	$-35/128$	×	×	×
	ρ^{-2}	×	×	×	×	$11/2$	$-11/2$	×	×
	ρ^{-3}	×	×	×	×	$-75/2$	$735/8$	$-3465/64$	×
	ρ^{-4}	×	×	×	×	$429/4$	$-2145/4$	$1573/2$	$-715/2$
	ρ^{-5}	×	×	×	×	-112	1330	$-32487/8$	$37191/8$
	ρ^{-6}	×	×	×	×	×	$-19305/16$	$143715/16$	$-173745/8$
	ρ^{-7}	×	×	×	×	×	×	-7280	43708
	ρ^{-8}	×	×	×	×	×	×	×	$-1033175/32$
6	ρ^{-1}	×	×	×	×	×	$-63/256$	×	×
	ρ^{-2}	×	×	×	×	×	$13/2$	$-13/2$	×
	ρ^{-3}	×	×	×	×	×	$-245/4$	$18375/128$	$-21021/256$
	ρ^{-4}	×	×	×	×	×	$2145/8$	-1170	$12675/8$
	ρ^{-5}	×	×	×	×	×	-560	4508	$-46011/4$
	ρ^{-6}	×	×	×	×	×	$7293/16$	$-133705/16$	$319345/8$
	ρ^{-7}	×	×	×	×	×	×	6048	-67032
	ρ^{-8}	×	×	×	×	×	×	×	$2817529/64$
7	ρ^{-1}	×	×	×	×	×	×	$-231/1024$	×
	ρ^{-2}	×	×	×	×	×	×	$15/2$	$-15/2$
	ρ^{-3}	×	×	×	×	×	×	$-735/8$	$6699/32$
	ρ^{-4}	×	×	×	×	×	×	$1105/2$	$-8925/4$
	ρ^{-5}	×	×	×	×	×	×	-1764	11907
	ρ^{-6}	×	×	×	×	×	×	$46189/16$	$-272935/8$
	ρ^{-7}	×	×	×	×	×	×	-1920	50400
	ρ^{-8}	×	×	×	×	×	×	×	$-969969/32$
8	ρ^{-1}	×	×	×	×	×	×	×	$-429/2048$
	ρ^{-2}	×	×	×	×	×	×	×	$17/2$
	ρ^{-3}	×	×	×	×	×	×	×	$-2079/16$
	ρ^{-4}	×	×	×	×	×	×	×	$8075/8$
	ρ^{-5}	×	×	×	×	×	×	×	-4410
	ρ^{-6}	×	×	×	×	×	×	×	$88179/8$
	ρ^{-7}	×	×	×	×	×	×	×	-14784
	ρ^{-8}	×	×	×	×	×	×	×	$1062347/128$

TABLE 8. Coefficients $d^{\cos}_{n,k,l}$ for $\cos(\phi)/\cos(\psi) - 1$ in terms of $\sin^{2n}(\psi)$.