Linear Algebra Fundamentals

Vectors, Matrices, Factorization, and Tensor Operations.

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AMMI BootCamp 2025

African Masters of Machine Intelligence

August 4, 2025

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The Mathematical Language of AI

- **Foundation:** Linear algebra is the math behind all machine learning
- **Data Structure:** Data becomes matrices and vectors we can work with
- **Speed:** Matrix operations run fast on modern computers (GPUs)
- **Pattern Finding:** Tools like PCA help find hidden patterns in data

Real AI Applications

- **Neural Networks:** Each layer multiplies matrices to learn
- **Training Models:** Uses vectors to improve AI performance
- **Decision Making:** Helps AI draw boundaries between different categories
- **Everyday Use:** Powers Netflix recommendations, self-driving cars, ChatGPT

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Vectors - Definition and Notation

- A *vector* is an ordered list of numbers.
- Written in **row representation** as:

$$[x_1, x_2, \cdots, x_n, \ldots] \text{ or } (x_1, x_2, \cdots, x_n, \ldots) \tag{2.1}$$

- Or in **column representation** as:

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ \vdots \end{bmatrix} \text{ or } \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ \vdots \end{pmatrix} \tag{2.2}$$

Vector Components and Dimensions

- Quantities $x_1, x_2, \ldots, x_n$ are called **components** (elements, coefficients, entries).
- The number of elements is the **dimension** (or length) of the vector.
- Examples of finite vectors:

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \text{ and } \begin{bmatrix} -2 \\ 1.1 \\ 0 \\ 0 \end{bmatrix} \tag{2.3}$$

are vectors of size 3 and 4, respectively.

Vector Notation Conventions

- **Scalars:** Numbers are vectors of dimension 0.
- **Notation:** Lowercase letters denote vectors: a, x, p, r or $\vec{x}$.
- **Indexing:** The i-th element of vector x is denoted x_i.
- **Equality:** Vectors x and y are equal $(x = y)$ if and only if $x_i = y_i$ for all i.
- **Dimension notation:**
  - n-vector: vector of dimension n
  - $(n, 1)$-column vector or $(1, n)$-row vector

Block Vectors and Concatenation

- **Stacked vector:** For vectors b, c, d of sizes m, n, p:

$$a = \begin{bmatrix} b \\ c \\ d \end{bmatrix} \quad (2.4)$$

- Also called **block vector** with block entries b, c, d.
- Result: a is an $(m + n + p)$-vector.
- **Expanded form:**

$$a = (b_1, b_2, \dots, b_m, c_1, c_2, \dots, c_n, d_1, d_2, \dots, d_p)$$

Special Vectors

- **Zero vector:** All entries are zero, denoted $\mathbf{0}$.
- **Ones vector:** All entries are 1, denoted $\mathbf{1}$.
- **Unit vectors:** One entry is 1, all others are 0, denoted e_i.
- **Example** - Unit vectors in dimension 3:

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{2.5}$$

- **Sparse vectors:** Many entries are zero. We denote $\text{nnz}(x)$ as the number of nonzero entries.

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Geometric Representation

- A 2D vector (x_1, x_2) represents a location or displacement in a plane:

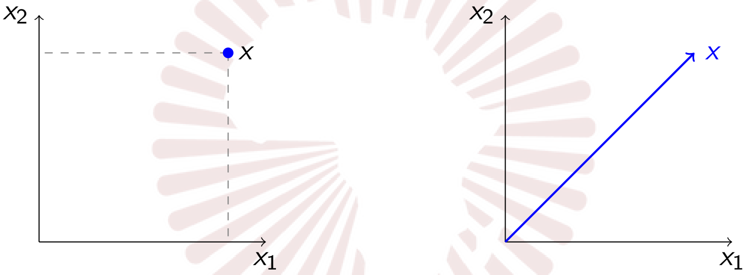

Figure: Point vs. Vector representation in 2D

Real-World Vector Examples

- **Economics:** Quantities of n different commodities
- **Education:** Student grades per subject/year
- **Finance:** Cash flow where x_i is payment in period i
- **Audio:** Acoustic pressure at sample time i
- **NLP:** Word count where x_i is frequency of word i in a document

Text Processing Example

"I have a dream that one day every valley shall be exalted, every hill and mountain shall be made low, the rough places will be made plain, and the crooked places will be made straight; and the glory of the Lord shall be revealed, and all flesh shall see it together."

— Martin Luther King Jr., **"I have a dream"** (excerpt)

Word Count Vector Example

- Dictionary and corresponding word count vector:

$$\begin{array}{r} \text{dream} \\ \text{every} \\ \text{be} \\ \text{shall} \\ \text{made} \end{array} \quad \rightarrow \quad \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 3 \end{bmatrix}$$

- In practice, dictionaries contain thousands of words.

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Vector Operations

- We can add and subtract two vectors a and b of the same size, with the sum denoted $a \pm b$
- To get the sum, we add/subtract corresponding entries:

$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \pm \begin{bmatrix} 9 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \pm 9 \\ 1 \pm 2 \\ 1 \pm (-1) \end{bmatrix}$$

- Addition of vectors is commutative, associative, and has the zero vector as the neutral element

Geometric Interpretation: Addition

- Geometrically, if a and b are vectors, their sum $a + b$ is represented as:

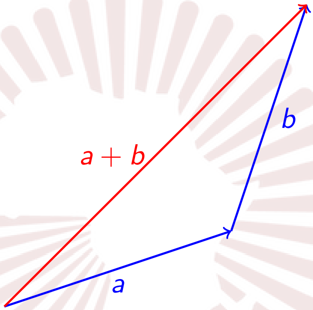


Figure: Addition of two vectors a and b

- The opposite of a vector a, denoted $-a$, is a vector in the opposite direction of a

Geometric Interpretation: Subtraction

- The representation of $a - b$:

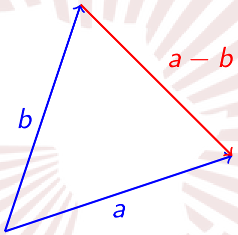
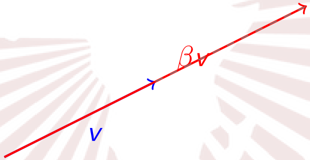


Figure: Subtraction of two vectors a and b

Scalar Multiplication

- We can multiply a vector a by a scalar β, denoted βa
- $\beta a = (\beta a_1, \beta a_2, \ldots, \beta a_n)$
- It is a scaling in the magnitude of the vector a:



- Scalar multiplication is associative and distributive

Element-wise Multiplication

- For two n-vectors a and b, the element-wise multiplication $a \circ b$ produces a vector where each component is the product of corresponding elements
- Formally:

$$a \circ b = (a_1 b_1, a_2 b_2, \ldots, a_n b_n)$$

- The symbol $\circ$ or $\star$ denotes *element-wise* or *Hadamard* multiplication

- Example:

$$(0,-1,2)\circ(1,-1,\frac{1}{2})=(0\cdot 1,(-1)\cdot(-1),2\cdot\frac{1}{2})=(0,1,1)$$

- Element-wise multiplication is not commutative, i.e., $a\circ b\neq b\circ a$ in general
- However, it is associative, i.e., $(a\circ b)\circ c=a\circ(b\circ c)$
- It is distributive over vector addition, i.e., $a\circ(b+c)=a\circ b+a\circ c$
- **Note:** This operation is intensively used in NumPy and machine learning for vectorization.

Transpose and Linear Combinations

- **Transpose:** For an $(n, 1)$-vector x, its transpose x^T is a $(1, n)$-vector
- Properties of transpose:
  - $(x^T)^T = x$ (Transpose of transpose is the original vector)
  - $(\beta x)^T = \beta x^T$ for a scalar β
  - $(x + y)^T = x^T + y^T$ for vectors x and y
- **Linear combination:** A linear combination of vectors $x_1, x_2, \ldots, x_k$ with scalars $\alpha_1, \alpha_2, \ldots, \alpha_k$ is:

$$y = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_k x_k$$

- This is a vector formed by scaling and adding the vectors x_i.

- Example: For $x_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $x_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, and $\alpha_1 = 2$, $\alpha_2 = -1$:

$$y = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} - 1\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

- The result y is a vector in the same dimension as x_1 and x_2.
- Linear combinations are fundamental in linear algebra, forming the basis for **vector spaces**.

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Vector Spaces

- A **vector space** is a set of vectors of the same dimension, equipped with vector addition and scalar multiplication operations, and satisfies specific axioms.
- **Fundamental axioms** of a vector space V (over the field $\mathbb{R}$) are:
  1. **Closure under addition**: $\mathbf{u} + \mathbf{v} \in V$ for all $\mathbf{u}, \mathbf{v} \in V$
  2. **Closure under scalar multiplication**: $c\mathbf{v} \in V$ for all $c \in \mathbb{R}$ and $\mathbf{v} \in V$
  3. **Associativity**: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
  4. **Commutativity**: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
  5. **Zero vector**: There exists $\mathbf{0} \in V$ such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$
  6. **Additive inverse**: For each $\mathbf{v} \in V$, there exists $-\mathbf{v} \in V$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$
  7. **Distributivity**: $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ and $(c + d)\mathbf{v} = c\mathbf{v} + d\mathbf{v}$
  8. **Scalar multiplication identity**: $1 \cdot \mathbf{v} = \mathbf{v}$

- To show that a set of vectors forms a vector space, we need to verify these axioms.
- A vector space can be defined over any field, such as $\mathbb{R}$ or $\mathbb{C}$, or even finite fields, such as $\mathbb{F}_p$.
- **Note:** We consider vector spaces over the field of real numbers $\mathbb{R}$, meaning that all the scalars are real numbers.

- Examples of vector spaces:
  - $\mathbb{R}^n$: The set of all n- vectors with real components forms a vector space.
  - The set of all polynomials of degree at most n forms a vector space, i.e. $P_n = \{p(x) = a_0 + a_1 x + \cdots + a_n x^n \mid a_i \in \mathbb{R}\}$ is a vector space.
  - The set of all continuous functions on a closed interval, i.e. $C[a, b]$, forms a vector space.
- We can easily show the examples above saisfy the axioms of a vector space.

- **Dimension:** The **dimension** of a vector space equals the number of components in each vector
- Example: $\mathbb{R}$ represents the real numbers and forms a 1-dimensional vector space
- Example: $\mathbb{R}^2$ consists of all ordered pairs:

$$\mathbb{R}^2 = \left\{ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mid x_1, x_2 \in \mathbb{R} \right\}$$

forming a 2-dimensional vector space that can be visualized as the Cartesian plane

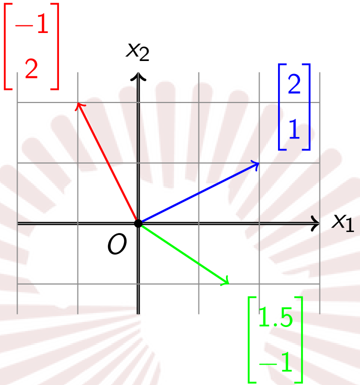

Figure: Visualization of $\mathbb{R}^2$ as a 2-dimensional vector space

- In general, $\mathbb{R}^n$ constitutes an n-dimensional vector space

Subspaces. Spannning set

- **Subspace:** A **subspace** is a subset of a vector space that is also a vector space itself, with respect to the same vector addition and scalar multiplication.
- A subspace must satisfy the same axioms as the parent vector space.
- Equivalently, a subspace is closed under vector addition and scalar multiplication, i.e. $\emptyset \neq S \subseteq V$ a subspace of V must satisfy

$$(i)\;\; \mathbf{x}, \mathbf{y} \in S \implies \mathbf{x} + \mathbf{y} \in S, \quad (ii)\;\; \lambda \in \mathbb{R}, \mathbf{x} \in S \implies \lambda \mathbf{x} \in S.$$

- Example: The set of all vectors in $\mathbb{R}^3$ that lie on the xy-plane is a subspace of $\mathbb{R}^3$.
- Example: The set of all vectors in $\mathbb{R}^3$ that lie on the line $x = y = z$ is a subspace of $\mathbb{R}^3$.
- Example: The set of all polynomials of degree at most n is a subspace of the vector space of all polynomials.
- Example: The set of all continuous functions on a closed interval $[a, b]$ is a subspace of the vector space of all functions defined on $[a, b]$.

- **Spanning Set:** A set of vectors $\{v_1, v_2, \ldots, v_k\}$ **spans** a vector space V if every vector in V can be expressed as a linear combination of the vectors in the set.
- Formally, $V = \text{span}\{v_1, v_2, \ldots, v_k\}$ means that any vector $v \in V$ can be written as:

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k$$

  where α_i are scalars.
- The set $\{v_1, v_2, \ldots, v_k\}$ is called a **spanning set** for V.

- Example: The set $\{e_1, e_2, e_3\}$ spans $\mathbb{R}^3$, where:

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- Indeed, any vector in $\mathbb{R}^3$ can be expressed as a linear combination of these vectors:

$$v = x_1 e_1 + x_2 e_2 + x_3 e_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Other examples of spanning sets include **lines** and **planes** in vector spaces:

- **Line:** Given a vector u, the line spanned by u is the set of all scalar multiples of u, i.e.,

$$l_u = \{v \mid v = \lambda u \text{ for some } \lambda \in \mathbb{R}\} = \text{span}\{u\}$$

- **Plane:** Given two vectors u and v, the plane spanned by u and v is the set of all linear combinations of u and v, i.e.,

$$\pi_{uv} = \{w \mid w = \lambda u + \beta v \text{ for some } \lambda, \beta \in \mathbb{R}\} = \text{span}\{u, v\}$$

- Example:

$$\mathbb{R}^2 = \{\lambda e_1 + \beta e_2 \mid \lambda, \beta \in \mathbb{R}\}$$

Dependence, Independence,and Basis

- **Dependence:** A set of vectors $\{a_1, \ldots, a_k\}$ (with $k \geq 1$) is **linearly dependent** if there exist coefficients $\beta_1, \ldots, \beta_k$, not all zero, such that

$$\beta_1 a_1 + \beta_2 a_2 + \cdots + \beta_k a_k = 0 \tag{2.6}$$

- Equivalently, at least one vector a_i can be expressed as a linear combination of the others.
- We say the vectors $a_1, \ldots, a_k$ are **linearly dependent**.

- A single vector $\{a_1\}$ is linearly dependent if and only if $a_1 = 0$.
- Two vectors $\{a_1, a_2\}$ are linearly dependent if and only if one is a scalar multiple of the other, i,.e., $a_1 = \lambda a_2$ for some scalar λ.
- For more than two vectors, it is not easy to state their dependence using the definition. The notion **determinant** will make it "easy ".

- Example: Consider the vectors

$$a_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \; a_2 = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \; a_3 = \begin{bmatrix} -2 \\ 4 \\ 6 \end{bmatrix} \tag{2.7}$$

These vectors are linearly dependent because we can find coefficients $\beta_1 = 1$, $\beta_2 = 2$, $\beta_3 = -3$ (not all zero) such that $\beta_1 a_1 + \beta_2 a_2 + \beta_3 a_3 = 0$.

- **Independence:** A set of $k \geq 1$ vectors $\{a_1, \ldots, a_k\}$ is **linearly independent** if the equation

$$\beta_1 a_1 + \beta_2 a_2 + \cdots + \beta_k a_k = 0 \tag{2.8}$$

has only the trivial solution $\beta_1 = \beta_2 = \cdots = \beta_k = 0$.

- Equivalently, none of the vectors can be written as a linear combination of the others.
- We say the vectors $a_1, \ldots, a_k$ are **linearly independent**.
- The standard unit vectors $e_1, \ldots, e_n$ form a linearly independent set.

- Example: Consider the vectors

$$a_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; a_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \; a_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{2.9}$$

These vectors are linearly independent because the only solution to the equation $\beta_1 a_1 + \beta_2 a_2 + \beta_3 a_3 = 0$ is $\beta_1 = \beta_2 = \beta_3 = 0$.

- In general, a set of n vectors in $\mathbb{R}^n$ is linearly independent if and only if the determinant of the matrix formed by these vectors is non-zero.
- We will see later how to make use of the assertion above.
- The dimension of a vector space is the number of vectors in any basis for that space.

- **Basis:** A basis for a vector space V is a set of vectors $\{b_1, b_2, \ldots, b_k\}$ such that:
  1. The vectors are linearly independent.
  2. The vectors span the vector space V, that is any vector in V can be expressed as a linear combination of the basis vectors.
- **Dimension:** The number of vectors in a basis is the **dimension** of the vector space.
- Example: In $\mathbb{R}^3$, the set $\{e_1, e_2, e_3\}$ is a basis where:

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{2.10}$$

- Any vector in $\mathbb{R}^3$ can be uniquely expressed as a linear combination of these basis vectors:

- The dimension of $\mathbb{R}^3$ is 3, as there are three basis vectors.
- The concept of basis extends to any vector space, not just $\mathbb{R}^n$.
- For example, the set of all polynomials of degree at most n has a basis consisting of the monomials $\{1, x, x^2, \ldots, x^n\}$.
- The dimension of this polynomial space is $n + 1$.

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Inner product

- The dot product (also known as inner product) between two *n*-vectors a and b is defined and denoted as the scalar:

$$a^T b = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \tag{2.12}$$

- Alternative notations include: (a, b) and $a \cdot b$. And we write it as: $(a, b) = a^T b$ or $a \cdot b = a^T b$.
- Illustration:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}^T \begin{bmatrix} 1 \\ \frac{1}{2} \\ -2 \end{bmatrix} = (1 \times 1) + (2 \times \frac{1}{2}) + (3 \times -2) = -4 \tag{2.13}$$

Key Properties

- Fundamental properties:
  - Commutativity: $a^T b = b^T a$
  - Scalar multiplication: $(\lambda a)^T b = \lambda(a^T b)$
  - Distributivity: $(a + b)^T c = a^T c + b^T c$
  - Zero vector: $0^T a = a^T 0 = 0$ for any vector a
- Practical applications:
  - $e_i^T a = a_i$ (extracts the i-th component)
  - $\mathbf{1}^T a = a_1 + a_2 + \cdots + a_n$ (computes sum of entries)
  - $a^T a = a_1^2 + \cdots + a_n^2$ (sum of squared entries)
- **Note :** An Euclidean space is a vector space with inner product.

Vector Norms

- The Euclidean norm of a vector x is defined as:

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \sqrt{x^T x} \tag{2.14}$$

- For $n = 1$, this simplifies to the absolute value.
- **Example:** For $x = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, the norm is:

$$\|x\| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

- The norm is a measure of the distance from the origin to the point represented by the vector in a vector space.

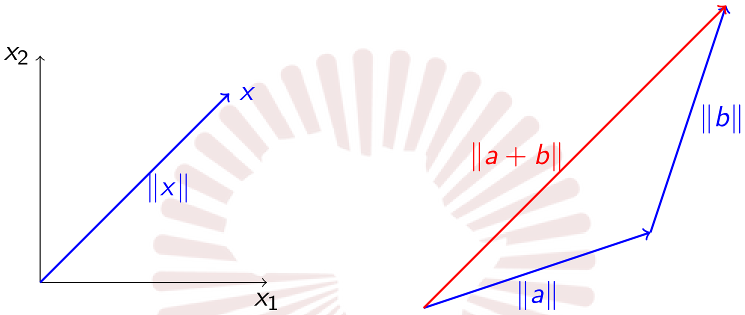


Figure: (Left) Geometric representation of 2-vector norms. (Right) Triangle inequality visualization

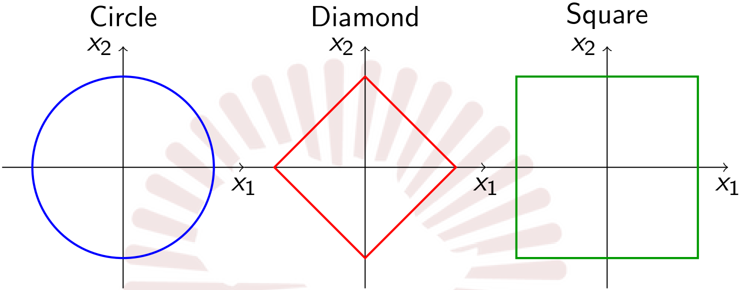
- The norm is also known as the **length** or **magnitude** of the vector.
- **Note :** There are other types of norms, such as the L_1 and L_∞ norms, which are defined as:

$$L_1 \text{ norm : } \|x\|_1 = |x_1| + |x_2| + \cdots + |x_n| \tag{2.15}$$

$$L_\infty \text{ norm: } \|x\|_\infty = \max(|x_1|, |x_2|, \ldots, |x_n|) \tag{2.16}$$

- The Euclidean(also known as L_2) norm is the most commonly used norm.

- Visual comparison of different norms for the same vector:



L_2 norm: $\|x\|_2 = 1$ L_1 norm: $\|x\|_1 = 1$$L_\infty$ norm: $\|x\|_\infty = 1$

Figure: Unit balls for different norms in 2D space

- Each shape represents all vectors with norm equal to 1 under the respective norm definition.

Norm Properties

All norms for n-vectors x and y and scalar λ must satisfy these axioms:

- **Scaling property**: $\|\lambda x\| = |\lambda|\|x\|$
- **Trinagual inequality**: $\|x + y\| \leq \|x\| + \|y\|$
- **Non-negativity**: $\|x\| \geq 0$
- **Zero condition**: $\|x\| = 0$ if and only if $x = 0$

Note: That is, for a function to be a norm, it must satisfy these properties.

Exercise: Verify these properties hold for the L_1, L_2, and L_∞ norms.

Exercice: Does the function $\|x\| = \sqrt{x_1^2 + x_2^2 + 1}$ satisfy the norm properties? Why or why not?

Norm of Block Vectors

- Consider block vector formed by concatenating vectors a, b, and c:
- The squared norm satisfies:

$$\|(a, b, c)\|^2 = a^T a + b^T b + c^T c = \|a\|^2 + \|b\|^2 + \|c\|^2 \tag{2.17}$$

- Therefore, the norm of the block vector is:

$$\|(a, b, c)\| = \sqrt{\|a\|^2 + \|b\|^2 + \|c\|^2} \tag{2.18}$$

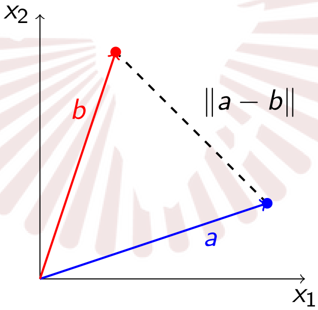
- This property extends to any number of block components.

Euclidean Distance Between Vectors

- The Euclidean distance between two n-vectors a and b is defined as:

$$\text{dist}(a, b) = \|a - b\| = \sqrt{(a_1 - b_1)^2 + \cdots + (a_n - b_n)^2} \tag{2.19}$$

- Geometric interpretation in 2D space:



- The distance is zero if and only if $a = b$.
- The distance satisfies the **triangle inequality**:

$$\mathrm{dist}(a, b) + \mathrm{dist}(b, c) \geq \mathrm{dist}(a, c) \tag{2.20}$$

- The distance can also be generalized to other norms, such as L_1 and L_∞:

$$\mathrm{dist}_1(a, b) = \|a - b\|_1 = |a_1 - b_1| + |a_2 - b_2| + \cdots + |a_n - b_n| \tag{2.21}$$

$$\mathrm{dist}_\infty(a, b) = \|a - b\|_\infty = \max(|a_1 - b_1|, |a_2 - b_2|, \ldots, |a_n - b_n|) \tag{2.22}$$

- Example: For vectors $a = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$:

$$\text{dist}(a, b) = \sqrt{(3-1)^2 + (4-2)^2} = \sqrt{2^2 + 2^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$$

- The distance can be interpreted as the length of the line segment connecting points a and b in 2D space.

Statistical Properties Using Inner Products and Norms

We can compute statstic measures of vector x in terms of its norm:

- For an n-vector x, the mean can be expressed using inner products:

$$\overline{x} = \frac{1}{n}\mathbf{1}^T x \tag{2.23}$$

- The centered (de-meaned) vector is obtained by:

$$\tilde{x} = x - \overline{x}\mathbf{1}$$

- The standard deviation can be computed using the norm:

$$\text{std}(x) = \frac{\|\tilde{x}\|}{\sqrt{n}} = \frac{\|x - \overline{x}\mathbf{1}\|}{\sqrt{n}}$$

- The mean square value of vector x is defined as:

$$\mathrm{MSV}(x) = \frac{x_1^2 + \cdots + x_n^2}{n} = \frac{\|x\|^2}{n} \tag{2.24}$$

- The root mean square (RMS) value of vector x is:

$$\mathrm{RMS}(x) = \sqrt{\frac{x_1^2 + \cdots + x_n^2}{n}} = \frac{\|x\|}{\sqrt{n}} \tag{2.25}$$

- The variance of vector x is defined as:

$$\mathrm{Var}(x) = \frac{1}{n}\|x - \overline{x}\mathbf{1}\|^2 = \frac{1}{n}(x - \overline{x}\mathbf{1})^T(x - \overline{x}\mathbf{1}) \tag{2.26}$$

- The covariance matrix of vector x is given by:

$$C(x) = \frac{1}{n}(x - \overline{x}\mathbf{1})(x - \overline{x}\mathbf{1})^T \tag{2.27}$$

- The covariance matrix captures the variance and covariance of the components of vector x.

Some Norms Inequalities

Cauchy-Schwarz Inequality

For any n-vectors a and b, the following inequality holds:

$$|a^T b| \leq \|a\| \|b\| \tag{2.28}$$

This fundamental result is known as the **Cauchy-Schwarz inequality**.

Proof:

- Consider the vector $c = a - \lambda b$ for some scalar λ.
- The norm of c must be non-negative:

$$\|c\|^2 = (a - \lambda b)^T (a - \lambda b) \geq 0$$

- Expanding this gives:

$$a^T a - 2\lambda a^T b + \lambda^2 b^T b \geq 0$$

- This is a quadratic in λ, which must have non-negative discriminant:

$$(a^T b)^2 - a^T a b^T b \geq 0$$

- Rearranging yields the Cauchy-Schwarz inequality:

$$|a^T b|^2 \leq (a^T a)(b^T b)$$

- Taking square roots gives:

$$|a^T b| \leq \|a\| \|b\|$$

Exercise: Use the Cauchy-Schwarz inequality to demonstrate that the triangle inequality holds for vector norms, i.e., show that:

$$\|a + b\| \leq \|a\| + \|b\|$$

Minkowski Inequality

For any n-vectors a and b, and for $p \geq 1$, the following holds:

$$\|a + b\|_p \leq \|a\|_p + \|b\|_p \tag{2.29}$$

This is known as the **Minkowski inequality**.

Proof:

- Consider the p-norm defined as:

$$\|x\|_p = \left(|x_1|^p + |x_2|^p + \cdots + |x_n|^p\right)^{1/p}$$

- The Minkowski inequality states that the p-norm satisfies the triangle inequality.
- This can be shown using the Cauchy-Schwarz inequality for p-norms.

- An other way to show the Minkowsky's inequality is to use the convexity of the function $f(x) = |x|^p$ for $p \geq 1$.
- The function $f(x)$ is convex, because its second derivative is non-negative:

$$f''(x) = p(p-1)|x|^{p-2}$$

- Convex functions satisfy the Jensen's inequality:

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

for all $\lambda \in [0, 1]$.

- Applying this to the p-norm gives:

$$\|a+b\|_p^p = (|a_1+b_1|^p + |a_2+b_2|^p + \cdots + |a_n+b_n|^p) \leq (|a_1|^p + |b_1|^p$$

- Taking the p-th root yields the Minkowski inequality:

$$\|a+b\|_p \leq \|a\|_p + \|b\|_p$$

Holder's Inequality

For any n-vectors a and b and for any $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, the following holds:

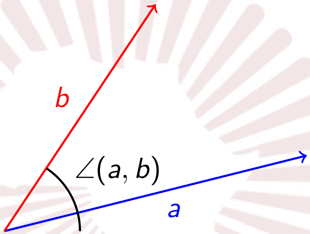
$$|a^T b| \leq \|a\|_p \|b\|_q \tag{2.30}$$

This is known as **Hölder's inequality**.

Proof: Exercice

Angle Between Vectors

- Two vectors sharing a common origin define an angle between them.



- The angle between two vectors a and b is given by:

$$\angle(a, b) = \arccos\left(\frac{\langle a, b\rangle}{\|a\| \times \|b\|}\right) \tag{2.31}$$

- The existence of the angle is guaranteed by the Cauchy-Schwarz inequality:

$$|\langle a, b\rangle| \leq \|a\|\|b\| \tag{2.32}$$

- For non-zero vectors a and b, we can normalize the inner product:

$$-1 \leq \frac{\langle a, b\rangle}{\|a\|\|b\|} \leq 1$$

- Since arccos is defined on the interval $[-1, 1]$ and maps bijectively to $[0, \pi]$, the angle $\angle(a, b)$ is well-defined.
- For more details, refer to the blog post *continuous inverse theorem*.

Properties of Angles

- The angle is always between 0 and π radians (or 0 and 180°).
- If the angle is 0, the vectors are in the same direction.
- If the angle is π, the vectors are in opposite directions.
- If the angle is $\frac{\pi}{2}$, the vectors are orthogonal (perpendicular).
- The angle can be computed using the inner product:

- From equation 2.31, we can express the dot product of vectors a and b as:

$$a^T b = \|a\| \|b\| \cos(\angle(a, b)) \tag{2.33}$$

- This gives us insight into the sign of the dot product:
  - $a^T b > 0$ when $\angle(a, b) < \frac{\pi}{2}$ (acute angle)
  - $a^T b = 0$ when $\angle(a, b) = \frac{\pi}{2}$ (orthogonal vectors)
  - $a^T b < 0$ when $\angle(a, b) > \frac{\pi}{2}$ (obtuse angle)

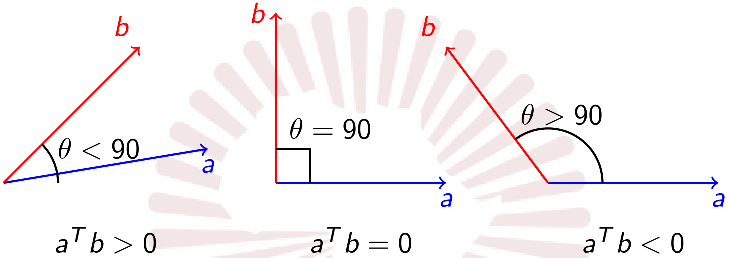


Figure: Relationship between dot product sign and angle between vectors

Example

- Consider vectors $a = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $b = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.
- Compute the angle between them:

$$\begin{aligned} a^T b &= 1 \cdot 3 + 2 \cdot 4 = 3 + 8 = 11 \\ \|a\| &= \sqrt{1^2 + 2^2} = \sqrt{5} \\ \|b\| &= \sqrt{3^2 + 4^2} = \sqrt{25} = 5 \end{aligned}$$

- The angle is:

$$\angle(a, b) = \arccos\left(\frac{11}{\sqrt{5} \cdot 5}\right) = \arccos\left(\frac{11}{5\sqrt{5}}\right)$$

Example: Consider vectors $a = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$ and $b = \begin{bmatrix} 4 \\ -3 \\ 5 \end{bmatrix}$.

- Compute the angle between them:

$$
\begin{aligned}
a^T b &= 3 \cdot 4 + 4 \cdot (-3) + 0 \cdot 5 = 12 - 12 + 0 = 0 \\
\|a\| &= \sqrt{3^2 + 4^2 + 0^2} = \sqrt{9 + 16} = 5 \\
\|b\| &= \sqrt{4^2 + (-3)^2 + 5^2} = \sqrt{16 + 9 + 25} = \sqrt{50} = 5\sqrt{2}
\end{aligned}
$$

- The angle is:

$$
\angle(a, b) = \arccos\left(\frac{0}{5 \cdot 5\sqrt{2}}\right) = \arccos(0) = \frac{\pi}{2}
$$

- **Exercise** (Pythagorean theorem): Prove that for any n-vectors a and b:

$$\|a+b\|^2 = \|a\|^2 + \|b\|^2 \quad \text{if and only if} \quad \angle(a,b) = \frac{\pi}{2} \tag{2.34}$$

- **Exercise**: Determine the angles between the standard unit vectors e_1, e_2, and e_3 in $\mathbb{R}^3$.

Orthogonal Vectors and Orthogonality

- **Orthonanility:** Two n-vectors a and b are orthogonal, denoted $a \perp b$, if their inner product is zero:

$$a^T b = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = 0 \tag{2.35}$$

- **Example:** The standard unit vectors e_1 and e_2 are orthogonal. For $n = 3$:

$$e_1^T e_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = (1)(0) + (0)(1) + (0)(0) = 0$$

- **Mutual orthogonaity:** A set of vectors $a_1, \dots, a_k$ are called **mutually orthogonal** if $a_i \perp a_j$ for all $i \neq j$.
- The standard unit vectors $e_1, \dots, e_n$ form a mutually orthogonal set, satisfying:

$$e_i^T e_j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{2.36}$$

- The quantity δ_{ij} is called the **Kronecker delta**.

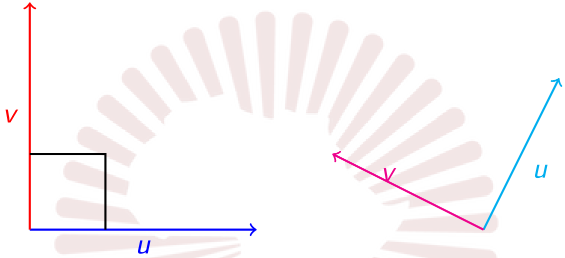


Figure: Examples of orthogonal vector pairs in $\mathbb{R}^2$

- **Orthonormality:** A set of vectors $a_1, \ldots, a_k$ are **orthonormal** if they are mutually orthogonal and each vector has unit norm:

$$\|a_i\| = 1 \quad \text{for all } i \tag{2.37}$$

- The standard unit vectors $e_1, \ldots, e_n$ form an orthonormal set:

$$\|e_i\| = 1 \quad \text{and} \quad e_i^T e_j = \delta_{ij} \quad \text{for all } i, j \tag{2.38}$$

Exercice: Show that orthogonal vectors are linearly independent.

Parseval's Identity

For any n-vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and any orthonormal basis $\{u_1, u_2, \ldots, u_n\}$, where $\|u_i\| = 1$ and $u_i^T u_j = \delta_{ij}$, the squared Euclidean norm satisfies:

$$\|x\|_2^2 = \langle x, x \rangle = \sum_{i=1}^{n} |(x, u_i)|^2 = \sum_{i=1}^{n} (x^T u_i)^2 \tag{2.39}$$

where $(\cdot, \cdot)$ denotes the standard inner product on $\mathbb{R}^n$.

Note: This shows that the squared norm of a vector is preserved under any orthonormal basis representation.

Orthonormal Expansion

- If $\{a_1, a_2, \dots, a_n\}$ forms an orthonormal basis for $\mathbb{R}^n$, then any vector $x \in \mathbb{R}^n$ can be uniquely expressed as:

$$x = (a_1^T x)a_1 + (a_2^T x)a_2 + \cdots + (a_n^T x)a_n = \sum_{i=1}^{n} (a_i^T x)a_i \qquad (2.40)$$

- This representation is called the **orthonormal expansion** of x with respect to the orthonormal basis $\{a_1, a_2, \dots, a_n\}$.
- The coefficients $(a_i^T x)$ represent the projections of x onto each basis vector a_i.

Exercice: Prove Parseval's identity by showing that the inner product of x with each basis vector u_i gives the projection of x onto u_i, and that the sum of the squares of these projections equals the squared norm of x.

Exercise: Prove this expansion 2.40 by showing that the right-hand side equals x using the orthonormality properties.

Gram-Schmidt (orthogonalization) algorithm

- An algorithm to construct orthogonal vectors from any vectors.
- **Given** k **vectors of dimension** n**,** $a_1, \ldots, a_k$
- The $G - S$ algorithm generates orthonormal vectors $q_1, q_2, \ldots, q_k$ as follows:

$$
\begin{aligned}
q_1 &= a_1 \\
q_2 &= a_2 - \frac{a_2^T q_1}{\|q_1\|^2} q_1 \\
\vdots &= \vdots \\
q_k &= a_k - \sum_{i=1}^{k-1} \frac{a_k^T q_i}{\|q_i\|^2} q_i
\end{aligned}
$$

- **Gram-Schmidt Algorithm:**
  1. **Input:** Linearly independent vectors $a_1, a_2, \dots, a_k \in \mathbb{R}^n$
  2. **Output:** Orthonormal vectors $q_1, q_2, \dots, q_k \in \mathbb{R}^n$
  3. **For** $i = 1$ to k:
     - Set $\tilde{q}_i \leftarrow a_i$
     - **For** $j = 1$ to $i-1$: $\tilde{q}_i \leftarrow \tilde{q}_i - (a_i^T q_j)q_j$
     - $q_i \leftarrow \frac{\tilde{q}_i}{\|\tilde{q}_i\|}$
- The algorithm produces orthonormal vectors where $q_i^T q_j = \delta_{ij}$.
- The newly obtained vectors $q_1, \dots, q_k$ are orthonormal. (Proof: exercise. Hint: induction).
- **Complexity:** $O(kn^2)$ operations for k vectors of dimension n.

Example: Gram-Schmidt Algorithm

Example: Let's consider the vectors

$$a_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad a_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

- These vectors are not orthogonal (verify by computing dot products).
- **Step 1:** $q_1 = \frac{a_1}{\|a_1\|} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

- **Step 2:** Orthogonalize a_2 against q_1:

$$\tilde{q}_2 = a_2 - (a_2^T q_1)q_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -2/3 \\ 1/3 \end{bmatrix}$$

Then normalize: $q_2 = \frac{\tilde{q}_2}{\|\tilde{q}_2\|}$

- **Step 3:** Orthogonalize a_3 against q_1 and q_2:

$$\tilde{q}_3 = a_3 - (a_3^T q_1)q_1 - (a_3^T q_2)q_2$$

Compute the projections and normalize to get q_3.

- The final orthonormal vectors are:

$$q_1, q_2, q_3$$

- **Verification:** Let us check that $q_i^T q_j = \delta_{ij}$ for all i, j.

- Compute the inner products:

$$q_1^T q_2 = \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} (1 \cdot 1 + 1 \cdot 0 + 1 \cdot 1) = \frac{1}{3} \cdot 2 = \frac{2}{3}$$
$$q_1^T q_3 = \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} (1 \cdot 1 + 1 \cdot 1 + 1 \cdot 0) = \frac{1}{3} \cdot 2 = \frac{2}{3}$$
$$q_2^T q_3 = \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} \left(\frac{1}{3} \cdot 1 + \left(-\frac{2}{3}\right) \cdot 1 + \frac{1}{3} \cdot 0\right) = \frac{1}{3} \cdot 0 = 0$$

- Since $q_1^T q_2 = 0$, $q_1^T q_3 = 0$, and $q_2^T q_3 = 0$, the vectors are orthogonal.
- Normalize each vector to ensure they have unit norm:

$$\|q_1\| = \|q_2\| = \|q_3\| = 1$$

- Therefore, the vectors q_1, q_2, q_3 form an orthonormal set.

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2. Vectors
   - Definitions and Notations
   - Vectors Operations
   - Vector Spaces
   - Inner Products and Norms
     - Inner product
     - Norms, Distance and Angle
     - Some Norms Inequalities
     - Angles Between Vectors
     - Orthogonality
3. Matrices
   - Matrices Definitions
   - Matrix Operations

Matrix Definitions and Notations

- A matrix is a rectangular arrangement of numbers organized in rows and columns, for example:

$$\begin{bmatrix} 1 & 2 & 4 & 0 \\ -2 & 0 & 1 & \pi \\ 2.1 & 10 & -9 & e \end{bmatrix}$$

- Matrix dimensions are denoted as (rows) $\times$ (columns)
  - The example above has dimensions 3×4
- Each value in a matrix is called an element, entry, or component.

- Convention: matrices are denoted with uppercase letters (A, B, C, M, etc.)
- B_{ij} represents the element at row i, column j of matrix B
- For example, in matrix B:

$$B = \begin{bmatrix} 0 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

- $B_{12} = 2$, $B_{21} = -1$, and $B_{22} = 1$
- The element B_{ij} is located at the intersection of row i and column j.
- The element B_{ij} is often referred to as the (i, j)-th entry of matrix B.

Tall, Wide, Square Matrices

- An $m \times n$ matrix A can be characterized as:
  - *tall* when $m > n$ (more rows than columns)
  - *wide* when $m < n$ (more columns than rows)
  - *square* when $m = n$ (equal rows and columns)
- Example:

$$(a) \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}, \quad (b) \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}, \quad (c) \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

- Special matrix categories:
  - An $m \times 1$ matrix forms a column vector. For instance: $\begin{bmatrix} 1.2 \\ 0 \end{bmatrix}$.
  - A $1 \times n$ matrix forms a *row* vector, for instance: $[-1, 2, 0, 3]$.
  - A 1×1 matrix represents a scalar value. For instance: $\begin{bmatrix} 3.14 \end{bmatrix}$.

Block Matrix Structure

- The i-th row of A forms an n-dimensional row vector:

$$[A_{i1}, \cdots, A_{in}]$$

- A block matrix contains matrices as elements, structured as:

$$A = \begin{bmatrix} B & C \\ D & E \end{bmatrix}$$

where B, C, D, and E are submatrices (blocks) within A.

- blocks within the same row must share identical row dimensions
- blocks within the same column must share identical column dimensions
- Example: given

$$B = [0\ 2\ 3],\ C = [-1]\ D = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 3 \end{bmatrix},\ E = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

This yields

$$\begin{bmatrix} B & C \\ D & E \end{bmatrix} = \begin{bmatrix} 0 & 2 & 3 & -1 \\ 2 & 2 & 1 & 3 \\ 1 & 3 & 3 & 2 \end{bmatrix}$$

Diagonal and Triangular Matrices

- diagonal matrix: square matrix with $A_{ij} = 0$ when $i \neq j$
- $\text{diag}(a_1, \ldots, a_n)$ denotes a diagonal matrix with $A_{ii} = a_i$ for $i = 1, \ldots, n$
- Examples of diagonal matrices:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.1}$$

- lower triangular matrix: $A_{ij} = 0$ when $i < j$
- upper triangular matrix: $A_{ij} = 0$ when $i > j$

- Examples:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{(upper triangular)}, \quad \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ .3 & 1 & 1 \end{bmatrix} \text{(lower triangular)},$$

Matrix Transpose

- Given matrix A, its **transpose** is the matrix A^T where the rows of A become columns, defined as

$$(A^T)_{ij} = A_{ji},\ i = 1, \dots, n, j = 1, \dots, m$$

- For instance, the transpose of $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ yields the lower triangular matrix $A^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

Transpose Properties

- For matrices A, B, and scalar λ:

$$(A+B)^T = A^T + B^T, \;\; (\lambda A)^T = \lambda A^T, \;\; (A^T)^T = A \qquad (3.2)$$

- $(A^T)^T = A$
- Matrix addition, subtraction, and scalar multiplication are well-defined operations.

- Specifically

$$(A \pm B)_{ij} = A_{ij} \pm B_{ij},\ i = 1, \ldots, m, j = 1, \ldots, n$$
$$(\alpha A)_{ij} = \alpha A_{ij},\ i = 1, \ldots, m, j = 1, \ldots, n$$

- These operations follow standard algebraic properties:

$$A + B = B + A,\ \alpha(A + B) = \alpha A + \alpha B,\ (A + B)^T = A^T + B^T \tag{3.3}$$

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Matrix-Vector Multiplication

- The product of an $m \times n$ matrix A and an n-dimensional vector x yields an m-dimensional vector $y = Ax$, where each entry is computed as:

$$y_i = A_{i1}x_1 + \cdots + A_{in}x_n, \;\; i = 1, \ldots, m$$

$$\begin{bmatrix} \boxed{a_{11}} & \boxed{a_{12}} & \boxed{a_{13}} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \boxed{x_1} \\ \boxed{x_2} \\ \boxed{x_3} \end{bmatrix} = \begin{bmatrix} \boxed{y_1} \\ y_2 \\ y_3 \end{bmatrix}$$

Figure: Computing the first element $y_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3$

- This operation can be viewed as computing dot products between matrix rows and the vector:

$$y_i = a_i^T x, \; i = 1, \dots, m \tag{3.4}$$

where $a_1^T, \dots, a_m^T$ denote the rows of matrix A, i.e., $a_i^T = [A_{i1}, A_{i2}, \dots, A_{in}]$.

- Example:

$$\begin{bmatrix} 0 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \cdot 1 + 2 \cdot 2 + (-1) \cdot (-1) \\ (-1) \cdot 1 + 1 \cdot 2 + 0 \cdot (-1) \end{bmatrix}$$

$$= \begin{bmatrix} 0 + 4 + 1 \\ -1 + 2 + 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

Special Cases and Alternative Interpretation

- When multiplying matrix A by the vector of ones $\mathbf{1}$, the result is a vector containing the row sums of A:

$$\begin{bmatrix} 0 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 + 2 + (-1) \\ -1 + 1 + 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- The matrix-vector product $y = Ax$ can alternatively be expressed as:

$$y = x_1 a_1 + \cdots + x_n a_n$$

where $a_1, \ldots, a_n$ represent the columns of matrix A.

- This interpretation shows that $y = Ax$ is a linear combination of the columns of A, with weights given by the components $x_1, \ldots, x_n$.

Matrix Multiplication

- For matrices A (dimensions $n \times m$) and B (dimensions $p \times q$)
- The matrix product $C = AB$ is defined if and only if the number of columns in A equals the number of rows in B, i.e., $m = p$
- When this condition is satisfied, the resulting matrix C has dimensions $n \times q$, with elements computed as:

$$C_{ij} = \sum_{k=1}^{m} A_{ik} B_{kj} \tag{3.5}$$

- The product AB is obtained by computing the dot product of each row of A with each column of B

- Equivalently, the matrix product can be expressed as:

$$C = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_n^T \end{bmatrix} \begin{bmatrix} b_1 & b_2 & \cdots & b_q \end{bmatrix} = \begin{bmatrix} a_1^T b_1 & a_1^T b_2 & \cdots & a_1^T b_q \\ a_2^T b_1 & a_2^T b_2 & \cdots & a_2^T b_q \\ \vdots & \vdots & \ddots & \vdots \\ a_n^T b_1 & a_n^T b_2 & \cdots & a_n^T b_q \end{bmatrix} \tag{3.6}$$

where a_i^T denotes the i-th row of matrix A and b_j represents the j-th column of matrix B.

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \begin{bmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{bmatrix} = \begin{bmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & c_9 \end{bmatrix}$$

Figure: Matrix multiplication visualization

Matrix Multiplication Example

- Consider matrices A and B:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix}$$

- The product $C = AB$ is computed as follows:

$$C_{11} = 1 \cdot 7 + 2 \cdot 9 + 3 \cdot 11 = 58, \quad C_{12} = 1 \cdot 8 + 2 \cdot 10 + 3 \cdot 12 = 64$$
$$C_{21} = 4 \cdot 7 + 5 \cdot 9 + 6 \cdot 11 = 139, \quad C_{22} = 4 \cdot 8 + 5 \cdot 10 + 6 \cdot 12 = 154$$

- Thus, the resulting matrix C is:

$$C = \begin{bmatrix} 58 & 64 \\ 139 & 154 \end{bmatrix}$$

- Note that matrix multiplication is not commutative, i.e., $AB \neq BA$ in general
- However, it is associative, i.e., $(AB)C = A(BC)$
- It is also distributive over addition, i.e., $A(B + C) = AB + AC$
- The identity matrix I_n serves as the multiplicative identity for matrices, i.e., $AI_n = A$ and $I_nA = A$ for any compatible matrix A
- The zero matrix O acts as the additive identity, satisfying $A + O = A$ and $O + A = A$ for any matrix A

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Linear Transformations

- A mapping $T : \mathbb{R}^m \to \mathbb{R}^n$ is called **linear** if it satisfies

$$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v), \quad \text{for all } u, v \in \mathbb{R}^m,\ \alpha, \beta \in \mathbb{R} \tag{4.1}$$

- Consider an $m \times n$ matrix A. It defines a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$ since

$$A(\lambda x + \beta y) = A(\lambda x) + A(\beta y) = \lambda Ax + \beta Ay \tag{4.2}$$

- Another example: given $a \in \mathbb{R}^n$, the inner product $T(u) = \langle a, u \rangle$ for all $u \in \mathbb{R}^n$ is linear. Indeed, $T(\lambda u) = \langle a, \lambda u \rangle = \lambda \langle a, u \rangle = \lambda T(u)$ and $T(u+v) = \langle a, u+v \rangle = \langle a, u \rangle + \langle a, v \rangle = T(u) + T(v)$.

Exercise

- Determine which of the following functions $T : \mathbb{R}^2 \to \mathbb{R}^2$ represent linear transformations:
  1. $T(x_1, x_2) = (1 + x_1, x_2)$
  2. $T(x_1, x_2) = (x_2, x_1)$
  3. $T(x_1, x_2) = (x_1 - x_2, 0)$
  4. $T(x_1, x_2) = (\sin x_1, x_2)$
  5. $T(x_1, x_2) = (x_1^2, x_2)$
- Given a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$, prove that $T(\mathbf{0}) = \mathbf{0}$. Does the property $T(\mathbf{1}) = \mathbf{1}$ hold for the vector of ones?

Matrix Representation of Linear Transformations

- A linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ can be represented by an $n \times m$ matrix A such that for any vector $x \in \mathbb{R}^m$:

$$T(x) = Ax$$

- The matrix A is constructed from the images of the standard basis vectors:

$$A = \begin{bmatrix} T(e_1) & T(e_2) & \cdots & T(e_m) \end{bmatrix}$$

where e_i are the standard basis vectors in $\mathbb{R}^m$.

- The columns of A are the transformed basis vectors, i.e., $A = [T(e_1), T(e_2), \dots, T(e_m)]$.
- For example, if $T(e_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $T(e_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then the matrix representation is:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- This matrix A represents the identity transformation in $\mathbb{R}^2$.
- The linear transformation can be visualized as a mapping of vectors in $\mathbb{R}^m$ to $\mathbb{R}^n$ through the matrix multiplication Ax.

- Example: For a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x_1, x_2) = (2x_1 + x_2, -x_1 + 3x_2)$, the matrix representation is:

$$A = \begin{bmatrix} T(e_1) & T(e_2) \\ \downarrow & \downarrow \\ 2 & 1 \\ -1 & 3 \end{bmatrix}$$

where $T(e_1) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $T(e_2) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

- The matrix A can be used to compute the transformation of any vector $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ in $\mathbb{R}^2$ as:

$$T(x) = Ax = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- This results in a new vector in $\mathbb{R}^2$, which is the image of x under the transformation T.
- The matrix representation allows for efficient computation of linear transformations, especially in higher dimensions.

Exercices:

- Prove that the matrix representation of a linear transformation is unique.
- Show that if T is a linear transformation, then $T(\mathbf{0}) = \mathbf{0}$.
- Given a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x_1, x_2, x_3) = (x_1 + 2x_2, 3x_3)$, find its matrix representation.
- For the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x_1, x_2) = (x_1 - x_2, 2x_1 + x_2)$, compute the image of the vector $x = (1, 2)$.
- If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $T(x_1, x_2) = (3x_1 + x_2, -x_1 + 4x_2)$, find the matrix representation and compute $T(e_1)$ and $T(e_2)$, where $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

Image, Kernel, and Rank

Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation. The following concepts are essential in understanding the properties of T:

- The **image** (or **range**) of T is the set of all vectors that can be obtained as outputs, i.e.

$$\operatorname{Im}(T) = \{T(x) \in \mathbb{R}^n : x \in \mathbb{R}^m\}$$

- The **kernel** (or **null space**) of T consists of all input vectors that are mapped to the zero vector, i.e.

$$\operatorname{Ker}(T) = \{x \in \mathbb{R}^m : T(x) = \mathbf{0}\}$$

- The **rank** of a linear transformation equals the dimension of its image, written as $\text{rank}(T)$.
- In other words, the rank is the number of linearly independent vectors in the image of T.
- The **nullity** of a linear transformation equals the dimension of its kernel, written as $\text{nullity}(T)$.

Example: Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by:

$$T(x_1, x_2, x_3) = (x_1 + 2x_2, 3x_3)$$

- The image of T is the set of all vectors in $\mathbb{R}^2$ that can be expressed as $(x_1 + 2x_2, 3x_3)$ for some $x_1, x_2, x_3 \in \mathbb{R}$.
- That is, the image consists of all vectors of the form (a, b) where a can take any value in $\mathbb{R}$ and b can take any value in $\mathbb{R}$, leading to $\text{Im}(T) = \mathbb{R}^2$.

- The kernel of T consists of all vectors (x_1, x_2, x_3) such that $T(x_1, x_2, x_3) = (0, 0)$, which leads to the equations $x_1 + 2x_2 = 0$ and $3x_3 = 0$.
- This implies that x_3 must be zero, and x_1 can be expressed in terms of x_2 as $x_1 = -2x_2$. Thus, the kernel can be described by the equation $x_1 + 2x_2 = 0$ and $x_3 = 0$, which is a line in $\mathbb{R}^3$.
- That is, $\text{Ker}(T) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = -2x_2, x_3 = 0\}$.
- Or, the kernel can be expressed as:

$$\text{Ker}(T) = \left\{ \begin{bmatrix} -2t \\ t \\ 0 \end{bmatrix} \in \mathbb{R}^3 : t \in \mathbb{R} \right\} = \text{span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right)$$

- The nullity of T is the dimension of the kernel, which is 1, as it is spanned by one vector only.
- The rank of T is the dimension of the image, which in this case is 2, since the image spans a plane in $\mathbb{R}^2$.

Rank-Nullity Theorem

For a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$, the following relationship holds:

$$\text{rank}(T) + \text{nullity}(T) = m$$

where $\text{rank}(T)$ is the dimension of the image of T and $\text{nullity}(T)$ is the dimension of the kernel of T.

Note: This theorem connects the dimensions of the image and kernel of a linear transformation to the dimension of the domain.

Example: For the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x_1, x_2, x_3) = (x_1 + 2x_2, 3x_3)$:

- The rank of T is 2 (dimension of the image $\mathbb{R}^2$).
- The nullity of T is 1 (dimension of the kernel, which is a line in $\mathbb{R}^3$).
- Thus, by the rank-nullity theorem:

$$\text{rank}(T) + \text{nullity}(T) = 2 + 1 = 3$$

which matches the dimension of the domain $\mathbb{R}^3$.

Exercise: Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation.

- Prove that $Ker(T)$ is a subspace of $\mathbb{R}^m$.
- Show that $Im(T)$ is a subspace of $\mathbb{R}^n$.

Exercice:

- Prove the rank-nullity theorem for a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$.
- Given a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x_1, x_2, x_3) = (x_1 + 2x_2, 3x_3)$, find the image and kernel of T.
- Determine the rank and nullity of the transformation T.
- Verify that the rank-nullity theorem holds for this transformation.

Exercise: For the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x_1, x_2, x_3) = (x_1 + 2x_2, 3x_3)$:

- Find the image and kernel of T.
- Determine the rank and nullity of T.
- Verify that the rank-nullity theorem holds, i.e., $\text{rank}(T) + \text{nullity}(T) = m$, where m is the dimension of the domain $\mathbb{R}^3$.

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Rotation Transformations in $\mathbb{R}^2$

- **Two-dimensional rotation:**

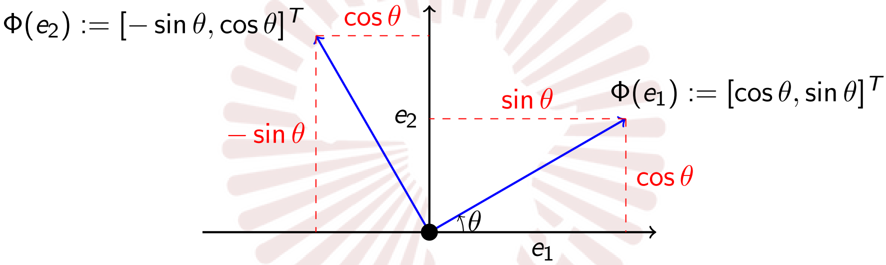

Figure: Standard basis rotation in $\mathbb{R}^2$ through angle θ

Matrix Representation of 2D Rotation

- **Derivation:** From the transformed basis vectors in the previous slide:

$$\Phi(e_1) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \quad \Phi(e_2) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \tag{4.3}$$

- The rotation transformation matrix R_θ has these as columns:

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{4.4}$$

- Given a vector $\mathbf{x}$, its rotation by angle θ yields $\mathbf{y}$:

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{4.5}$$

Rotation Transformations in $\mathbb{R}^3$

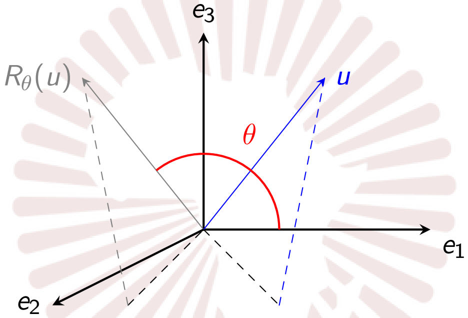

Figure: Vector rotation in $\mathbb{R}^3$ around the e_3-axis by angle θ

Rotation Matrices in Three Dimensions

- **Rotation around the e_1-axis:**

$$R_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \tag{4.6}$$

- **Rotation around the e_2-axis:**

$$R_2(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \tag{4.7}$$

Additional 3D Rotation Matrix and Key Properties

- **Rotation around the e_3-axis:**

$$R_3(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.8}$$

- In each rotation, the corresponding axis e_i remains invariant.

Exercises on Rotation Matrices

Problem: Demonstrate the following fundamental properties of rotation matrices:

1. Distance preservation: For any angle θ, verify that:

$$\|\mathbf{x} - \mathbf{y}\| = \|R_\theta(\mathbf{x}) - R_\theta(\mathbf{y})\|$$

   holds for all vectors $\mathbf{x}, \mathbf{y}$.

2. Rotation composition: Prove that $R_{\theta+\phi} = R_\theta \cdot R_\phi$ for all $\theta, \phi \in [0, 2\pi)$.

- Identity and inverse relations: Establish that $R_0 = I$ and $R_{-\theta} = R_\theta^{-1}$.
- Angular preservation: Demonstrate that rotations maintain inner products:

$$\langle R_\theta \mathbf{x}, R_\theta \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle$$

for all vectors $\mathbf{x}, \mathbf{y}$.

Exercise: Consider the rotation matrix R_θ defined as:

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{4.9}$$

- Show that R_θ is orthogonal, i.e., $R_\theta^T R_\theta = I$.
- Verify that the determinant of R_θ is 1, confirming that it represents a rotation.
- Prove that the inverse of R_θ is its transpose, i.e., $R_\theta^{-1} = R_\theta^T$.

Exercise: Given a vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, compute its rotation by angle θ using the matrix R_θ:

$$\mathbf{y} = R_\theta \mathbf{x} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{4.10}$$

- Calculate the resulting vector $\mathbf{y}$.
- Discuss how the rotation affects the coordinates of $\mathbf{x}$.

Exercise: Consider the rotation matrix R_θ and a vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

- Compute the rotated vector $\mathbf{y} = R_\theta \mathbf{x}$.
- Verify that the angle between $\mathbf{x}$ and $\mathbf{y}$ is θ.
- Discuss the geometric interpretation of this rotation in the context of the standard basis vectors.

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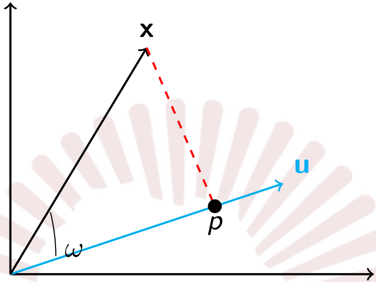
Projection onto a line

- **Vector projection onto a line**: Given a vector x, its projection p onto a line spanned by vector u is defined as

$$p = \frac{x^T u}{\|u\|^2} u \tag{4.11}$$

- The residual $e = x - p$ is orthogonal to u, satisfying $u^T e = 0$.
- **Projection matrix**: The projection can be expressed as $p = Px$ where P is the projection matrix:

$$P = \frac{uu^T}{\|u\|^2} \tag{4.12}$$



- **Example**: Consider vectors x and u:

$$x = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- Since $\|u\|^2 = 1$ and $x^T u = 2 \cdot 1 + 3 \cdot 0 = 2$

- The projection is $p = 2\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix}$
- **Matrix example**: For projection onto the line spanned by $b = [1,2,2]^T$:

$$P = \frac{bb^T}{\|b\|^2} = \frac{1}{9}\begin{bmatrix}1\\2\\2\end{bmatrix}\begin{bmatrix}1 & 2 & 2\end{bmatrix} = \frac{1}{9}\begin{bmatrix}1 & 2 & 2\\2 & 4 & 4\\2 & 4 & 4\end{bmatrix} \tag{4.13}$$

- For any vector x, its projection onto this line is:

$$p = Px = \frac{1}{9}\begin{bmatrix}1 & 2 & 2\\2 & 4 & 4\\2 & 4 & 4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \frac{1}{9}\begin{bmatrix}x_1 + 2x_2 + 2x_3\\2x_1 + 4x_2 + 4x_3\\2x_1 + 4x_2 + 4x_3\end{bmatrix} \tag{4.14}$$

Exercise

1. Calculate the projection vector p and determine the projection matrix P for the given vectors:

$$a)\;\; x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\; u = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \quad b)\;\; x = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, u = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \tag{4.15}$$

2. For both cases above, evaluate P^2 and P^3. What pattern do you notice?
3. Verify that the residual $e = x - p$ is orthogonal to u, i.e., $u^T e = 0$
4. Demonstrate that $\|p\| = \|x\| \cos\theta$, where θ represents the angle between x and u.

Projection onto a Plane

- **Projection of a vector onto a plane**: Given a vector x and a 2D plane π, the projection of x onto π is a vector x' given by

$$x' = \frac{x^T u}{\|u\|^2} u + \frac{x^T v}{\|v\|^2} v \tag{4.16}$$

  where u and v are orthogonal basis vectors spanning the plane.

- In matrix notation, the projection x' of x is:

$$x' = \left(I - nn^T\right) x, \tag{4.17}$$

  where n is the unit normal vector to the plane, and $P = I - nn^T$ is the orthogonal projection matrix onto the plane.

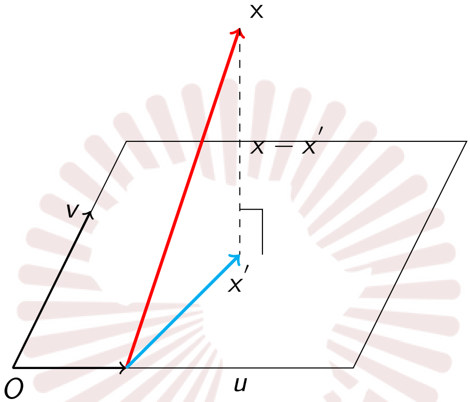


Figure: Projection onto a plane spanned by u and v. The projection x' can be expressed as a linear combination of u and v.

- **Example**: Consider a plane with normal vector n and a vector x to be projected:

$$n = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

- First, verify that n is a unit vector: $\|n\| = 1$ (which is already the case).
- Calculate the outer product nn^T:

$$nn^T = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Determine the projection matrix P:

- The projection matrix P is:

$$P = I - nn^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- The projection of x onto the plane is:

$$x' = Px = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

Projection onto a Subspace

- **General subspace projection**: Given n linearly independent vectors $a_1, \ldots, a_n \in \mathbb{R}^m$, we seek the projection p of vector b onto the subspace spanned by these vectors:

$$p = \hat{p}_1 a_1 + \cdots + \hat{p}_n a_n \tag{4.18}$$

  where the coefficients $\hat{p}_i$ are determined by the orthogonality condition.

- Let $A = [a_1 \; a_2 \; \cdots \; a_n]$ be the matrix with columns a_i. The projection coefficients satisfy:

$$A^T(b - A\hat{p}) = 0 \quad \Rightarrow \quad A^T A\hat{p} = A^T b \tag{4.19}$$

Projection onto General Subspaces

- **Setting**: Let $x \in \mathbb{R}^n$ and $U \subset \mathbb{R}^n$ be a subspace with $\dim(U) = m \geq 1$.
- Let $\{b_1, b_2, \ldots, b_m\}$ be a basis for U, and define $B = [b_1 \ b_2 \ \cdots \ b_m]$.
- Any vector in U can be written as:

$$x' = \sum_{i=1}^{m} \lambda_i b_i = B\lambda$$

for some coefficient vector $\lambda = [\lambda_1, \ldots, \lambda_m]^T$.

- The orthogonal projection of x onto U is:

$$x' = B(B^T B)^{-1} B^T x \tag{4.21}$$

where $P = B(B^T B)^{-1} B^T$ is the projection matrix onto U.

Exercices

1. Given a vector $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and a subspace spanned by $b_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $b_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, compute the projection of x onto the subspace.
2. For the same vectors, calculate the projection matrix P and verify that $P^2 = P$.
3. Show that the error vector $e = x - x'$ is orthogonal to both b_1 and b_2.
4. Prove that the projection x' satisfies $\|x'\| = \|b_1\| \cos\theta_1 + |b_2| \cos\theta_2$, where θ_1 and θ_2 are the angles between x' and b_1, and x' and b_2, respectively.

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Determinants

- For any square matrix $A \in \mathbb{R}^{n\times n}$, the determinant $\det(A)$ (also written as $|A|$) is a scalar value given by

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \tag{5.1}$$

- Determinants are only defined for square matrices.

- For any $n \times n$ matrix A, we can compute $\det A$ using cofactor expansion:

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij} \quad \text{or} \quad \det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij} \tag{5.2}$$

  where a_{ij} is the element at row i and column j, and A_{ij} is the (i, j)-minor of matrix A.

- For a 2×2 matrix A, we have:

$$\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \tag{5.3}$$

- For a 3×3 matrix A, applying Eq. 5.2 gives:

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

- The determinant serves as a test for matrix invertibility.

Determinant and Matrix Invertibility

- A square matrix $A \in \mathbb{R}^{n\times n}$ is invertible if and only if $\det(A) \neq 0$.
- Proof: Left as an exercise.
- Example: Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix} \tag{5.4}$$

- We calculate:
  $\det A = 1(1\cdot 0 - 4\cdot 6) - 2(0\cdot 0 - 4\cdot 5) + 3(0\cdot 6 - 1\cdot 5) = -24 + 40 - 15 = 1.$
- Since $\det A = 1 \neq 0$, the matrix A is invertible.

Properties

- Matrix determinants satisfy the following properties:
  1. $\det(AB) = \det(A) \times \det(B)$
  2. $\det(A) = \det(A^T)$
  3. If A is invertible, then $\det(A^{-1}) = \frac{1}{\det(A)}$.
  4. $\det(\lambda A) = \lambda^n \det(A)$
  5. Exchanging two rows or columns negates $\det(A)$.

Trace

- The trace of $A \in \mathbb{R}^{n\times n}$ is the scalar defined as

$$\mathrm{tr}(A) := \sum_{i=1}^{n} a_{ii} \tag{5.5}$$

- In other words, the trace equals the sum of all diagonal entries of A.
- It has the following properties: (Exercise)
  1. $\mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B)$ for $A, B \in \mathbb{R}^{n\times n}$
  2. $\mathrm{tr}(\lambda A) = \lambda \mathrm{tr}(A)$
  3. $\mathrm{tr}(I_n) = n$
  4. $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for all $A \in \mathbb{R}^{n\times k}$, $B \in \mathbb{R}^{k\times n}$.
- The trace remains unchanged under cyclic permutations:

$$\mathrm{tr}(AKL) = \mathrm{tr}(KLA) \tag{5.6}$$

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Eigenvalues and eigenvectors

- In this section, we work exclusively with square matrices.
- A scalar $\lambda \in \mathbb{R}$ is an **eigenvalue of** A if there exists a non-zero vector x such that

$$Ax = \lambda x \tag{7.1}$$

- For example, $\lambda = 1$ is an eigenvalue of the identity matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Also, $\lambda = 5$ is an eigenvalue of $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$ since $A \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 5 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Eigenvectors

- Any vector x that satisfies $Ax = \lambda x$ for some scalar λ is called an **eigenvector of** A associated with eigenvalue λ.
- We refer to equation 7.1 as **the eigenvalue equation**.
- As an example, $x = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$ corresponding to eigenvalue $\lambda = 5$.
- Finding eigenvalues and eigenvectors for a given matrix A requires systematic computation.

Computing eigenvalues

- For any matrix A, these statements are equivalent:
  1. λ is an eigenvalue of A
  2. There exists a non-zero vector $x \in \mathbb{R}^n$ such that $Ax = \lambda x$
  3. $\det(A - \lambda I) = 0$
- Define $p_A(\lambda) = \det(A - \lambda I)$. This function p_A is a polynomial in λ.
- **The eigenvalues of matrix A correspond exactly to the roots of polynomial p_A.**

Exercise

- Determine the eigenvalues of these matrices:

$$a)\ \ A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \quad b)\ \ I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad c)\ \ C = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \tag{7.2}$$

$$d)\ \ U = \begin{bmatrix} 3 & 5 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 7 \end{bmatrix} \quad e)\ \ L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & -3 & 0 \\ 2 & 5 & 6 \end{bmatrix} \tag{7.3}$$

What common property do you notice for matrices d) and e)?

- For any matrix A:
  1. Show that if x is an eigenvector of A with eigenvalue λ, then any non-zero scalar multiple cx is also an eigenvector of A with the same eigenvalue λ.
  2. Show that if λ is an eigenvalue of A, then λ^k is an eigenvalue of A^k for any positive integer k.
  3. For a 2×2 triangular matrix A, show that the eigenvalues are the diagonal elements.
  4. For any 2×2 matrix A, find the characteristic polynomial p_A and verify that:

$$p_A(\lambda) = \lambda^2 - \lambda \mathrm{tr}(A) + \det(A) \tag{7.4}$$

  5. Show that if A is symmetric, then all eigenvalues of A are real.

Finding Eigenvectors

- For a matrix A, these statements are equivalent:
  1. x is an eigenvector of A with eigenvalue λ
  2. x satisfies the equation $Ax = \lambda x$
  3. x is a non-zero solution to the homogeneous system $(A - \lambda I)x = 0$
  4. x lies in the nullspace of $A - \lambda I$, i.e., $x \in \ker(A - \lambda I)$
  5. The matrix $A - \lambda I$ is singular (non-invertible), which means $\det(A - \lambda I) = 0$
- **Eigenvectors corresponding to λ are non-zero solutions to** $(A - \lambda I)x = 0$.

Example: Finding Eigenvectors

1. Consider $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$. We will find its eigenvectors.
2. First, we compute the eigenvalues using the characteristic equation:

$$\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = 0 \tag{7.5}$$

$$= (4-\lambda)(3-\lambda) - 2 \tag{7.6}$$

$$= \lambda^2 - 7\lambda + 10 = 0 \Longrightarrow \lambda_1 = 5, \ \lambda_2 = 2 \tag{7.7}$$

3. Next, we find eigenvectors by solving $(A - \lambda I)x = 0$.

- **For eigenvalue** $\lambda = 5$:

$$(A - 5I)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \implies x = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

- **For eigenvalue** $\lambda = 2$:

$$(A - 2I)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \implies x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Exercise

1. Compute the eigenvectors corresponding to the eigenvalues of matrices $(a) - (e)$ from the previous slide.
2. If x and y are eigenvectors of matrix A with the same eigenvalue λ, prove that any linear combination $\alpha x + \beta y$ is also an eigenvector for λ.
3. Can the zero vector serve as an eigenvector? Explain your reasoning.
4. Is it possible for a single eigenvector to correspond to two distinct eigenvalues? Provide justification.
5. If matrix A has eigenvalue $\lambda = 0$, what can you conclude about the properties of A, particularly its invertibility?

Algebraic Multiplicity

- An eigenvalue λ of matrix A corresponds to a root of the characteristic polynomial $p_A(\lambda)$.
- The algebraic multiplicity of eigenvalue λ_i refers to how many times λ_i appears as a root in p_A.
- Example: For a 3×3 matrix A with characteristic polynomial

$$p_A(\lambda) = (\lambda - 1)^2(\lambda - 3) \tag{7.8}$$

eigenvalue $\lambda = 1$ has algebraic multiplicity 2, while $\lambda = 3$ has multiplicity 1.

Exercise

- Determine the algebraic multiplicity of each eigenvalue for the following matrices:

$$a)\;\; A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \quad b)\;\; B = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \quad c)\;\; C = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \tag{7.9}$$

Geometric Multiplicity

- For an eigenvalue λ of matrix A, the **geometric multiplicity** equals $\dim(\ker(A - \lambda I))$.
- Equivalently, it represents the **maximum number of linearly independent eigenvectors associated with** λ.
- Example: For matrix $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, eigenvalue $\lambda = 2$ has geometric multiplicity 2.

Eigenspace and Spectrum

- Given matrix A and eigenvalue λ, the **eigenspace** E_λ contains all corresponding eigenvectors:

$$E_\lambda = \{x \in \mathbb{R}^n \mid (A - \lambda I)x = 0\} = \ker(A - \lambda I) \tag{7.10}$$

- The eigenspace E_λ forms a subspace spanned by eigenvectors of λ.
- From our earlier example with $\lambda = 5$, the eigenspace is:

$$E_5 = \text{span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\} = \left\{\begin{bmatrix} 2\alpha \\ \alpha \end{bmatrix} \mid \alpha \in \mathbb{R}\right\} \tag{7.11}$$

- Note: **geometric multiplicity = dimension of eigenspace**.
- The **spectrum** (or **eigenspectrum**) of A, denoted $\sigma(A)$, is the **complete set of eigenvalues**:

$$\sigma(A) = \{\lambda \in \mathbb{R} \mid \det(A - \lambda I) = 0\} \tag{7.12}$$

- Example: For $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ with eigenvalues $\lambda = 1$ and $\lambda = 3$, we have $\sigma(A) = \{1, 3\}$.

Computing Eigenvalues, Eigenvectors, and Eigenspaces

- Consider the 2×2 matrix:

$$A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} \tag{7.13}$$

- **Characteristic polynomial**:

$$p_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = (4-\lambda)(3-\lambda) - 2 \tag{7.14}$$

$$= \lambda^2 - 7\lambda + 10 \tag{7.15}$$

$$= (\lambda - 2)(\lambda - 5) \tag{7.16}$$

- **Eigenvalues**: The roots are $\lambda_1 = 2$ and $\lambda_2 = 5$, each with algebraic

- **Eigenvectors and Eigenspaces**: For eigenvalue $\lambda = 5$, we find eigenvectors by solving:

$$(A - \lambda I)x = 0 \implies \begin{bmatrix} 4-5 & 2 \\ 1 & 3-5 \end{bmatrix} x = 0 \tag{7.17}$$

- This gives us the system:

$$\begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \tag{7.18}$$

- From the first row: $-x_1 + 2x_2 = 0 \Rightarrow x_1 = 2x_2$
- Setting $x_2 = t$, the eigenspace is:

$$E_5 = \text{span}\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\} = \left\{ t \begin{bmatrix} 2 \\ 1 \end{bmatrix} \mid t \in \mathbb{R}, t \neq 0 \right\} \tag{7.19}$$

- The eigenvector and eigenspace associated to $\lambda = 5$ are

$$v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, E_5 = \text{span}\{v_1\} := \{cv_1 \mid c \in \mathbb{R}\}$$

- For $\lambda = 2$, we find analogously the equations

$$\begin{bmatrix} 4-2 & 2 \\ 1 & 3-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Longleftrightarrow \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

which gives $x_2 = -x_1$, such as $v_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$, is an eigenvector. The corresponding eigenspace is $E_2 = \text{span}\{v_2\}$.

Back to Matrix Norms: Spectral Norm

- For any matrix A (not necessarily square), the **spectral norm** $\|A\|_2$ is defined as the square root of the largest eigenvalue of A^TA:

$$\|A\|_2 = \sqrt{\lambda_{\max}(A^TA)} \tag{7.20}$$

where $\lambda_{\max}$ denotes the maximum eigenvalue.

- Example: Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$. First, compute A^TA:

$$A^TA = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix} \tag{7.21}$$

- To find the eigenvalues of $A^T A$, we solve the characteristic equation:

$$\det(A^T A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 8-\lambda \end{vmatrix} = (1-\lambda)(8-\lambda) - 4 = 0 \quad (7.22)$$

- Solving yields eigenvalues $\lambda_1 \approx 8.531$ and $\lambda_2 \approx 0.469$.
- Therefore, the spectral norm is:

$$\|A\|_2 = \sqrt{\lambda_{\max}} = \sqrt{8.531} \approx 2.921 \quad (7.23)$$

- Compare this with other norms: What are the Frobenius norm, 1-norm, and ∞-norm of A? What patterns do you observe?

Spectral theorem

- Racall that a symmetric matrix A is such that $A^T = A$.
- For example,the following matrices are symmetric

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = A^T \quad \text{and} \quad D = \begin{bmatrix} -2 & 0 & 1 \\ 0 & 3 & 2 \\ 1 & 2 & 0 \end{bmatrix} = D^T \tag{7.24}$$

- What are the eigenvalues of A and D and the eigenvectors for each? What do you observe?
- Let's consider A. A quick computation yields

$$\lambda_1 = 1, \quad \lambda_2 = 3, \; v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \text{and} \quad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{7.25}$$

- Property 1: All eigenvalues of A are real numbers.
- Property 2: The eigenvectors of A are orthogonal to each other, as shown by:

$$v_1^T v_2 = \begin{bmatrix}1 & -1\end{bmatrix} \begin{bmatrix}1 \\ 1\end{bmatrix} = 1 \cdot 1 + (-1) \cdot 1 = 0 \tag{7.26}$$

- By normalizing v_1 and v_2 to unit vectors $x_1 = \frac{v_1}{\|v_1\|}$ and $x_2 = \frac{v_2}{\|v_2\|}$, we obtain the orthogonal matrix Q satisfying $Q^T Q = I$:

$$Q = \begin{bmatrix}x_1 & x_2\end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix}1 & 1 \\ -1 & 1\end{bmatrix} \tag{7.27}$$

- Property 3: Matrix A can be decomposed as:

$$A = Q\Lambda Q^T = \frac{1}{\ } \begin{bmatrix}1 & 1\end{bmatrix} \begin{bmatrix}1 & 0\end{bmatrix} \frac{1}{\ } \begin{bmatrix}1 & -1\end{bmatrix} \tag{7.28}$$

Spectral Theorem for Symmetric Matrices

- **Spectral Theorem**: If A is a symmetric matrix (i.e., $A = A^\top$), then:
  1. Every eigenvalue of A is real
  2. $\mathbb{R}^n$ has an orthonormal basis formed entirely by eigenvectors of A
- From property 2, we can construct an orthogonal matrix Q such that

$$A = QDQ^T \tag{7.29}$$

where D contains the eigenvalues of A on its diagonal, and the columns of Q are the corresponding normalized eigenvectors.

- **Eigenvalues are real**. For any complex eigenpair (λ, x):

$$Ax = \lambda x \Rightarrow x^T Ax = \lambda x^T x \Rightarrow \lambda = \overline{\lambda} \Rightarrow \lambda \in \mathbb{R}$$

- **Orthonormal eigenvector basis exists**:
  1. Choose an eigenvalue λ of A with corresponding unit eigenvector $v \in \mathbb{R}^n$. Let $V_1 = \text{Span}(v)$ and consider its orthogonal complement $V_1^\perp$
  2. **Key insight:** $V_1^\perp$ is A-invariant, meaning $A(V_1^\perp) \subseteq V_1^\perp$
  3. **Induction:** Apply the theorem to the restriction of A on $V_1^\perp$ (dimension $n-1$), which remains symmetric

Proof, continued

5 **Step 5:** By the induction hypothesis, there exists an orthonormal basis of eigenvectors for the subspace $V_1^\perp$

6 **Conclusion:** By combining the unit eigenvector v with the orthonormal eigenvectors from $V_1^\perp$, we construct a complete orthonormal basis of $\mathbb{R}^n$ consisting entirely of eigenvectors of A

Example

- Consider the 2×2 symmetric matrix:

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

This matrix admits the following spectral decomposition:

$$A = QDQ^T \iff \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}^T$$

- Here, $\lambda_1 = 4$ and $\lambda_2 = 2$ are the eigenvalues of A, while the columns of Q contain the corresponding normalized eigenvectors.
- What is the determinant of A?

Relationship Between Determinants, Traces, and Eigenvalues

- **Determinant formula:** For any matrix A, its determinant equals the product of all eigenvalues:

$$\det(A) = \prod_{i=1}^{n} \lambda_i \tag{7.30}$$

where λ_i denote the eigenvalues of A.

- **Invertibility criterion:** Matrix A is invertible if and only if all eigenvalues are non-zero.
- **Trace formula:** The trace of matrix A equals the sum of all eigenvalues:

n

Eigenvalue Localization: Gershgorin Circle Theorem

- **Theorem:** For any matrix A, all eigenvalues lie within the union of Gershgorin discs:

$$\sigma(A) \subseteq \bigcup_{i=1}^{n} \left\{ \lambda \in \mathbb{C} \mid |\lambda - a_{ii}| \leq \sum_{j=1,\, j\neq i}^{n} |a_{ij}| \right\} \tag{7.32}$$

- Each disc D_i is centered at diagonal element a_{ii} with radius equal to the sum of absolute values of off-diagonal elements in row i.
- This theorem provides bounds for eigenvalue locations without computing them explicitly.

Example: Applying Gershgorin's Theorem

- Consider the matrix:

$$A = \begin{bmatrix} 5 & 1 & 0 \\ 2 & 4 & 1 \\ 1 & 2 & 6 \end{bmatrix}$$

- We construct three Gershgorin discs, one for each row.

- **Disc 1 (Row 1):**

$$\text{Center: } a_{11} = 5, \quad \text{Radius: } R_1 = |a_{12}| + |a_{13}| = |1| + |0| = 1$$

$$D_1 = \{\lambda \in \mathbb{C} : |\lambda - 5| \leq 1\}$$

- **Disc 2 (Row 2):**

$$\text{Center: } a_{22} = 4, \quad \text{Radius: } R_2 = |a_{21}| + |a_{23}| = |2| + |1| = 3$$

$$D_2 = \{\lambda \in \mathbb{C} : |\lambda - 4| \leq 3\}$$

- **Disc 3 (Row 3):**

$$\text{Center: } a_{33} = 6, \quad \text{Radius: } R_3 = |a_{31}| + |a_{32}| = |1| + |2| = 3$$

$$D_3 = \{\lambda \in \mathbb{C} : |\lambda - 6| \leq 3\}$$

- **Conclusion:** All eigenvalues of A must lie within $D_1 \cup D_2 \cup D_3$.
- For real eigenvalues, this corresponds to the interval $[1, 9]$.

Exercise

1. Prove or explain why the geometric multiplicity of an eigenvalue is always less than or equal to its algebraic multiplicity.
2. Give an example of a matrix where the algebraic multiplicity is greater than the geometric multiplicity.
3. Can a matrix have an eigenvalue with algebraic multiplicity 2 and geometric multiplicity 1? What does this imply about the matrix?
4. Is it possible for a matrix to have an eigenvalue whose geometric multiplicity is greater than its algebraic multiplicity? Why or why not?
5. Let $A \in \mathbb{R}^{n\times n}$. If an eigenvalue λ has algebraic multiplicity n, what does this imply about the characteristic polynomial?
6. If A is diagonalizable, how are the algebraic and geometric multiplicities of its eigenvalues related?

$\begin{bmatrix} 2 & 1 \end{bmatrix}$

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Cholesky decomposition

- **Any symmetric, positive definite matrix** A can be uniquely factorized as $A = LL^T$, where L is a **lower triangular matrix with positive diagonal elements:**

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} l_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ l_{n1} & \cdots & l_{nn} \end{bmatrix} \begin{bmatrix} l_{11} & \cdots & l_{n1} \\ 0 & \ddots & \vdots \\ 0 & \cdots & l_{nn} \end{bmatrix} \tag{8.1}$$

- The lower triangular matrix L is called the Cholesky factor of A.

Cholesky Decomposition: Example

- **Example:** Consider the symmetric, positive-definite matrix A:

$$A = \begin{bmatrix} 4 & 2 \\ 2 & 5 \end{bmatrix}$$

- Let $L = \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix}$. Then:

$$LL^T = \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} \\ 0 & l_{22} \end{bmatrix} = \begin{bmatrix} l_{11}^2 & l_{11} l_{21} \\ l_{21} l_{11} & l_{21}^2 + l_{22}^2 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 5 \end{bmatrix}$$

- **Solving for L:**

1. From $a_{11} = l_{11}^2$:

$$l_{11}^2 = 4 \implies l_{11} = 2 \quad (\text{since } l_{11} > 0)$$

2. From $a_{12} = l_{11} l_{21}$:

$$2 l_{21} = 2 \implies l_{21} = 1$$

3. From $a_{22} = l_{21}^2 + l_{22}^2$:

$$l_{21}^2 + l_{22}^2 = 5 \implies 1^2 + l_{22}^2 = 5 \implies l_{22}^2 = 4 \implies l_{22} = 2 \quad (\text{since } l_{22} > 0)$$

Thus:

$$L = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$$

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Eigendecomposition/Diagonalization

- Consider a matrix A possessing n eigenvalues $\lambda_1, \ldots, \lambda_n$ and n linearly independent eigenvectors $x_1, \ldots, x_n$. We can construct:
  - Diagonal matrix: $D = \text{diag}(\lambda_1, \ldots, \lambda_n)$
  - Matrix of eigenvectors: $P = [x_1 \; x_2 \; \cdots \; x_n]$

  This allows us to represent A as:

$$A = PDP^{-1} = P \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} P^{-1} \tag{8.2}$$

- A matrix that allows such a decomposition is termed "diagonalizable". **A matrix is diagonalizable if and only if it can be expressed as $A = PDP^{-1}$**

Example

- **All symmetric matrices are diagonalizable.** The spectral theorem guarantees the existence of a matrix P such that:

$$A = PDP^{-1} \tag{8.3}$$

- Consider the matrix A:

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

- We aim to find the eigendecomposition $A = PDP^{-1}$, where P has eigenvectors as columns and D is the diagonal matrix of corresponding eigenvalues.

- **Step 1: Find the Eigenvalues**. Solve $\det(A - \lambda I) = 0$:

$$p_A(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix} = (3-\lambda)^2 - 1$$
$$= \lambda^2 - 6\lambda + 8 = (\lambda - 4)(\lambda$$

- The eigenvalues are: $\sigma(A) = \{\lambda_1 = 4, \lambda_2 = 2\}$.
- **Step 2: Find the Eigenvectors**

  For $\lambda_1 = 4$:

  Solve $(A - 4I)\begin{bmatrix} x \\ y \end{bmatrix} = 0 \iff \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$. This yields:

$$-x + y = 0 \implies y = x. \text{ Thus: } v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- For $\lambda_2 = 2$:
  Solve $(A - 2I)\begin{bmatrix} x \\ y \end{bmatrix} = 0 \iff \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0.$ This gives:

$$x + y = 0 \implies y = -x. \text{ Thus: } \quad v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

- **Step 3: Form D and P:**

$$P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$$

Conditions for Diagonalization

- Not all matrices are symmetric. We must therefore establish conditions for a matrix to be diagonalizable. Let A be an $n \times n$ matrix.
- **Condition 1:** A is diagonalizable if there exists a basis of $\mathbb{R}^n$ composed of eigenvectors of A, i.e., A **possesses** n **linearly independent eigenvectors.**
- **Condition 2:** If A **has** n **distinct eigenvalues**, then it is diagonalizable (since eigenvectors corresponding to distinct eigenvalues are linearly independent).
- **Condition 3:** A is diagonalizable if A **satisfies** $GM = AM$**.**

Exercises: Diagonalizability

- Determine which of the following matrices are diagonalizable over $\mathbb{R}$. Provide justification for your answer.

a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ b) $B = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ c) $C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

d) $D = \begin{bmatrix} 5 & 4 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{bmatrix}$ e) $E = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix}$ f) $F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

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SVD Theorem

- Let $A \in \mathbb{R}^{m \times n}$ be a matrix of rank $r \leq \min(m, n)$. The Singular Value Decomposition (SVD) of A expresses it as:

$$A = U \Sigma V^T$$

where:

- $U \in \mathbb{R}^{m \times m}$ is an orthogonal matrix with columns u_i (left singular vectors)
- $V \in \mathbb{R}^{n \times n}$ is an orthogonal matrix with columns v_j (right singular vectors)
- $\Sigma \in \mathbb{R}^{m \times n}$ has diagonal entries $\sigma_i \geq 0$ and zeros elsewhere

Properties of SVD

- The diagonal entries $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0$ are called **singular values** of A.
- The columns u_i are **left singular vectors**, and columns v_j are **right singular vectors**.
- The singular values are conventionally ordered in decreasing order.
- The matrix Σ is uniquely determined. For $m > n$:

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \quad (8.4)$$

- For $m < n$, Σ has the structure:

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_m & 0 & \cdots & 0 \end{bmatrix} \tag{8.5}$$

- **Key fact:** The SVD exists for *any* matrix $A \in \mathbb{R}^{m \times n}$, regardless of rank or shape.

Computing the SVD

- The SVD can be constructed using the eigendecomposition of $A^T A$:

$$A^T A = Q \Lambda Q^T = Q \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} Q^T \tag{8.6}$$

  where Q contains orthonormal eigenvectors and $\lambda_i \geq 0$ are eigenvalues of $A^T A$.

- Starting from the SVD form $A = U \Sigma V^T$, we compute:

$$A^T A = (U \Sigma V^T)^T (U \Sigma V^T) = V \Sigma^T U^T U \Sigma V^T \tag{8.7}$$

- Since U is orthogonal ($U^T U = I$), we get:

$$A^T A = V\Sigma^T \Sigma V^T = V \begin{bmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_n^2 \end{bmatrix} V^T \tag{8.8}$$

- This shows that the singular values are $\sigma_i = \sqrt{\lambda_i}$, where λ_i are eigenvalues of $A^T A$.

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