APMA 922: Homework Set 03

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Problem 0

Part A

From the formula for the Discrete Fourier Transform (DFT)

$$\hat{v}_k = h \sum_{j=1}^{N} e^{-ikx_j} v_j, \quad k = -\frac{N}{2} + 1, \dots, \frac{N}{2} \tag{1}$$

if we wish to take the DFT of the periodic Kronecker

$$\delta_j = \begin{cases} 1, & j \equiv 0 \mod N \\ 0, & j \not\equiv 0 \mod N \end{cases} \tag{2}$$

then we have that

$$\hat{\delta}_k = h \sum_{j=1}^{N} e^{-ikx_j} \delta_j = h. \tag{3}$$

Now from the definition of the band-limited interpolant we have that

$$\begin{aligned}
p(x) &= \frac{h}{2\pi} \sum_{k=-N/2}^{N/2} c_k e^{ikx} && \text{(4a)} \\
&= \frac{h}{2\pi} \left(\frac{1}{2} \sum_{k=-N/2}^{N/2-1} e^{ikx} + \frac{1}{2} \sum_{k=-N/2+1}^{N/2} e^{ikx} \right) && \text{(4b)} \\
&= \frac{h}{2\pi} \cos\left(\frac{x}{2}\right) \sum_{k=-N/2+1/2}^{N/2-1/2} e^{ikx} && \text{(4c)} \\
&= \frac{h}{2\pi} \cos\left(\frac{x}{2}\right) \left[\frac{e^{i(-N/2+1/2)x} - e^{i(N/2+1/2)x}}{1 - e^{ix}} \right] && \text{(4d)} \\
&= \frac{h}{2\pi} \cos\left(\frac{x}{2}\right) \left[\frac{e^{i(-N/2)x} - e^{i(N/2)x}}{e^{-ix/2} - e^{ix/2}} \right] && \text{(4e)}
\end{aligned}$$

$$= \frac{h}{2\pi} \cos\left(\frac{x}{2}\right) \frac{\sin\left(Nx/2\right)}{\sin\left(x/2\right)}. \tag{4f}$$

Using the definition of the grid-spacing

$$h = \frac{2\pi}{N}, \tag{5}$$

we can then rewrite (4f) as

$$\begin{aligned} p(x) &= \cos\left(\frac{x}{2}\right) \frac{\sin\left(Nx/2\right)}{N\sin\left(x/2\right)} && (6a)\\ &= \frac{\sin\left(\pi x/h\right)}{\left(2\pi/h\right)\tan\left(x/2\right)} && (6b)\\ &= S_N(x). && (6c) \end{aligned}$$

An expansion of a periodic grid function v using the shifted periodic delta function as a basis takes the form,

$$v_j = \sum_{m=1}^{N} v_m \delta_{j-m}. \tag{7}$$

So the band-limited interpolant of (4a) is given by

$$p(x) = \sum_{m=1}^{N} v_m S_N(x - x_m). \tag{8}$$

Part B

The periodic sinc function, for even N, for non-bounded x, has the definition

$$S_N(x) = \begin{cases} 1, & x = 0 \\ \dfrac{\sin\left(\pi x/h\right)}{\left(2\pi/h\right)\tan\left(x/2\right)}, & x \neq 0. \end{cases} \tag{9}$$

Taking derivatives of the periodic sinc function $S_N(x)$ and evaluating at the grid points x_j, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} S_N(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\sin\left(\pi x/h\right)}{\left(2\pi/h\right)\tan\left(x/2\right)}\right]\Bigg|_{x=jh} && (10a)\\ &= \frac{h\sin\left(\frac{\pi x}{h}\right)}{2\pi\cos(x) - 2\pi} + \frac{\cos\left(\frac{\pi x}{h}\right)}{2\tan\left(\frac{x}{2}\right)}\Bigg|_{x=jh} && (10b)\\ &= \frac{h\sin(\pi j)}{2\pi\cos(hj) - 2\pi} + \frac{1}{2}\cos(\pi j)\cot\left(\frac{hj}{2}\right) && (10c)\\ &= \frac{(-1)^j}{2\tan\left(jh/2\right)}. && (10d) \end{aligned}$$

To obtain the behaviour near $x = 0$ we take the limit of (10c) as $j \to 0$ and find that this limit is 0. Therefore, we have that

$$S'_N(x) = \begin{cases} 0, & j \equiv 0 \mod N \\ \dfrac{(-1)^j}{2\tan\left(jh/2\right)}, & j \not\equiv 0 \mod N. \end{cases} \tag{11}$$

Taking another derivative of the sinc function, and again evaluating at the grid points we acquire

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}x^2} S_N(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}}{\mathrm{d}x} S_N(x)\right]\Bigg|_{x=jh} && \text{(12a)} \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{h\sin\left(\frac{\pi x}{h}\right)}{2\pi\cos(x) - 2\pi} + \frac{\cos\left(\frac{\pi x}{h}\right)}{2\tan\left(\frac{x}{2}\right)}\right]\Bigg|_{x=jh} && \text{(12b)} \\
&= \frac{\csc^2\left(\frac{x}{2}\right)\left(\cot\left(\frac{x}{2}\right)\sin\left(\frac{\pi x}{h}\right)\left(h^2 + \pi^2\cos(x) - \pi^2\right) - 2\pi h\cos\left(\frac{\pi x}{h}\right)\right)}{4\pi h}\Bigg|_{x=jh} && \text{(12c)} \\
&= \frac{\left(\sin(\pi j)\left(h^2 + \pi^2\cos(hj) - \pi^2\right)\cot\left(\frac{hj}{2}\right) - 2\pi h\cos(\pi j)\right)}{4\pi h\sin^2\left(\frac{hj}{2}\right)} && \text{(12d)} \\
&= \frac{(-1)^j}{2\sin^2\left(jh/2\right)}. && \text{(12e)}
\end{aligned}$$

Again to obtain the behaviour near $x = 0$ we take the limit of (12d) as $j \to 0$ and find that this limit is $-\left[\frac{1}{6} + \frac{\pi^2}{3h^2}\right]$. So we have that

$$S''_N(x_j) = \begin{cases} -\left[\frac{1}{6} + \frac{\pi^2}{3h^2}\right], & j \equiv 0 \mod N \\ \dfrac{(-1)^j}{2\sin^2\left(jh/2\right)}, & j \not\equiv 0 \mod N. \end{cases} \tag{13}$$

In the code `A3Q0.ipynb` I show that, indeed, $D_N^2 \neq D_N D_N$.

Part C

Now assume that we have an odd number $N' = 2N + 1$. From the definition of the band limited interpolant we have

$$\begin{aligned}
p(x) &= \frac{h'}{2\pi}\sum_{k=-N}^{N} e^{ikx} && \text{(14a)} \\
&= \frac{h'}{2\pi}\sum_{k=-N'/2+1/2}^{N'/2-1/2} e^{ikx} && \text{(14b)} \\
&= \frac{h'}{2\pi}\left[\frac{e^{i(-N'/2+1/2)x} - e^{i(N'/2+1/2)x}}{1 - e^{ix}}\right] && \text{(14c)}
\end{aligned}$$

$$= \frac{h'}{2\pi}\left[\frac{e^{i(-N'/2)x} - e^{i(N'/2)x}}{e^{-ix/2} - e^{ix/2}}\right] \tag{14d}$$

$$= \frac{h'}{2\pi}\frac{\sin\left(N'x/2\right)}{\sin\left(x/2\right)}. \tag{14e}$$

Using the definition of the grid spacing

$$h' = \frac{2\pi}{N'}, \tag{15}$$

we can rewrite (14e) as

$$p(x) = \frac{\sin\left(N'x/2\right)}{N'\sin\left(x/2\right)} \tag{16a}$$

$$= \frac{\sin\left(\pi x/h'\right)}{\left(2\pi/h'\right)\sin\left(x/2\right)} \tag{16b}$$

$$= S_{N'}(x). \tag{16c}$$

The periodic sinc function, for odd N', for unbounded x, has definition

$$S_{N'}(x) = \begin{cases} 1, & x = 0 \\ \dfrac{\sin\left(\pi x/h'\right)}{\left(2\pi/h'\right)\sin\left(x/2\right)}, & x \neq 0. \end{cases} \tag{17}$$

Now taking derivatives and evaluating at the grid points

$$\frac{\mathrm{d}}{\mathrm{d}x}S_{N'}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\sin\left(\pi x/h'\right)}{\left(2\pi/h'\right)\sin\left(x/2\right)}\right]\Bigg|_{x=jh'} \tag{18a}$$

$$= \frac{\left(2\pi\cos\left(\frac{\pi x}{h'}\right) - h'\cot\left(\frac{x}{2}\right)\sin\left(\frac{\pi x}{h'}\right)\right)}{4\pi\sin\left(x/2\right)}\Bigg|_{x=jh'} \tag{18b}$$

$$= \frac{2\pi\cos(\pi j) - h'\sin(\pi j)\cot\left(\frac{jh'}{2}\right)}{4\pi\sin\left(\frac{jh'}{2}\right)}. \tag{18c}$$

To obtain the behaviour at $x = 0$ we take the limit as $j \to 0$ and see that this limit is 0. Therefore we have that

$$\frac{\mathrm{d}}{\mathrm{d}x}S_{N'}(x) = \begin{cases} 0, & j \equiv 0 \mod N' \\ \dfrac{(-1)^j}{2\sin\left(jh'/2\right)}, & j \not\equiv 0 \mod N'. \end{cases} \tag{19}$$

Taking another derivative we have

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}S_{N'}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}}{\mathrm{d}x}S_{N'}(x)\right]\Bigg|_{x=jh'} \tag{20a}$$

$$= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\left(2\pi\cos\left(\frac{\pi x}{h'}\right) - h'\cot\left(\frac{x}{2}\right)\sin\left(\frac{\pi x}{h'}\right)\right)}{4\pi\sin\left({}^{x}/{}_{2}\right)}\right]\Bigg|_{x=jh'} \tag{20b}$$

$$= \frac{\left(-\sin\left(\frac{\pi x}{h'}\right)\left(\frac{h'^2(\cos(x)+3)}{\cos(x)-1} + 4\pi^2\right) - 4\pi h'\cot\left(\frac{x}{2}\right)\cos\left(\frac{\pi x}{h'}\right)\right)}{8\pi h'\sin\left({}^{x}/{}_{2}\right)}\Bigg|_{x=jh'} \tag{20c}$$

$$= -\frac{\sin(\pi j)\left(-2h'^2\csc^2\left(\frac{jh'}{2}\right) + h'^2 + 4\pi^2\right) + 4\pi h'\cos(\pi j)\cot\left(\frac{jh'}{2}\right)}{8\pi h'\sin\left({}^{jh'}/{}_{2}\right)}. \tag{20d}$$

To find the behaviour at $x = 0$ we take the limit as $j \to 0$ and see that this limit is $\frac{1}{12} - \frac{\pi^2}{3h'^2}$. Therefore we have that

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}S_{N'}(x) = \begin{cases} \dfrac{1}{12} - \dfrac{\pi^2}{3h'^2}, & j \equiv 0 \mod N' \\ -\dfrac{(-1)^j\cos\left(\frac{jh'}{2}\right)}{2\sin^2\left({}^{jh'}/{}_{2}\right)}, & j \not\equiv 0 \mod N'. \end{cases} \tag{21}$$

In the code `A3Q0.ipynb` I show that, indeed, $D^2_{N'} = D_{N'}D_{N'}$.

Part D

For a toeplitz circulant matrix the eigenvalues λ_j are computed via the relation

$$\lambda_j = c_0 + \sum_{m=1}^{N-1} c_{N-m}e^{i\frac{2\pi j}{N}m} \tag{22}$$

where the circulant matrix C is defined as,

$$C = \begin{bmatrix} c_0 & c_{n-1} & \cdots & c_2 & c_1 \\ c_1 & c_0 & c_{n-1} & & c_2 \\ \vdots & c_1 & c_0 & \ddots & \vdots \\ c_{n-2} & & \ddots & \ddots & c_{n-1} \\ c_{n-1} & c_{n-2} & \cdots & c_1 & c_0 \end{bmatrix}. \tag{23}$$

For $N = 2$ you have that,

$$\lambda_j = 0, \; j = 0, 1. \tag{24}$$

Therefore, the spectral radius of this matrix is $\rho(D_{N=2}) = 0$.
For $N = 4$ you have that,

$$\lambda_j = c_3e^{i\frac{\pi j}{2}} + c_2e^{i\frac{\pi j}{2}2} + c_1e^{i\frac{\pi j}{2}3}, \tag{25}$$

where here $c_1 = -\frac{1}{2}$, $c_2 = 0$, and $c_3 = \frac{1}{2}$. And so we have that $\lambda_1 = i$, $\lambda_2 = 0$, $\lambda_3 = -i$, and $\lambda_4 = 0$.
In the code `A3Q0.ipynb` I compute numerically the spectral radii of the differentiation matrices for both even and odd cases and indeed show that $\rho(D_{N=2}) = 0$, and that $\rho(D_{N=3}) = \rho(D_{N=4}) = 1$.

Problem 1

Part A

We observe from Fig. 1 that this method, for the given RHS, is of spectral order. The RHS function given is continuous on the interval $[-1, 1]$ and is also continuous when periodically extended across the real line. The RHS is in C^∞ and it's derivatives are 2-periodic on $[-1, 1]$ and thus we expect that we should acquire spectral order for the solution $v_j \approx u(x_j)$.

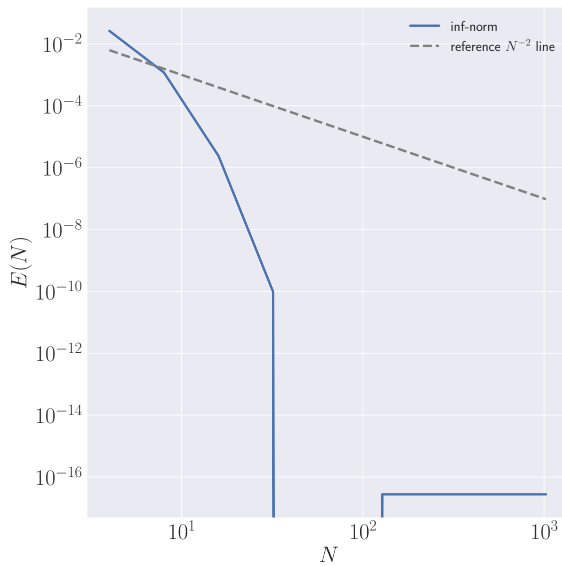

Figure 1: Error of the solution of the differential equation $u'' = 5\,(3 + 2\cos(\pi x))^{-1}$ as a function of the number of intervals on the mesh.

Part B

We observe from Fig. 2 that this method, for the given RHS, is of order 1. The RHS function is continuous on the interval $[-1, 1]$; however, its periodic extension across the real line is discontinuous at the edges. The first integral of the RHS function however is continuous on $[-1, 1]$ and also has a continuous periodic extension on the real line. From this we infer that the function u will also have continuous periodic extensions on the real line. From **[Tr]** we have, using Theorem 4 in Chapter 4, that

Theorem 1 *If u has p-1 continuous derivatives in $L^2(\mathbb{R})$ for some $p \geq \nu + 1$, where ν denotes the νth spectral derivative of u on the grid $h\mathbb{Z}$, and a pth derivative of bounded variation, then*

$$|w_j - u^{(\nu)}(x_j) = \mathcal{O}(h^{p-v}) \quad \text{as } h \to 0.$$

In this case we have $\nu = 1$ and $p = 2$, and thus we expect an error scaling $\mathcal{O}(h^1)$, which agrees with our observation from Fig. 2.

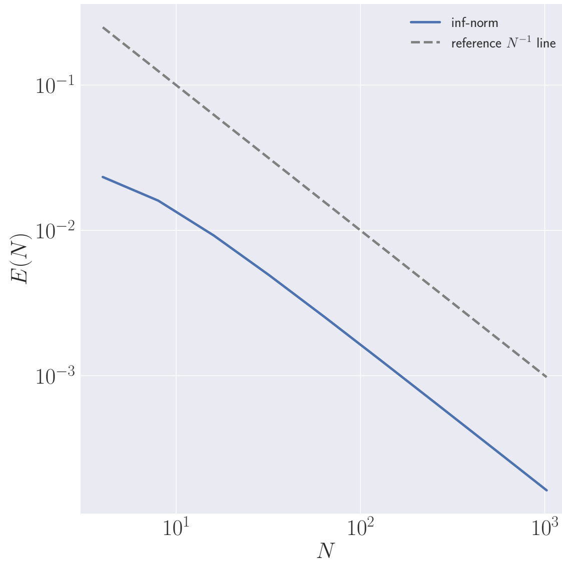

Figure 2: Error of the solution of differential equation $u'' = \sin(3\pi x/2)$ as a function of the number of intervals on the mesh.

Problem 2

Part A

The Discrete Fourier Transform is given by

$$u(x_j) = v_j = \sum_{k=-\frac{N}{2}+1}^{N/2} \hat{v}_k e^{ikx_j}. \tag{26}$$

The Fourier Expansion is given by

$$u(x_j) = \sum_{k'=-\infty}^{\infty} \hat{u}_{k'} e^{ik'x} \tag{27}$$

Writing the infinite sums as two sums, one over a finite range of wavenumbers k and another over the range of wavenumbers m not in that range (ie. write $k' = k + mN$),

$$u(x_j) = \sum_{k=-\frac{N}{2}+1}^{N/2} \sum_{m=-\infty}^{\infty} \hat{u}_{k+mN} e^{i[k+mN]x_j} \tag{28a}$$

$$= \sum_{k=-\frac{N}{2}+1}^{N/2} \sum_{m=-\infty}^{\infty} \hat{u}_{k+mN} e^{ikx_j} e^{imNx_j} \tag{28b}$$

$$= \sum_{k=-\frac{N}{2}+1}^{N/2} \left[\sum_{m=-\infty}^{\infty} \hat{u}_{k+mN} \right] e^{ikx_j}, \tag{28c}$$

where we used that $e^{imNx_j} = e^{im\frac{2\pi}{h}jh} = e^{i(mj)2\pi} = 1 \; \forall \; mj \in \mathbb{Z}$ in going from the penultimate to the last step. Thus equality between the Fourier Series Expansion and the Discrete Fourier Transform is held provided that

$$\hat{v}_k = \sum_{m=-\infty}^{\infty} \hat{u}_{k+mN}. \tag{29}$$

Computing the aliasing error

$$\begin{aligned}
R_N u(x) &= I_N u(x) - T_N u(x) && \text{(30a)}\\
&= \sum_{k=-M}^{M} \hat{v}_k e^{ikx} - \sum_{k=-M}^{M} \hat{u}_k e^{ikx} && \text{(30b)}\\
&= \sum_{k=-M}^{M} \left[\hat{v}_k - \hat{u}_k\right] e^{ikx} && \text{(30c)}\\
&= \sum_{k=-M}^{M} \left[\left(\sum_{m=-\infty}^{\infty} \hat{u}_{k+mN}\right) - \hat{u}_k\right] e^{ikx} && \text{(30d)}\\
&= \sum_{k=-M}^{M} \left(\sum_{m=-\infty, m\neq 0}^{\infty} \hat{u}_{k+mN}\right) e^{ikx}. && \text{(30e)}
\end{aligned}$$

To show orthogonality of the aliasing and truncation errors, we compute their inner product

$$\begin{aligned}
\int_0^{2\pi} \mathrm{d}x\, R_N u(x) \left[u(x) - T_N u(x)\right] &= \int_0^{2\pi} \mathrm{d}x \left\{ \sum_{k=-M}^{M} \left(\sum_{m=-\infty, m\neq 0}^{\infty} \hat{u}_{k+mN}\right) e^{ikx} \right\} \left\{ u(x) - \sum_{k=-M}^{M} \hat{u}_k e^{ikx} \right\} && \text{(31a)}\\
&= \int_0^{2\pi} \mathrm{d}x \left\{ \sum_{k=-M}^{M} \left(\sum_{m=-\infty, m\neq 0}^{\infty} \hat{u}_{k+mN}\right) e^{ikx} \right\} \left\{ \sum_{k=-\infty}^{\infty} \hat{u}_k e^{ikx} - \sum_{k=-M}^{M} \hat{u}_k e^{ikx} \right\} && \text{(31b)}\\
&= \int_0^{2\pi} \mathrm{d}x \left\{ \sum_{k=-M}^{M} \left(\sum_{m=-\infty, m\neq 0}^{\infty} \hat{u}_{k+mN}\right) e^{ikx} \right\} \left\{ \sum_{k'=-\infty}^{\infty} \hat{u}_{k'} e^{ik'x} \right\}, && \text{(31c)}
\end{aligned}$$

where we have that $|k| < |k'|$ (ie. the sum in the second $\{\cdot\}$ doesn't include any modes in the interval $|k| \leq M$). Expanding gives,

$$\begin{aligned}
&\int_0^{2\pi} \mathrm{d}x \left\{ \sum_{k=-M}^{M} \left(\sum_{m=-\infty, m\neq 0}^{\infty} \hat{u}_{k+mN}\right) e^{ikx} \right\} \left\{ \sum_{k'=-\infty}^{\infty} \hat{u}_{k'} e^{ik'x} \right\} && \text{(32a)}\\
&= \int_0^{2\pi} \mathrm{d}x \left\{ \sum_{k=-M}^{M} \sum_{k'=-\infty}^{\infty} \sum_{m=-\infty, m\neq 0}^{\infty} \hat{u}_{k+mN} \hat{u}_{k'} e^{i(k+k')x} \right\} && \text{(32b)}
\end{aligned}$$

$$= \sum_{k=-M}^{M} \sum_{k'=-\infty}^{\infty} \sum_{m=-\infty, m\neq 0}^{\infty} \int_0^{2\pi} \mathrm{d}x\, \hat{u}_{k+mN} \hat{u}_{k'} e^{i(k+k')x}. \tag{32c}$$

The integral will be nonzero in value only when $k = -k'$ by orthogonality of complex exponentials; however, this will never be satisfied as the domain of k is disjoint from the domain of k' and therefore the integral is always zero. Thus, we have that

$$\int_0^{2\pi} \mathrm{d}x\, R_N u(x) \left[u(x) - T_N u(x)\right] = 0 \tag{33}$$

and thereby the two errors are orthogonal. Now if we take the equality

$$u(x) - I_N u(x) = (u(x) - T_N u(x)) - R_N u(x), \tag{34}$$

square both sides,

$$\left[u(x) - I_N u(x)\right]^2 = \left[(u(x) - T_N u(x)) - R_N u(x)\right]^2 \tag{35a}$$

$$u(x)^2 - 2\left[I_N u(x)\right] u(x) + \left[I_N u(x)\right]^2 = (u(x) - T_N u(x))^2 - 2(u(x) - T_N u(x)) R_N u(x) + \left[R_N u(x)\right]^2, \tag{35b}$$

$$\tag{35c}$$

and then integrate on the domain,

$$\int_0^{2\pi} \mathrm{d}x\, \left[u(x) - I_N u(x)\right]^2 = \int_0^{2\pi} \mathrm{d}x\, (u(x) - T_N u(x))^2 - 2\int_0^{2\pi} \mathrm{d}x\, (u(x) - T_N u(x)) R_N u(x) + \int_0^{2\pi} \mathrm{d}x\, \left[R_N u(x)\right]^2 \tag{36a}$$

$$||u(x) - I_N u(x)||^2 = ||u(x) - T_N u(x)||^2 + ||R_N u(x)||^2, \tag{36b}$$

where the cross-term on the right-hand side vanishes via the orthogonality.

Part B

We see in Fig. 3, for the first and second functions ($u_1(x)$ and $u_2(x)$, respectively) that the interpolant is behaving as expected. It is going through the grid points; however, for lower values of N it tends to not be a very good approximation for the function. For the last function u_3 we observe that the interpolant is very badly behaved. This behaviour is arising due to aliasing of the underlying function. The underlying function is sinusoidal with wavenumber $k = 32$. Thus, in order to represent this using its Fourier series, the series must contain at least this mode and all others beneath it.

From Fig. 4, we observe that for the functions $u_1(x)$ and $u_2(x)$ that while the interpolant is directly equivalent to the function u on the grid, the truncated Fourier series doesn't interpolate those points exactly.

In `A3Q2.ipynb` I show that $\hat{v}_k \neq \hat{u}_k$ for the first case of $u_1(x)$; however, I am unable to show numerically the folding of the frequencies outside the Nyquist bandwidth.

In Fig. 5 we see the interpolation error (blue lines) and the truncation error (green lines) for the functions $u_1(x)$, $u_2(x)$, and $u_3(x)$. In the left subplot we observe that for $u_1(x)$ the

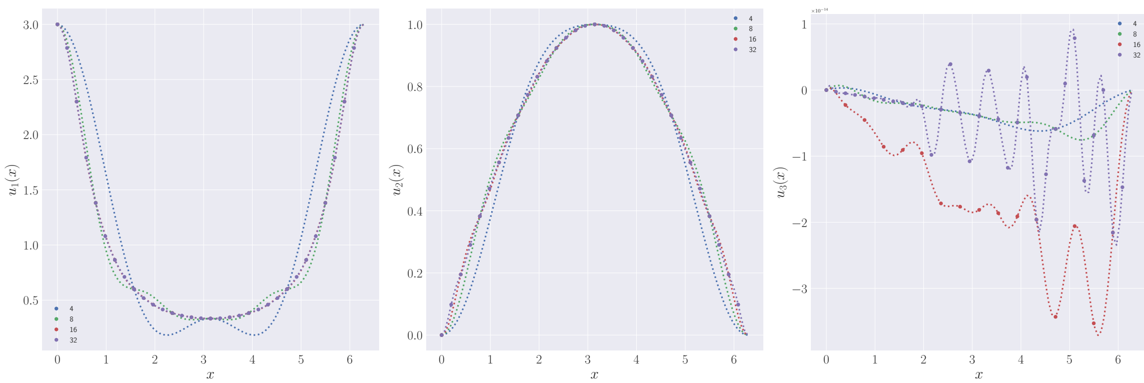

Figure 3: Interpolant calculated for the function $u_1(x) = 3\,(5 - 4\cos x)^{-1}$ [left subplot], $u_2(x) = \sin(x/2)$ [middle subplot], $u_3(x) = \sin(32x)$ [right subplot] for number of grid points $N = 4, 8, 16, 32$. Data markers indicate the function evaluated on the grid points. Dotted lines denote the interpolant evaluated on a grid of 100 points.

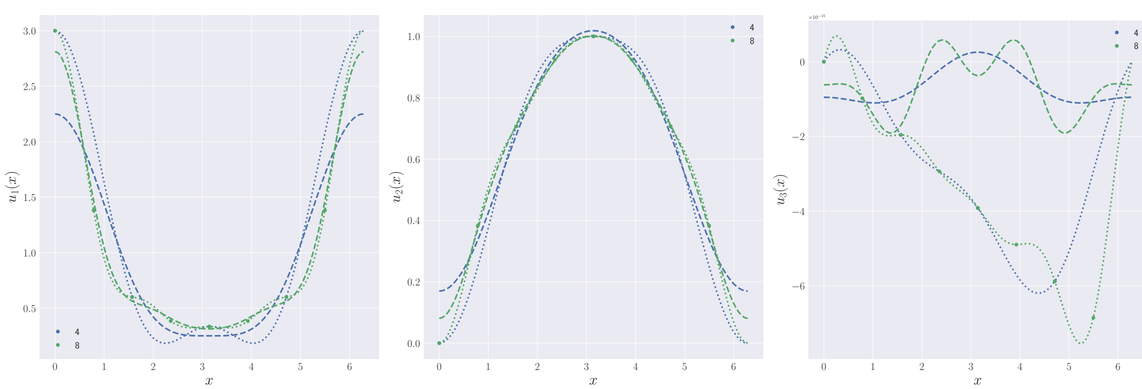

Figure 4: Interpolant and truncated Fourier Series calculated for the functions functions $u_1(x) = 3\,(5 - 4\cos x)^{-1}$ [left subplot], $u_2(x) = \sin(x/2)$ [middle subplot], $u_3(x) = \sin(32x)$ [right subplot] for number of grid points $N = 4, 8$. Data markers indicate the function evaluated on the grid points. Dotted lines denote the interpolant. Dashed lines denote the truncated Fourier Series. Different colors denote different number of *grid* points. Both interpolant and truncated Fourier Series are evaluated on a finer grid of 100 points.

error in both the interpolation and truncation decrease at the same rate initially; however, at $N > 100$ the interpolation error remains low while the truncation error increases drastically. This is a manifestation of spectral blocking where noise accumulates heavily in the largest wavenumbers. In the middle subplot, for $u_2(x)$ we observe that the error scaling is polynomial (the behaviour does change at larger N). This is due to the fact that the function's periodic extension is discontinuous and thus we expect that the error would not have spectral convergence. In the right subplot, for $u_3(x)$ we see that the interpolation and the truncation error stay constant for until $N > 64$ when the Nyquist criterion becomes fulfilled and the interpolation error decreases significantly; however, the truncation error stays constant.

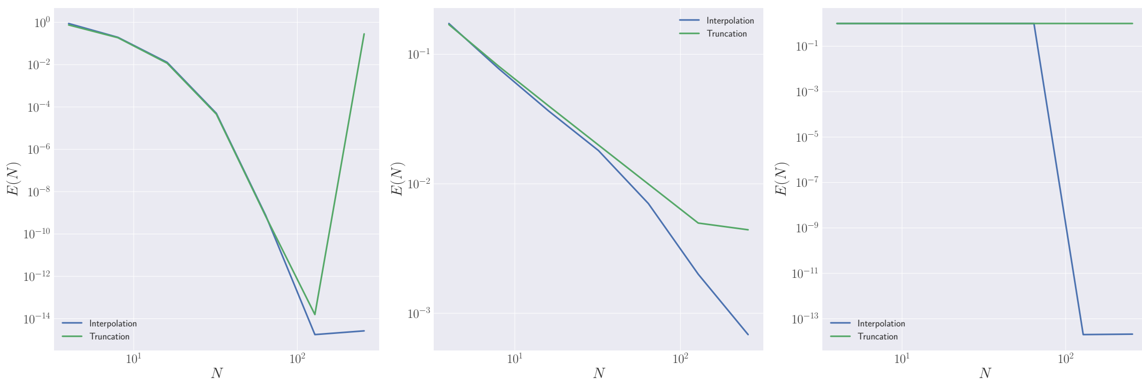

Figure 5: Error of the interpolation (blue line) and truncated Fourier Series (green line) calculated for the functions $u_1(x) = 3\,(5 - 4\cos x)^{-1}$ [left subplot], $u_2(x) = \sin(x/2)$ [middle subplot], $u_3(x) = \sin(32x)$ [right subplot].

Problem 3

Part A

Taking the logarithm of the Lagrange Polynomial we get

$$\log(L_j(x)) = \log\left[\frac{\prod_{k=0,k\neq j}^{N}(x - x_k)}{\prod_{k=0,k\neq j}^{N}(x_j - x_k)}\right] \tag{37a}$$

$$= \log\left[\prod_{k=0,k\neq j}^{N}\frac{(x - x_k)}{(x_j - x_k)}\right] \tag{37b}$$

$$= \sum_{k=0,k\neq j}^{N}\log\left(\frac{x - x_k}{x_j - x_k}\right). \tag{37c}$$

Now taking derivatives,

$$\frac{1}{L_j(x)}L_j'(x) = \sum_{k=0,k\neq j}^{N}\frac{x_j - x_k}{x - x_k}\left[\frac{1}{x_j - x_k}\right] \tag{38a}$$

$$\implies L_j'(x) = L_j(x)\sum_{k=0,k\neq j}^{N}\frac{1}{x - x_k}. \tag{38b}$$

Evaluating at points x_i to acquire the definitions for the entries of the differentiation matrix D. When $i \neq j$ we have

$$D_{ij} = L_j'(x_i) = L_j(x_i)\sum_{k=0,k\neq j}^{N}\frac{1}{x_i - x_k} \tag{39a}$$

$$= \left[\prod_{k=0,k\neq j}^{N} \frac{(x_i - x_k)}{(x_j - x_k)} \right] \sum_{k=0,k\neq j}^{N} \frac{1}{x_i - x_k} \tag{39b}$$

$$= \left[\prod_{k=0,k\neq j}^{N} \frac{1}{(x_j - x_k)} \right] \sum_{k=0,k\neq j}^{N} \left[\prod_{k=0,k\neq j}^{N} (x_i - x_k) \right] \frac{1}{x_i - x_k} \tag{39c}$$

$$= \frac{1}{a_j} \prod_{k=0,k\neq i,j}^{N} (x_i - x_k) \tag{39d}$$

$$= \frac{1}{a_j} \prod_{k=0,k\neq i}^{N} \frac{(x_i - x_k)}{(x_i - x_j)} \tag{39e}$$

$$= \frac{1}{a_j} \frac{\prod_{k=0,k\neq i}^{N} (x_i - x_k)}{(x_i - x_j)} \tag{39f}$$

$$= \frac{1}{a_j} \frac{a_i}{(x_i - x_j)} \tag{39g}$$

When $i = j$ we have

$$D_{ii} = L_i(x_i) = L_i(x_i) \sum_{k=0,k\neq j}^{N} \frac{1}{x_i - x_k} \tag{40a}$$

$$= \left[\prod_{k=0,k\neq j}^{N} \frac{(x_i - x_k)}{(x_i - x_k)} \right] \sum_{k=0,k\neq j}^{N} \frac{1}{x_i - x_k} \tag{40b}$$

$$= \sum_{k=0,k\neq j}^{N} \frac{1}{x_i - x_k} \tag{40c}$$

Taking a summation over the rows gives

$$\sum_{j=0,j\neq i}^{N} D_{ij} = \sum_{j=0,j\neq i}^{N} \frac{1}{a_j} \frac{a_i}{(x_i - x_j)} \tag{41a}$$

$$= a_i \sum_{j=0,j\neq i}^{N} \left[\prod_{k=0,k\neq j}^{N} \frac{1}{(x_j - x_k)} \right] \frac{1}{(x_i - x_j)} \tag{41b}$$

Finding the interpolant to the vector of ones, $\mathbf{a}$,

$$p(x) = \sum_{j=0}^{N} L_j(x) a_j \tag{42a}$$

$$= \sum_{j=0}^{N} L_j(x). \tag{42b}$$

Evaluating at any given grid point x_i, gives

$$p(x_i) = \sum_{j=0}^{N} L_j(x_i) \tag{43a}$$

$$= \sum_{j=0}^{N} \delta_{ij} \tag{43b}$$

$$= 1. \tag{43c}$$

This has to be the case, as if the matrix D is associated with differentiation, then a differentiation of a constant function must be zero. Since any constant function, evaluated on a grid, can be represented by $f(x_i) = c\mathbf{a}$ then it follows that for D to be an approximation to a differentiation matrix, that $D\mathbf{a} = \mathbf{0}$.

Part C

For the $N = 5$ the spectral radius is calculated to be ≈ 0 modulo roundoff error; however, for $N = 20$ the spectral radius of the Chebyshev matrix is ≈ 3.38. Since the spectral radius provides a lower bound on the magnitude of the 2-norm, then this shows that the Chebyshev matrix will never achieve a matrix-2norm value of less than 3.38 and therefore, will not satisfy $D_{20}^{21} = 0$. This arises from the fact that the matrix is poorly-conditioned.