Quadrotor Dynamic Modeling

UAVs

MEAer - Spring Semester – 2020/2021

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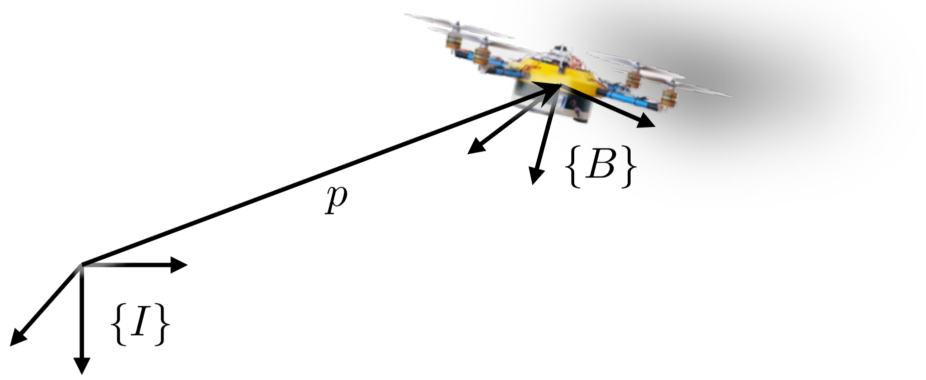
Quadrotor Dynamic Modeling

- Goal: derive the equations of motion for a quadrotor vehicle
- Starting point: rigid-body equations of motion
  - Kinematics

$$\dot{p} = Rv$$
$$\dot{R} = RS(\omega)$$



  - Dynamics

$$m\dot{v} = -S(\omega)mv + f$$
$$J\dot{\omega} = -S(\omega)J\omega + n$$

- 1st question: what is the input?

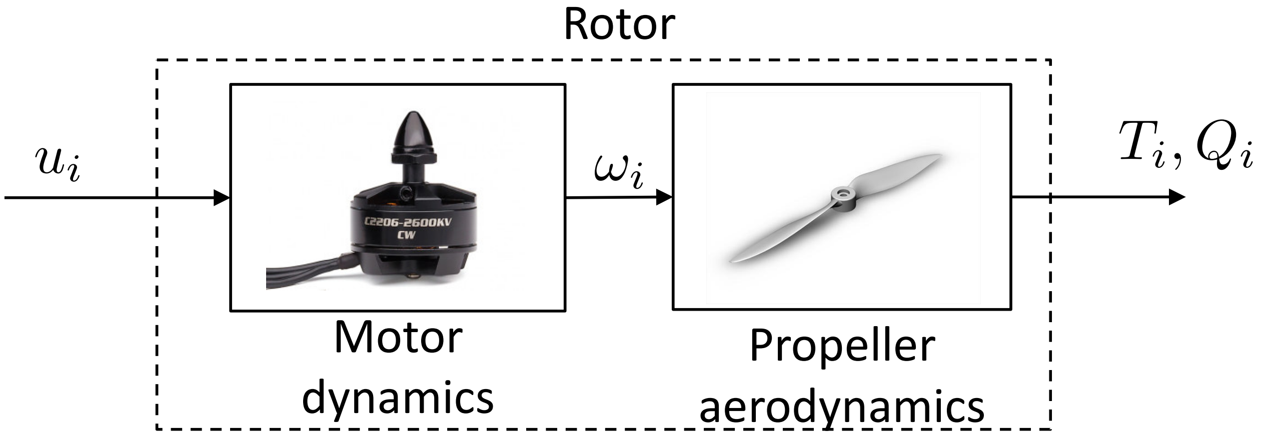
$$u = ?$$

- 2nd question: what are the generated forces and moments?

$$f = f(p, R, v, \omega, u) = ?$$
$$n = n(p, R, v, \omega, u) = ?$$

Quadrotor actuation – the inputs

- Four rotors: electric motor + propeller



  - Input: voltage u_i
  - Output: generated thrust and reaction torque T_i, Q_i

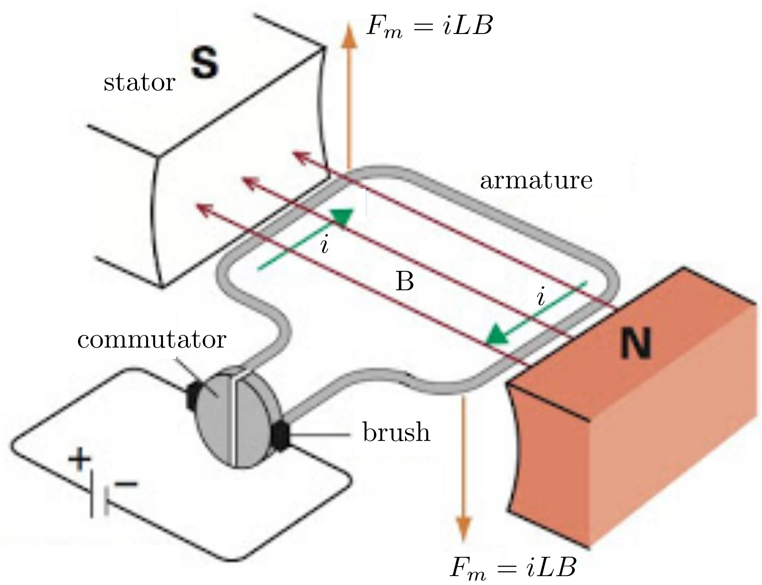
Motor Dynamics

- DC Motor Model
  - Mechanical part

$$I\frac{d\omega}{dt}(t) = k_t i(t) - \tau_{load}(t)$$

Load torque

Input torque: proportional to current flowing through the coil



  - Electrical part

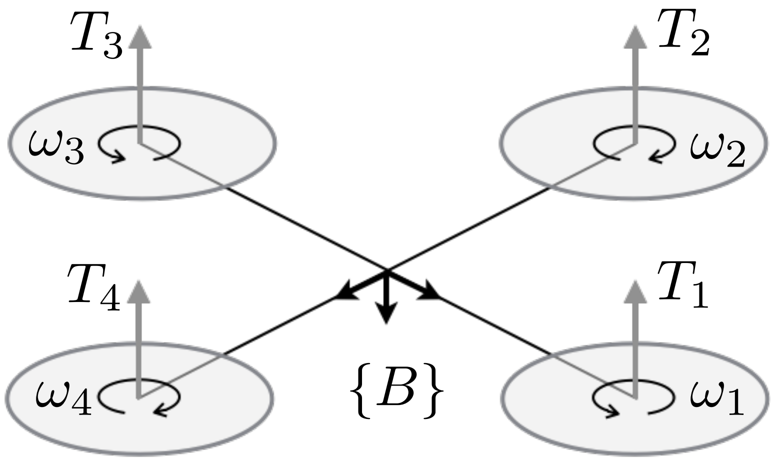
$$L\frac{di}{dt}(t) = u(t) - Ri(t) - k_b\omega(t)$$

Back electromotive force: proportional to angular velocity

Input voltage

Quadrotor actuation – the inputs

- Two pairs of counter-rotating rotors



- Thrust and reaction torque
  - Approximately proportional to square of angular velocity

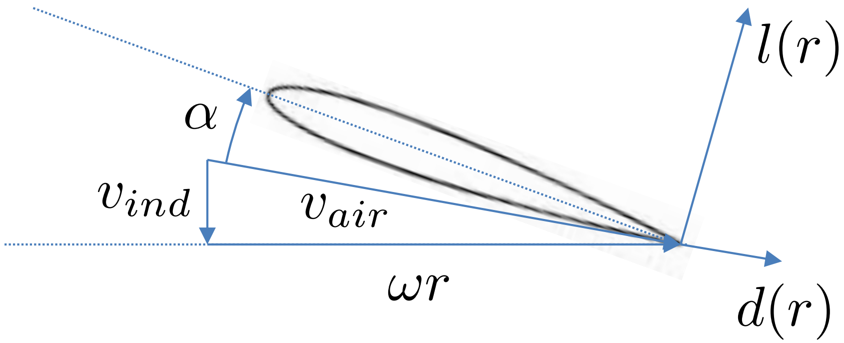
$$T_i = c_T \omega_i^2$$

$$Q_i = c_Q \omega_i^2 (-1)^{i+1}$$

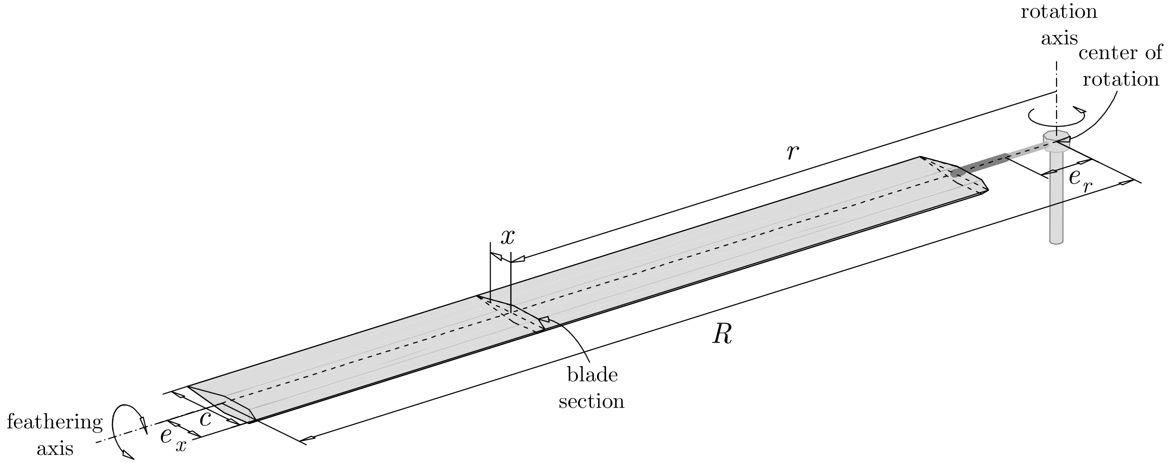
Lift and Drag at Blade Element

- Lift and Drag at Blade Element

$$l(r) = \frac{\rho}{2} v_{air}^2 c c_l$$
$$d(r) = \frac{\rho}{2} v_{air}^2 c c_d$$

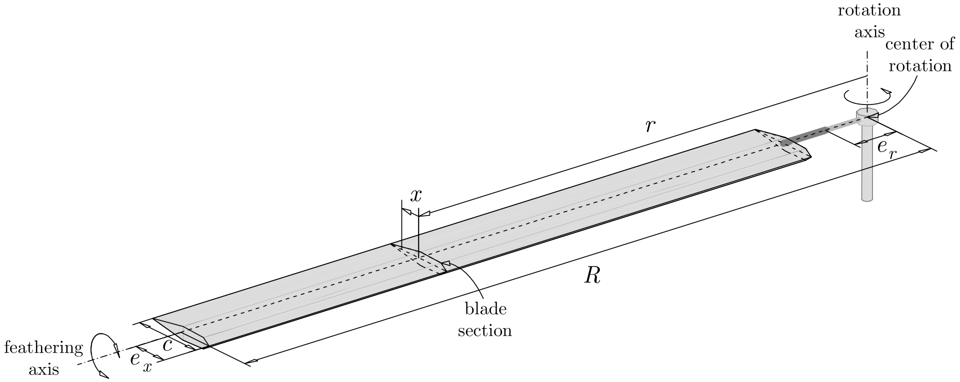


- ρ - air density
- $v_{air} = [\omega_i r \quad v_{ind}]^T$
- c - chord
- $c_l \approx a_0 \alpha$ - lift coefficient
- c_d - drag coefficient



Propeller Thrust and Reaction Torque

- Integrate lift and drag components over r
- Sum over number of blades N_b



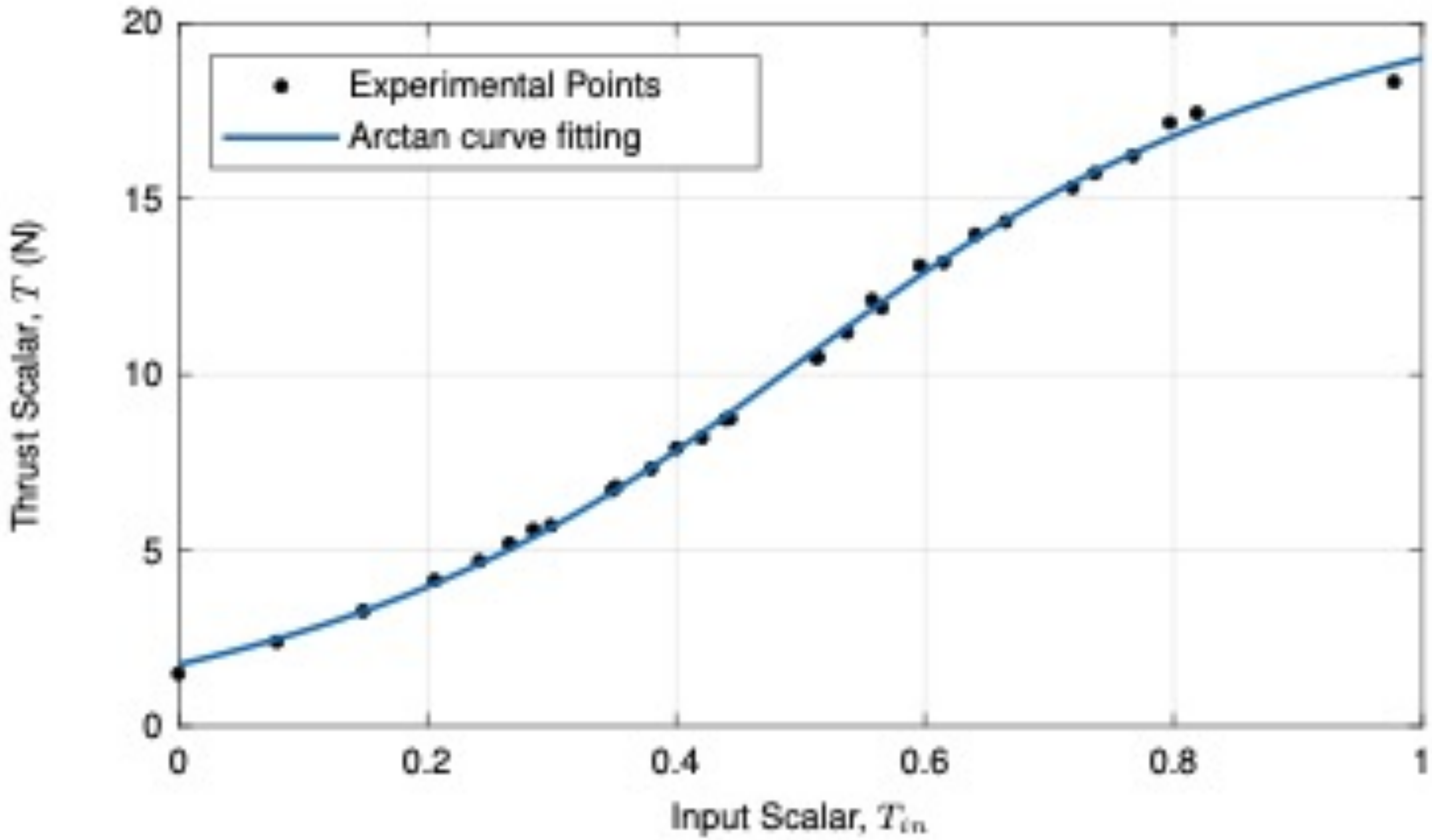
$$T_i = \sum_{j=1}^{N_b} \int_0^R l_j dr = N_b \frac{\rho}{2} R^3 a_0 \left(\frac{\theta}{3} - \frac{v_{ind}}{2} \right) \omega_i^2 = c_T \omega_i^2$$

$$Q_i = \sum_{j=1}^{N_b} \int_0^R (-l_j \alpha_j + d_j) r dr = c_Q \omega_i^2$$

- Thrust and torque constants can be identified experimentally – static thrust tests with different payloads

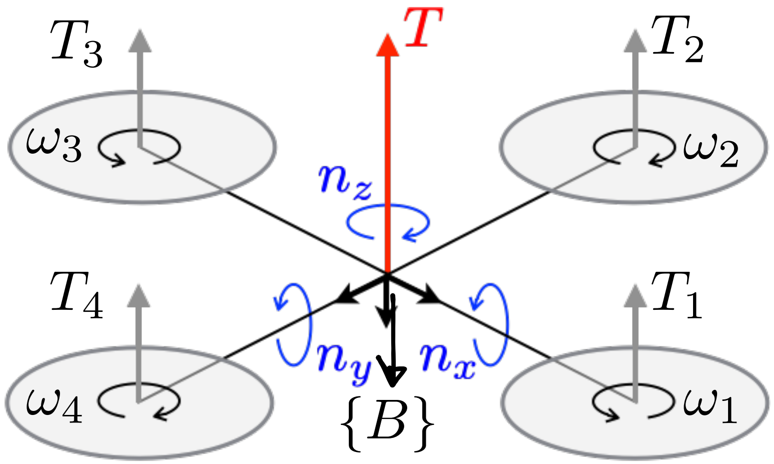
Experimental Thrust Curve Fitting

- Intel Aero drone



Quadrotor input forces and moments

- Two pairs of counter-rotating rotors



$$T = \sum_i T_i$$

$$n_x = l(T_2 - T_4)$$

$$n_y = l(T_1 - T_3)$$

$$T_i = c_T \omega_i^2$$

$$Q_i = (-1)^{i+1} c_Q \omega_i^2$$

$$Q_i = (-1)^{i+1} \frac{c_Q}{c_T} T_i$$

$$n_z = \frac{c_Q}{c_T}(T_1 - T_2 + T_3 + T_4)$$

$$\begin{bmatrix} T \\ n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & l & 0 & -l \\ l & 0 & -l & 0 \\ c & -c & c & -c \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix}$$

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Quadrotor input forces and moments

$e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

- Thrust T and moments n_x, n_y, and n_z expressed in {B}
- Rigid-body dynamics expressed in {B}
  - Near hover condition: negligible aerodynamic drag

$$m\dot{v} = \boxed{-S(\omega)mv} + f$$
$$J\dot{\omega} = -S(\omega)J\omega + n$$

$$\underline{f = -Te_3 + mgR^T e_3}$$
$$= \begin{bmatrix} 0 \\ 0 \\ -T \end{bmatrix} + {}^B_I R \underline{\begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix}}$$

$$n = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}$$

  - Exercise: express the translational dynamics in {I}

$$m\ddot{p} = ? \quad = {}^I f = mge_3 - TRe_3$$

Quadrotor dynamic model

- Translational dynamics

(p, v)

$$\dot{p} = Rv$$
$$m\dot{v} = -S(\omega)mv - Te_3 + mgR^T e_3$$

or

$(p, \dot{p})$

$$m\ddot{p} = -TRe_3 + mge_3$$

- Rotational dynamics

$$\dot{R} = RS(\omega)$$
$$J\dot{\omega} = -S(\omega)J\omega + n$$

- Compact form

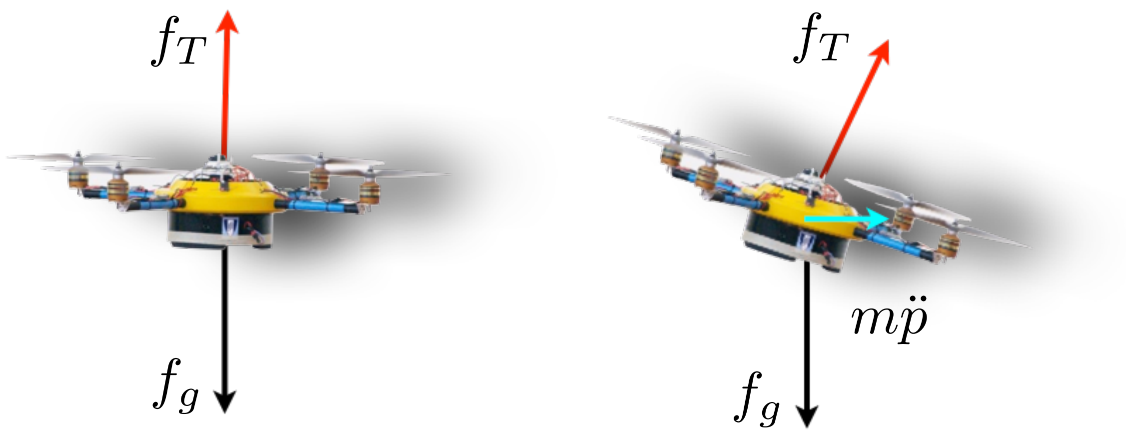
$$\dot{x} = f(x, u) \qquad x = (p, v, R, \omega)$$
$$u = (T, n_x, n_y, n_z)$$

Underactuated System:
6 DoFs, 4 inputs

Quadrotor dynamic model

- Underactuated system
  - 6 DoFs, 4 inputs
  - What trajectories are feasible? Can only prescribe 4 DoFs.

$$m\ddot{p} = f_T + f_g \qquad f_T = -TRe_3 \qquad f_g = mge_3$$



$$m\ddot{p} = 0$$
$$f_T + f_g = 0$$

$$m\ddot{p} \neq 0$$
$$f_T + f_g = m\ddot{p}$$

The quadrotor as a flat system

- What is a flat system?
  - State and inputs $(x(t), u(t))$ are completely characterized by an output and its time derivatives

    $$(x(t), u(t)) = f(y(t), \dot{y}(t), \ddot{y}(t), \ldots)$$

  - The *flat* output has the same size of the input

    $$u(t) \in \mathbb{R}^m \;\Rightarrow\; y(t) \in \mathbb{R}^m$$

- Flat output for a quadrotor

  $$y = (p, \psi) \in \mathbb{R}^4$$

  - Recall that $\quad x = (p, v, R, \omega) \qquad u = (T, n_x, n_y, n_z)$
  - It can be shown that $\;(x, u) = f(p, \dot{p}, \ddot{p}, p^{(3)}, p^{(4)}, \psi, \dot{\psi}, \ddot{\psi})$

The quadrotor as a flat system - Exercise

- Flat output for a quadrotor

$$y = (p, \psi) \in \mathbb{R}^4$$

  - Quadrotor dynamic model

$$\begin{aligned} \dot{p} &= Rv \\ m\dot{v} &= -S(\omega)mv - Te_3 + mgR^T e_3 \\ \dot{R} &= RS(\omega) \\ J\dot{\omega} &= -S(\omega)J\omega + n \end{aligned}$$

  - Show that the thrust and rotation matrix can be written as functions of $(y, \dot{y}, \ddot{y}, \ldots)$

$$T = f_1(y, \dot{y}, \ddot{y}, \ldots) = ? \qquad R = f_2(y, \dot{y}, \ddot{y}, \ldots) = ?$$

Suggestion: use $m\ddot{p} = -TRe_3 + mge_3$

The quadrotor as a flat system - Exercise

- Circular motion:

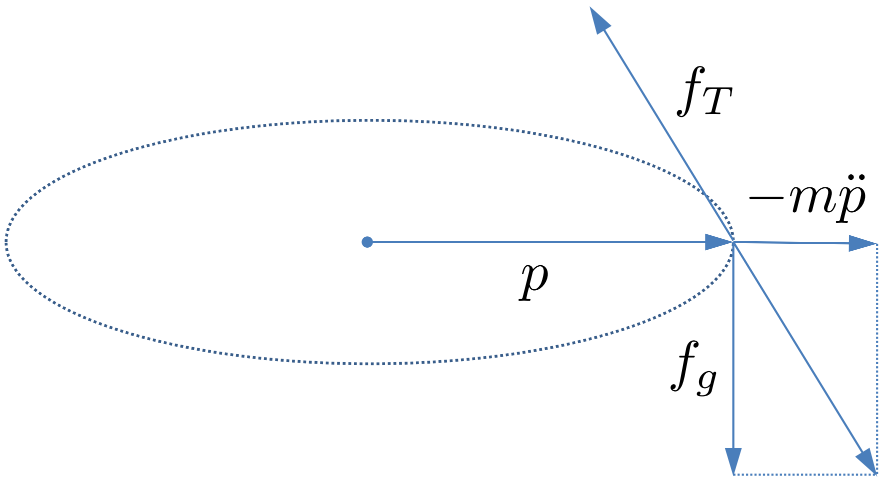
$$p(t) = {}^I p_B(t) = r \begin{bmatrix} \cos\psi(t) \\ \sin\psi(t) \\ 0 \end{bmatrix}$$

$$R(t) = {}^I_B R(t) = R_z(\psi(t)), \qquad \dot{\psi}(t) = c$$

  – Determine the roll and pitch Euler angles
    Suggestion: use

$$m\ddot{p} = f_T + f_g$$
$$f_T = -TRe_3$$
$$f_g = mge_3$$

$$R_y(\theta)R_x(\phi)e_3 = \begin{bmatrix} \cos\phi\sin\theta \\ -\sin\phi \\ \cos\phi\cos\theta \end{bmatrix}$$



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Design Considerations

The SWAP (Size, Weight, and Power) tradeoff

- Thrust/Weight Ratio
- Power consumption and Flight time
- Size vs Agility

Design considerations

- Maximum Thrust
  - Limited by maximum motor torque $Q_{\max}$

$$T = c_T \omega^2$$

$$|Q| = c_Q \omega^2$$

$$T_{\max} = \frac{c_T}{c_Q} Q_{\max}$$

- Thrust/Weight Ratio

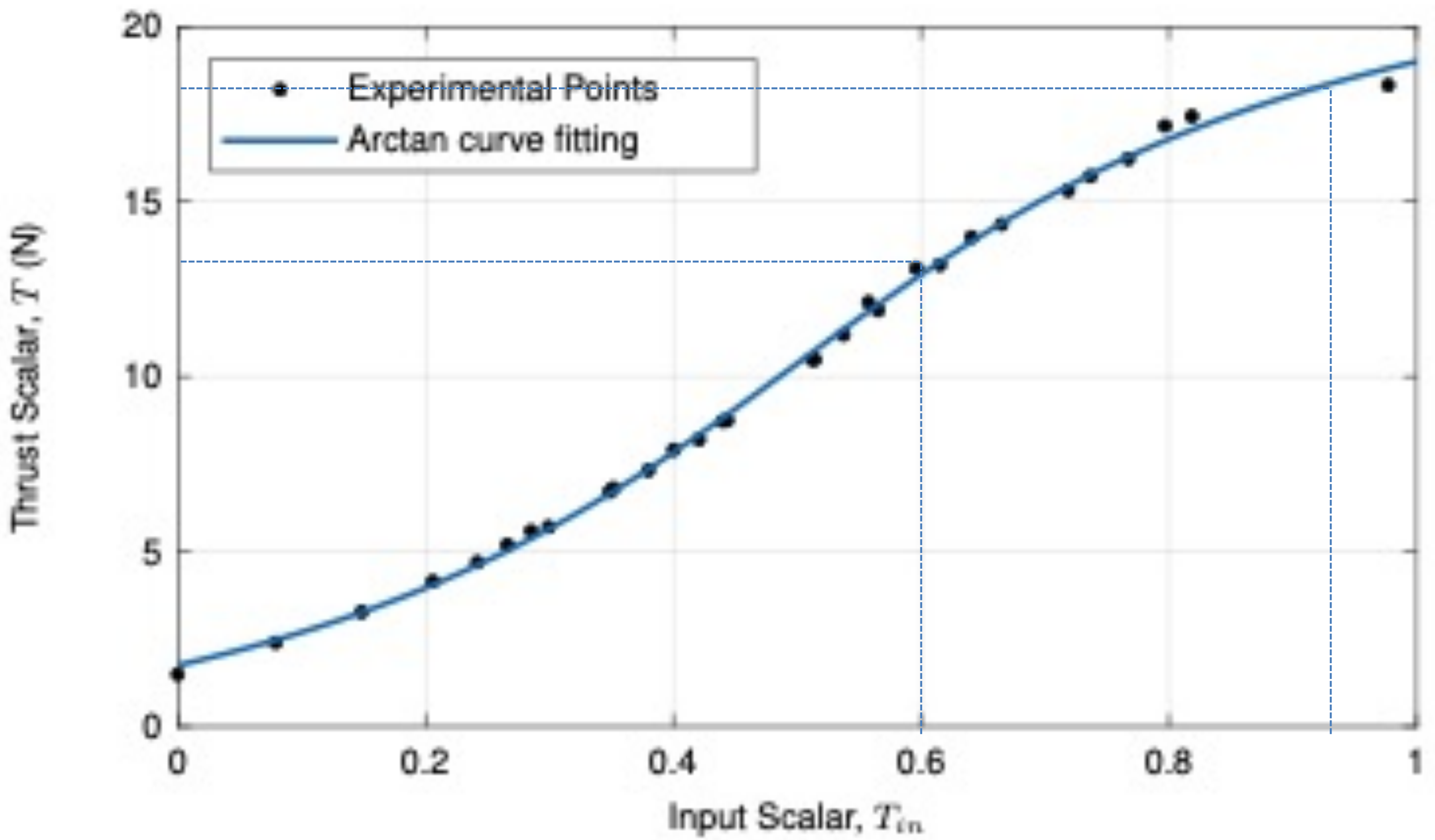
$$T_{hover} = \frac{1}{4} mg$$

$$\frac{T_{\max}}{T_{hover}} = \frac{4T_{\max}}{mg} = 4\frac{c_T}{c_Q}\frac{Q_{\max}}{mg}$$

  - Typically no greater than 2
  - Lower values -> slower response, less control authority

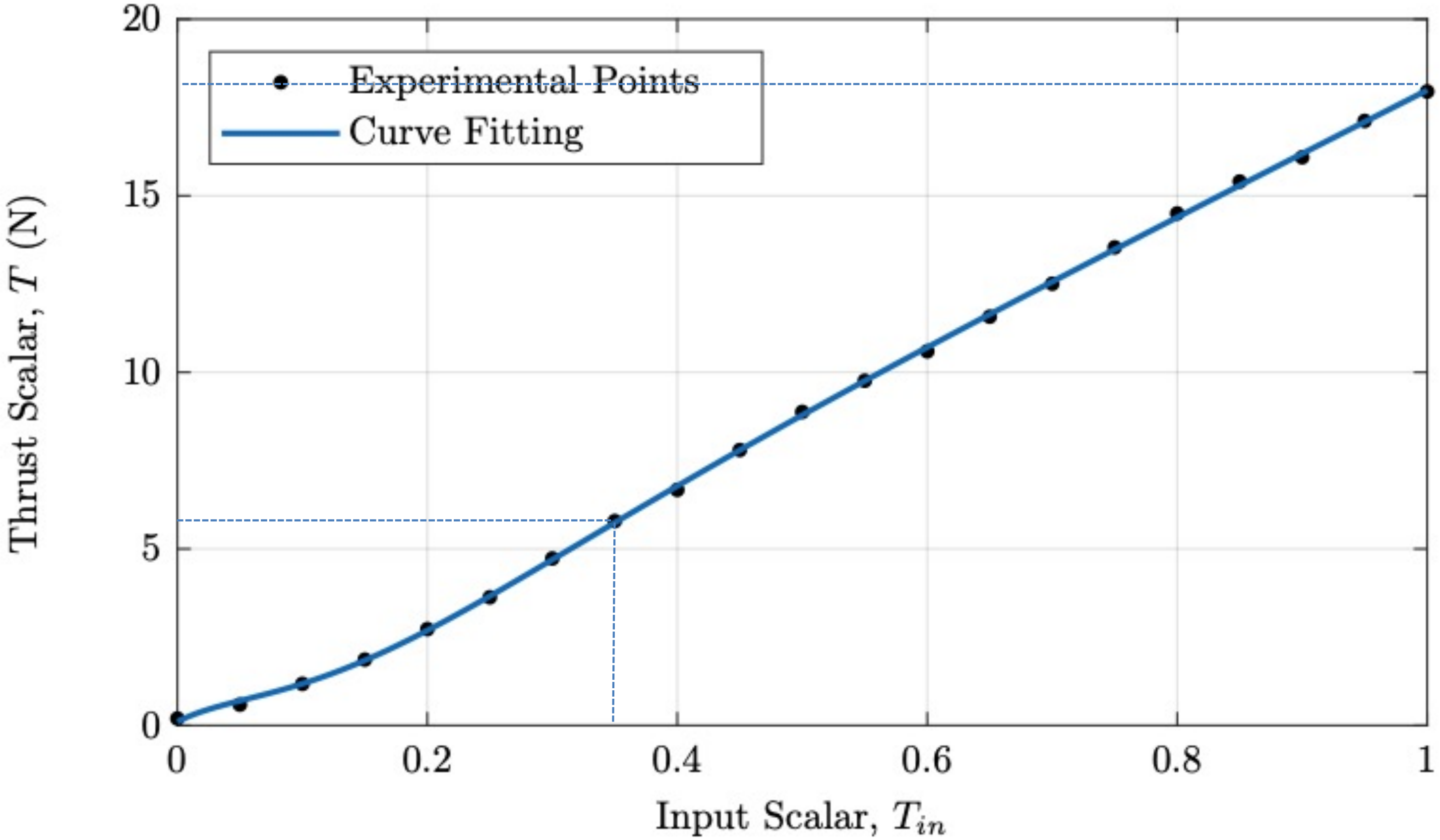
Experimental Thrust Curve Fitting

- Intel Aero drone
  - Maximum Thrust to Weight ratio $$\frac{T_{\max}}{T_{hover}} \approx \frac{18}{mg} = 1.36$$



Experimental Thrust Curve Fitting

- Snapdragon
  - Maximum Thrust to Weight ratio $\dfrac{T_{\max}}{T_{hover}} \approx \dfrac{18}{5.6} = 3.2$



Design considerations

- Power consumption (for each rotor)

$$T = c_T \omega^2$$
$$|Q| = c_Q \omega^2$$

$$\omega = \sqrt{\frac{T}{c_T}}$$
$$|Q| = \frac{c_Q T}{c_T}$$

$$p_{motor} = \omega Q = c_Q \left(\frac{1}{c_T} T\right)^{3/2}$$

Power proportional to $T^{3/2}$

– Maximum flight time

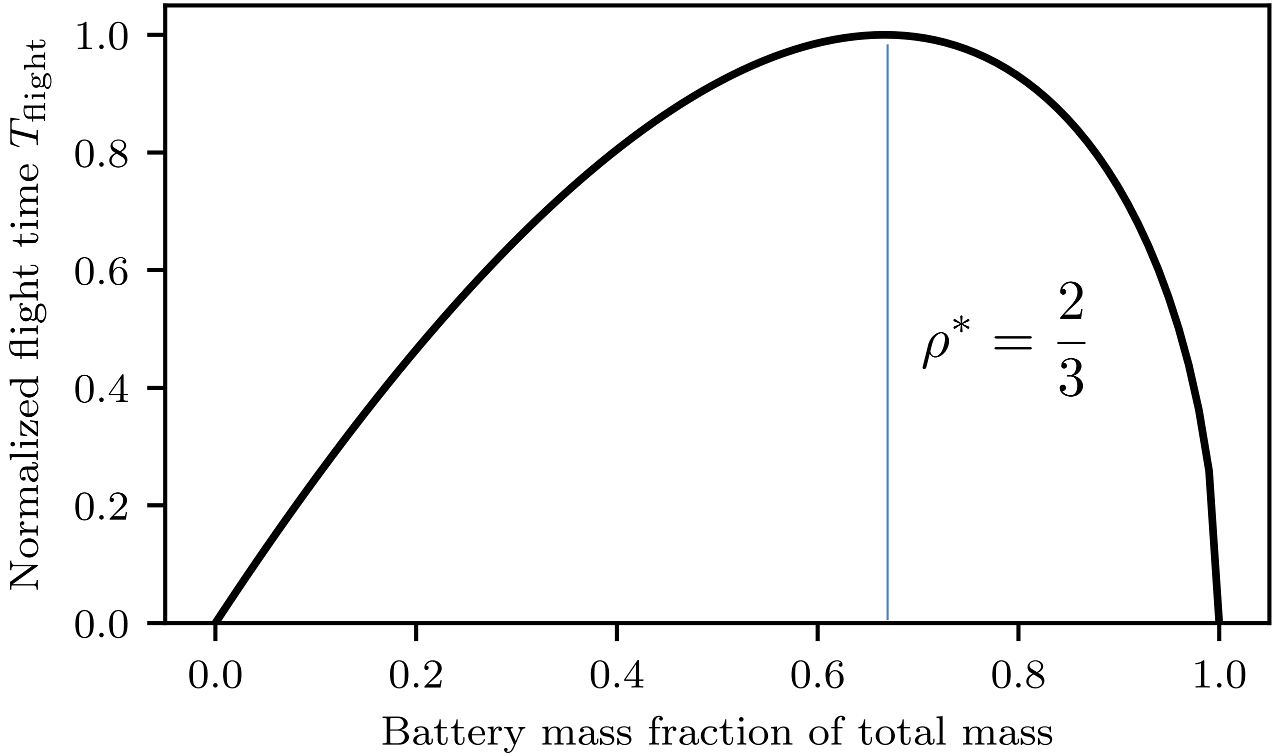
$$t_{flight} = \frac{E_{batt}}{4p_{motor} + p_{payload}}$$

- Neglecting payload consumption and assuming hovering condition

$$t_{flight} = \frac{E_{batt}}{p_{hover}} = \frac{E_{batt}}{\frac{c_Q}{2}(\frac{1}{c_T} mg)^{3/2}}$$

$$t_{flight} \propto \frac{m_{batt}}{m^{3/2}}$$

Power consumption and battery selection



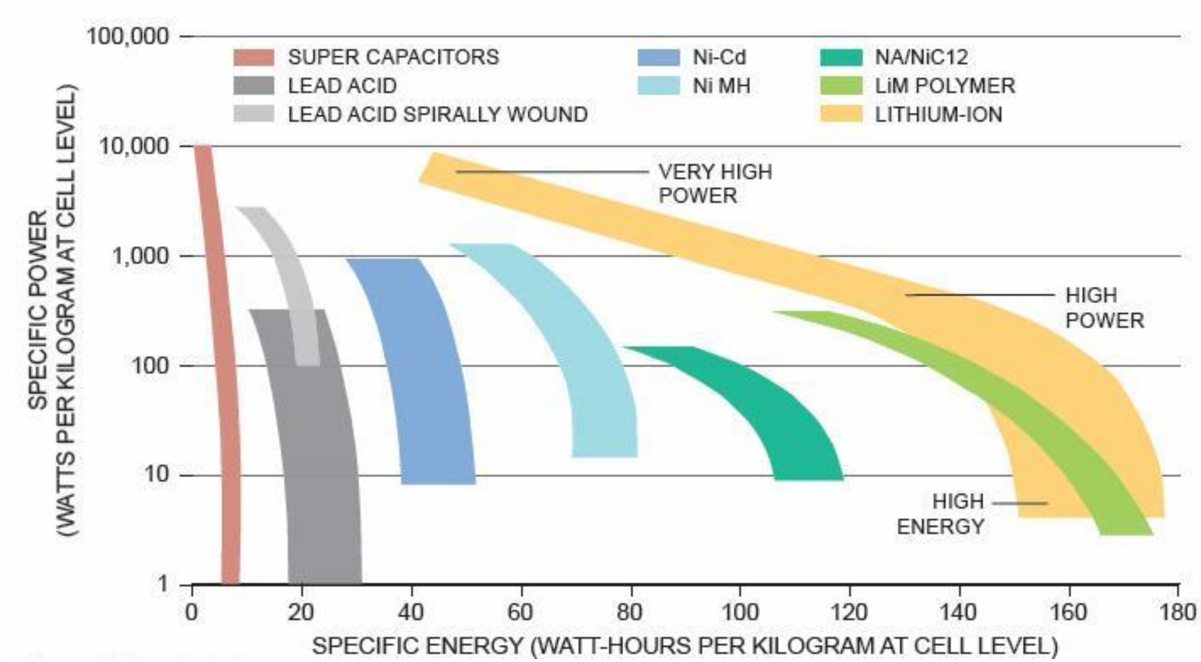
$$t_{flight} \propto \frac{m_{batt}}{m^{3/2}} \qquad \rho = \frac{m_{batt}}{m} \Rightarrow t_{flight} \propto \frac{\rho\sqrt{1-\rho}}{m_0},\ m_0 = m - m_{batt}$$

Source: https://spectrum.ieee.org/automaton/robotics/drones/swappable-flying-batteries-keep-drones-aloft-almost-forever

Design considerations

- Battery specs
  - Specific power vs specific energy

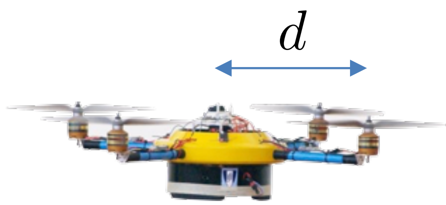
Increases
Thrust/Weight
ratio



Increases
Flight time

Design considerations

- Size
  - Characteristic length d



  - Inertia $m \propto d^3, \; J \propto d^5$
  - Rotor speed $r \propto d$
    - Mach scaling $\longrightarrow$ $\omega_i \propto \frac{1}{\sqrt{d}}$
    - Froude scaling $\longrightarrow$ $\omega_i \propto \frac{1}{d}$
  - Angular acceleration $\dot{\omega} \propto \frac{Td}{J} \propto \frac{(\omega_i^2 d^4)d}{d^5} = \omega_i^2$

$$\dot{\omega} \propto \frac{1}{d^2}$$

$$\dot{\omega} \propto \frac{1}{d}$$

  - Conclusion: reducing size increases agility

Source: R. Mahony, V. Kumar and P. Corke, "Multirotor Aerial Vehicles: Modeling, Estimation, and Control of Quadrotor," in IEEE Robotics & Automation Magazine, vol. 19, no. 3, pp. 20-32, Sept. 2012.