Sensors and State Estimation

(additional notes)

UAVs

MEAer - Spring Semester – 2020/2021

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Accelerometers – basic principle

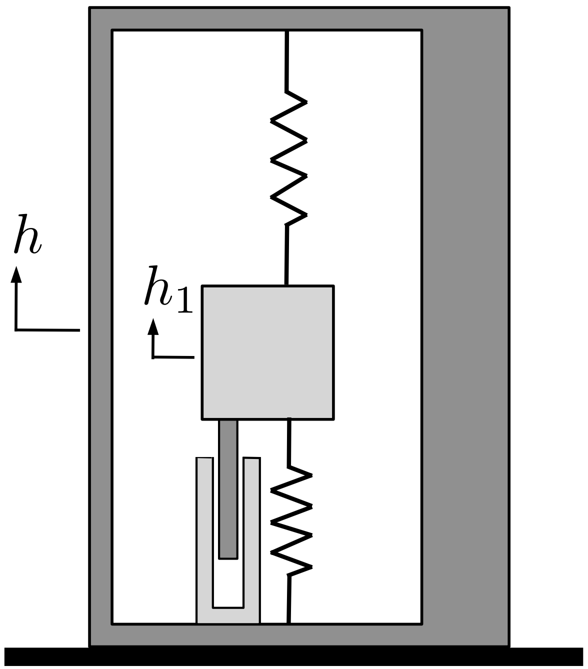
- h Inertial height of casing
- $h_1 = h + \delta$ Inertial height of proof mass
- δ Relative displacement

$$m\ddot{h}_1 = -k\delta(h_1 - h) - \beta(\dot{h}_1 - \dot{h}) - mg$$

  – change of variables

$$\delta = h_1 - h$$
$$z = -h$$

$$m(\ddot{h} + \ddot{\delta}) = -k\delta - \beta\dot{\delta} - mg$$
$$m\ddot{\delta} + \beta\dot{\delta} + k\delta = m(\ddot{z} - g)$$

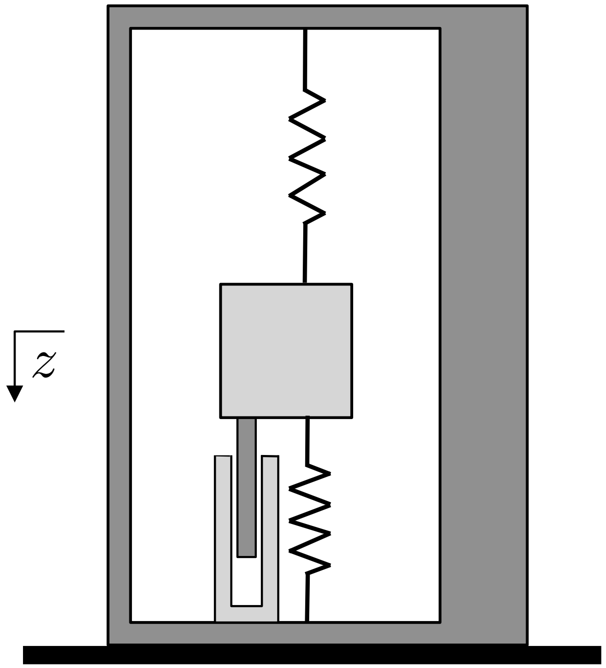


How smartphone knows up from down (accelerometer)
https://www.youtube.com/watch?v=KZVgKu6v808

Accelerometers – basic principle

$$m\ddot{\delta} + \beta\dot{\delta} + k\delta = m\ddot{z} - mg$$

$$D(s) = \frac{1}{s^2 + \frac{\beta}{m}s + \frac{k}{m}}(A_z(s) - g\frac{1}{s})$$



- Within the low frequency range

$$\delta \approx \frac{m}{\beta}(\ddot{z} - g)$$

- 3-axis accelerometers attached to the vehicle (body-fixed frame {B})

$$\delta \propto R^T(\ddot{p} - ge_3) = (S(\omega)v + \dot{v}) - gR^T e_3$$

Accelerometers – basic principle

- 3-axis accelerometers attached to the vehicle (body-fixed frame {B})

$$\delta \propto R^T(\ddot{p} - g e_3) = (S(\omega)v + \dot{v}) - g R^T e_3$$

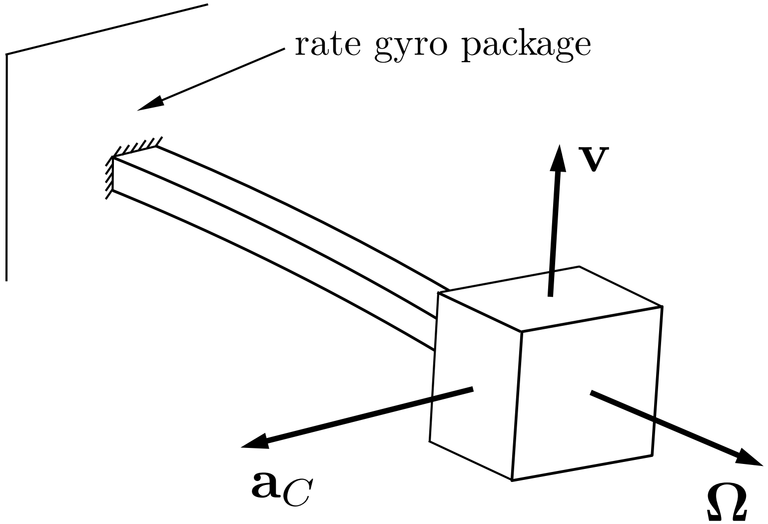
- Near the hover condition

$$\delta \propto -g R^T e_3 = -g \begin{bmatrix} -\sin\theta \\ \cos\theta \sin\phi \\ \cos\theta \cos\phi \end{bmatrix}$$

Accelerometers are typically used for attitude estimation in combination with other sensors.

Rate gyros – basic principle

- Proof mass with position p and actuated vibration velocity v
- Angular velocity to be measured from resulting Coriolis acceleration



$$p_1 = R^T p$$

$$\dot{p}_1 = -S(\omega)p_1 + R^T \dot{p}$$

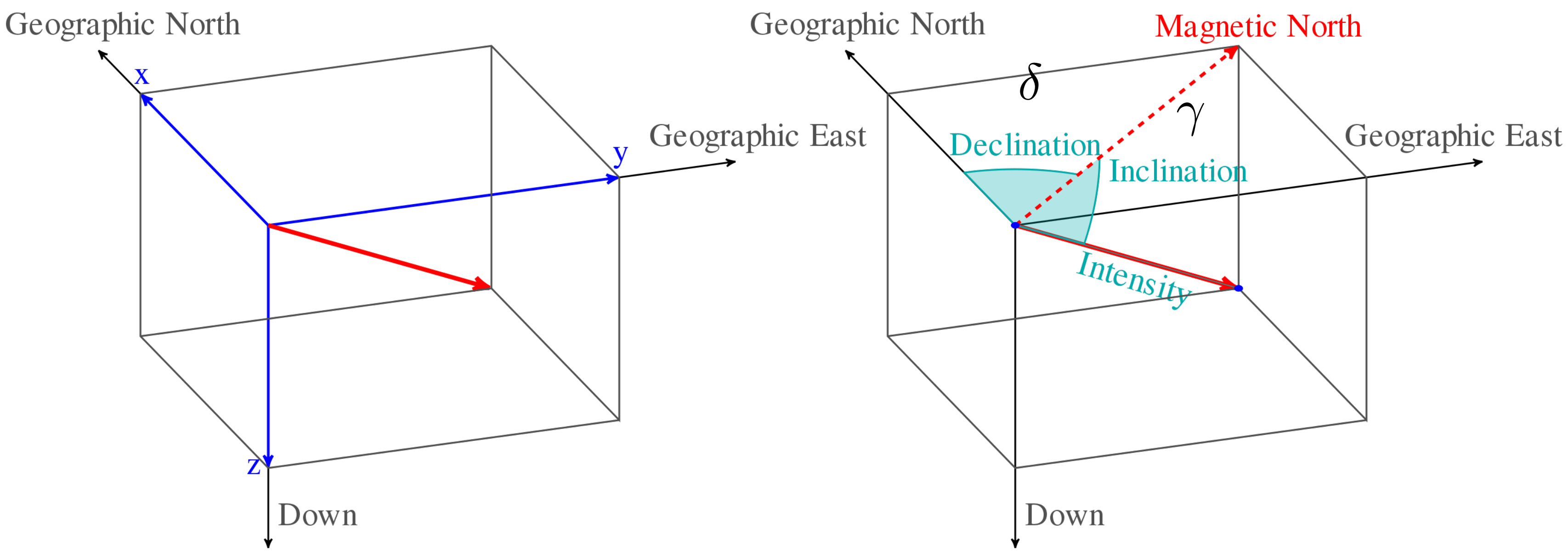
$$\ddot{p}_1 = -S(\dot{\omega})p_1 - 2S(\omega)\dot{p}_1 - S(\omega)^2 p_1 + R^T \ddot{p}$$

$$\omega = \begin{bmatrix} 0 \\ 0 \\ \Omega \end{bmatrix} \quad \dot{p}_1 = \begin{bmatrix} v_x \\ 0 \\ 0 \end{bmatrix}$$

$$\ddot{p}_1 = -\begin{bmatrix} 0 \\ 2\Omega v_x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \Omega^2 y_1 \end{bmatrix} + R^T \ddot{p}$$

Proportional to the angular velocity that we would like to measure

Magnetometers



- Available data: $^{I}m_0 = \begin{bmatrix} \cos\delta\cos\gamma \\ \sin\delta\cos\gamma \\ \sin\gamma \end{bmatrix}, \quad {}^{B}m_0 = {}^{B}_{I}R^{I}m_0, \quad \phi, \quad \theta$

  Depends on location

  Provided by 3-axis magnetometer

  Estimated using other sensors

- Unknown: ψ

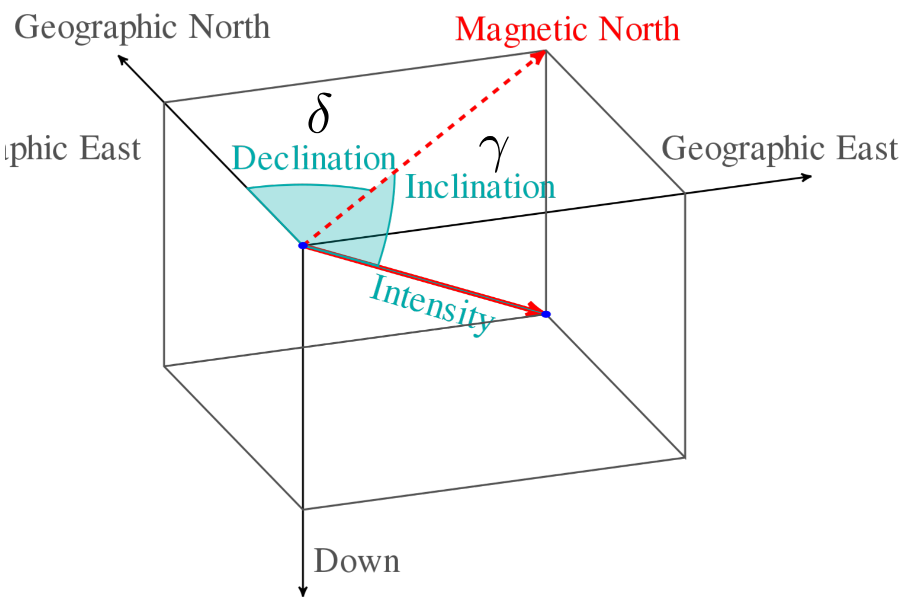
Magnetometers

- Available data: $\quad {}^{I}m_0 = \begin{bmatrix} \cos\delta\cos\gamma \\ \sin\delta\cos\gamma \\ \sin\gamma \end{bmatrix}, \quad {}^{B}m_0 = {}^{B}_{I}R\,{}^{I}m_0, \quad \phi, \quad \theta$
- Unknown: $\quad \psi$

$$ {}^{B}m_0 = {}^{B}_{I}R\,{}^{I}m_0 = R_x(-\phi)R_y(-\theta)R_z(-\psi)\,{}^{I}m_0 $$

$$ R_y(\theta)R_x(\phi)\,{}^{B}m_0 = R_z(-\psi)\,{}^{I}m_0 $$

$$ {}^{B_1}m_0 = \begin{bmatrix} \cos(-\psi+\delta)\cos\gamma \\ \sin(-\psi+\delta)\cos\gamma \\ \sin\gamma \end{bmatrix} $$



$$ (\psi - \delta) = \mathrm{atan2}(-{}^{B_1}m_y, {}^{B_1}m_x) $$

Kalman Filter

- State observer for

$$\dot{x} = Ax + Bu + w$$
$$y = Cy + n$$

  - the disturbance *w* and the measurement noise *n* are zero-mean white noise stochastic processes with covariance matrices

$$E[w(t)w(\tau)^T] = \delta(t-\tau)W$$
$$E[n(t)n(\tau)^T] = \delta(t-\tau)N$$

- Kalman filter design provides state estimate that minimizes

$$J = \lim_{t \to +\infty} E[\tilde{x}(t)^T \tilde{x}(t)]$$

  - Estimation error $\tilde{x}(t) = x(t) - \hat{x}(t)$

Kalman Filter

- Kalman filter design provides state estimate that minimizes

$$J = \lim_{t\to+\infty} E[\tilde{x}(t)^T \tilde{x}(t)] = \lim_{t\to+\infty} tr[\Sigma(t)]$$

  – Error covariance $\Sigma(t) = E[\tilde{x}(t)\tilde{x}(t)^T]$

- Solution takes the form a state observer

$$\dot{\hat{x}} = A\hat{x} + Bu - L(y - C\hat{x})$$

  – The gain L is given by $L = \Sigma_\infty C^T N^{-1}$

  where Σ_∞ is the steady-state error covariance, given by the solution of

$$(A - LC)\Sigma_\infty + \Sigma_\infty(A^T - C^T L^T) + W + LNL^T = 0$$

Kalman Filter

- Kalman filter design provides state estimate that minimizes

$$J = \lim_{t \to +\infty} E[\tilde{x}(t)^T \tilde{x}(t)] = \lim_{t \to +\infty} tr[\Sigma(t)]$$

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Duality between LQR and Kalman Filter

LQR design

$$\dot{x} = Ax + Bu$$

$$u = -Kx$$

$$\dot{x} = (A - BK)x$$

$$J = \int_0^{+\infty} (x^T Q x + u^T R u) dt$$

$$Q \geq 0, R > 0$$

(A, B) Stabilizable

$$K = R^{-1} B^T P$$

$$PA + A^T P + Q - PBR^{-1} B^T P = 0$$

Kalman filter design

$$\dot{x} = Ax + Bu + w$$

$$y = Cx + n$$

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

$$\dot{\tilde{x}} = (A - LC)\tilde{x} + w + Ln$$

$$J = \lim_{t \to \infty} E[\|\tilde{x}(t)\|^2]$$

$$W \geq 0, N > 0$$

(A, C) Detectable

$$L = \Sigma C^T N^{-1}$$

$$\Sigma A^T + A\Sigma + W - \Sigma C^T N^{-1} C \Sigma = 0$$

Kalman Filter

- Kalman filter design provides state estimate that minimizes

$$J = \lim_{t \to +\infty} E[\tilde{x}(t)^T \tilde{x}(t)] = \lim_{t \to +\infty} tr[\Sigma(t)]$$

  - Error covariance $\Sigma(t) = E[\tilde{x}(t)\tilde{x}(t)^T]$

- Solution takes the form of a state observer

$$\dot{\hat{x}} = A\hat{x} + Bu - L(y - C\hat{x})$$

  - The gain L is given by $L = \Sigma_\infty C^T N^{-1}$

  where Σ_∞ is the steady-state error covariance, given by the solution to

$$(A - LC)\Sigma_\infty + \Sigma_\infty(A^T - C^T L^T) + W + LNL^T = 0$$

Kalman filter - example

- Roll angle estimation for an aircraft (fixed-wing)
  - Assumption: coordinated turn maneuver, no wind, no sideslip

$$F_{lift}\cos\phi = mg$$

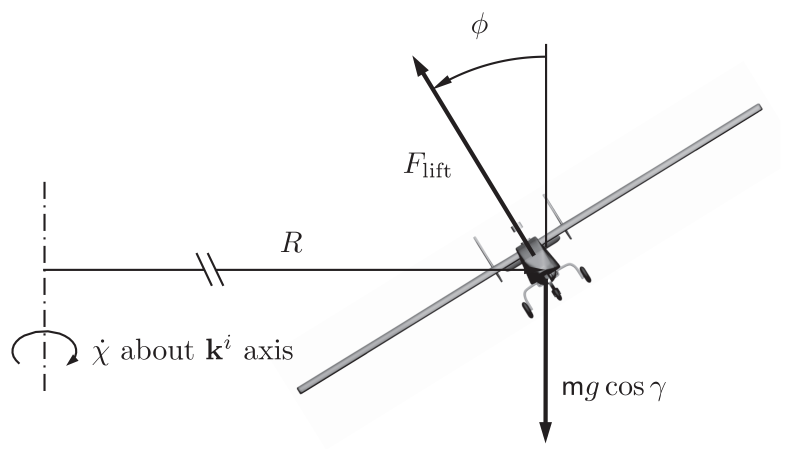
$$F_{lift}\sin\phi = mV^2/R = mV\dot{\psi}$$

$$\downarrow$$

$$\tan\phi = \frac{V\dot{\psi}}{g}$$

$$\downarrow$$

$$\dot{\psi} = \frac{g}{V}\tan\phi$$



  - Sensors: rate gyros
    (maybe not enough …)

Kalman filter - example

- Roll angle estimation for an aircraft
  - Assumption: coordinated turn maneuver, no wind, no sideslip

$$\dot{\psi} = \frac{g}{V} \tan\phi$$

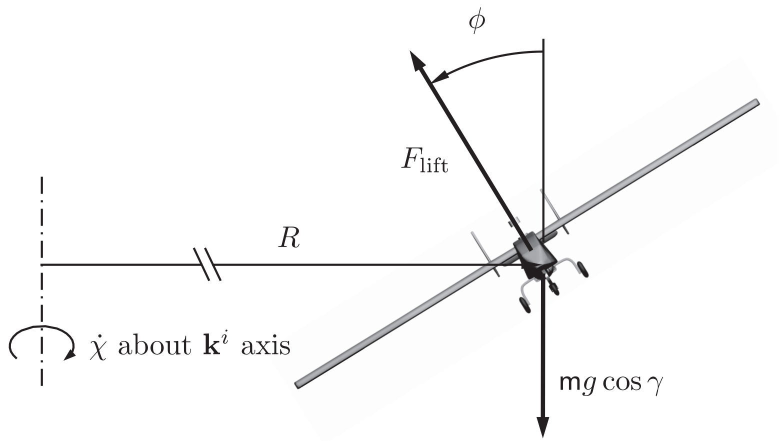
  - Sensors: rate gyros
  - Roll rate measurement

$$p_m \approx \dot{\phi} + b_p + n_p$$

  - Yaw rate measurement

$$r_m \approx \frac{g}{V}\phi + b_r + n_r$$

  - Bias terms in both measurements



Kalman filter - example

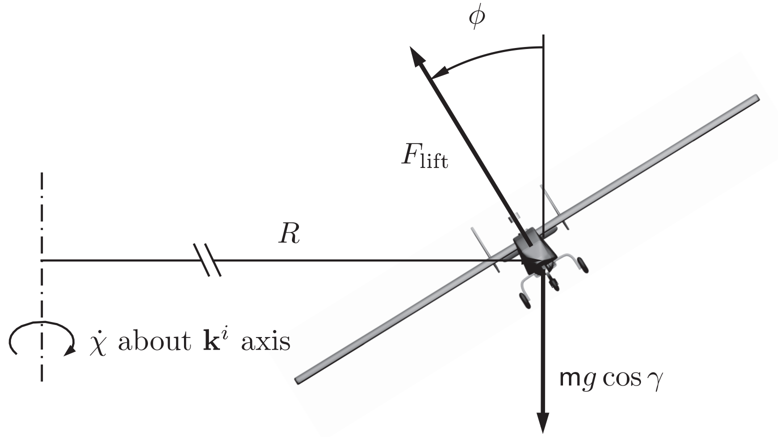
- Kalman filter setup
  - Estimate roll angle and bias terms

$$\hat{\phi}(t) = ? \quad \hat{b}_p(t) = ? \quad \hat{b}_r(t) = ?$$

  - Available measurements

$$p_m = \dot{\phi} + b_p + n_p$$

$$r_m = \frac{g}{V}\phi + b_r + n_r$$



  - How does this fit the state space description for Kalman filter design?

$$\dot{x} = Ax + Bu + w \qquad E[w(t)w(\tau)^T] = \delta(t-\tau)W$$

$$y = Cx + n \qquad E[n(t)n(\tau)^T] = \delta(t-\tau)N$$

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

Kalman filter - example

- Kalman filter setup
  - Estimate roll angle and bias terms

    $$\hat{\phi}(t) = ? \quad \hat{b}_p(t) = ? \quad \hat{b}_r(t) = ?$$

  - Starting point for model description

    $$\dot{\phi} = p \qquad p_m = \dot{\phi} + b_p + n_p$$

    $$r_m = \frac{g}{V}\phi + b_r + n_r$$

  - Identify state, input, and output for state-spate description

$$\dot{x} = Ax + Bu + w$$
$$y = Cx + n$$

Try $x = \begin{bmatrix} \phi \\ b_p \\ b_r \end{bmatrix} \quad y = r_m$

Kalman filter - example

- Kalman filter setup
  - Identify state, input, and output for state-spate description

$$\dot{x} = Ax + Bu + w$$
$$y = Cx + n$$

Try $x = \left[\phi \ b_p \ b_r\right]' \quad y = r_m$

  - Resulting equations

$$\dot{\phi} = p_m - b_p - n_p$$
$$y = r_m = \frac{g}{V}\phi + b_r + n_r$$
$$\dot{b}_p = n_1$$
$$\dot{b}_r = n_2$$

- Remaining questions

$$A = ? \quad B = ? \quad C = ?$$
$$u = ? \quad w \sim ? \quad n \sim ?$$

- Is the system observable?
  Check pair (A,C).

Kalman filter - example

- Kalman filter setup
  - Identify state, input, and output for state-spate description

$$\dot{x} = Ax + Bu + w$$
$$y = Cx + n$$

Try $x = [\phi \; b_p \; b_r]'$ $\quad y = r_m$

  - Resulting equations

$$\dot{\phi} = p_m - b_p - n_p$$
$$y = r_m = \frac{g}{V}\phi + b_r + n_r$$
$$\dot{b}_p = n_1$$
$$\dot{b}_r = n_2$$

- Remaining questions

$$A = ? \quad B = ? \quad C = ?$$
$$u = ? \quad w \sim ? \quad n \sim ?$$

- Is the system observable?

No.

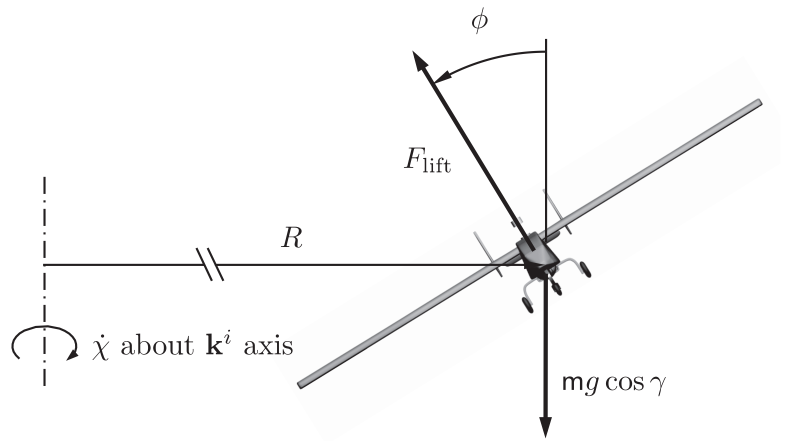
Additional sensor is needed.

Kalman filter - example

- Roll angle estimation for an aircraft (fixed-wing)
  - Assumption: coordinated turn maneuver, no wind, no sideslip

$$F_{lift} \cos\phi = mg$$

$$F_{lift} \sin\phi = mV^2/R = mV\dot{\psi}$$



  - Accelerometer measures

$$a_m = R^T(\ddot{p} - ge_3) + n_a$$

Neglecting $\ddot{p}$ and noise

$$a_m = -R^T ge_3 = -g \begin{bmatrix} 0 \\ \sin\phi \\ \cos\phi \end{bmatrix}$$

In fact,

$$a_m = -\frac{1}{m}\begin{bmatrix} 0 \\ 0 \\ F_{lift} \end{bmatrix} = -\begin{bmatrix} 0 \\ 0 \\ \sqrt{g^2 + (V\dot{\psi})^2} \end{bmatrix} = -g\begin{bmatrix} 0 \\ 0 \\ \sqrt{1 + \tan^2\phi} \end{bmatrix}$$

Kalman filter - example

- Kalman filter setup (rate gyros + accelerometers)
  - Identify state, input, and output for state-spate description

$$\dot{x} = Ax + Bu + w$$
$$y = Cx + n$$

Try $x = [\phi \; p \; b_p \; b_r]'$ $\quad y = [r_m \; p_m \; \phi_m]'$

  - Resulting equations

$$\begin{bmatrix} \dot{\phi} \\ \dot{p} \\ \dot{b}_p \\ \dot{b}_r \end{bmatrix} = \begin{bmatrix} p \\ w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

$$y = \begin{bmatrix} \frac{g}{V}\phi + b_r + n_r \\ p + b_p + n_p \\ \phi + n_\phi \end{bmatrix}$$

- Remaining questions

$$A = ? \quad B = ? \quad C = ?$$
$$u = ? \quad w \sim ? \quad n \sim ?$$

- Is the system observable?

Yes!

Kalman filter - example

- Kalman filter setup (2nd alternative)
  - Identify state, input, and output for state-spate description

$$\dot{x} = Ax + Bu + w$$
$$y = Cx + n$$

Try $x = \left[\phi \; p \; b_p \; b_r\right]' \quad y = \left[r_m \; p_m \; \phi_m\right]'$

  - Resulting equations

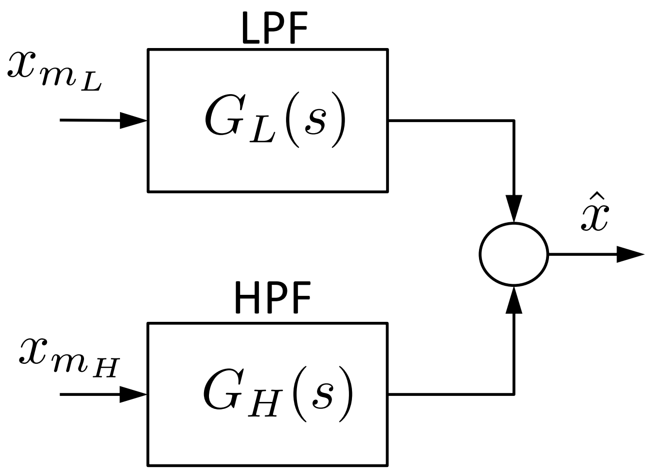
Same as before,
but using dynamic model for
time evolution of $\dot{\phi}$

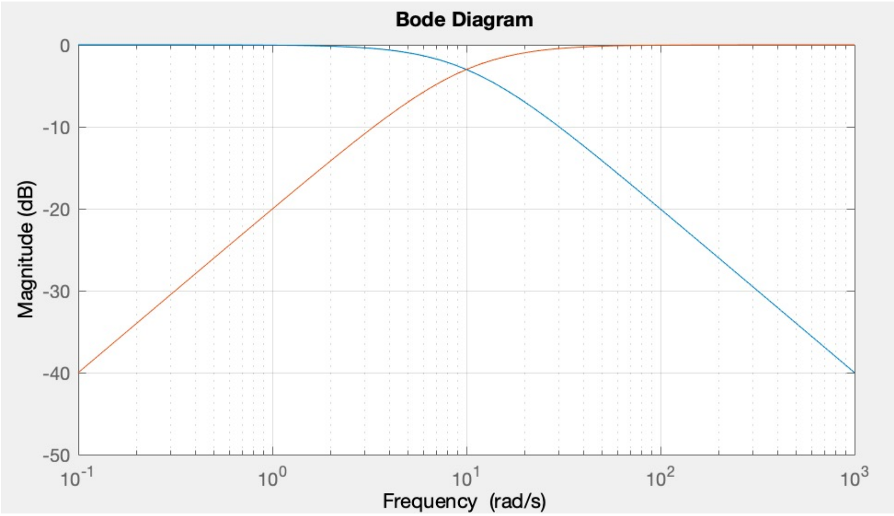
- Remaining questions

$$A = ? \quad B = ? \quad C = ?$$
$$u = ? \quad w \sim ? \quad n \sim ?$$

Complementary filters

- Context:
  - Different sensors provide reliable measurements over different (complementary) frequency ranges.
- Strategy (for a simple example):





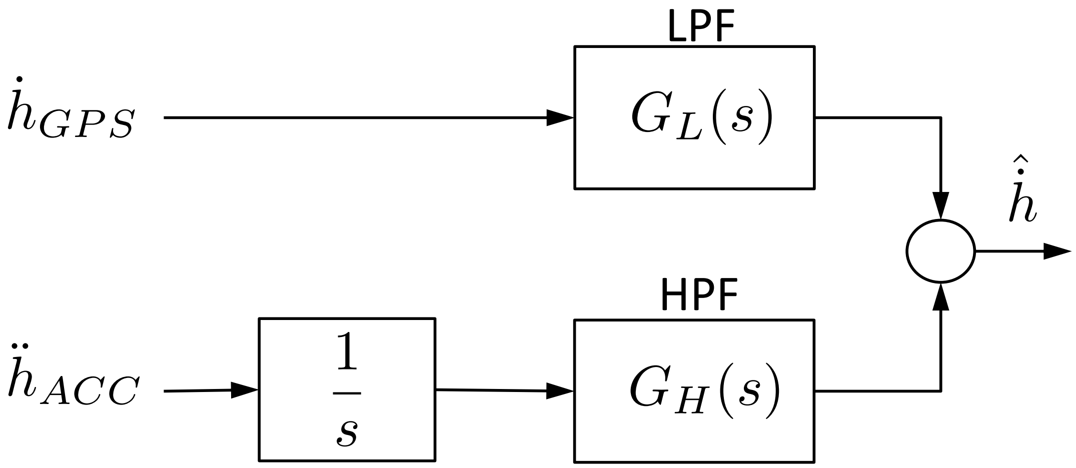
$$G_L(s) = \frac{l}{s+l} \qquad G_H(s) = \frac{s}{s+l} \qquad G_L(s) + G_H(s) = 1$$

Complementary filters

- Example: altitude rate estimation
  - low frequency from GPS receiver $\dot{h}_{GPS}$
  - high frequency integration of acceleration along the inertial z-direction

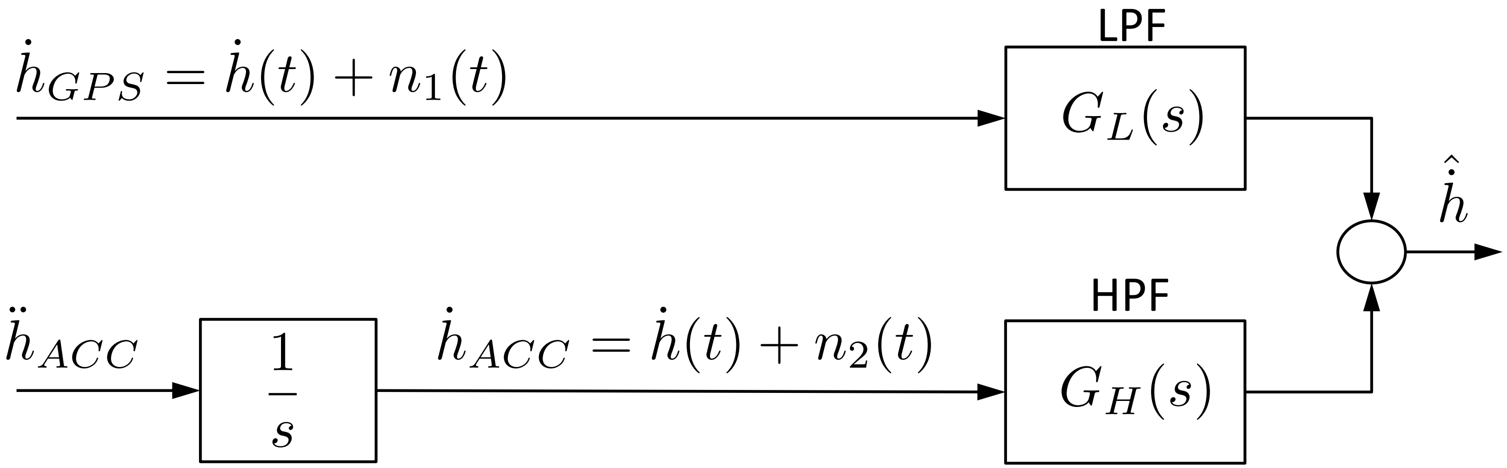
$$a_m = R(\ddot{p} - g e_3)$$

$$\ddot{h}_{ACC} = -e_3^T R_X(\hat{\phi}) R_Y(\hat{\theta}) a_m - g$$



Complementary filters

- Example: altitude rate estimation



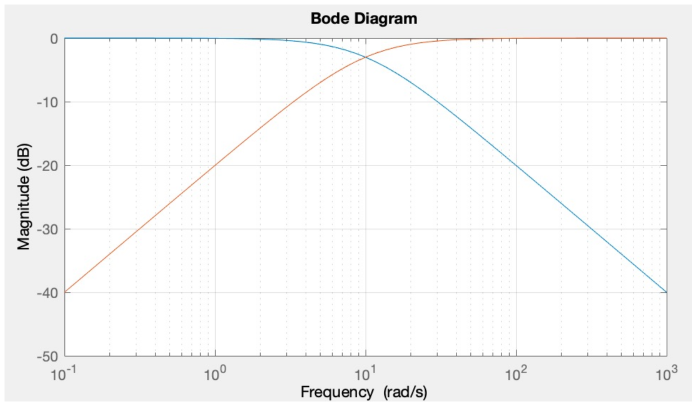
$$\hat{\dot{H}}(s) = \frac{l}{s+l}\dot{H}_1(s) + \frac{s}{s+l}\dot{H}_2(s)$$

$$= \dot{H}(s) + \frac{l}{s+l}N_1(s) + \frac{s}{s+l}N_2(s)$$

Keeps whole signal

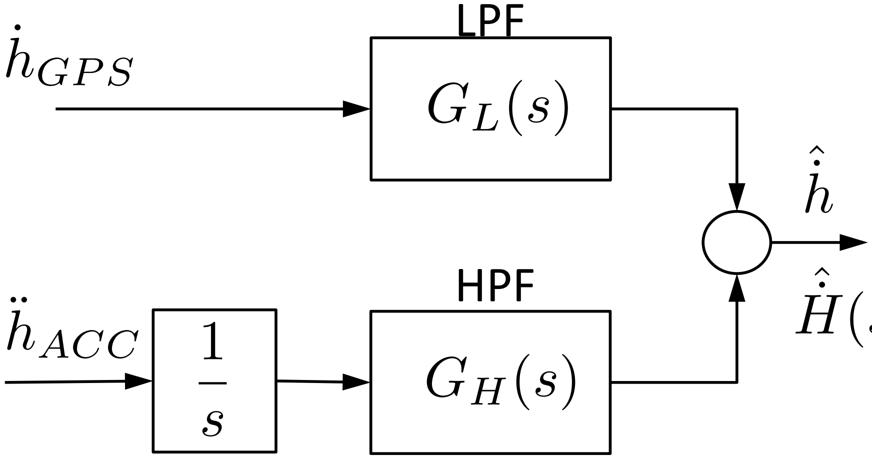
Attenuation in the high frequencies

Attenuation in the low frequencies



Complementary filters and Kalman filters

- Is there a connection?



$$\hat{\dot{H}}(s) = \frac{l}{s+l}\dot{H}_{GPS}(s) + \frac{s}{s+l}\frac{\ddot{H}_{ACC}}{s}(s)$$

  - Yes!

$$\dot{x} = Ax + Bu + w \qquad x = \dot{h} \quad u = \ddot{h}_{ACC}$$

$$y = Cx + n \qquad \hat{x} = \hat{\dot{h}} \quad \hat{y} = \dot{h}_{GPS}$$

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

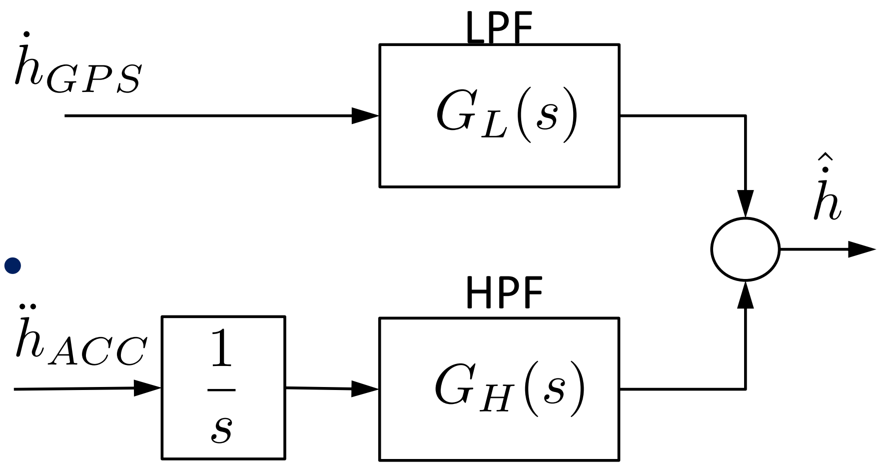
  - Check the connection between

$$\hat{Y}(s) = \left[C(sI - A + LC)^{-1}L\right]Y(s) + \left[C(sI - A + LC)^{-1}B\right]U(s)$$

Complementary filters and Kalman filters

- Is there a connection?

Complementary filter



Kalman filter

$$\dot{x} = Ax + Bu + w$$

$$y = Cx + n$$

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

with

$$x = \dot{h} \quad u = \ddot{h}_{ACC}$$

$$\hat{x} = \hat{\dot{h}} \quad \hat{y} = \dot{h}_{GPS}$$

– Yes! Check the connection between

$$\hat{\dot{H}}(s) = \frac{l}{s+l}\dot{H}_{GPS}(s) + \frac{s}{s+l}\frac{\ddot{H}_{ACC}}{s}(s)$$

and

$$\hat{Y}(s) = \left[C(sI - A + LC)^{-1}L\right]Y(s) + \left[C(sI - A + LC)^{-1}B\right]U(s)$$

Complementary filters and Kalman filters

- Also applies to more evolved cases
  - State observer for roll angle and roll rate bias estimation

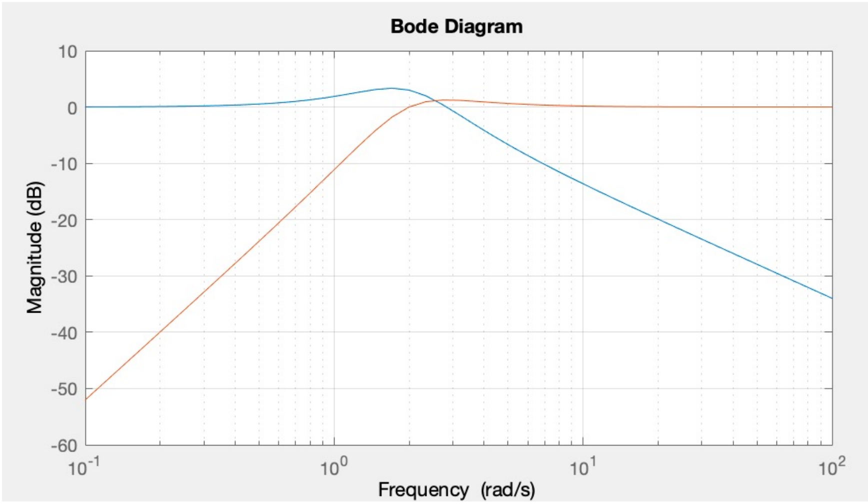
$$x = [\phi \; b_p]' \qquad u = p_m$$

$$\hat{x} = [\hat{\phi} \; \hat{b}_p]' \qquad \hat{y} = \hat{\phi}$$

$$\dot{x} = Ax + Bu + w$$

$$y = Cx + n$$

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$



$$\hat{\Phi}(s) = \frac{l_1 s + l_2}{s^2 + l_1 s + l_2}\Phi_m(s) + \frac{s}{s^2 + l_1 s + l_2}P_m(s)$$

$$= G_L(s)\Phi_m(s) + G_H(s)\frac{P_m(s)}{s} \qquad G_L(s) + G_H(s) = 1$$