LQR Design Example

UAVs

MEAer - Spring Semester – 2020/2021

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LQR Design Example

- Design an LQR controller for the linearized model of an aircraft roll dynamics[1]
  - Gain selection
  - Analyse effect of changing weights in the frequency domain
  - Connection with loop-shaping

[1]Example extracted from
J. Hespanha, Linear Systems Theory, Princeton University Press, 2009.

LQR Design Example

- Linearized aircraft roll dynamics

$$\dot{\phi} = \omega$$

$$\dot{\omega} = 0.8\omega - 20\tau$$

$$\dot{\tau} = -50\tau + 50u$$

  - In state space form

$$x = \begin{bmatrix} \phi \\ \omega \\ \tau \end{bmatrix} \qquad y = \phi \qquad \begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -0.8 & -20 \\ 0 & 0 & -50 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 50 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

LQR Design Example

- Optimal LQR Problem: find u(t) that minimizes

$$J = \int_0^\infty x(t)' Q x(t) + \rho u(t)^2 dt$$

  - Choose

    $$Q = G'G \qquad G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \gamma & 0 \end{bmatrix} \qquad \rho > 0, \ \gamma > 0$$

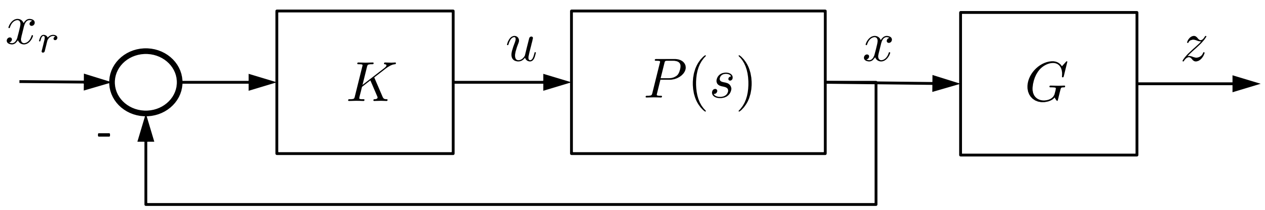
    such that

    $$J = \int_0^\infty z(t)' z(t) + \rho u(t)^2 dt, \quad z(t) = G x(t) = \begin{bmatrix} \phi \\ \gamma \dot{\phi} \end{bmatrix}$$

  - LQR solution $u = -Kx$
  - Analyse the effect of changing the weights (ρ, γ) in the frequency domain –> loop shaping
    - Which transfer functions should be considered?

LQR Design Example

- Block diagram and Transfer functions



  - Open-loop transfer function from u to x

$$P(s) = (sI - A)^{-1}B$$

  - Open-loop negative-feedback transfer function (scalar)

$$L(s) = KP(s) = K(sI - A)^{-1}B$$

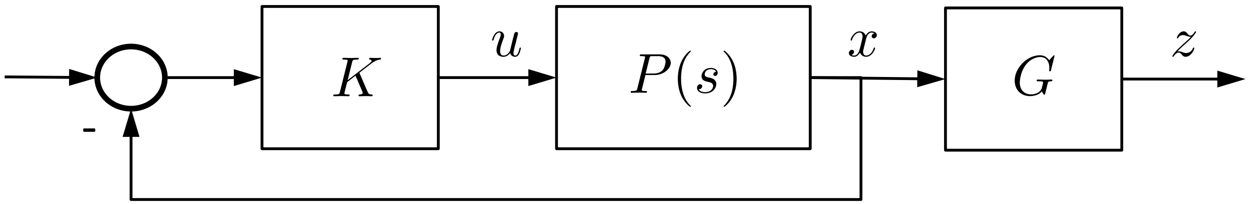
  - Closed-loop transfer function from x_r to x

$$\dot{x} = (A - BK)x + BKx_r$$

$$X(s) = (sI - (A - BK))^{-1}BKX_r(s)$$

LQR Design Example

- Block diagram and Transfer functions

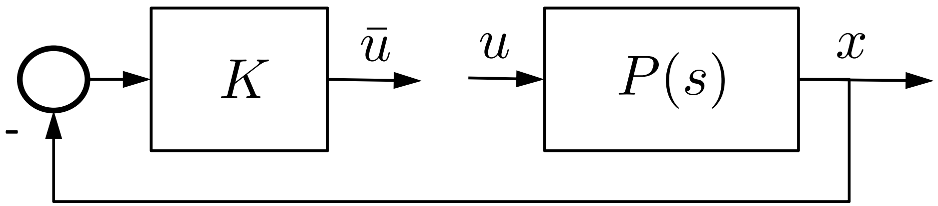


  - Open-loop transfer function from u to x

    $$P(s) = (sI - A)^{-1}B$$

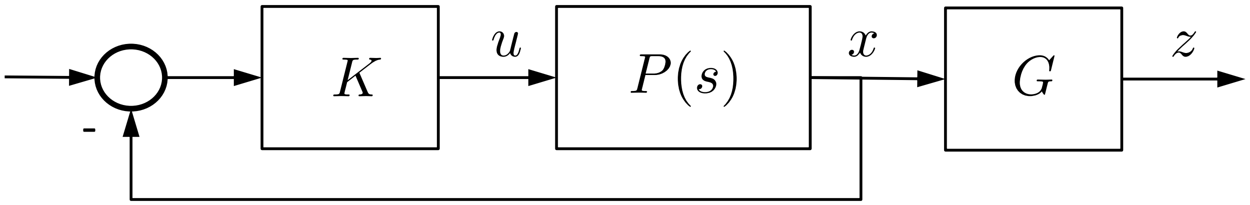
  - Open-loop transfer function from u to $-\bar{u}$ (scalar)

$$KP(s) = K(sI - A)^{-1}B = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} (sI_3 - A)^{-1} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$



LQR Design Example

- Block diagram and Transfer functions



- Open-loop gain transfer function (scalar)

  $L(s) = KP(s) = K(sI - A)^{-1}B$

  can be used as in traditional loop shaping
- G_M = ?
- P_M = ?
- Noise attenuation (high frequencies)
- Disturbance rejection, reference tracking (low frequencies)

LQR Design Example

- Kalman's equality (can be derived from ARE)

$$|1+L(j\omega)|^2 = |1+KP(j\omega)|^2 = 1 + \frac{\|GP(j\omega)\|^2}{\rho}$$

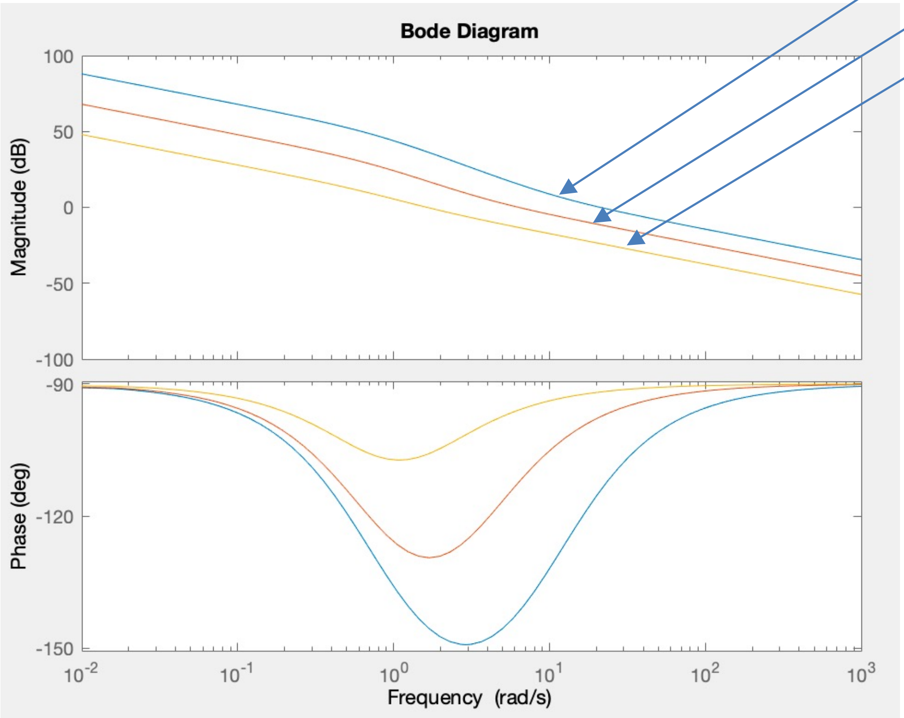
  - With $G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \gamma & 0 \end{bmatrix}$ and $P(s) = \begin{bmatrix} P_1(s) \\ sP_1(s) \\ P_3(s) \end{bmatrix}$ in the low freq.

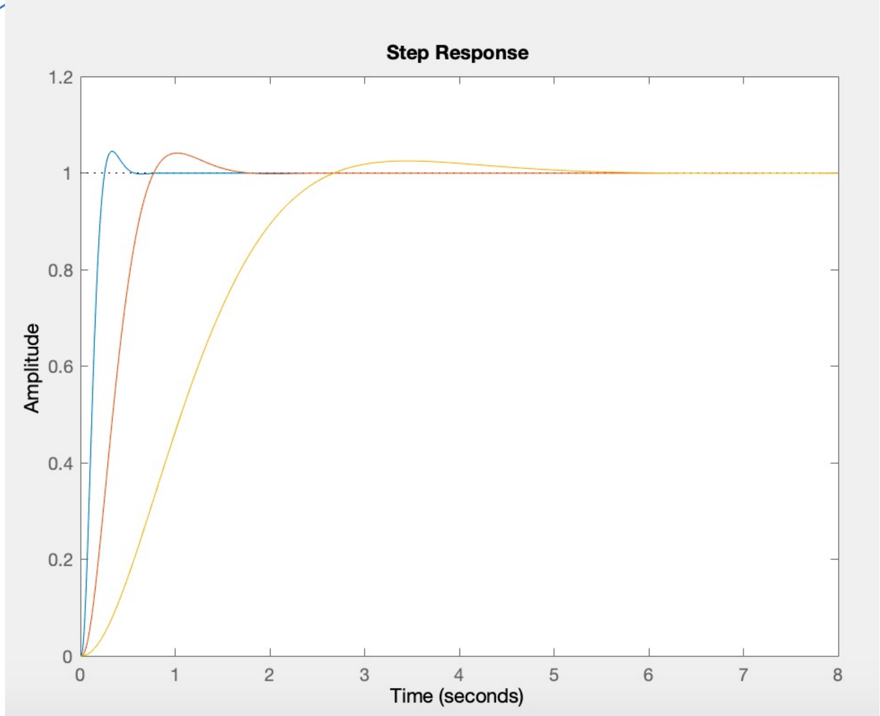
$$|L(j\omega)| \approx \frac{|1+j\gamma\omega||P_1(j\omega)|}{\sqrt{\rho}}$$

- Effect of changing ρ
  - decreasing ρ moves Bode magnitude plot up
- Effect of changing γ
  - large values of γ lead to a low frequency zero

LQR Design Example

- Example with $\gamma = 0.01$ $\rho \in \{0.01, 1, 100\}$

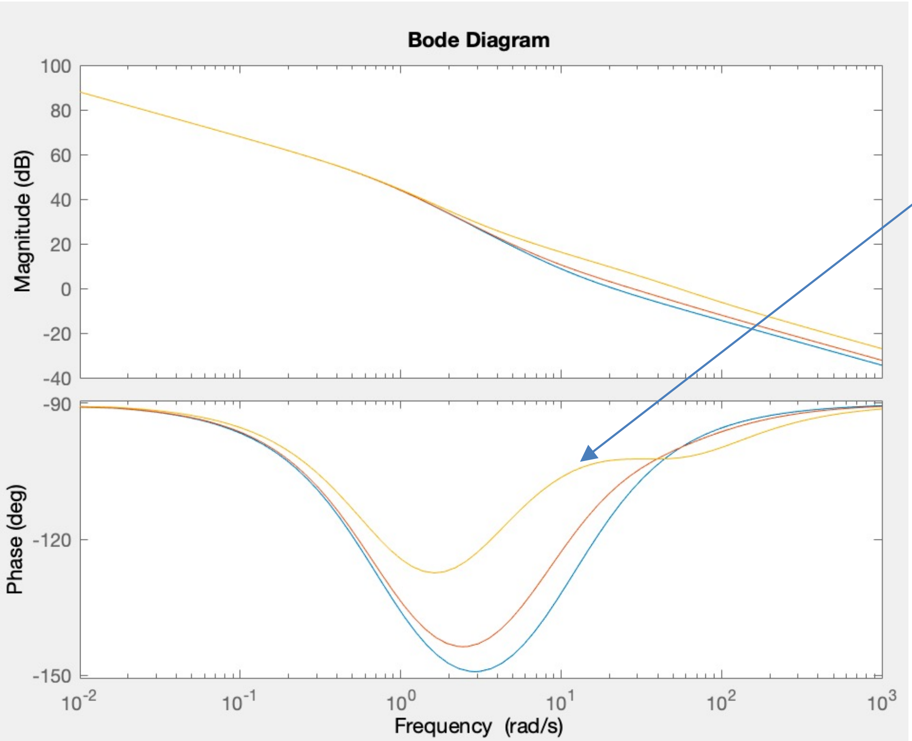


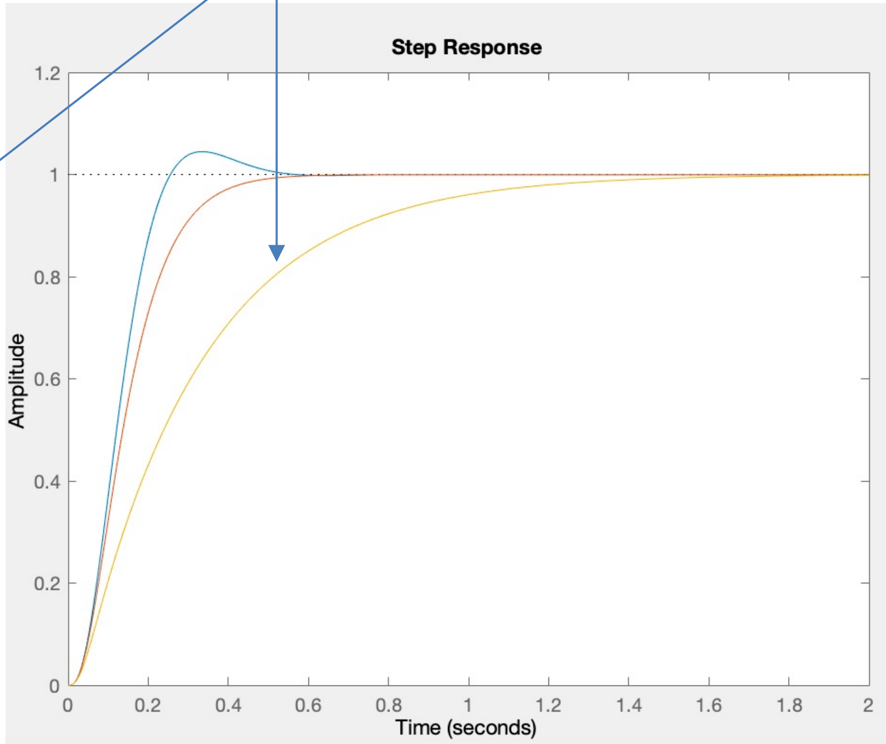


  – decreasing ρ moves Bode magnitude plot up
    - faster response, less sensitivity to disturbances
    - reasoning: input becomes “cheaper”

LQR Design Example

- Example with $\rho = 0.01 \quad \gamma \in \{0.01, 0.1, 0.3\}$





  – increasing γ increases the phase margin
    - less overshoot, slower response
    - reasoning: increasing the cost of angular velocity