Rigid-Body Kinematics and Dynamics

UAVs
MEAer - Spring Semester – 2020/2021

Rita Cunha

Rigid Motion in 3-D - Kinematics

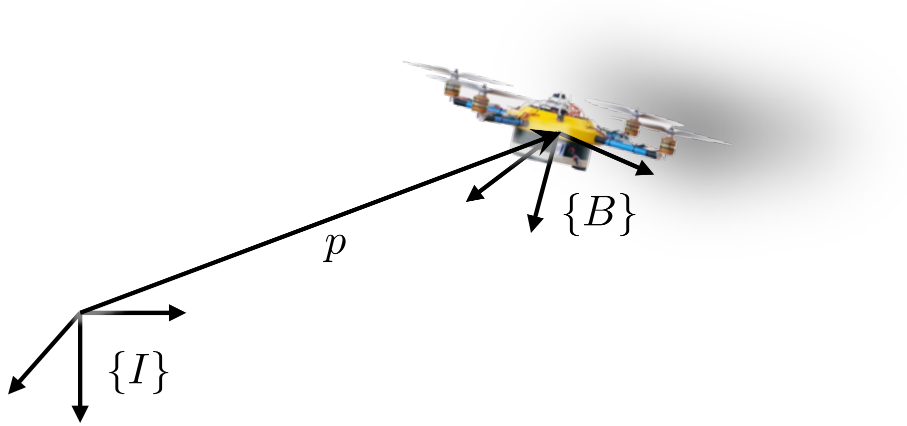
- Goal: derive the equations of motion for a 3-D rigid body
  - Kinematics

$$\dot{p} = Rv$$

$$\dot{R} = RS(\omega)$$

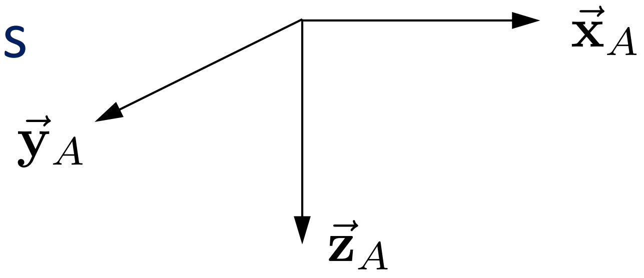
  - Dynamics

$$m\dot{v} = -S(\omega)mv + f$$
$$J\dot{\omega} = -S(\omega)J\omega + n$$



Reference frames and configurations

- A **reference frame** {A} is defined by its
  - Origin
  - 3 orthonormal axes (right-hand convention)



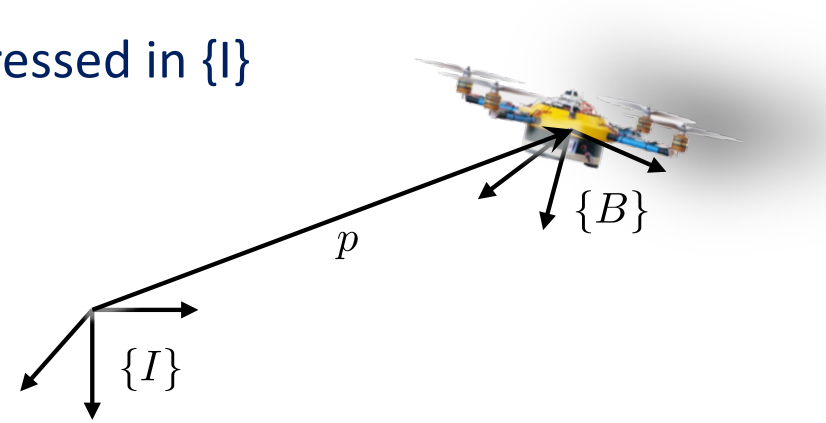
- The **configuration** of {B} w. r. t. {I} is defined by
  - The position (of the origin) of {B} expressed in {I}

    $^{I}p_B \in \mathbb{R}^3$

  - The orientation (of the axes) of {B} expressed in {I}

  Rotation matrix representation

  $^{I}_{B}R \in \mathbb{SO}^3$



Notation: the subscript indicates that the vector is expressed in the specified reference frame

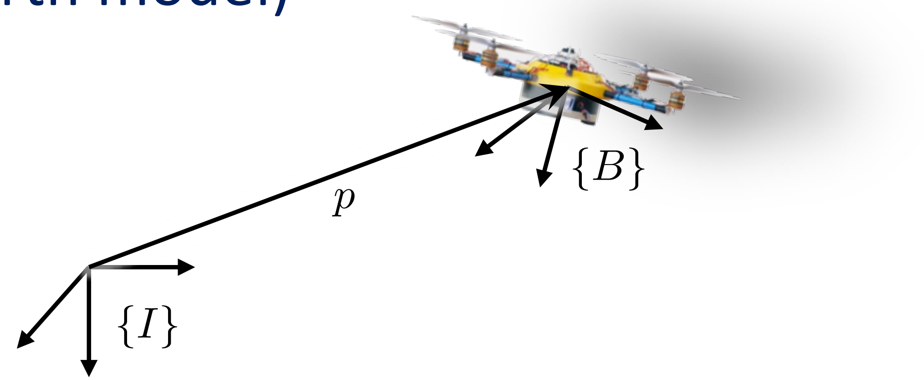
Reference frames and configurations

- Inertial reference frame (flat earth model)

  $\{I\}$

- Body-fixed reference frame

  $\{B\}$



- Configuration of $\{B\}$ w. r. t. $\{I\}$

  $({}^{I}p_B, {}^{I}_{B}R) \in \mathbb{R}^3 \times \mathbb{SO}(3)$

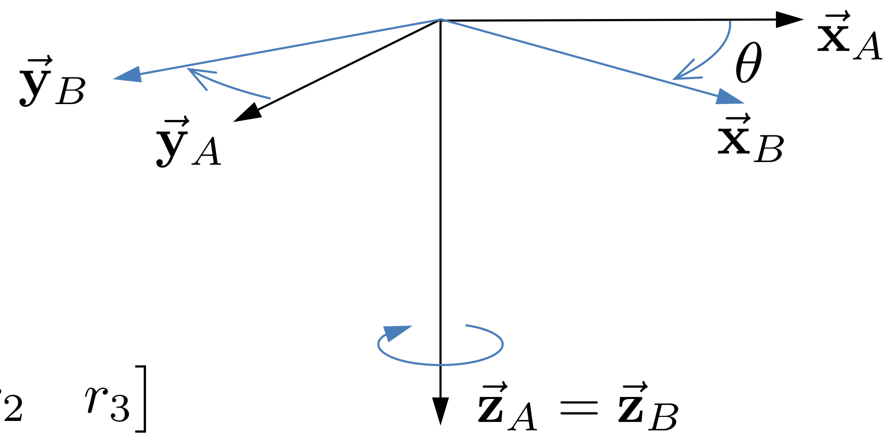
  ${}^{I}p_B \in \mathbb{R}^3$ position of the origin of $\{B\}$ expressed in $\{I\}$

  ${}^{I}_{B}R \in \mathbb{SO}^3$ rotation matrix from $\{B\}$ to $\{I\}$

Rotation Matrices and reference frames

Rotation from frame $\{B\}$ to frame $\{A\}$

$$ {}^A_B R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} $$



In general,

$$ {}^A_B R = \begin{bmatrix} {}^A x_B & {}^A y_B & {}^A z_B \end{bmatrix} = \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix} $$

Concatenates the coordinates of the principal axes of $\{B\}$ expressed in $\{A\}$

In this case,

$$ {}^A x_B = r_1 = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \\ 0 \end{bmatrix} \quad {}^A y_B = r_2 = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \\ 0 \end{bmatrix} \quad {}^A z_B = r_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} $$

Rotation Matrices and reference frames

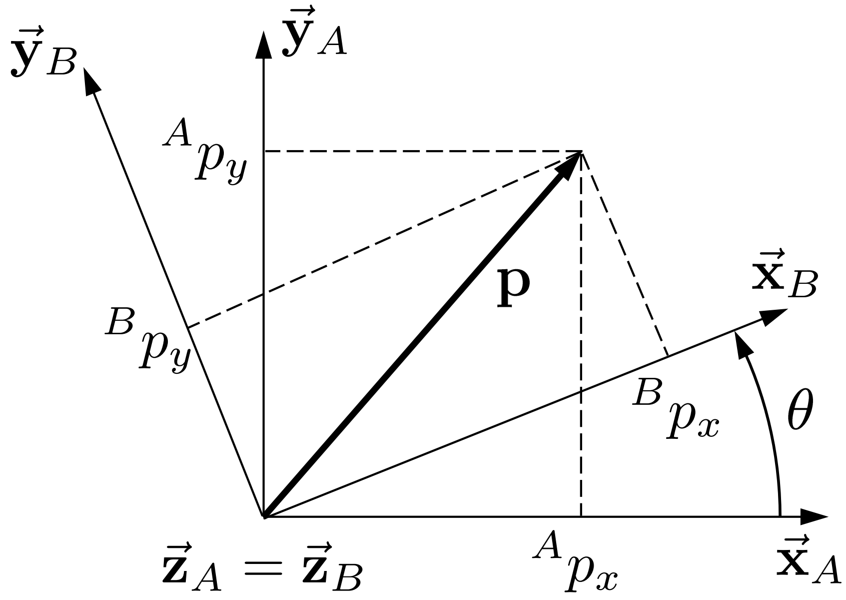
Rotation from frame $\{B\}$ to frame $\{A\}$

- Coordinates expressed in {B} are transformed to coordinates expressed in {A}

$$^{A}p = {}^{A}_{B}R\,{}^{B}p$$

$$\begin{bmatrix} {}^{A}p_x \\ {}^{A}p_y \\ {}^{A}p_z \end{bmatrix} = {}^{A}_{B}R \begin{bmatrix} {}^{B}p_x \\ {}^{B}p_y \\ {}^{B}p_x \end{bmatrix}$$

$$\begin{bmatrix} p\cos(\theta+\alpha) \\ p\sin(\theta+\alpha) \\ 0 \end{bmatrix} = {}^{A}_{B}R \begin{bmatrix} p\cos(\alpha) \\ p\sin(\alpha) \\ 0 \end{bmatrix}$$



Rotation about z axis

$${}^{A}_{B}R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation Matrices – *Special Orthogonal Group*

Properties of $R = \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix} \in \mathbb{R}^{3\times 3}$

- Columns are mutually *orthonormal* $r_i^T r_i = 1,\ r_i^T r_j = 0,\ j \neq i$

  $\det(R) = r_1^T (r_2 \times r_3) = \pm 1$

- Right-handed coordinate system -> *special* $\det(R) = 1$
- *Group* (under matrix multiplication)
  - Closure: $R_1, R_2 \in \mathbb{SO}(3) \Rightarrow R_1 R_2 \in \mathbb{SO}(3)$
  - Identity: $RI_3 = I_3 R = R,\ \forall R \in \mathbb{SO}(3)$
  - Inverse: $R^{-1} = R^T$ inverse is unique and satisfies $RR^T = R^T R = I_3$
  - Associativity: $(R_1 R_2) R_3 = R_1 (R_2 R_3)$
- *Special Orthogonal Group* of order 3

  $\mathbb{SO}(3) = \{R \in \mathbb{R}^{3\times 3} : RR^T = R^T R = I_3, \det(R) = 1\}$

Rotation Matrices - Properties

Properties of $R = \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix} \in \mathbb{R}^{3\times 3}$

- Inverse

$$ {}^{B}_{A}R = ({}^{A}_{B}R)^{-1} = ({}^{A}_{B}R)^{T} $$

$$ {}^{A}p = {}^{A}_{B}R\,{}^{B}p \;\Leftrightarrow\; {}^{B}p = ({}^{A}_{B}R)^{T}\;{}^{A}p $$

- Composition rule

$$ {}^{A}R_{B}\,{}^{B}R_{C} = {}^{A}R_{C} $$

Rotation Matrices - Properties

Properties of $R = \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix} \in \mathbb{R}^{3\times 3}$

- Rotations are rigid body transformations
  - R preserve distances between points:

$$\|R(p-q)\| = \|p-q\|$$

  - R preserves angles between vectors:

$$R(p \times q) = (Rp) \times (Rq)$$

The skew-symmetric matrix

- Alternative way to express the cross-product

$$S(\mathbf{a})\mathbf{b} = \mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

$$S(\mathbf{a}) = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

- Returns a skew-symmetric matrix

$$S := S(\mathbf{a})$$

$$S^T = -S$$

$$S^T + S = 0$$

- R preserves orientation $RS(\mathbf{p}) = S(R\,\mathbf{p})R$

Rotation Representations

Exponential map

- Any rotation can be obtained by rotating an angle $\theta \in [0, 2\pi)$ about an axis $\mathbf{n} \in \mathbb{R}^3, \ \|\mathbf{n}\| = 1$

$$R(\theta, \mathbf{n}) = e^{\theta S(\mathbf{n})} = I_3 + \theta S(\mathbf{n}) + \frac{\theta^2}{2} S(\mathbf{n})^2 + \frac{\theta^3}{3!} S(\mathbf{n})^3 + \ldots$$

- Rodrigues' formula

$$R(\theta, \mathbf{n}) = I_3 + \sin\theta S(\mathbf{n}) + (1 - \cos\theta) S(\mathbf{n})^2$$

$$R(\theta, \mathbf{n})\mathbf{p} = \mathbf{n}\mathbf{n}^T\mathbf{p} + \sin\theta S(\mathbf{n})\mathbf{p} - \cos\theta S(\mathbf{n})^2\mathbf{p}$$

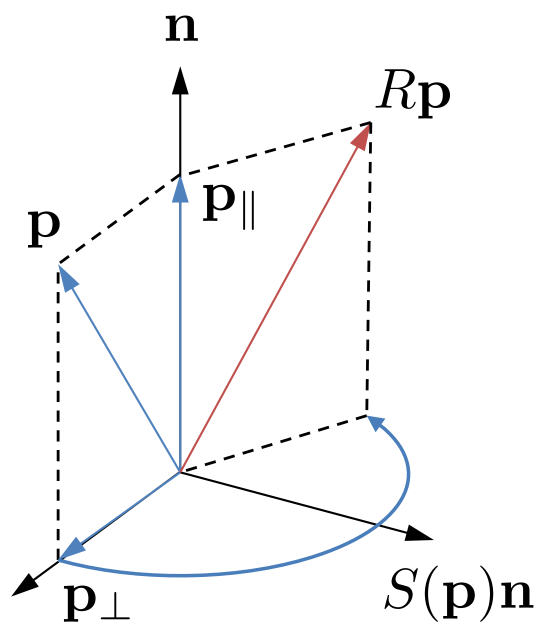
Rotation Representations

Exponential map and Rodrigues' formula

$$
\begin{aligned}
R(\theta, \mathbf{n}) &= I_3 + \sin\theta S(\mathbf{n}) + (1-\cos\theta) S(\mathbf{n})^2 \\
&= \mathbf{n}\mathbf{n}^T + \sin\theta S(\mathbf{n}) - \cos\theta S(\mathbf{n})^2
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{p} &= \mathbf{p}_{\|} + \mathbf{p}_{\perp} \\
&= \mathbf{n}\mathbf{n}^T\mathbf{p} - S(\mathbf{n})^2\mathbf{p}
\end{aligned}
$$

$$
R\mathbf{p} = \mathbf{p}_{\|} + \sin\theta S(\mathbf{n})\mathbf{p} + \cos\theta\, \mathbf{p}_{\perp}
$$



Rotation Representations

Euler angles

- Any rotation can be decomposed into three elementary rotations, e.g. Z-Y-Z, Y-Z-X, Z-Y-X.
- Z-Y-X parametrization, also known as, yaw ψ, pitch θ, roll ϕ Euler angles

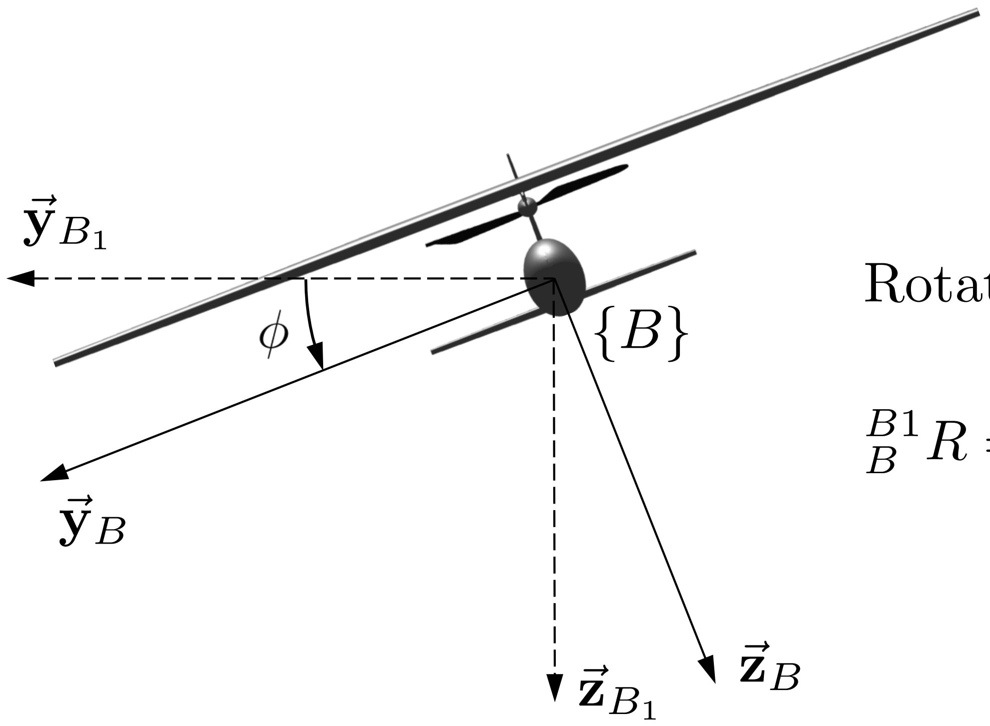
$$\lambda = \begin{bmatrix} \phi & \theta & \psi \end{bmatrix} \in R^3$$

$$
\begin{aligned}
{}^I_B R = R(\lambda) &= R_z(\psi) R_y(\theta) R_x(\phi) \\
&= \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}
\end{aligned}
$$

- 3-dimensional parametrization means that singularities are always present

Z-Y-X Euler angles – Intermediate frames

Rotation from {B} to {B$_1$}



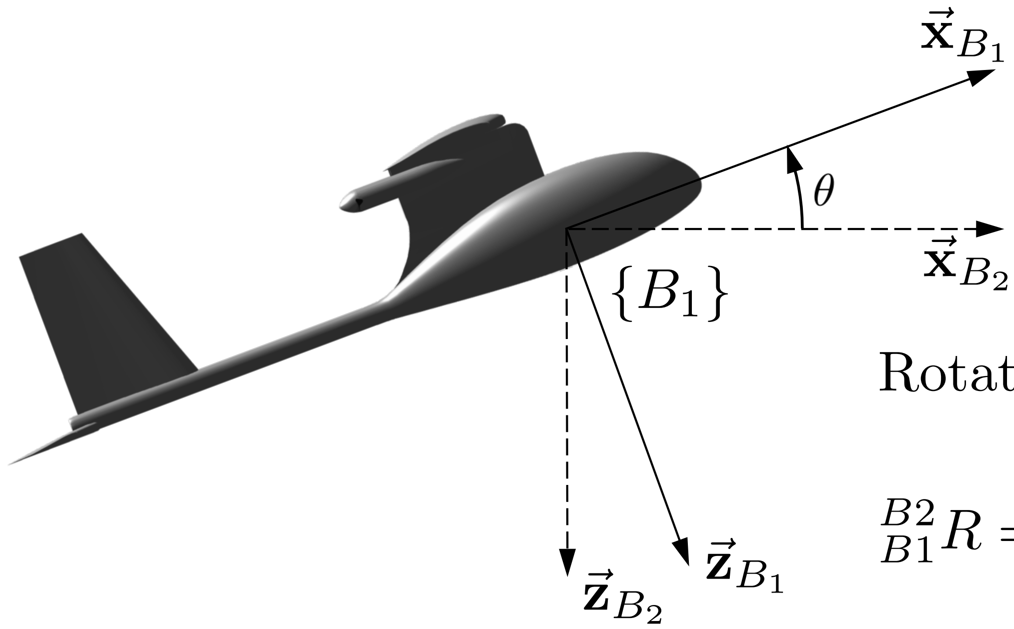
Rotation about x-axis of body frame (roll)

$$ {}^{B1}_{B}R = R_x(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} $$

Beard & McLain, “Small Unmanned Aircraft,” *Princeton University Press,* 2012

Z-Y-X Euler angles – Intermediate frames

Rotation from {B$_1$} to {B$_2$}



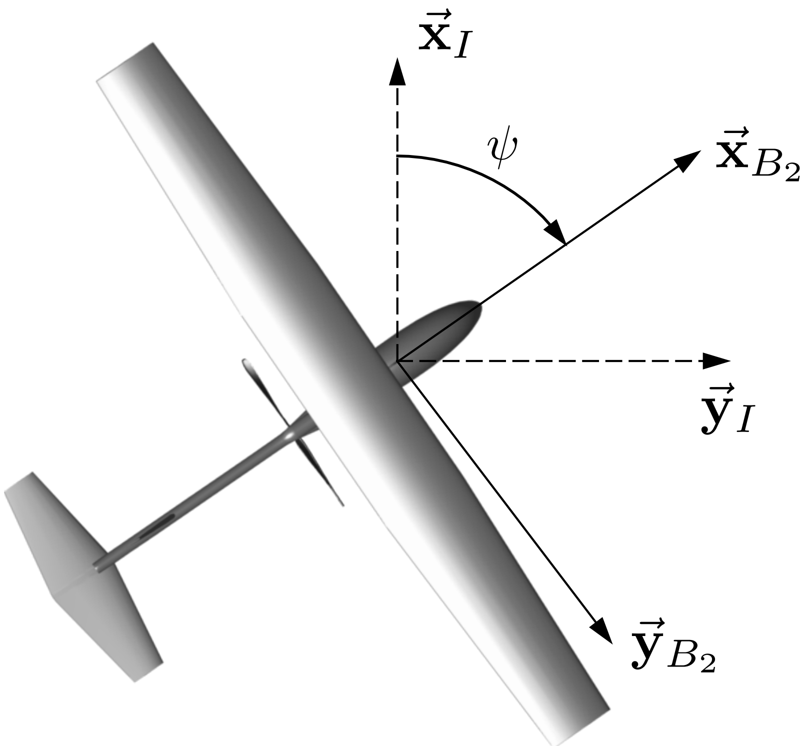
Rotation about y-axis of frame $\{B_1\}$ (pitch)

$$ {}^{B2}_{B1}R = R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} $$

Beard & McLain, "Small Unmanned Aircraft," *Princeton University Press,* 2012

Z-Y-X Euler angles – Intermediate frames

Rotation from {B_2} to {B_3}
{B_3} and {I}: same orientation, different origins



Rotation about z-axis of frame $\{B_2\}$ (yaw)

$$
{}^{I}_{B2}R = R_z(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

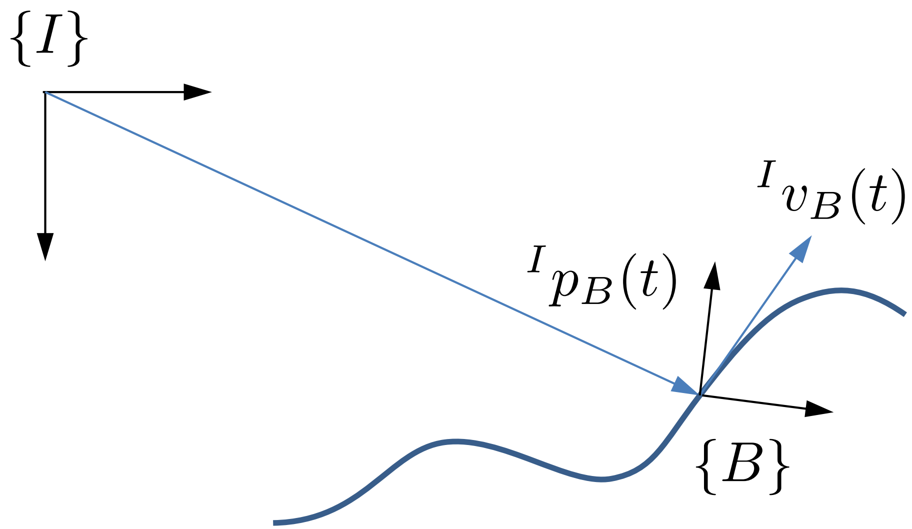
Beard & McLain, "Small Unmanned Aircraft," *Princeton University Press,* 2012

Rigid Body Kinematics – Linear Motion

- Linear motion: position evolves as a function of time

$$ {}^I\dot{p}_B(t) = {}^I v_B(t) $$

${}^I v_B(t)$ linear velocity expressed in inertial frame tangent to the curve at ${}^I p_B(t)$



Rigid Body Kinematics – Angular Motion

- Angular motion: orientation evolves as a function of time

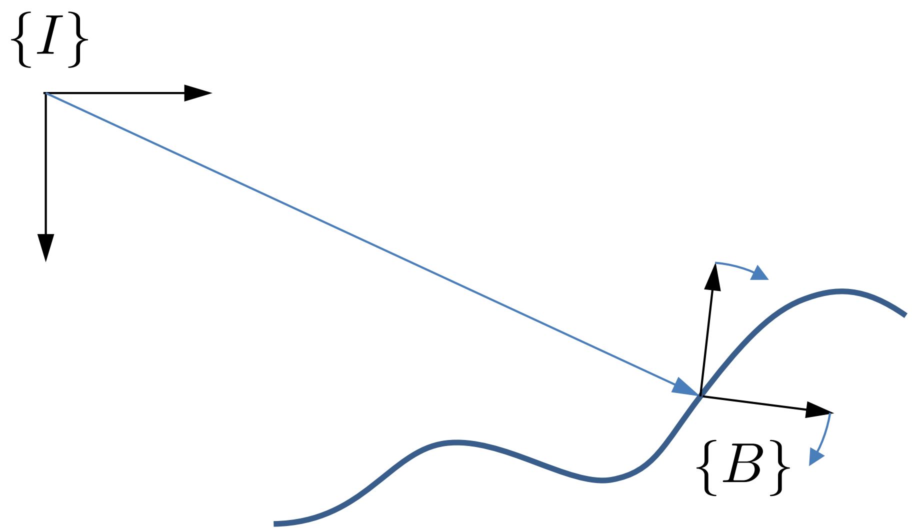
$$ {}^{I}_{B}\dot{R} = S({}^{I}\omega_B)\,{}^{I}_{B}R $$

${}^{I}\omega_B(t)$ angular velocity expressed in inertial frame

Recall that

$$ S(a)b = a \times b $$

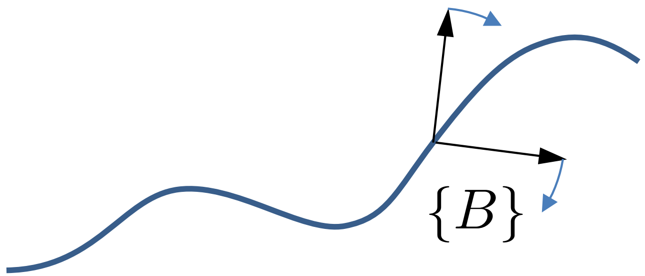
$$ S(a) = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} $$



Rigid Body Kinematics – Angular Motion

- Angular motion

$$ {}^I_B\dot{R} = S({}^I\omega_B)\,{}^I_BR $$

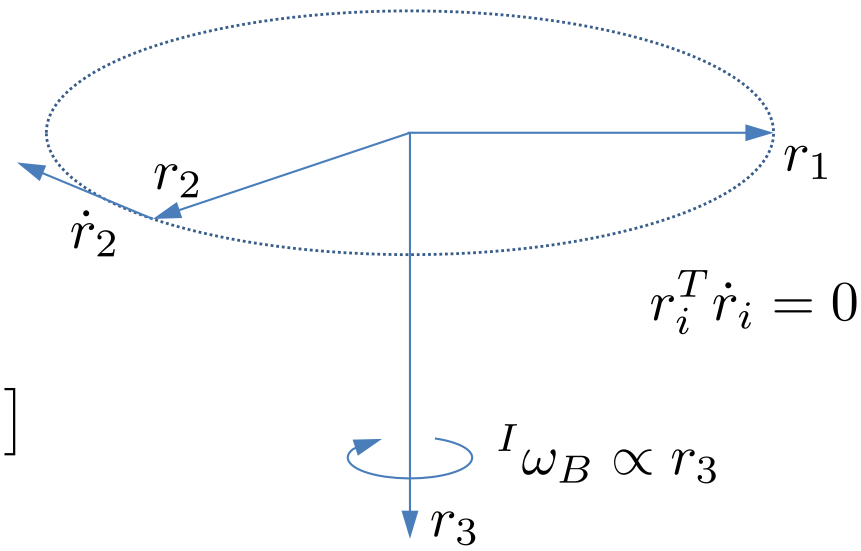


Why is that so?

$$ {}^I_BR = R = \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix} \in \mathbb{SO}(3) $$

$$ r_i^T r_i = 1 $$
$$ r_i^T \dot{r}_i = 0 $$
$$ \dot{r}_i = {}^I\omega_B \times r_i $$

$$ \begin{bmatrix} \dot{r}_1 & \dot{r}_2 & \dot{r}_3 \end{bmatrix} = S({}^I\omega_B)\begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix} $$



Rigid Body Kinematics – Angular Motion

- Angular motion

$$ {}^I_B\dot{R} = S({}^I\omega_B)\,{}^I_BR \qquad \text{and} \qquad {}^I_B\dot{R} = {}^I_BR\,S({}^B_IR\,{}^I\omega_B) $$

Using “lighter” notation

$$ R := {}^I_BR, \;\; \omega := {}^B_IR\,{}^I\omega_B $$

$$ \dot{R} = S({}^I\omega_B)R \qquad \text{and} \qquad \dot{R} = RS(\omega) $$

Expressed in inertial frame — Expressed in body frame

$$ \begin{aligned} RR^T &= I \\ \dot{R}R^T &= -R\dot{R}^T = S({}^I\omega_B) \\ \dot{R} &= S({}^I\omega_B)R \end{aligned} \qquad \begin{aligned} R^TR &= I \\ \dot{R}^TR &= -R^T\dot{R} = S(\omega) \\ R^T\dot{R} &= S(\omega) \end{aligned} $$

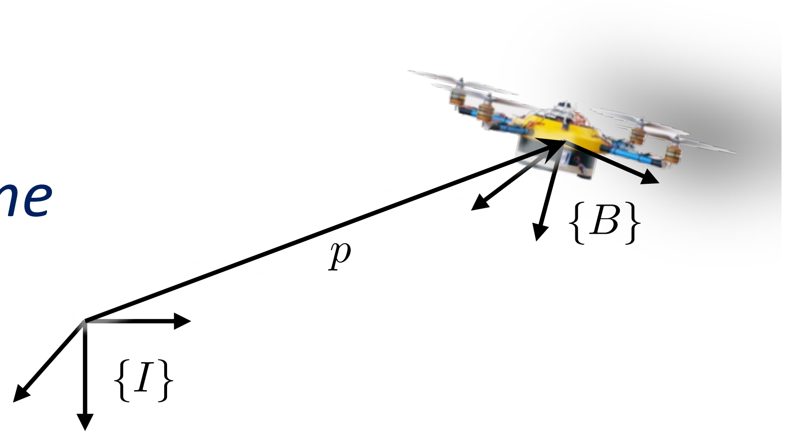
Rigid Body Kinematics

- With velocities *expressed in inertial frame*

$$\dot{p} = {}^{I}v_B$$

$$\dot{R} = S({}^{I}\omega_B)R$$

- With velocities *expressed in body frame*



$$\dot{p} = Rv$$

$$\dot{R} = RS(\omega)$$

- With position also expressed in body frame ${}^{B}p := {}^{B}_{I}R\,p$

$${}^{B}\dot{p} := -S(\omega)\,{}^{B}p + v$$

Rigid Body Kinematics - Exercise

- Circular motion: given

$$p(t) = {}^I p_B(t) = r \begin{bmatrix} \cos\psi(t) \\ \sin\psi(t) \\ 0 \end{bmatrix}$$

$$R(t) = {}^I_B R(t) = R_z(\psi(t))$$

determine

$$^B p = {}^B_I R\, p$$

$$v = {}^B_I R \dot{p}$$

$$\omega = {}^B_I R^I \omega_B$$

Rigid Body Kinematics

- Attitude kinematics (angular velocity expressed in body frame)

$\dot{R} = RS(\omega)$ Rotation matrix $R \in \mathbb{SO}(3)$

Or

$\dot{\lambda} = Q(\lambda)\omega$ Z-Y-X Euler angles $\lambda = \begin{bmatrix} \phi & \theta & \psi \end{bmatrix} \in \mathbb{R}^3$

$$Q(\lambda) = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \frac{\sin\phi}{\cos\theta} & \frac{\cos\phi}{\cos\theta} \end{bmatrix}$$

Rigid Body Kinematics – Exercise

- Given $\lambda = \begin{bmatrix}\phi & \theta & \psi\end{bmatrix} \in \mathbb{R}^3$

$$R(\lambda) = R_z(\psi)R_y(\theta)R_x(\phi)$$

$$\dot{R} = RS(\omega)$$

derive

$$\dot{\lambda} = Q(\lambda)\omega \qquad Q\left(\lambda\right) = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \frac{\sin\phi}{\cos\theta} & \frac{\cos\phi}{\cos\theta} \end{bmatrix}$$

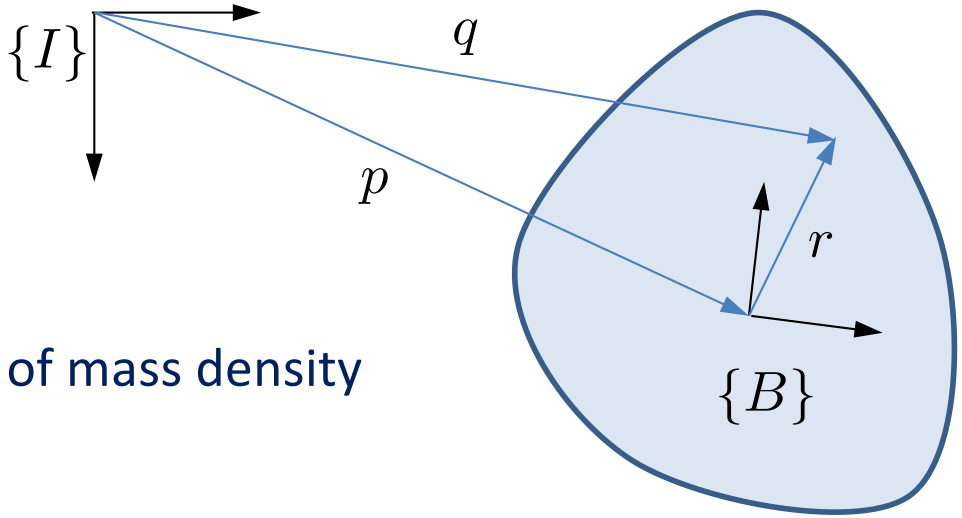
Intermediate step: derive

$$\omega = R_x^T(\phi)R_y^T(\theta)\dot{\psi}e_3 + R_x^T(\phi)\dot{\theta}e_2 + \dot{\phi}e_1 \qquad e_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}\ e_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}\ e_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

Rigid Body Dynamics

- Particles in the rigid body

$$ {}^{I}q = {}^{I}p_B + {}^{I}_{B}R\, {}^{B}r $$
$$ q = p + R\, r $$



- Mass

$$ m = \int \rho(r) dV $$

Volume integral of mass density

- Center of mass

$$ \bar{q} = \frac{1}{m} \int q(r)\rho(r) dV $$

Weighted average of the density

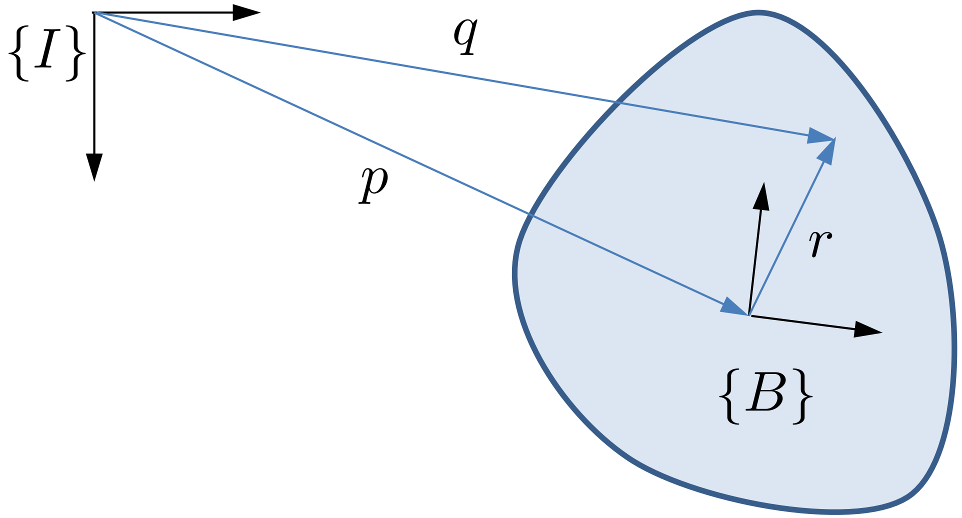
If origin of {B} coincides with center of mass

$$ p = \bar{q} \quad \Rightarrow \quad \int r\rho(r) dV = 0 $$

Rigid Body Dynamics

- Particles in the rigid body

$$ {}^{I}q = {}^{I}p_B + {}^{I}_{B}R\,{}^{B}r $$
$$ q = p + R\,r $$



- Velocity of particles

$$ \dot{q} = \dot{p} + RS(\omega)r $$

- Linear momentum

$$ \int \dot{q}(r)\rho(r)dV = m\dot{p} $$

- Angular momentum relative to center of mass

$$ \int \rho(r)(q(r) - \bar{q}) \times \dot{q}(r)dV = -R\int \rho(r)S(r)^2 dV\omega = RJ\omega $$

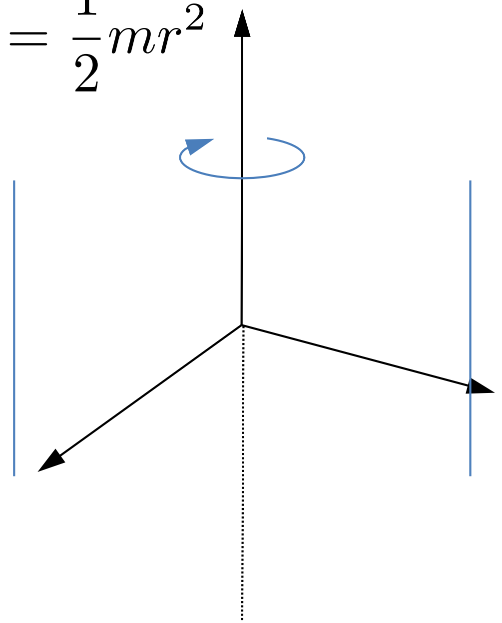
Tensor of inertia $J = -\int \rho(r)S(r)^2 dV$

Rigid Body Dynamics – Tensor of Inertia

- Tensor of Inertia

$$
\begin{aligned}
J &= -\int \rho(r) S(r)^2 dV \\
&= \int \rho(r)(r^T r I_3 - r r^T) dV \\
&= \int \rho(r) \begin{bmatrix} r_y^2 + r_z^2 & -r_x r_y & -r_x r_z \\ -r_x r_y & r_x^2 + r_z^2 & -r_y r_z \\ -r_x r_z & -r_y r_z & r_x^2 + r_y^2 \end{bmatrix} dV \\
&= \begin{bmatrix} J_{xx} & J_{xy} & J_{xz} \\ J_{xy} & J_{yy} & J_{yz} \\ J_{xz} & J_{yz} & J_{zz} \end{bmatrix}
\end{aligned}
$$

$$J_{zz} = \frac{1}{2} m r^2$$



$$J_{xx} = J_{yy} = \frac{1}{12} m (3r^2 + h^2)$$

$$J_{xy} = J_{xz} = J_{yz} = 0$$

$$u_i = e_i$$

Principal axes are given by the eigenvalues of J, $J u_i = \alpha_i u_i$

If the body rotates about a principal axis, angular momentum keeps the direction of rotation.

TÉCNICO LISBOA

Rigid Body Dynamics

- Newton's 2nd law
  - Conservation of momentum (holds in inertial frames)
- Translational motion
  - Linear momentum

    $$m\dot{p}$$

  - Conservation of linear momentum

    $$\frac{d}{dt}(m\dot{p}) = m\ddot{p} = {}^I f$$

    Applied forces, expressed in {I}

  - Transformation to body frame

$$\frac{d}{dt}(mRv) = {}^I f$$

$$mR\dot{v} + mRS(\omega)v = R\,{}^I f$$

$$m\dot{v} + mS(\omega)v = f, \quad f := R^T\,{}^I f$$

Rigid Body Dynamics

- Newton's 2nd law
  - Conservation of momentum (holds in inertial frames)
- Rotational motion
  - Angular momentum

    $RJ\omega$
  - Conservation of angular momentum

    $$\frac{d}{dt}(RJ\omega) = RJ\dot{\omega} + RS(\omega)J\omega = {}^{I}\tau$$

    Applied torques, expressed in {I}
  - Transformation to body frame

    $$J\dot{\omega} + S(\omega)J\omega = \tau, \quad \tau = R^{T\ I}\tau$$

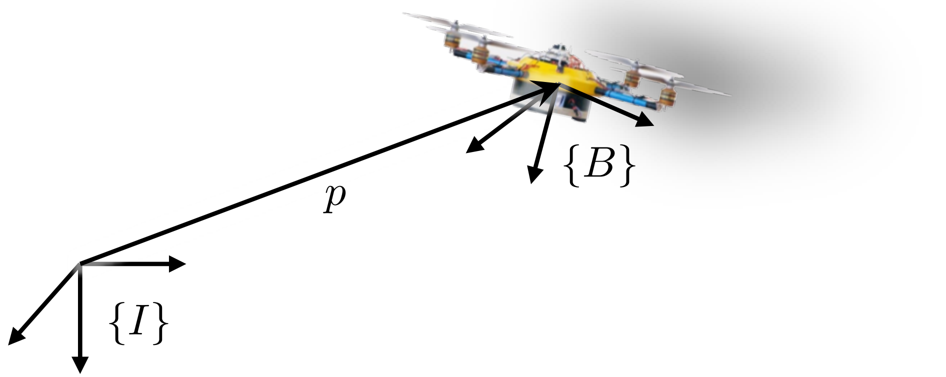
Rigid Body Equations of Motion

- Goal achieved:
  derive the equations of motion for a 3-D rigid body

  - Kinematics

$$\dot{p} = Rv$$

$$\dot{R} = RS(\omega)$$



  - Dynamics

$$m\dot{v} = -S(\omega)mv + f$$

$$J\dot{\omega} = -S(\omega)J\omega + \tau$$