Unmanned Aerial Vehicles

Example Questions June 2020

1. Dynamic Modeling

1.1 3D $\longrightarrow$ 2D

$$p = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \longrightarrow p = \begin{bmatrix} x \\ 0 \\ z \end{bmatrix}$$

$$R = R_z(\psi) R_y(\theta) R_x(\phi) \longrightarrow R = R_y(\theta)$$

$$m\ddot{p} = -TRe_3 + mge_3 \longrightarrow m\begin{bmatrix} \ddot{x} \\ 0 \\ \ddot{z} \end{bmatrix} = -T\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} m\ddot{x} = -T\sin\theta \\ m\ddot{z} = T\cos\theta + mg \end{cases}$$

$$\dot{R} = RS(\omega) \longrightarrow \dot{R}_y(\theta) = R_y(\theta) S\left(\dot{\theta}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$$

$$\Leftrightarrow \dot{\theta} = \omega_y$$

$$J\dot{\omega} = -JS(\omega)J\omega + \tau \longrightarrow J_{yy}\dot{\omega}_y = J_{yy}\ddot{\theta} = \underline{\tau_y}$$

1.2 From figure 1

$T = f_1 + f_2$ (sum of the forces along the negative body z-axis)

$\tau_y = l(f_1 - f_2)$ (difference between the torques produced by the forces f_1 and f_2)

1.3 State vector $\quad$ Input vector $\quad$ Flat output

$$\begin{bmatrix} x \\ \dot{x} \\ z \\ \dot{z} \\ \theta \\ \dot{\theta} \end{bmatrix} \qquad \begin{bmatrix} T \\ Z_y \end{bmatrix} \qquad \begin{bmatrix} x \\ z \end{bmatrix}$$

Write state and input as functions of $\begin{bmatrix} x \\ z \end{bmatrix}, \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix}, \begin{bmatrix} \ddot{x} \\ \ddot{z} \end{bmatrix}, \ldots$

Immediate for $\begin{bmatrix} x \\ z \end{bmatrix}$ and $\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix}$

From (4) and (5) it follows that

$$T \begin{bmatrix} \sin\theta \\ \cos\theta \end{bmatrix} = -m \begin{bmatrix} \ddot{x} \\ \ddot{z} - g \end{bmatrix}$$

Then $T = m\sqrt{(\ddot{x})^2 + (\ddot{z} - g)^2}$ ✓

$\theta = \operatorname{atan2}(-\ddot{x}, -\ddot{z} + g)$ ✓

$\dot{\theta}$ and $Z_y = \frac{\ddot{\theta}}{J_{yy}}$ can be obtained from the time-derivatives of θ, and depend on the time-derivatives of x and z up to the fourth-order.

1.4 In forward level flight $\dot{z} = \ddot{z} = 0$

$$|\theta| < \frac{\pi}{4} \Leftrightarrow |\tan\theta| < 1 \Leftrightarrow \frac{|\ddot{x}|}{g} < 1$$ ✓

2 Control and Observer Design

2.1 Since $\begin{bmatrix} u_x^* \\ u_z^* \end{bmatrix} = \begin{bmatrix} \ddot{x} \\ \ddot{z} \end{bmatrix}$, it follows from the derivations in 1.3 that $T = m\sqrt{(u_x^*)^2 + (u_z^* - g)^2}$

$$\theta_d = \mathrm{atan2}(-u_x^*, g - u_z^*)$$

2.2

$$\begin{cases} x_1 = \theta - \theta_d \\ x_2 = \dot{\theta} - \dot{\theta}_d \end{cases} \qquad \begin{aligned} \dot{x}_1 &= \dot{\theta} - \dot{\theta}_d = x_2 \\ \dot{x}_2 &= \ddot{\theta} - \ddot{\theta}_d = \frac{1}{J_{yy}} Z_y - \ddot{\theta}_d = \frac{1}{J_{yy}} u_e \end{aligned}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/J_{yy} \end{bmatrix} u_e \qquad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1/J_{yy} \end{bmatrix}$$

2.3

The pair (A, B) must be stabilizable and the pair (A, G) must be detectable

These conditions also hold if (A,B) is controllable and (A,G) is observable

Controllability matrix $\mathcal{C} = [B \ \ AB] = \begin{bmatrix} 0 & 1/J_{yy} \\ 1/J_{yy} & 0 \end{bmatrix}$

Full rank

Observability matrix $\mathcal{O} = \begin{bmatrix} G \\ GA \end{bmatrix}$

i) $\mathcal{O} = \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_1 \end{bmatrix}$ ✓ ii) $\mathcal{O} = \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \\ 0 & \gamma_1 \\ 0 & 0 \end{bmatrix}$ ✓

2.4 $\quad G = [\gamma_1 \quad 0] \qquad \rho = J_{yy}^{-2}$.

$$K = \frac{1}{\rho} B^T P = J_{yy}^2 \begin{bmatrix} 0 & 1/J_{yy} \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = J_{yy} \begin{bmatrix} p_{12} & p_{22} \end{bmatrix}$$

$$A^T P + PA + Q - \frac{1}{\rho} P B B^T P = 0 \qquad Q = G^T G$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} P + P \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \gamma_1 \\ 0 \end{bmatrix} \begin{bmatrix} \gamma_1 & 0 \end{bmatrix} - \cancel{J_{yy}^2} P \begin{bmatrix} 0 \\ 1/J_{yy} \end{bmatrix} \begin{bmatrix} 0 & 1/J_{yy} \end{bmatrix} P = 0$$

$$\begin{bmatrix} 0 & 0 \\ p_{11} & p_{12} \end{bmatrix} + \begin{bmatrix} 0 & p_{11} \\ 0 & p_{12} \end{bmatrix} + \begin{bmatrix} \gamma_1^2 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} p_{12} \\ p_{22} \end{bmatrix} \begin{bmatrix} p_{12} & p_{22} \end{bmatrix} = 0$$

$$\begin{cases} \gamma_1^2 - p_{12}^2 = 0 \\ p_{11} - p_{12} p_{22} = 0 \\ 2 p_{12} - p_{22}^2 = 0 \end{cases} \quad \Rightarrow K = J_{yy} [p_{12} \quad p_{22}] = J_{yy} [\gamma_1 \quad \sqrt{2\gamma_1}]$$

2.5

$$\dot{x}_e = (A - BK) x_e \Leftrightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1/J_{yy} \end{bmatrix} \begin{bmatrix} K_1 & K_2 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{K_1}{J_{yy}} x_1 - \frac{K_2}{J_{yy}} x_2 \end{bmatrix} \Rightarrow J_{yy}(\ddot{\theta} - \ddot{\theta}_d) = -K_1(\theta - \theta_d) - K_2(\dot{\theta} - \dot{\theta}_d)$$

$$\cancel{J_{yy}} (\ddot{\theta} - \ddot{\theta}_d) = -\cancel{J_{yy}} \gamma_1 (\theta - \theta_d) - \cancel{J_{yy}} \sqrt{2\gamma_1} (\dot{\theta} - \dot{\theta}_d)$$

2nd order system with

natural frequency $\omega_n = \sqrt{\gamma_1}$

damping coefficient $\qquad \xi = \frac{\sqrt{2}}{2} \qquad 2\xi\omega_n = \sqrt{2\gamma_1}$

Increasing γ_1 increases the natural frequency but has no effect on the damping.

2.6

$x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$ $\dot{x} = Ax + Bu$ $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$y = Cx$

$u = \underline{Z}y$ $B = \begin{bmatrix} 0 \\ 1/J_{yy} \end{bmatrix}$

$y = \theta$

$C = \begin{bmatrix} 1 & 0 \end{bmatrix}$

State observer

$\dot{\hat{x}} = A\hat{x} + Bu - L(C\hat{x} - y)$ $\hat{x} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}$ $\leftarrow$ estimate of θ / $\leftarrow$ estimate of $\hat{\theta}$

The observer is ~~stable~~ stabilizing if the error system with state $\tilde{x} = x - \hat{x}$ is stable.

$\dot{\tilde{x}} = Ax + \cancel{Bu} - A\hat{x} - \cancel{Bu} - L(y - C\hat{x})$

$= (A - LC)\tilde{x}$

$A - LC = \begin{bmatrix} -l_1 & 1 \\ -l_2 & 0 \end{bmatrix}$

~~det (A-LC)~~

$\det(\lambda I - (A - LC)) = 0 \iff \lambda^2 + l_1\lambda + l_2 = 0$

If l_1 and l_2 are positive, the eigenvalues of $A - LC$ have negative real part.

2.7

$u_e = -K(\hat{x} - x_d)$ instead of $u_e = -K(x - x_d)$

$$u_e = -K(x - x_d) - K(\hat{x} - x)$$
$$= -K x_e + K\tilde{x}$$

$$\dot{x}_e = A x_e + B u_e = (A - BK) x_e + BK\tilde{x}$$
$$\dot{\tilde{x}} = (A - LC)\tilde{x}$$

$$\begin{bmatrix} \dot{x}_e \\ \dot{\tilde{x}} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x_e \\ \tilde{x} \end{bmatrix} = A_{cl} \begin{bmatrix} x_e \\ \tilde{x} \end{bmatrix}$$

$$\mathrm{eig}(A_{cl}) = \mathrm{eig}(A - BK) \cup \mathrm{eig}(A - LC)$$

Separation Principle

controller and observer can be designed independently