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Unmanned Aerial Vehicles

Modern Control Design

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$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t) \end{cases}$$

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The term $\mathbf{D}(t)\mathbf{u}(t)$ corresponds to a direct effect of the input on the output. That is not possible in general, hence we will always consider $\mathbf{D}(t) = \mathbf{0}$.

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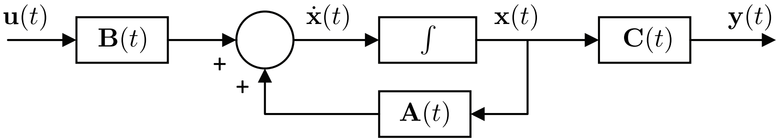
A special case occurs when the system matrices are constant, yielding the so-called linear time invariant (LTI) system:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \end{cases}$$

State space vector block diagram



1. Take the Laplace transform of both sides considering zero initial conditions:

$$\begin{cases} s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) \\ \mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) \end{cases}$$

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2. Determine $\mathbf{Y}(s)$ as a function of $\mathbf{U}(s)$:

$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) = \mathbf{C}\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B}\mathbf{U}(s)$$

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$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \end{cases} \Leftrightarrow \quad \mathbf{G}(s) = \mathbf{C}\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B}$$

Theorem

For any t_0 and $\mathbf{x}_0$, with $\mathbf{A}(t)$ continuous, the linear state equation

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) \\ \mathbf{x}(t_0) = \mathbf{x}_0 \end{cases}$$

has a unique solution given by

$$\mathbf{x}(t) = \mathbf{\Phi}(t, t_0)\,\mathbf{x}_0,$$

where $\mathbf{\Phi}(t, t_0)$ is the transtion matrix associated with $\mathbf{A}(t)$, given by the Peano-Baker series

$$\mathbf{\Phi}(t, t_0) := \mathbf{I} + \int_{t_0}^{t} \mathbf{A}(\sigma_1)\, d\sigma_1 + \int_{t_0}^{t} \mathbf{A}(\sigma_1) \int_{t_0}^{\sigma_1} \mathbf{A}(\sigma_2)\, d\sigma_2 d\sigma_1 + \ldots$$

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) \\ \mathbf{x}\left(t_0\right) = \mathbf{x}_0 \end{cases}$$

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) \\ \mathbf{x}(t_0) = \mathbf{x}_0 \end{cases}$$

$$\begin{cases} \mathbf{x}(t) = \mathbf{\Phi}(t, t_0)\,\mathbf{x}_0 + \int_{t_0}^{t} \mathbf{\Phi}(t, \sigma)\,\mathbf{B}(\sigma)\,\mathbf{u}(\sigma)\,d\sigma \\ \mathbf{y}(t) = \mathbf{C}(t)\mathbf{\Phi}(t, t_0)\,\mathbf{x}_0 + \int_{t_0}^{t} \mathbf{C}(t)\mathbf{\Phi}(t, \sigma)\,\mathbf{B}(\sigma)\,\mathbf{u}(\sigma)\,d\sigma \end{cases}$$

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$$\begin{cases} \mathbf{x}(t) = \mathbf{\Phi}(t, t_0)\, \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{\Phi}(t, \sigma)\, \mathbf{B}(\sigma)\, \mathbf{u}(\sigma)\, d\sigma \\ \mathbf{y}(t) = \mathbf{C}(t)\mathbf{\Phi}(t, t_0)\, \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{C}(t)\mathbf{\Phi}(t, \sigma)\, \mathbf{B}(\sigma)\, \mathbf{u}(\sigma)\, d\sigma \end{cases}$$

Check the solution of $\mathbf{x}(t)$ by direct differentiation, noting that

$$\frac{d}{dt}\mathbf{\Phi}(t, t_0) = \mathbf{A}(t)\mathbf{\Phi}(t, t_0)\,.$$

It is also possible to check this by direct computation.

1. $\frac{d}{dt}\mathbf{\Phi}\left(t,t_0\right) = \mathbf{A}(t)\mathbf{\Phi}\left(t,t_0\right)$

$$
\begin{aligned}
\frac{d}{dt}\mathbf{\Phi}\left(t,t_0\right) = {} & \frac{d}{dt}\left(\mathbf{I}\right) + \frac{d}{dt}\left(\int_{t_0}^{t}\mathbf{A}\left(\sigma_1\right)d\sigma_1\right) \\
& + \frac{d}{dt}\left(\int_{t_0}^{t}\mathbf{A}\left(\sigma_1\right)\int_{t_0}^{\sigma_1}\mathbf{A}\left(\sigma_2\right)d\sigma_2 d\sigma_1\right) \\
& + \frac{d}{dt}\left(\int_{t_0}^{t}\mathbf{A}\left(\sigma_1\right)\int_{t_0}^{\sigma_1}\mathbf{A}\left(\sigma_2\right)\int_{t_0}^{\sigma_2}\mathbf{A}\left(\sigma_3\right)d\sigma_3 d\sigma_2 d\sigma_1\right) \\
& + \ldots
\end{aligned}
$$

1. $\frac{d}{dt}\boldsymbol{\Phi}(t,t_0) = \mathbf{A}(t)\boldsymbol{\Phi}(t,t_0)$

$$\begin{aligned}
\frac{d}{dt}\boldsymbol{\Phi}(t,t_0) &= \frac{d}{dt}(\mathbf{I}) + \frac{d}{dt}\left(\int_{t_0}^{t}\mathbf{A}(\sigma_1)\,d\sigma_1\right) \\
&\quad + \frac{d}{dt}\left(\int_{t_0}^{t}\mathbf{A}(\sigma_1)\int_{t_0}^{\sigma_1}\mathbf{A}(\sigma_2)\,d\sigma_2 d\sigma_1\right) \\
&\quad + \frac{d}{dt}\left(\int_{t_0}^{t}\mathbf{A}(\sigma_1)\int_{t_0}^{\sigma_1}\mathbf{A}(\sigma_2)\int_{t_0}^{\sigma_2}\mathbf{A}(\sigma_3)\,d\sigma_3 d\sigma_2 d\sigma_1\right) \\
&\quad + \dots \\
&= \mathbf{A}(t) + \mathbf{A}(t)\int_{t_0}^{t}\mathbf{A}(\sigma_1)\,d\sigma_1 \\
&\quad \mathbf{A}(t)\int_{t_0}^{t}\mathbf{A}(\sigma_1)\int_{t_0}^{\sigma_1}\mathbf{A}(\sigma_2)\,d\sigma_2 d\sigma_1 + \dots
\end{aligned}$$

1. $\frac{d}{dt}\mathbf{\Phi}\left(t,t_0\right) = \mathbf{A}(t)\mathbf{\Phi}\left(t,t_0\right)$

$$\begin{aligned}
\frac{d}{dt}\mathbf{\Phi}\left(t,t_0\right) &= \frac{d}{dt}\left(\mathbf{I}\right) + \frac{d}{dt}\left(\int_{t_0}^{t}\mathbf{A}\left(\sigma_1\right)d\sigma_1\right) \\
&\quad + \frac{d}{dt}\left(\int_{t_0}^{t}\mathbf{A}\left(\sigma_1\right)\int_{t_0}^{\sigma_1}\mathbf{A}\left(\sigma_2\right)d\sigma_2 d\sigma_1\right) \\
&\quad + \frac{d}{dt}\left(\int_{t_0}^{t}\mathbf{A}\left(\sigma_1\right)\int_{t_0}^{\sigma_1}\mathbf{A}\left(\sigma_2\right)\int_{t_0}^{\sigma_2}\mathbf{A}\left(\sigma_3\right)d\sigma_3 d\sigma_2 d\sigma_1\right) \\
&\quad + \ldots \\
&= \mathbf{A}(t) + \mathbf{A}(t)\int_{t_0}^{t}\mathbf{A}\left(\sigma_1\right)d\sigma_1 \\
&\quad \mathbf{A}(t)\int_{t_0}^{t}\mathbf{A}\left(\sigma_1\right)\int_{t_0}^{\sigma_1}\mathbf{A}\left(\sigma_2\right)d\sigma_2 d\sigma_1 + \ldots \\
&= \mathbf{A}(t)\mathbf{\Phi}\left(t,t_0\right)
\end{aligned}$$

2. Complete solution:

$$\dot{\mathbf{x}}(t) = \frac{d}{dt}\left(\mathbf{\Phi}\left(t, t_0\right)\mathbf{x}_0 + \int_{t_0}^{t} \mathbf{\Phi}\left(t, \sigma\right)\mathbf{B}\left(\sigma\right)\mathbf{u}\left(\sigma\right) d\sigma\right)$$

2. Complete solution:

$$\begin{aligned}
\dot{\mathbf{x}}(t) &= \frac{d}{dt}\left(\mathbf{\Phi}\left(t,t_0\right)\mathbf{x}_0 + \int_{t_0}^{t}\mathbf{\Phi}\left(t,\sigma\right)\mathbf{B}\left(\sigma\right)\mathbf{u}\left(\sigma\right)d\sigma\right)\\
&= \mathbf{A}(t)\mathbf{\Phi}\left(t,t_0\right)\mathbf{x}_0 + \frac{d}{dt}(t)\left(\mathbf{\Phi}\left(t,\sigma\right)\mathbf{B}\left(\sigma\right)\mathbf{u}\left(\sigma\right)\big|_{\sigma=t}\right)\\
&\quad -\frac{d}{dt}(t_0)\left(\mathbf{\Phi}\left(t,\sigma\right)\mathbf{B}\left(\sigma\right)\mathbf{u}\left(\sigma\right)\big|_{\sigma=t_0}\right)\\
&\quad +\int_{t_0}^{t}\frac{d}{dt}\left(\mathbf{\Phi}\left(t,\sigma\right)\mathbf{B}\left(\sigma\right)\mathbf{u}\left(\sigma\right)\right)d\sigma
\end{aligned}$$

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\begin{aligned}
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&= \mathbf{A}(t)\mathbf{\Phi}\left(t,t_0\right)\mathbf{x}_0 + \frac{d}{dt}\left(t\right)\left(\left.\mathbf{\Phi}\left(t,\sigma\right)\mathbf{B}\left(\sigma\right)\mathbf{u}\left(\sigma\right)\right|_{\sigma=t}\right) \\
&\quad -\frac{d}{dt}\left(t_0\right)\left(\left.\mathbf{\Phi}\left(t,\sigma\right)\mathbf{B}\left(\sigma\right)\mathbf{u}\left(\sigma\right)\right|_{\sigma=t_0}\right) \\
&\quad + \int_{t_0}^{t}\frac{d}{dt}\left(\mathbf{\Phi}\left(t,\sigma\right)\mathbf{B}\left(\sigma\right)\mathbf{u}\left(\sigma\right)\right)d\sigma \\
&= \mathbf{A}(t)\mathbf{\Phi}\left(t,t_0\right)\mathbf{x}_0 + \mathbf{\Phi}\left(t,t\right)\mathbf{B}\left(t\right)\mathbf{u}\left(t\right) \\
&\quad + \int_{t_0}^{t}\mathbf{A}(t)\mathbf{\Phi}\left(t,\sigma\right)\mathbf{B}\left(\sigma\right)\mathbf{u}\left(\sigma\right)d\sigma
\end{aligned}
$$

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\begin{aligned}
\dot{\mathbf{x}}(t) &= \frac{d}{dt}\left(\mathbf{\Phi}\left(t,t_0\right)\mathbf{x}_0 + \int_{t_0}^{t}\mathbf{\Phi}\left(t,\sigma\right)\mathbf{B}\left(\sigma\right)\mathbf{u}\left(\sigma\right)d\sigma\right) \\
&= \mathbf{A}(t)\mathbf{\Phi}\left(t,t_0\right)\mathbf{x}_0 + \frac{d}{dt}(t)\left(\mathbf{\Phi}\left(t,\sigma\right)\mathbf{B}\left(\sigma\right)\mathbf{u}\left(\sigma\right)|_{\sigma=t}\right) \\
&\quad -\frac{d}{dt}(t_0)\left(\mathbf{\Phi}\left(t,\sigma\right)\mathbf{B}\left(\sigma\right)\mathbf{u}\left(\sigma\right)|_{\sigma=t_0}\right) \\
&\quad +\int_{t_0}^{t}\frac{d}{dt}\left(\mathbf{\Phi}\left(t,\sigma\right)\mathbf{B}\left(\sigma\right)\mathbf{u}\left(\sigma\right)\right)d\sigma \\
&= \mathbf{A}(t)\mathbf{\Phi}\left(t,t_0\right)\mathbf{x}_0 + \mathbf{\Phi}\left(t,t\right)\mathbf{B}\left(t\right)\mathbf{u}\left(t\right) \\
&\quad +\int_{t_0}^{t}\mathbf{A}(t)\mathbf{\Phi}\left(t,\sigma\right)\mathbf{B}\left(\sigma\right)\mathbf{u}\left(\sigma\right)d\sigma \\
&= \mathbf{A}(t)\left(\mathbf{\Phi}\left(t,t_0\right)\mathbf{x}_0 + \int_{t_0}^{t}\mathbf{\Phi}\left(t,\sigma\right)\mathbf{B}\left(\sigma\right)\mathbf{u}\left(\sigma\right)d\sigma\right) + \mathbf{B}\left(t\right)\mathbf{u}\left(t\right)
\end{aligned}
$$

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$$
\begin{aligned}
\dot{\mathbf{x}}(t) &= \frac{d}{dt}\left(\mathbf{\Phi}\left(t,t_0\right)\mathbf{x}_0 + \int_{t_0}^{t}\mathbf{\Phi}\left(t,\sigma\right)\mathbf{B}\left(\sigma\right)\mathbf{u}\left(\sigma\right)d\sigma\right) \\
&= \mathbf{A}(t)\mathbf{\Phi}\left(t,t_0\right)\mathbf{x}_0 + \frac{d}{dt}\left(t\right)\left(\mathbf{\Phi}\left(t,\sigma\right)\mathbf{B}\left(\sigma\right)\mathbf{u}\left(\sigma\right)|_{\sigma=t}\right) \\
&\quad -\frac{d}{dt}\left(t_0\right)\left(\mathbf{\Phi}\left(t,\sigma\right)\mathbf{B}\left(\sigma\right)\mathbf{u}\left(\sigma\right)|_{\sigma=t_0}\right) \\
&\quad + \int_{t_0}^{t}\frac{d}{dt}\left(\mathbf{\Phi}\left(t,\sigma\right)\mathbf{B}\left(\sigma\right)\mathbf{u}\left(\sigma\right)\right)d\sigma \\
&= \mathbf{A}(t)\mathbf{\Phi}\left(t,t_0\right)\mathbf{x}_0 + \mathbf{\Phi}\left(t,t\right)\mathbf{B}\left(t\right)\mathbf{u}\left(t\right) \\
&\quad + \int_{t_0}^{t}\mathbf{A}(t)\mathbf{\Phi}\left(t,\sigma\right)\mathbf{B}\left(\sigma\right)\mathbf{u}\left(\sigma\right)d\sigma \\
&= \mathbf{A}(t)\left(\mathbf{\Phi}\left(t,t_0\right)\mathbf{x}_0 + \int_{t_0}^{t}\mathbf{\Phi}\left(t,\sigma\right)\mathbf{B}\left(\sigma\right)\mathbf{u}\left(\sigma\right)d\sigma\right) + \mathbf{B}\left(t\right)\mathbf{u}\left(t\right) \\
&= \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)
\end{aligned}
$$

For LTI systems, the transition matrix can be easily computed...

$$
\begin{aligned}
\mathbf{\Phi}(t,t_0) &:= \mathbf{I} + \int_{t_0}^{t} \mathbf{A} d\sigma_1 + \int_{t_0}^{t} \mathbf{A} \int_{t_0}^{\sigma_1} \mathbf{A} d\sigma_2 d\sigma_1 \\
&\quad + \int_{t_0}^{t} \mathbf{A} \int_{t_0}^{\sigma_1} \mathbf{A} \int_{t_0}^{\sigma_2} \mathbf{A} d\sigma_3 d\sigma_2 d\sigma_1 + \dots \\
&= \mathbf{I} + (t-t_0)\mathbf{A} + \frac{(t-t_0)^2}{2}\mathbf{A}^2 + \frac{(t-t_0)^3}{3}\mathbf{A}^3 + \dots \\
&= \mathbf{e}^{(t-t_0)\mathbf{A}}
\end{aligned}
$$

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \\ \mathbf{x}\left(t_0\right) = \mathbf{x}_0 \end{cases}$$

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$$\begin{cases} \mathbf{x}(t) = \mathbf{e}^{\mathbf{A}(t-t_0)}\mathbf{x}_0 + \int_{t_0}^{t} \mathbf{e}^{\mathbf{A}(t-\sigma)}\mathbf{B}\mathbf{u}(\sigma)\, d\sigma \\ \mathbf{y}(t) = \mathbf{C}\mathbf{e}^{\mathbf{A}(t-t_0)}\mathbf{x}_0 + \int_{t_0}^{t} \mathbf{C}\mathbf{e}^{\mathbf{A}(t-\sigma)}\mathbf{B}\mathbf{u}(\sigma)\, d\sigma \end{cases}$$

Eigenstructure of $\mathbf{A} \in \mathbb{R}^{n\times n}$:

$$\mathbf{A}_i\mathbf{v}_i = \lambda_i\mathbf{v}_i \qquad \mathbf{W}_i^T\mathbf{A} = \lambda_i\mathbf{w}_i^T$$

- i-th eigenvalue (<u>all distinct</u>): $\lambda_i \in \mathbb{C}$
- Right eigenvector: $\mathbf{v}_i$
- Left eigenvector: $\mathbf{w}_i^T$

Diadic formula:

$$\mathbf{A} = \sum_{i=1}^{n} \lambda_i\mathbf{v}_i\mathbf{w}_i^T$$

Orthonormal vectors:

$$\mathbf{w}_i^T\mathbf{v}_i = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Exponential matrix:

$$e^{\mathbf{A}t} = \sum_{i=1}^{n} e^{\lambda_i t}\mathbf{v}_i\mathbf{w}_i^T$$

Unforced LTI dynamics modal response:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) \\ \mathbf{x}\left(t_0\right) = \mathbf{x}_0 \end{cases}$$

$$\mathbf{x}(t) = e^{(t-t_0)\mathbf{A}}\mathbf{x}_0 = \sum_{i=1}^{n} e^{\lambda_i(t-t_0)}\mathbf{v}_i\left(\mathbf{w}_i^T\mathbf{x}_0\right)$$

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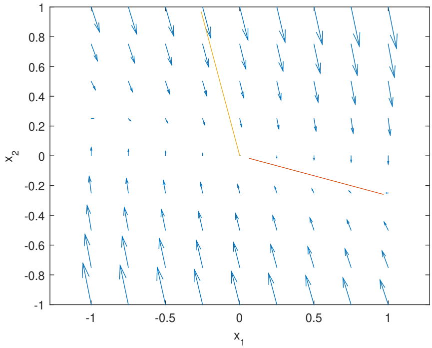
Definition

Consider the unforced system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$, with $\mathbf{x}(t_0) = \mathbf{x}_0$. The i-th mode is $e^{\lambda_i (t-t_0)}\mathbf{v}_i$, defined by the right eigenvector $\mathbf{v}_i$ and the associated eigenvalue λ_i.

Unforced LTI dynamics modal response:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & -4 \end{bmatrix}$$

$$\mathbf{A}\mathbf{V} = \mathbf{V}\boldsymbol{\Lambda},\ \boldsymbol{\Lambda} = \begin{bmatrix} -0.2679 & 0 \\ 0 & -3.7321 \end{bmatrix},\ \mathbf{V} = \begin{bmatrix} 0.9659 & -0.2588 \\ -0.2588 & 0.9659 \end{bmatrix}$$



Can I fly, or park, an airplane laterally?

Can I fly, or park, an airplane laterally?

Given the linear state equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$$

with initial condition $\mathbf{x}(0) = \mathbf{0}$ and given an arbitrary state $\mathbf{x}_f$, is it possible to make

$$\mathbf{x}(t_f) = \mathbf{x}_f$$

by appropriate choice of $\mathbf{u}(t)$?

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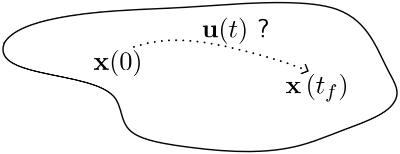

Given the linear state equation

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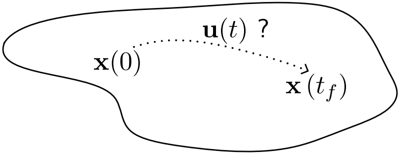

The answer depends on the pair of matrices $\mathbf{A}(t)$ and $\mathbf{B}(t)$.

Definition of controllability

The linear state equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$$

is called *controllable on* $[t_0, t_f]$ if, given any initial state $\mathbf{x}(0) = \mathbf{x}_0$, there exists a continuous input signal $\mathbf{u}(t)$ such that the corresponding solution satisfies $\mathbf{x}\left(t_f\right) = \mathbf{0}$.

For LTI system there exists a simple controllability test!

Theorem

The LTI system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

is controllable if and only if the controllability matrix

$$\boldsymbol{\mathcal{C}} := \begin{bmatrix}\mathbf{B} & \mathbf{A}\mathbf{B} & \mathbf{A}^2\mathbf{B} & \dots & \mathbf{A}^{n-1}\mathbf{B}\end{bmatrix}$$

has rank equal to the number of states of the system, i.e.,

$$\operatorname{rank} \boldsymbol{\mathcal{C}} = n,\ \mathbf{x} \in \mathbb{R}^n.$$

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

- Controllable system:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathcal{C} := [\mathbf{B}\ \mathbf{AB}] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- Uncontrollable system:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\mathcal{C} := [\mathbf{B}\ \mathbf{AB}] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Definition of observability

The linear state equation

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) \\ \mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) \end{cases}$$

is called *observable on* $[t_0, t_f]$ if any initial state $\mathbf{x}(0) = \mathbf{x}_0$ is uniquely determined by the corresponding response $\mathbf{y}(t)$ for $t \in [t_0, t_f]$.

Question: Does anything change if we consider

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) \end{cases}$$

with a known input $\mathbf{u}(t)$, $t \in [t_0, t_f]$?

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with a known input $\mathbf{u}(t)$, $t \in [t_0, t_f]$?

No!

When the input is known, we can always subtract from the output the component that depends on the input, and thus compute the output corresponding to the zero-input case.

Conclusion: For LTV systems, the observability is independent of the input. In general, that is not the case for nonlinear systems, which makes the analysis much more intricate...

For LTI system there exists a simple observability test!

Theorem

The LTI system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

is observable if and only if the observability matrix

$$\boldsymbol{\mathcal{O}} := \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \\ \vdots \\ \mathbf{C}\mathbf{A}^{n-1} \end{bmatrix}$$

has rank equal to the number of states of the system, i.e.,

$$\operatorname{rank} \boldsymbol{\mathcal{C}} = n, \; \mathbf{x} \in \mathbb{R}^n.$$

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \end{cases}$$

- Observable system:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\mathcal{O} := \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Unobservable system:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$\mathcal{O} := \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Controllability and observability matrices

```
>> A = [0 1; 0 0]; B = [0; 1]; CM = ctrb(A, B)
CM =
    0 1
    1 0
>> A = [0 1; 0 0]; C = [1, 0]; OM = obsv(A, C)
OM =
    1 0
    0 1
>> rank(CM), rank(OM)
ans =
    2
ans =
    2
```

For LTV systems there also exist powerful tools for controllability and observability analysis.

Definition of controllability Gramian

The controllability Gramian associated with the pair $(\mathbf{A}(t), \mathbf{B}(t))$ on $[t_0, t_f]$ is given by

$$\boldsymbol{\mathcal{W}}_C\left(t_0, t_f\right) := \int_{t_0}^{t_f} \boldsymbol{\phi}\left(t_0, t\right) \mathbf{B}(t)\mathbf{B}^T(t)\boldsymbol{\phi}^T\left(t_0, t\right) dt.$$

Definition of observability Gramian

The observability Gramian associated with the pair $(\mathbf{A}(t), \mathbf{C}(t))$ on $[t_0, t_f]$ is given by

$$\boldsymbol{\mathcal{W}}_O\left(t_0, t_f\right) := \int_{t_0}^{t_f} \boldsymbol{\phi}^T\left(t, t_0\right) \mathbf{C}^T(t)\mathbf{C}(t)\boldsymbol{\phi}^T\left(t, t_0\right) dt.$$

Theorem

The LTV system

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) \end{cases}$$

is controllable (observable) if and only if the controllability Gramian $\boldsymbol{\mathcal{W}}_C(t_0, t_f)$ (observability Gramian $\boldsymbol{\mathcal{W}}_O(t_0, t_f)$) is invertible.

Theorem

The LTV system

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For nonlinear systems there are some local results but no general results...

Is it possible to control the "behavior" of a system using linear state feedback?

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Let us focus on LTI systems only, for now... What is linear state feedback?

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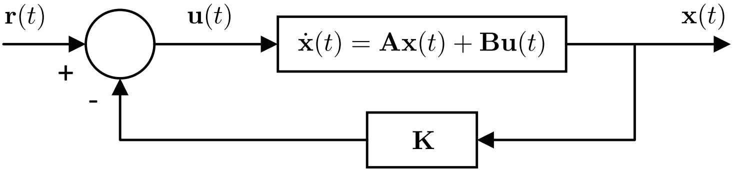

Linear state feedback:

$$\mathbf{u}(t) = \mathbf{r}(t) - \mathbf{K}\mathbf{x}(t)$$

Is it possible to assign the poles of a LTI system using linear state feedback?

Is it possible to assign the poles of a LTI system using linear state feedback?

Theorem: Pole placement for LTI systems

Suppose that the LTI system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

is controllable. Given any real, monic polynomial $p(\lambda)$ of degree-n, there exists a constant state feedback gain $\mathbf{K}$ such that

$$\det\left(\lambda\mathbf{I} - \mathbf{A} - \mathbf{B}\mathbf{K}\right) = p(\lambda).$$

1. By direct computation (comparison of polinomials)
2. Bass-Gura formula
3. Ackermann's formula

Matlab example

```
>> A = [0 1; 0 0]; B = [0; 1]; p = [-1+i -1-i];
>> place(A, B, p)
ans =
    2.0000 2.0000
```

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Desired poles:
$-1 + i,\ -1 - i$

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1. Desired characteristic polynomial:

$$p(\lambda) = [\lambda - (-1+i)]\,[\lambda - (-1-i)] = \lambda^2 + 2\lambda + 2$$

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$$p(\lambda) = [\lambda - (-1+i)]\,[\lambda - (-1-i)] = \lambda^2 + 2\lambda + 2$$

2. Closed-loop system matrix:

$$\mathbf{A} - \mathbf{BK} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix}$$

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3. Closed-loop polynomial:

$$\det\left(\lambda \mathbf{I} - (\mathbf{A} - \mathbf{BK})\right) = \lambda^2 + k_2\lambda + k_1$$

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3. Closed-loop polynomial:

$$\det\left(\lambda\mathbf{I} - (\mathbf{A} - \mathbf{BK})\right) = \lambda^2 + k_2\lambda + k_1$$

4. By comparison, $k_1 = k_2 = 2$.

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State observers

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State observers

Problem: Given the dynamic system

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) \end{cases},$$

design a state observer that yields a *"good"* estimate of $\mathbf{x}(t)$.

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design a state observer that yields a *"good"* estimate of $\mathbf{x}(t)$.

What is a *"good"* estimate of $\mathbf{x}(t)$?

Why not just buy sensors to measure the state?

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Idea: Replica of the system with feedback of the output estimation error

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}(t)\hat{\mathbf{x}}(t) + \mathbf{B}(t)\mathbf{u}(t) + \mathbf{L}(t)\left[\mathbf{y}(t) - \mathbf{C}(t)\hat{\mathbf{x}}(t)\right]$$

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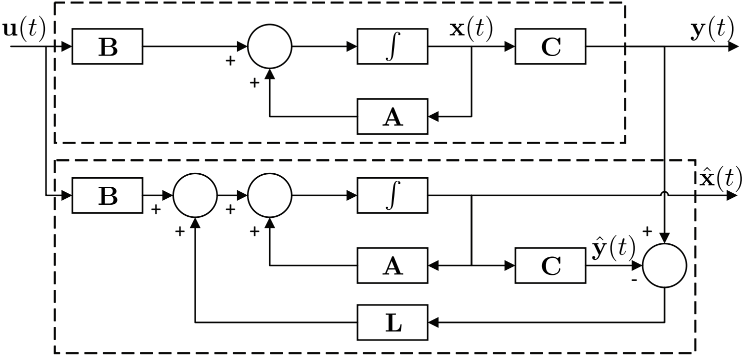

Observer estimation:

$$\tilde{\mathbf{x}}(t) := \mathbf{x}(t) - \hat{\mathbf{x}}(t)$$

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Observer estimation error dynamics:

$$\begin{aligned}
\dot{\tilde{\mathbf{x}}}(t) &= \dot{\mathbf{x}}(t) - \dot{\hat{\mathbf{x}}}(t) \\
&= \mathbf{A}(t)\mathbf{x}(t) - \mathbf{A}(t)\hat{\mathbf{x}}(t) - \mathbf{L}(t)\left[\mathbf{y}(t) - \mathbf{C}(t)\hat{\mathbf{x}}(t)\right] \\
&= \mathbf{A}(t)\tilde{\mathbf{x}}(t) - \mathbf{L}(t)\left[\mathbf{C}(t)\mathbf{x}(t) - \mathbf{C}(t)\hat{\mathbf{x}}(t)\right] \\
&= \left[\mathbf{A}(t) - \mathbf{L}(t)\mathbf{C}(t)\right]\tilde{\mathbf{x}}(t)
\end{aligned}$$

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Observer estimation error dynamics:

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&= \mathbf{A}(t)\tilde{\mathbf{x}}(t) - \mathbf{L}(t)\left[\mathbf{C}(t)\mathbf{x}(t) - \mathbf{C}(t)\hat{\mathbf{x}}(t)\right] \\
&= \left[\mathbf{A}(t) - \mathbf{L}(t)\mathbf{C}(t)\right]\tilde{\mathbf{x}}(t)
\end{aligned}$$

Question: How does one design the observer gain $\mathbf{L}(t)$?

For linear systems, the control and estimation problems are **dual!**

This is very clear in the LTI case...

The eigenvalues of

$$\mathbf{A} - \mathbf{L}\mathbf{C}$$

are the same of

$$(\mathbf{A} - \mathbf{L}\mathbf{C})^T = \mathbf{A}^T - \mathbf{C}^T\mathbf{L}^T.$$

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Duality

$$\mathbf{A} \longleftrightarrow \mathbf{A}^T$$
$$\mathbf{B} \longleftrightarrow \mathbf{C}^T$$
$$\mathbf{K} \longleftrightarrow \mathbf{L}^T$$

Linear state feedback

$+$

State observers

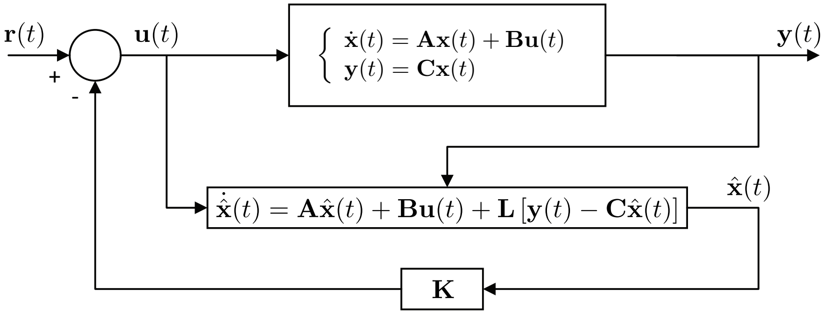
Linear state feedback

$+$

State observers

Does it work?

Can they be designed independently?


$\mathbf{r}(t)$
+
-
$\mathbf{u}(t)$
$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \end{cases}$
$\mathbf{y}(t)$
$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{L}\left[\mathbf{y}(t) - \mathbf{C}\hat{\mathbf{x}}(t)\right]$
$\hat{\mathbf{x}}(t)$
$\mathbf{K}$

$$\left\{\begin{array}{l}\dot{\mathbf{x}}(t)=\mathbf{A}\mathbf{x}(t)-\mathbf{B}\mathbf{K}\hat{\mathbf{x}}(t)+\mathbf{B}\mathbf{r}(t)\\ \mathbf{y}(t)=\mathbf{C}\mathbf{x}(t)\\ \dot{\hat{\mathbf{x}}}(t)=\mathbf{A}\hat{\mathbf{x}}(t)-\mathbf{B}\mathbf{K}\hat{\mathbf{x}}(t)+\mathbf{B}\mathbf{r}(t)+\mathbf{L}\left[\mathbf{y}(t)-\mathbf{C}\hat{\mathbf{x}}(t)\right]\end{array}\right.$$

Recall that, for the observer, we shape the dynamics of the estimation error $\tilde{\mathbf{x}}(t)=\mathbf{x}(t)-\hat{\mathbf{x}}(t)$.

$$\left\{\begin{array}{l}\dot{\mathbf{x}}(t)=\left(\mathbf{A}-\mathbf{B}\mathbf{K}\right)\mathbf{x}(t)+\mathbf{B}\mathbf{K}\tilde{\mathbf{x}}(t)+\mathbf{B}\mathbf{r}(t)\\ \dot{\tilde{\mathbf{x}}}(t)=\left(\mathbf{A}-\mathbf{L}\mathbf{C}\right)\tilde{\mathbf{x}}(t)\\ \mathbf{y}(t)=\mathbf{C}\mathbf{x}(t)\end{array}\right.$$

$$\begin{cases} \begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\tilde{\mathbf{x}}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{B}\mathbf{K} & \mathbf{B}\mathbf{K} \\ \mathbf{0} & \mathbf{A} - \mathbf{L}\mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \tilde{\mathbf{x}}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} \mathbf{r}(t) \\ \mathbf{y}(t) = \begin{bmatrix} \mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \tilde{\mathbf{x}}(t) \end{bmatrix} \end{cases}$$

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Theorem

The eigenvalues of

$$\begin{bmatrix} \mathbf{A} - \mathbf{B}\mathbf{K} & \mathbf{B}\mathbf{K} \\ \mathbf{0} & \mathbf{A} - \mathbf{L}\mathbf{C} \end{bmatrix}$$

are the union of the eigenvalues of $\mathbf{A} - \mathbf{B}\mathbf{K}$ and $\mathbf{A} - \mathbf{L}\mathbf{C}$.

$$\begin{cases} \begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\tilde{\mathbf{x}}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A}-\mathbf{B}\mathbf{K} & \mathbf{B}\mathbf{K} \\ \mathbf{0} & \mathbf{A}-\mathbf{L}\mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \tilde{\mathbf{x}}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} \mathbf{r}(t) \\ \mathbf{y}(t) = \begin{bmatrix} \mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \tilde{\mathbf{x}}(t) \end{bmatrix} \end{cases}$$

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are the union of the eigenvalues of $\mathbf{A}-\mathbf{B}\mathbf{K}$ and $\mathbf{A}-\mathbf{L}\mathbf{C}$.

Conclusion: The eigenvalues of the controller and the observer may be assigned independently.

Separation theorem

- The separation principle concerns the design of controllers and observers for LTI systems by pole assignement and basically states that these processes can be carried out independently.
- Separation theorem: for linear systems, a much stronger property can be shown, which concerns the design of **optimal** controllers and observers, which can be carried out, again, independently! More on this later...
- For nonlinear systems there are no general results.

Consider the LTI system

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases} .$$

- We know how to *"shape"* the closed-loop dynamics by pole placement.

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- We know how to *"shape"* the closed-loop dynamics by pole placement.
- Is it possible to do better, to optimize some kind of cost functional?

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- We know how to *"shape"* the closed-loop dynamics by pole placement.
- Is it possible to do better, to optimize some kind of cost functional?

LQR problem

Given the LTI state equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t),$$

consider the LQR optimization criterion

$$J := \int_0^{+\infty} \mathbf{x}^T(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}^T(t)\mathbf{R}\mathbf{u}(t),$$

where $\mathbf{Q}$ is positive semi-definite and $\mathbf{R}$ is positive definite.
Problem: What is the control law that minimizes the LQR criterion?

Theorem

Suppose that the pair $(\mathbf{A}, \mathbf{B})$ is stabilizable and the pair $(\mathbf{A}, \mathbf{G})$ is detectable, with $\mathbf{Q} = \mathbf{G}^T\mathbf{G}$. Then:

- there exists a unique, positive definite solution $\mathbf{P}$ of the algebraic Riccati equation

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{Q} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} = \mathbf{0};$$

- The feedback law

$$\mathbf{u}(t) := -\mathbf{K}\mathbf{x}(t),\ \mathbf{K} := \mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}$$

  minimizes the LQR criterion and $J = \mathbf{x}^T(0)\mathbf{P}\mathbf{x}(0)$; and
- The eigenvalues of $\mathbf{A} - \mathbf{B}\mathbf{K}$ all have negative real part.

It is sufficient that the pair $(\mathbf{A}, \mathbf{B})$ is stabilizable and the pair $(\mathbf{A}, \mathbf{G})$ is detectable.

- Stabilizability: A system is controllable if all the uncontrollable modes are stable.
- Detectability: A system is detectable if all the unobservable modes are stable.

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Do these conditions make sense?

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- Stabilizability: A system is controllable if all the uncontrollable modes are stable.
- Detectability: A system is detectable if all the unobservable modes are stable.

Do these conditions make sense?

- If the system is not stabilizable there are modes that we cannot stabilize!
- If the pair $(\mathbf{A}, \mathbf{G})$ is not detectable, it could be possible to make J very small, yet the state would be exploding!

Trade-off between performance and control cost:

- If $\mathbf{Q} \gg \mathbf{R}$, the easiest way to minimize J is to use a lot of control resulting in a small controlled state.
- If $\mathbf{Q} \ll \mathbf{R}$, the easiest way to minimize J is to use little control, resulting in a large controlled state.

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Bryson's rule:

$$\mathbf{Q}_{ii} = \frac{1}{\text{maximum acceptable value of } x_i^2(t)}$$
$$\mathbf{R}_{ii} = \frac{1}{\text{maximum acceptable value of } u_i^2(t)}$$

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Trial and error...

LQR design

```
>> A = [0 1; 0 0]; B = [0; 1]; Q = eye(2); R = 0.1; K
= lqr(A, B, Q, R)
K =
    3.1623 4.0404
```

LQR design

```
>> A = [0 1; 0 0]; B = [0; 1]; Q = eye(2); R = 0.1; K
= lqr(A, B, Q, R)
K =
    3.1623 4.0404
>> A = [0 1; 0 0]; B = [0; 1]; Q = eye(2); R = 10; K
= lqr(A, B, Q, R)
K =
    0.3162 0.8558
```

References

- W. Rugh, *Linear System Theory*, 2^{nd} edition, Prencice Hall, 1995.
- C.-T. Chen, *Linear System Theory and Design*, 4^{th} edition, Oxford University Press, 2012.
- J. Hespanha, *Linear Systems Theory*, Princeton University Press, 2009.