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Integrated Masters in Aerospace Engineering
Unmanned Aerial Vehicles

Example Questions

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1 Dynamic Modeling

Consider the simplified 2-D model of a quadrotor shown in the Figure 1, which can only move in the (x, z) plane. The position and orientation of the body frame $\{B\}$ with respect to the inertial frame $\{I\}$ are described by the coordinates $(x, z) \in \mathbb{R}^2$ and $\theta \in \mathbb{R}$, respectively. Let m denote the mass of the vehicle, g the gravitational acceleration, J_{yy} the moment of inertia about the body's y axis, and f_1 and f_2 the force inputs generated by each of the rotors, placed at a distance l from the vehicle's center of mass.

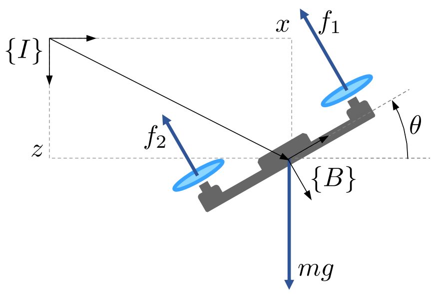

Figura 1: *2D quadrotor*

1.1 Consider the equations of motion for the 3-D model of a quadrotor

$$m\ddot{p} = -TRe_3 + mge_3 \tag{1}$$
$$\dot{R} = RS(\omega) \tag{2}$$
$$J\dot{\omega} = -JS(\omega)J\omega + \tau \tag{3}$$

where $p \in \mathbb{R}^3$ denotes the position of $\{B\}$ with respect to $\{I\}$, $R \in SO(3)$ the rotation matrix from $\{B\}$ to $\{I\}$, and $\omega \in \mathbb{R}^3$ the angular velocity of $\{B\}$ with respect to $\{I\}$, expressed in $\{B\}$, $T \in \mathbb{R}$ the thrust input, $\boldsymbol{\tau} \in \mathbb{R}^3$ the torque input, and $e_3 = [0\ 0\ 1]^T$. Apply the necessary simplifications and show that the simplified 2-D model can be written as

$$m\ddot{x} = -T\sin\theta \tag{4}$$
$$m\ddot{z} = -T\cos\theta + mg \tag{5}$$
$$J_{yy}\ddot{\theta} = \tau_y \tag{6}$$

1.2 Write expressions for T and τ_y as functions of f_1 and f_2 (see Figure 1).

1.3 Identify a possible state vector for the system described by (4)-(6) and show that the system is differentially flat, with flat output given by (x, z). More specifically, show that the state and input vectors can be written as functions of the flat output and its time derivatives up to some order. Write the explicit expressions for T and θ as functions of (x, z) and its derivatives.

1.4 Show that to keep the absolute value of the pitch angle below 45^o in forward level flight, the absolute value forward acceleration $|\ddot{x}|$ cannot exceed the gravitational acceleration g.

2 Control and Observer Design

2.1 From now on, with a slight abuse of notation, the position vector p is redefined as a two-dimensional vector such that $p = \begin{bmatrix} x & z \end{bmatrix}^T$. Consider the model described by (4)-(6) and a trajectory tracking controller with a hierarchical structure, comprising the modules shown in Figure 2. For control design purposes, the position dynamics is approximated by $\ddot{p} = u^*$, where $u^* = [u_x^*\ u_z^*]^T$ is a virtual control input. Determine expressions for the thrust T and the desired pitch angle θ_d (as functions of u^*) to be tracked by the inner-loop controller.

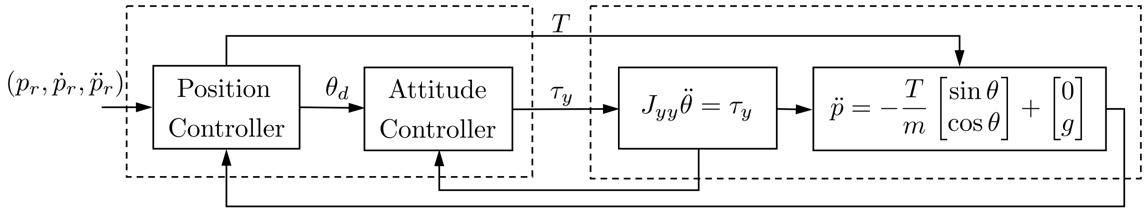

Figura 2: Hierarchical Control System.

2.2 As an initial step to design a tracking controller for the inner loop system (so that tracking of θ_d can be achieved), write in state-space form the error system with state $x_e = x - x_d$, where $x = \begin{bmatrix} \theta & \dot{\theta} \end{bmatrix}^T$ and $x_d = \begin{bmatrix} \theta_d & \dot{\theta}_d \end{bmatrix}^T$, and input $u_e = \tau_y - J_{yy}\ddot{\theta}_d$, such that $\dot{x}_e = Ax_e + Bu_e$.

2.3 Given the error system defined in **2.2**, consider the problem of minimizing the LQR criterion

$$J = \int_0^\infty x_e(t)^T G^T G x_e(t) + \rho u_e(t)^2 dt \tag{7}$$

where $G^T G \geq 0$ and $\rho > 0$. Check whether or not there exists a solution to this problem when i) $G = [\gamma_1 \quad 0]$ and ii) $G = \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix}$, with γ_1, $\gamma_2 > 0$. Justify your answers.

2.4 For $G = [\gamma_1 \quad 0]$ and $\rho = J_{yy}^{-2}$, compute the gain matrix $K = [k_1 \quad k_2]$ for the LQR control law $u_e = -Kx_e$.

2.5 Show that the closed-loop system described by $\dot{x}_e = (A - BK)x_e$ can also take the form $J_{yy}\ddot{\theta} = -k_1(\theta - \theta_d) - k_2(\dot{\theta} - \dot{\theta}_d) + J_{yy}\ddot{\theta}_d$. Based on this expression, discuss the impact of changing γ_1 on the performance of the system and show that keeping a single tuning parameter γ_1 reduces the ability to meet certain performance specifications.

2.6 Assume that the only sensor available for attitude estimation is an inclinometer, providing measurements of the pitch angle $\theta_m(t)$. Design a stabilizing observer for the original attitude system that provides estimates for all state variables.

2.7 Redefining the error input u_e such that $u_e = -K(\hat{x} - x_d)$, where $\hat{x}$ is the state estimate provided by the observer, comment on the stability of the overall closed-loop system combining the controller designed in **2.4** and observer designed in **2.6**.