ENAE414 Prerequisites Review

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1 Complex Analysis

Complex numbers and the complex plane

Complex numbers are numbers of the form

$$z = a + bi$$

where a and b are **real** numbers, and $\sqrt{i} = -1$.

Complex numbers can be represented on the complex plane, where the x-axis contains the real component a and the y-axis contains the imaginary component b.

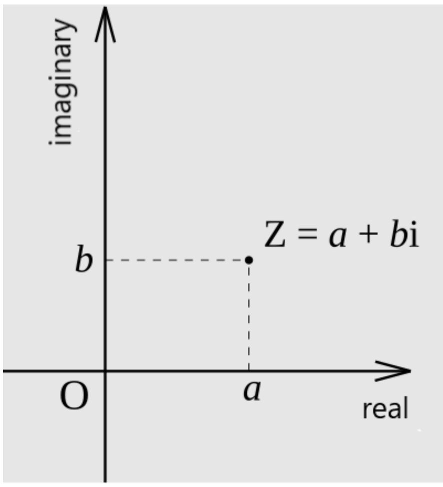

Figure 1: Complex plane

Euler's identity

Euler's identity is given by:

$$e^{i\pi} + 1 = 0$$

It follows from Euler's formula, viz.,

$$e^{i\theta} = \cos\theta + i\sin\theta$$

by plugging in $\theta = \pi$.

Complex conjugate

Consider a complex number $z = a + bi$. It's complex conjugate is

$$\overline{z} = a - bi$$

The product of a complex number and its conjugate is a real number:

$$z\overline{z} = (a+bi)(a-bi) = a^2 + b^2$$

Complex algebra

Addition, subtraction, multiplication, and division of complex numbers. For two complex numbers $z_1 = a + bi$ and $z_2 = c + di$:

$$z_1 + z_2 = (a+c) + (b+d)i$$

$$z_1 - z_2 = (a-c) + (b-d)i$$

$$z_1 \cdot z_2 = (ac - bd) + (ad + bc)i$$

$$\frac{z_1}{z_2} = \frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} = \frac{(ac+bd) + (bc-ad)i}{c^2 + d^2}$$

2 Vector and Matrix Algebra

Matrix-matrix product

The product of two matrices $\mathbf{A}$ and $\mathbf{B}$ is defined if the number of columns in $\mathbf{A}$ is equal to the number of rows in $\mathbf{B}$. For $\mathbf{A}$ of size $m \times n$ and $\mathbf{B}$ of size $n \times p$:

$$(\mathbf{AB})_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

Matrix-vector product

The product of a matrix $\mathbf{A}$ and a vector $\mathbf{v}$ (considered as a column vector) is:

$$(\mathbf{Av})_i = \sum_{j=1}^{n} A_{ij} v_j$$

Transpose

The transpose of a matrix $\mathbf{A}$, denoted $\mathbf{A}^T$, is obtained by swapping its rows with its columns:

$$(\mathbf{A}^T)_{ij} = A_{ji}$$

Trace

The trace of a square matrix $\mathbf{A}$ is noted by tr $\mathbf{A}$ and is the sum of the elements on the leading diagonal. If $\mathbf{A}$ is a 3×3 matrix,

$$\text{tr } \mathbf{A} = A_{11} + A_{22} + A_{33} = \sum_{i=1}^{3} A_{ii}$$

Determinants

The determinant of a 2 X 2 square matrix $\mathbf{A}$ is a **scalar**, given by

$$\det(\mathbf{A}) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Consider a 3×3 matrix square matrix $\mathbf{B}$ below,

$$\mathbf{B} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

The determinant of **B** is a **scalar**, given by

$$\det(\mathbf{B}) = a \cdot \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \cdot \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \cdot \begin{vmatrix} d & e \\ g & h \end{vmatrix} = a(ei - fh) - b(di - fg) + c(dh - eg)$$

Dot Products

The dot product is an **operation** on two vectors **a** and **b**, and is performed as follows:

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n} a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3 \qquad \text{(for n = 3)}$$

Note that the result is a **scalar**.

From the law of cosines, the dot product may also be computed as

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$$

where $|\mathbf{a}|$ and $|\mathbf{b}|$ denote the magnitudes of **a** and **b** respectively, and θ is the angle between the two vectors. Remember the magnitude of a vector **v** in 3-D space is

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^3}$$

If the vectors **a** and **b** above are **parallel**, then the dot product outputs 1. If the vectors are **perpendicular**, then the output is 0.

The dot product can also be used to calculate the length of the "shadow" cast by one vector on another, i.e., the magnitude of the projection of one vector onto another. The scalar (or component) projection of **a** onto **b** is

$$\text{Comp}_{\mathbf{b}}\ \mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} = |\mathbf{a}| \cos\theta$$

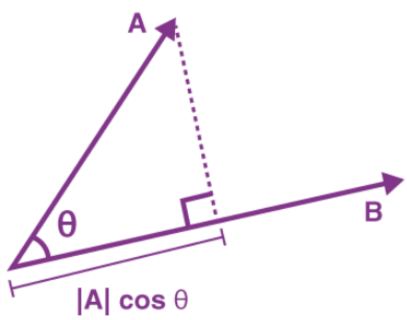

Figure 2: Complex plane

Note that if **b** is a unit vector, then the scalar projection is just $\mathbf{a} \cdot \mathbf{b}$. This means that the **component** of a vector in any direction is the dot product of the vector with the **unit** vector in that direction.

Cross Products

The cross product of two vectors **a** and **b** in $\mathbb{R}^3$ is given by:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i}(a_2 b_3 - a_3 b_2) - \mathbf{j}(a_1 b_3 - a_3 b_1) + \mathbf{k}(a_1 b_2 - a_2 b_1)$$

The method for evaluating the determinant here is the exact same as for the determinant of a 3×3 matrix. The cross product of **a** and **b** results in a **vector** that is perpendicular to both **a** and **b** with a magnitude equal to the area of the parallelogram formed by **a** and **b**.

3 Alternate Coordinate Systems

Polar Coordinates

In polar coordinates, a point in the plane is represented by (r, θ), where r is the radial distance from the origin, and θ is the angle from the positive x-axis. Note the following transformations between polar and Cartesian coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta$$
$$r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Cylindrical Coordinates

Cylindrical coordinates are an extension of polar coordinates to 3 dimensions. A point in space is represented by (r, θ, z), where (r, θ) are the polar coordinates of the projection of the point onto the xy-plane, and z is the height above the xy-plane. The transformations to Cartesian coordinates are:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

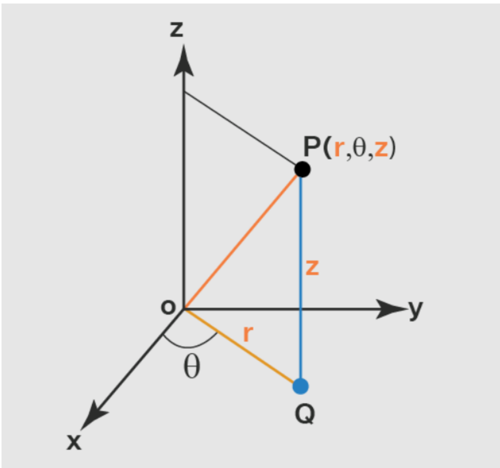

Figure 3: Cylindrical Coordinates

Spherical Coordinates

In spherical coordinates, a point in space is represented by (ρ, θ, ϕ), where ρ is the radial distance from the origin, θ is the azimuthal angle in the xy-plane from the positive x-axis, and ϕ is the polar angle from the positive z-axis. The transformations to Cartesian coordinates are:

$$x = \rho\sin\phi\cos\theta$$

$$y = \rho \sin\phi \sin\theta$$

$$z = \rho \cos\phi$$

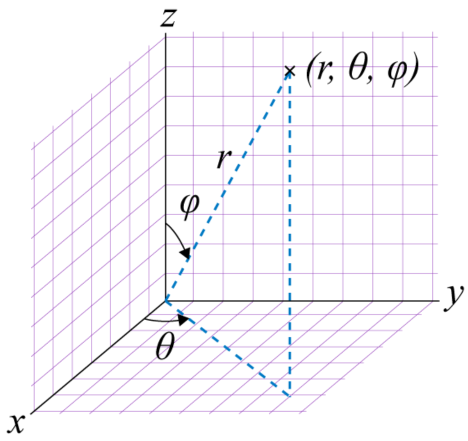

Figure 4: Spherical Coordinates

4 Single Variable Calculus

Product Rule

The product rule states that if $u(x)$ and $v(x)$ are differentiable functions, then the derivative of their product is given by:

$$\frac{d}{dx}[u(x) \cdot v(x)] = u'(x) \cdot v(x) + u(x) \cdot v'(x)$$

Quotient Rule

The quotient rule is used for the derivative of the quotient of two functions. If $u(x)$ and $v(x)$ are differentiable and $v(x) \neq 0$, then:

$$\frac{d}{dx}\left[\frac{u(x)}{v(x)}\right] = \frac{u'(x) \cdot v(x) - u(x) \cdot v'(x)}{[v(x)]^2}$$

Note that the quotient rule is derived directly from the product rule. Express a quotient of two functions as a product:

$$\frac{u(x)}{v(x)} = u(x) \cdot [v(x)]^{-1}$$

Then apply the product rule:

$$\frac{d}{dx}\left[\frac{u(x)}{v(x)}\right] = u'(x) \cdot [v(x)]^{-1} + u(x) \cdot \frac{d}{dx}\left[v(x)^{-1}\right]$$

The remainder of the derivation will be left to you.

Integration with Polynomial Substitution

We will review integration with a change of variables by performing the following integral

$$\int_{x=0}^{3} \frac{dx}{\sqrt{x^2+9}}$$

using the transformation $x = 3\tan\theta$. From this transformation, we can derive:

$$\theta = \tan^{-1}\left(\frac{x}{3}\right)$$

$$dx = 3\sec^2(\theta)\ d\theta$$

The limits of integration become:

$$\theta_1 = \theta(0) = \tan^{-1}\left(\frac{0}{3}\right) = \tan^{-1}(0) = 0$$

$$\theta_2 = \theta(3) = \tan^{-1}\left(\frac{3}{3}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

Making the previous substitutions for x and dx and replacing the limits of the integral,

$$\begin{aligned}
\int_{x=0}^{3} \frac{dx}{\sqrt{x^2+9}} &= \int_{\theta_1=0}^{\theta_2=\pi/4} \frac{3\sec^2\theta d\theta}{\sqrt{(3\tan^2\theta)+3^2}} \\
&= \int_0^{\pi/4} \frac{3\sec^2\theta d\theta}{\sqrt{3^2(1+\tan^2\theta)}} \\
&= \int_0^{\pi/4} \frac{3\sec^2\theta d\theta}{\sqrt{3^2\sec^2\theta}} \\
&= \int_0^{\pi/4} \sec\theta d\theta \\
&= \Big[\ln|\sec\theta + \tan\theta|\Big]_{\theta=0}^{\theta=\pi/4} \\
&= \ln\left|\sec\left(\frac{\pi}{4}\right) + \tan\left(\frac{\pi}{4}\right)\right| - \ln|\sec(0)+\tan(0)| \\
&= \ln\left|\sqrt{2}+1\right| - \ln|1+0| \\
&= \boxed{\ln\left|\sqrt{2}+1\right|}
\end{aligned}$$

5 Vector Calculus

In this section, results will be given for Cartesian coordinates only. Results for other coordinate systems can be referenced in textbooks.

Gradient (of a scalar field)

If $p(x, y, z)$ is a scalar function of the coordinates, then

$$\text{grad } p = \nabla p = \frac{\partial p}{\partial x}\,\mathbf{i} + \frac{\partial p}{\partial y}\,\mathbf{j} + \frac{\partial p}{\partial z}\,\mathbf{k}$$

The gradient is a **vector** quantity, and is evaluated at a **single point**. The **magnitude** is the maximum rate of change of $p(x, y, z)$ per unit length of the coordinate space at the given point. The **direction** is towards the maximum rate of change of $p(x, y, z)$ at a given point.

∇ itself is an **operator**, and NOT a vector. In 3D space, it is defined as

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

Directional Derivative

Suppose s is some arbitrary direction away from a given point, and let $\mathbf{n}$ be the unit vector in the s direction. Let $p(x, y, z)$ be a scalar function of the coordinates. Then at a given point, $\partial p/\partial s$ is the **directional derivative** in the s direction and is given by

$$\frac{\partial p}{\partial s} = \nabla p \cdot \mathbf{n}$$

The above equation means that the rate of change of p in any direction is the component of the **gradient** of p in that direction.

Chain Rule and Substantial Derivative

The chain rule is used when differentiating composite functions. Suppose we have two functions $u(t)$ and $f(u)$. Then we can say $f = f(u(t))$, and the derivative of $f(u(t))$ with respect to t is given by the **chain rule** as follows:

$$\frac{df}{dt} = \frac{df}{du}\frac{du}{dt}$$

In aerodynamics, we often deal with functions like $\mathbf{V}(\mathbf{x}, t)$ which describes a velocity field, where $\mathbf{x} = (x, y, z)$. Such a velocity field may change in time and space. The **substantial (or material) derivative**, D/Dt, is used to describe the **time rate of change of a physical quantity** (like velocity). The definition is a direct implication of the **chain rule**. The substantial derivative is an operator defined as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla$$

where:

1) $\frac{\partial}{\partial t}$ is the **local** or **temporal** derivative and represents the rate of change with time
2) $\mathbf{V} \cdot \nabla$ is the **convective** derivative and represents the rate of change due to the motion of the fluid.

When applied to a velocity field $\mathbf{V}(x, y, z, t) = (u, v, w, t)$, it describes the total acceleration of a fluid particle as it moves through the velocity field.

Total Derivative and Total Differential

The total derivative and total differential are also based on the **chain rule**. For example, the total derivative and total differential for a function of two variables are:

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} \qquad , \qquad df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

The total differential represents how a function changes as its variables change. Note that df/dt is NOT the same as $\partial f/\partial t$.

Divergence (of a vector field)

If $\mathbf{V}(x, y, z)$ is a vector function of the coordinates, such that $\mathbf{V} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$, then

$$\text{div } \mathbf{V} = \nabla \cdot \mathbf{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

The divergence is a **scalar** quantity, and is evaluated at a **single point**. If $\mathbf{V}$ is a velocity field, then physically the divergence represents the time rate of change of volume of a fluid element of **fixed mass**, normalized by the volume of the element.

Curl (of a vector field)

If $\mathbf{V}(x, y, z)$ is a vector function of the coordinates, such that $\mathbf{V} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$, then

$$\text{curl } \mathbf{V} = \nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\mathbf{i} - \left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right)\mathbf{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathbf{k}$$

If $\mathbf{V}$ is a velocity field, then physically the curl represents the rotation of the fluid element.

NOTE: Divergence and curl are fundamentally just two different ways to take the derivative of a vector field.

Differentiation Under the Integral Sign

Given $f(x, t)$, we define:

$$I(t) = \int_a^b f(x, t)dx$$

What is $I'(t)$? Is it just $\int_a^b \frac{\partial f}{\partial t}(x, t)dt$? Not always, but yes under **reasonable assumptions**:

Theorem:
Let a, b, c, and d be constants. If the functions $f(x, t)$ and $\frac{\partial f}{\partial t}(x, t)$ are continuous for all (x, t) such that $a \leq x \leq b$ and $c \leq t \leq d$, then the following holds for $c \leq t \leq d$:

$$\frac{d}{dt}\left(\int_a^b f(x, t)\, dx\right) = \int_a^b \frac{\partial f}{\partial t}(x, t)\, dx$$

Essentially, if it's reasonable to assume $f(x, t)$ is continuous on some finite domain, we can move the partial derivative under the integral. See **Leibniz Integral Rule** if you're interested.

Integrals

Line Integrals

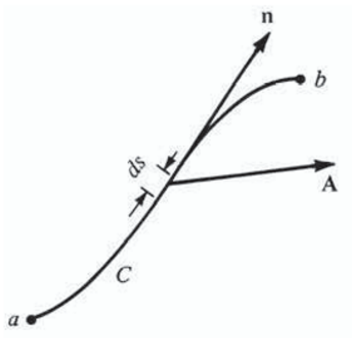

Figure 5: Line integral representation

If **A** is a vector field and C is a curve parameterized from point a to point b, the line integral of **A** along C is given by:

$$\int_a^b \mathbf{A} \cdot d\mathbf{s} = \int_a^b \mathbf{A} \cdot \mathbf{n}\, ds$$

where **n** is a unit vector function which gives unit vectors that are tangent to the curve C. By convention, integration is considered positive in the counter-clockwise (CCW) direction.

Notation differences when C is an open or closed curve:

$$\text{Open curve: } \int_C \mathbf{A} \cdot d\mathbf{s} \qquad \text{Closed curve: } \oint_C \mathbf{A} \cdot d\mathbf{s}$$

Fundamentally, line integrals are a summation of some quantity along a curve.

Ex. Compute $\oint_C \mathbf{V} \cdot d\mathbf{s}$ where $\mathbf{V} = (V_x, V_y)$ and $V_x = -Ax^2y$, $V_y = Axy^2$ and C is a rectangular path with corners at (0,0), (6,0), (6,3), and (0,3) as shown below.

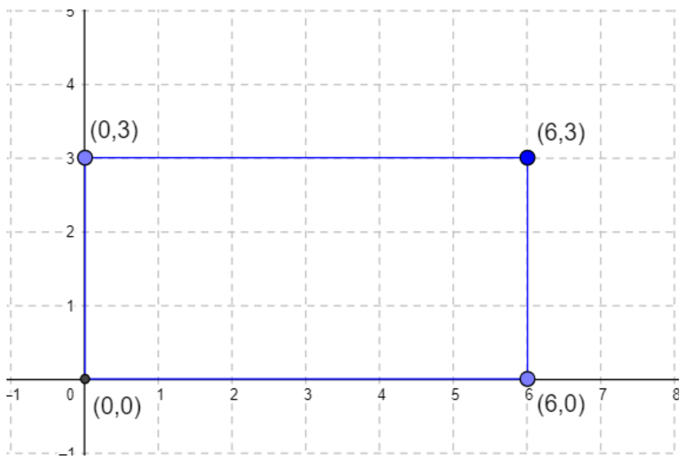

Figure 6: Rectangular path C

Solution:

Split the rectangular path into 4 lines and integrate in the CCW direction (right hand rule):

$$\oint_C \mathbf{V} \cdot d\mathbf{s} = \int_B \mathbf{V} \cdot d\mathbf{l_B} + \int_R \mathbf{V} \cdot d\mathbf{l_R} + \int_T \mathbf{V} \cdot d\mathbf{l_T} + \int_L \mathbf{V} \cdot d\mathbf{l_L}$$

But $\int_B \mathbf{V} \cdot d\mathbf{l_B} = 0$ because $V_x(y=0) = 0$, and $\int_L \mathbf{V} \cdot d\mathbf{l_L} = 0$ because $V_y(x=0) = 0$

$$\begin{aligned}
\implies \oint_C \mathbf{V} \cdot d\mathbf{s} &= \int_R \mathbf{V} \cdot d\mathbf{l_R} + \int_T \mathbf{V} \cdot d\mathbf{l_T} \\
&= \int_0^3 Axy^2\, dy + \int_0^6 -Ax^2y(-dx) \\
&= \left[\frac{A}{3}xy^3\right]_{y=0,x=6}^{y=3,x=6} + \left[\frac{A}{3}x^3y\right]_{x=0,y=3}^{x=6,y=3} \\
&= 54A + 216A \\
&= \boxed{270A}
\end{aligned}$$

Surface Integrals

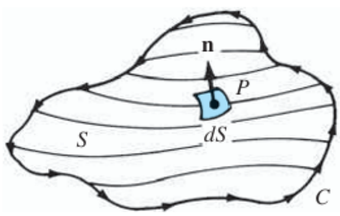

Figure 7: Surface integral representation

Here, dS is an elemental area, and $d\mathbf{S} = \mathbf{n}dS$ where $\mathbf{n}$ is normal to the surface, in the general direction that the thumb points when curling right hand fingers in the direction of movement (i.e. integration) around curve C.

3 ways to define a surface integral:

$$\text{open surface: } \iint_S \rho\, d\mathbf{S} \qquad \text{closed surface: } \oiint_S \rho\, d\mathbf{S}$$

$$\text{open surface: } \iint_S \mathbf{A} \cdot d\mathbf{S} \qquad \text{closed surface: } \oiint_S \mathbf{A} \cdot d\mathbf{S}$$

$$\text{open surface: } \iint_S \mathbf{A} \times d\mathbf{S} \qquad \text{closed surface: } \oiint_S \mathbf{A} \times d\mathbf{S}$$

Fundamentally, surface integrals are a summation of some quantity over a surface.

Volume Integrals

2 ways to define a volume integral:

$$\iiint_V p\, dV \qquad \text{or} \qquad \iiint_V \mathbf{V}\, dV$$

Fundamentally, volume integrals are a summation of some quantity within a volume.

Fundamental Theorems of Vector Calculus

Only the results are relevant, and can be accepted without proof.

Stokes' Theorem

$$\oint_C \mathbf{A} \cdot d\mathbf{s} = \iint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S}$$

The LHS must be a closed curve, which is inherently related to a surface.

Divergence Theorem (aka Gauss' Theorem)

$$\oiint_S \mathbf{A} \cdot d\mathbf{S} = \iiint_V (\nabla \cdot \mathbf{A})\, dV$$

The LHS must be a closed surface, which is inherently related to a volume.

Gradient Theorem

$$\oiint_S \nabla p \cdot d\mathbf{S} = \iiint_V \nabla p \, dV$$

The LHS must be a closed surface, which is inherently related to a volume.