

3d TQFT from 6d SCFT

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ABSTRACT: We study the six-dimensional $(2, 0)$ superconformal field theory on $S^1 \times S^2 \times M$ via compactification to five dimensions, where M is a three-manifold. Twisted along M , the five-dimensional theory has a half of $\mathcal{N} = (2, 2)$ supersymmetry on S^2 , the other half being broken by a superpotential. We show that in the limit where M is infinitely large, the twisted theory reduces to a three-dimensional topological quantum field theory which is closely related to Chern-Simons theory for the complexified gauge group.

KEYWORDS: Extended Supersymmetry, Field Theories in Higher Dimensions, Topological Field Theories

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1 Introduction

In the past several years various correspondences have been discovered between supersymmetric theories in d dimensions and nonsupersymmetric theories in $6 - d$ dimensions. The most famous example is the AGT correspondence [1] between $\mathcal{N} = 2$ supersymmetric gauge theories in four dimensions and Liouville (or more generally, Toda [2–4]) conformal field theory, which has now been extensively investigated. A less studied example is the 3d/3d correspondence [5] between $\mathcal{N} = 2$ superconformal field theories (SCFTs) in three dimensions and Chern-Simons theory with complex gauge group. The aim of the present paper is to gain a deeper understanding of the 3d/3d correspondence, via five-dimensional maximally supersymmetric Yang-Mills theory (5d MSYM).

It has been argued that these intriguing correspondences originate from the $\mathcal{N} = (2, 0)$ SCFT in six dimensions, formulated on the product $X \times M$ of a d -dimensional space X and a $(6 - d)$ -dimensional space M . Typically, one allows M to be a fairly general manifold and twists the theory along M , but chooses X to be very specific so that the theory admits supersymmetry without twisting along X . On the one hand, compactification of the theory on M produces a supersymmetric theory $T[M]$ on X . On the other hand, upon “supersymmetric localization on X ,” the theory reduces to a nonsupersymmetric theory $\mathbb{T}[X]$ on M . Identifying protected quantities after these different procedures, one establishes a correspondence between $T[M]$ and $\mathbb{T}[X]$.

The AGT correspondence arises by taking $X = S^4$ (either round or squashed [6]) and M to be punctured Riemann surfaces; $T[M]$ is an $\mathcal{N} = 2$ theory of class \mathcal{S} [7, 8] and $\mathbb{T}[X]$ is Toda theory. In one version of 3d/3d correspondence, one takes $X = S^3$ and M to be three-manifolds. Then $T[M]$ is an $\mathcal{N} = 2$ SCFT and $\mathbb{T}[X]$ is analytically continued Chern-Simons theory [9–11].

As one can see from these examples, the theory $\mathbb{T}[X]$ is rather rigid in the sense that it has few free parameters, reflecting the highly constrained geometry of X . Furthermore, it is often conformal or even topological since the twist along M decouples some part or the whole of the dependency on the metric chosen on M to formulate the $(2, 0)$ theory, by making the corresponding components of the stress tensor Q -exact with respect to some supercharge Q . Because of this decoupling, quantities involving only Q -invariant operators and states are protected under the compactification described above. These are also the quantities captured by the localization method.

For the 3d/3d correspondence of our interest, one takes

$$X = S^1 \times S^2, \quad (1.1)$$

with S^1 a circle and S^2 a round two-sphere. On $S^1 \times S^2$, the $\mathcal{N} = 2$ SCFT $T[M]$ has a class of correlation functions that take the form of an index. This class in particular includes the superconformal index, which is the partition function with a twisted boundary condition as the fields go around the S^1 .¹ Being indices, these correlators are invariant under continuous changes in the parameters of the theory. This suggests that they come from six-dimensional indices that are protected under deformations of the geometry of M . So these should be the quantities $\mathbb{T}[X]$ on M captures, and we deduce that $\mathbb{T}[X]$ is a topological quantum field theory (TQFT). The expectation is that $\mathbb{T}[X]$ is complex Chern-Simons theory [5].

While the geometric structures that emerge when M is varied provide important hints for identifying $T[M]$, the rigidity of X prevents one from learning about $\mathbb{T}[X]$ in a similar manner. So far the identification of $\mathbb{T}[X]$ has mainly been achieved through more indirect routes, for example, by looking for structures in the partition function of $T[M]$ on X , or by relating the setup to another case where the correspondence has already been established. Clearly it is desirable to find a more direct derivation of $\mathbb{T}[X]$ starting from six dimensions and following the above line of reasoning. Our goal is to actually do this for $X = S^1 \times S^2$.

There is an obvious obstacle, however: localization computations require a Lagrangian description, but the $(2, 0)$ theory has none known.

Happily, the problem can be avoided in our case. Since X contains an S^1 factor, one can first consider compactifying the $(2, 0)$ theory on the S^1 down to five dimensions. This gives 5d MSYM on $S^2 \times M$ with gauge coupling

$$e^2 \propto R, \quad (1.2)$$

where R is the radius of the S^1 . The latter theory does have a Lagrangian, so one can take this as a starting point and hope to derive $\mathbb{T}[X]$ by localization of the path integral. A similar strategy was employed in [13, 14] to approach the conjecture [15, 16] that $\mathbb{T}[S^1 \times S^3]$ is the zero area limit of q -deformed Yang-Mills theory in two dimensions.

With this in mind, in section 2 we construct 5d MSYM on $S^2 \times M$, twisted along M in the way that descends from the aforementioned twist in six dimensions. The idea is to look

¹Boundary conditions relevant in the present context are given by the combined action of an R-symmetry rotation and flavor symmetry transformations, which preserve a half of the eight supersymmetries of the $\mathcal{N} = 2$ superconformal group on $\mathbb{R} \times S^2$ [12]. By dimensional reduction the four preserved supersymmetries generate $\mathcal{N} = (2, 2)$ supersymmetry on S^2 .

at the dimensional reduction of the theory on M . An analysis of the twisting reveals that this is an $\mathcal{N} = (2, 2)$ supersymmetric gauge theory on S^2 with adjoint chiral multiplets, for which we know the general form of Lagrangian [17, 18]. Then we lift this Lagrangian to five dimensions. This step can be done straightforwardly, except that we have to determine the superpotential which reproduces the standard Lagrangian on \mathbb{R}^5 after we set $M = \mathbb{R}^3$ and replace the S^2 metric by the flat \mathbb{R}^2 metric. The required superpotential is given by the Chern-Simons functional for a complexified gauge field \mathcal{A} on M .

This superpotential violates the vector R-symmetry and, as a consequence, breaks $\mathcal{N} = (2, 2)$ supersymmetry on S^2 . It is therefore not possible to realize on $S^2 \times M$ all of the four supersymmetries. We will see, however, that a half of them can still be preserved by adding appropriate correction terms. That the curvature of the S^2 halves the number of supersymmetries is not totally surprising; if one instead tried to realize the theory on a stack of D4-branes wrapped on $S^2 \times M \subset T^*S^2 \times T^*M$, then one would get a slightly different situation where the theory is twisted along both the S^2 and M , and the twisting along the S^2 would also halve the number.

To connect to the story of the 3d/3d correspondence, we propose that the $(2, 0)$ theory can be formulated on $S^1 \times S^2 \times M$ in such a way that when the S^1 is small, it reduces at low energies to the twisted 5d MSYM on $S^2 \times M$ constructed as above, including these corrections to the F-term. One piece of evidence is that the five-dimensional theory is invariant under the simultaneous rescaling of the coupling e^2 , the radius r of the S^2 , and the length scale of M . In view of the relation (1.2), this is nothing but the scale invariance of the $(2, 0)$ theory.

From the consideration on the nature of protected quantities in six dimensions, we expect that the twisted theory on $S^2 \times M$ becomes topological along M in the limit $R \rightarrow 0$ where the $(2, 0)$ theory reduces to 5d MSYM. Actually, because of the relation (1.2), this limit is the zero coupling limit and the physics is not very interesting there. So we instead consider the limit where R and r are both sent to zero with r/R kept fixed, or equivalently, the metric g_M on M is rescaled by an infinitely large factor. Since the twisted $(2, 0)$ theory is presumably topological on M , we can do this without affecting the protected quantities.

In section 3, we show that the g_M -dependence of the twisted theory indeed decouples in the Q -invariant sector when M becomes infinitely large, with Q being one of the two preserved supercharges. Therefore, the infinite volume limit of the twisted theory is a TQFT on M , and we identify it with $\mathbb{T}[X]$.² Also in this limit the vector R-symmetry and the full $\mathcal{N} = (2, 2)$ supersymmetry are restored. This is an expected behavior since the $(2, 0)$ theory on $S^1 \times S^2 \times M$ must preserve four supersymmetries in order for $T[M]$ to have $\mathcal{N} = 2$ supersymmetry.

Encouraged by these findings, in section 4 we attempt to determine $\mathbb{T}[X]$. By suitably deforming the action in the infinite volume limit, we localize the path integral and express it as the sum of path integrals labeled by the magnetic flux B through the S^2 . If G is

²In fact, the choice of Q used in “localizing” the $(2, 0)$ theory may not correspond to Q we use for the twisted theory. In particular, one can choose it to be one of the supercharges restored in the infinite volume limit. The difference does not matter as far as the superconformal index is concerned, since the twisted boundary condition can be chosen to be invariant under either supercharge.

the gauge group of the twisted theory and $G_B \subset G$ is the stabilizer of B (the subgroup consisting of all $g \in G$ such that $g^{-1}Bg = B$), then each summand is the path integral for a variant of Chern-Simons theory on M whose gauge group is the complexification $(G_B)_{\mathbb{C}}$ of G_B :

$$\sum_B \int_{\mathcal{M}_B^{\text{ss}}} \widetilde{\mathcal{D}}\mathcal{A}_B \exp(iS_{\text{CS}}(\mathcal{A}_B)). \quad (1.3)$$

We find the Chern-Simons level $k = 0$, which is the correct value for the 3d/3d correspondence. The differences from the ordinary complex Chern-Simons theory are that the measure is modified by one-loop contributions, and that the integration is performed over the space $\mathcal{M}_B^{\text{ss}}$ of semistable $(G_B)_{\mathbb{C}}$ -orbits in the space of complex gauge fields, rather than all $(G_B)_{\mathbb{C}}$ -orbits.

Unfortunately, in this work we will not be able to write down an explicit formula for the modified measure. The difficulty lies in that we cannot compute the one-loop determinants explicitly, as we keep M to be a general three-manifold. We hope to address this issue in future research.

2 5d MSYM on $S^2 \times M$

From now on we forget about the 3d/3d correspondence and the six-dimensional physics underlying it. Our goal here is to formulate 5d MSYM on $S^2 \times M$, where S^2 is a two-sphere equipped with the round metric h of radius r and M is a three-manifold with metric g_M . The metric of the total product space is $g = h \oplus g_M$. Our conventions for spinors on S^2 are summarized in appendix A.

The theory consists of a gauge field A , five adjoint scalars X^I ($I = 1, \dots, 5$), and sixteen adjoint spinors Ψ . The R-symmetry group of the theory is $\text{Spin}(5)_R$. Suppose that the theory is formulated on a generic spin five-manifold Y , for which the structure group of the spin bundle is $\text{Spin}(5)_Y$. Under $\text{Spin}(5)_Y \times \text{Spin}(5)_R$, the fields transform as follows:

$$\begin{aligned} A &: (\mathbf{5}, \mathbf{1}), \\ X &: (\mathbf{1}, \mathbf{5}), \\ \Psi &: (\mathbf{4}, \mathbf{4}). \end{aligned} \quad (2.1)$$

Here $\mathbf{4}$ is a spinor representation of $\text{Spin}(5)$. On the flat space \mathbb{R}^5 , the theory has sixteen supersymmetries whose generators transform as $(\mathbf{4}, \mathbf{4})$.

We are interested in the case when $Y = S^2 \times M$. In this case the supersymmetries are completely broken generically, but we can preserve a fraction of them with a suitable twist along M .

The relevant twist is performed in the following way. On $S^2 \times M$, the structure group of the spin bundle can be reduced to

$$\text{Spin}(2)_{S^2} \times \text{Spin}(3)_M \cong \text{U}(1)_{S^2} \times \text{SU}(2)_M. \quad (2.2)$$

We want to replace $\text{SU}(2)_M$ by a different group $\text{SU}(2)'_M$ that acts trivially on some of the supercharges. To this end we split $\text{Spin}(5)_R$ as

$$\text{Spin}(2)_R \times \text{Spin}(3)_R \cong \text{U}(1)_R \times \text{SU}(2)_R, \quad (2.3)$$

and take $SU(2)'_M$ to be the diagonal subgroup of $SU(2)_M \times SU(2)_R$. Under $SU(2)'_M \times U(1)_{S^2} \times U(1)_R$, the fields transform as

$$\begin{aligned} A &: \mathbf{1}^{(\pm 2, 0)} \oplus \mathbf{3}^{(0, 0)}, \\ X &: \mathbf{1}^{(0, \pm 2)} \oplus \mathbf{3}^{(0, 0)}, \\ \Psi &: \mathbf{1}^{(\pm 1, \pm 1)} \oplus \mathbf{3}^{(\pm 1, \pm 1)}. \end{aligned} \quad (2.4)$$

We see that after the twisting four components of Ψ become scalars on M , hence so do four supercharges.

As a preliminary step to placing this twisted theory on $S^2 \times M$, let us consider the situation where the S^2 is replaced by \mathbb{R}^2 . On $\mathbb{R}^2 \times M$, the twisted theory has four supersymmetries generated by the supercharges that are scalars on M . Along the \mathbb{R}^2 , two of these supercharges are spinors of chirality positive and the other two negative. Thus the twisted theory has $\mathcal{N} = (2, 2)$ supersymmetry from the two-dimensional point of view.

We can readily figure out the field content of this $\mathcal{N} = (2, 2)$ supersymmetric theory. There are two kinds of fields.

First, we have those that are scalars on M . These are the components A_μ of the gauge field along the \mathbb{R}^2 , two real scalars σ_1, σ_2 or one complex scalar $\sigma = \sigma_1 - i\sigma_2$, and two Dirac fermions $\lambda, \bar{\lambda}$. Supplemented with a real auxiliary field D , they form a vector multiplet:

$$(A_\mu, \sigma, \lambda, \bar{\lambda}, D). \quad (2.5)$$

Second, we have those that are one-forms on M . These are the components A_m of the gauge field along M and the remaining scalars X_m which combine into a complex gauge field

$$\mathcal{A}_m = A_m + iX_m, \quad (2.6)$$

and the remaining fermions ψ_m . Together with complex auxiliary fields F_m , they form chiral multiplets:

$$(\mathcal{A}_m, \psi_m, F_m). \quad (2.7)$$

Therefore, upon dimensional reduction on M , the twisted theory becomes an $\mathcal{N} = (2, 2)$ gauge theory with three adjoint chiral multiplets. The $U(1)_R$ symmetry rotates the vector multiplet scalars, and is identified with the axial R-symmetry $U(1)_A$.

Now we want to place the theory on $S^2 \times M$, without twisting it any further. The dimensional reduction of this system on M would give an $\mathcal{N} = (2, 2)$ gauge theory on S^2 . Here we briefly review the basic features of $\mathcal{N} = (2, 2)$ supersymmetry on S^2 ; more details can be found in appendix B.

$\mathcal{N} = (2, 2)$ supersymmetry transformations on S^2 are parametrized by a pair of conformal Killing spinors $(\varepsilon, \bar{\varepsilon})$ obeying the equations

$$\nabla_\mu \varepsilon = +\frac{1}{2r} \gamma_\mu \gamma_{\hat{3}} \varepsilon, \quad \nabla_\mu \bar{\varepsilon} = -\frac{1}{2r} \gamma_\mu \gamma_{\hat{3}} \bar{\varepsilon}. \quad (2.8)$$

We take $\varepsilon, \bar{\varepsilon}$ to be commuting spinors. Then supersymmetry transformations are anticommuting.

Each of the above equations has two independent solutions, ε_α or $\bar{\varepsilon}_\alpha$ ($\alpha = 1, 2$), so we have four supercharges in total:

$$\bar{Q}_\alpha, Q_\alpha. \quad (2.9)$$

They generate the supersymmetry transformations with $(\varepsilon, \bar{\varepsilon}) = (\varepsilon_\alpha, 0)$ and $(0, \bar{\varepsilon}_\alpha)$, respectively. We can choose the parameters in such a way that

$$\bar{\varepsilon}_\alpha = \gamma_3 \varepsilon_\alpha \quad (2.10)$$

and $\varepsilon_1 \varepsilon_2 = -\bar{\varepsilon}_1 \bar{\varepsilon}_2 = 1$. The vector R-symmetry $U(1)_V$ rotates \bar{Q}_α and Q_α by opposite phases. Our normalization of the vector R-charge is that \bar{Q}_α has charge $q = +1$ and Q_α has $q = -1$. As we will see shortly, $U(1)_V$ is actually broken to \mathbb{Z}_2 in our theory.

The commutator of two supersymmetry transformations generates bosonic continuous symmetries, which are $U(1)_V$, rotations of S^2 , and gauge transformations. Since none of these has $q = \pm 2$, it follows immediately that

$$\{\bar{Q}_\alpha, \bar{Q}_\beta\} = \{Q_\alpha, Q_\beta\} = 0. \quad (2.11)$$

These relations will be useful later when we analyze the topological property of the twisted theory.

To write down the supersymmetry transformation rules and the Lagrangian for the twisted theory, we can lift their two-dimensional counterparts to five dimensions. This is done by promoting the fields from functions on S^2 to those on $S^2 \times M$, and replacing

$$A_m \rightarrow iD_m, \quad [A_m, A_n] \rightarrow iF_{mn}, \quad (2.12)$$

where $D_m = \nabla_m - iA_m$ are the covariant derivatives along M . (For the purpose of writing down the five-dimensional Lagrangian, we can assume that the gauge group is nonabelian.) This lifting operation commutes with any derivation commuting with ∇ , especially supersymmetry transformations. The lift of a supersymmetric action is therefore invariant under the lifted supersymmetry.

Following this procedure, we find that the supersymmetry transformation of the vector multiplet is

$$\begin{aligned} \delta A_\mu &= -\frac{i}{2}(\bar{\varepsilon} \gamma_\mu \lambda + \varepsilon \gamma_\mu \bar{\lambda}), \\ \delta \sigma &= \bar{\varepsilon} \gamma_- \lambda - \varepsilon \gamma_+ \bar{\lambda}, \\ \delta \bar{\sigma} &= \bar{\varepsilon} \gamma_+ \lambda - \varepsilon \gamma_- \bar{\lambda}, \\ \delta \lambda &= i \left(F_{12} \gamma_3 + \gamma_- \not{D} \sigma + \gamma_+ \not{D} \bar{\sigma} + \frac{1}{2} [\sigma, \bar{\sigma}] \gamma_3 + iD \right) \varepsilon - \frac{i}{r} (\sigma \gamma_- - \bar{\sigma} \gamma_+) \varepsilon, \\ \delta \bar{\lambda} &= i \left(F_{12} \gamma_3 - \gamma_+ \not{D} \sigma - \gamma_- \not{D} \bar{\sigma} - \frac{1}{2} [\sigma, \bar{\sigma}] \gamma_3 - iD \right) \bar{\varepsilon} + \frac{i}{r} (\sigma \gamma_+ - \bar{\sigma} \gamma_-) \bar{\varepsilon}, \\ \delta D &= -\frac{i}{2} \bar{\varepsilon} (\not{D} \lambda + [\sigma, \gamma_+ \lambda] + [\bar{\sigma}, \gamma_- \lambda]) + \frac{i}{2r} \lambda \gamma_3 \bar{\varepsilon} \\ &\quad + \frac{i}{2} \varepsilon (\not{D} \bar{\lambda} - [\sigma, \gamma_- \lambda] - [\bar{\sigma}, \gamma_+ \lambda]) - \frac{i}{2r} \bar{\lambda} \gamma_3 \varepsilon. \end{aligned} \quad (2.13)$$

The complex gauge field \mathcal{A}_m has $q = 0$, for otherwise $U(1)_V$ would not commute with the gauge symmetry which acts on \mathcal{A}_m inhomogeneously. Then the supersymmetry transformation of the chiral multiplets is

$$\begin{aligned}
 \delta \mathcal{A}_m &= \bar{\varepsilon} \psi_m, \\
 \delta \bar{\mathcal{A}}_m &= \varepsilon \bar{\psi}_m, \\
 \delta \psi_m &= i(\mathcal{F}_{\mu m} \gamma^\mu - i\mathcal{D}_m \sigma \gamma_+ - i\mathcal{D}_m \bar{\sigma} \gamma_-) \varepsilon + F_m \bar{\varepsilon}, \\
 \delta \bar{\psi}_m &= i(\bar{\mathcal{F}}_{\mu m} \gamma^\mu + i\bar{\mathcal{D}}_m \sigma \gamma_- + i\bar{\mathcal{D}}_m \bar{\sigma} \gamma_+) \bar{\varepsilon} + \bar{F}_m \varepsilon, \\
 \delta F_m &= i\varepsilon(\not{D}\psi_m - [\sigma, \gamma_- \psi_m] - [\bar{\sigma}, \gamma_+ \psi_m] + i\mathcal{D}_m \lambda), \\
 \delta \bar{F}_m &= i\bar{\varepsilon}(\not{D}\bar{\psi}_m + [\sigma, \gamma_+ \psi_m] + [\bar{\sigma}, \gamma_- \psi_m] + i\bar{\mathcal{D}}_m \bar{\lambda}).
 \end{aligned} \tag{2.14}$$

Here $\mathcal{D}_m = \nabla_m - i\mathcal{A}_m$ and $\mathcal{F}_{\mu m} = \partial_\mu \mathcal{A}_m - \partial_m A_\mu - i[A_\mu, \mathcal{A}_m]$; $\bar{\mathcal{D}}_m$ and $\bar{\mathcal{F}}_{\mu m}$ are defined by similar formulas with \mathcal{A}_m replaced by $\bar{\mathcal{A}}_m = A_m - iX_m$.

Next, we lift the two-dimensional Lagrangian to five dimensions. A standard $\mathcal{N} = (2, 2)$ supersymmetric action on S^2 takes the form

$$S = S_V + S_C + S_W, \tag{2.15}$$

where S_V and S_C are the vector and chiral multiplet actions, and S_W is an F-term. It is straightforward to write down the five-dimensional versions of the first two; the vector multiplet action is

$$\begin{aligned}
 S_V = \frac{1}{2e^2} \int \sqrt{g} d^5x \operatorname{Tr} \Big[& \left(F_{12} + \frac{\sigma_1}{r} \right)^2 + D^\mu \bar{\sigma} D_\mu \sigma + \frac{1}{4} [\sigma, \bar{\sigma}]^2 \\
 & + i\lambda(\not{D}\bar{\lambda} + [\sigma, \gamma_+ \bar{\lambda}] + [\bar{\sigma}, \gamma_- \bar{\lambda}]) + \left(D + \frac{\sigma_2}{r} \right)^2 \Big], \tag{2.16}
 \end{aligned}$$

and the chiral multiplet action is

$$\begin{aligned}
 S_C = \frac{1}{2e^2} \int \sqrt{g} d^5x \operatorname{Tr} \Big(& \bar{\mathcal{F}}^{\mu m} \mathcal{F}_{\mu m} + \frac{1}{2} \bar{\mathcal{D}}^m \bar{\sigma} \mathcal{D}_m \sigma + \frac{1}{2} \mathcal{D}^m \bar{\sigma} \bar{\mathcal{D}}_m \sigma + 2i D \mathcal{D}^m X_m \\
 & - i\bar{\psi}^m (\not{D}\psi_m - [\sigma, \gamma_- \psi_m] - [\bar{\sigma}, \gamma_+ \psi_m]) + \bar{\psi}^m \mathcal{D}_m \lambda + \psi^m \bar{\mathcal{D}}_m \bar{\lambda} + \bar{F}^m F_m \Big). \tag{2.17}
 \end{aligned}$$

It turns out that the superpotential of our theory is given by the Chern-Simons functional

$$W = \frac{1}{2} \int_M \operatorname{CS}(\mathcal{A}) = \frac{1}{2} \int_M \left(\mathcal{A} \wedge d\mathcal{A} - \frac{2i}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right). \tag{2.18}$$

This choice gives

$$S_W = \frac{i}{2e^2} \int_{S^2} \sqrt{h} d^2x \int_M \operatorname{Tr} \left(F \wedge \mathcal{F} - \frac{1}{2} \psi \wedge d\mathcal{A} \psi + \bar{F} \wedge \bar{\mathcal{F}} - \frac{1}{2} \bar{\psi} \wedge d\bar{\mathcal{A}} \bar{\psi} \right), \tag{2.19}$$

where $d\mathcal{A} = d - i\mathcal{A}$ and $\mathcal{F} = d\mathcal{A} - i\mathcal{A} \wedge \mathcal{A}$ is the curvature of \mathcal{A} .

To see that this is the right choice, take $M = \mathbb{R}^3$ and replace h by the flat metric of \mathbb{R}^2 , and drop from the action all the curvature correction terms which depend explicitly on r .

One can integrate out the auxiliary fields and show that the bosonic part of the resulting expression can be written as

$$\frac{1}{e^2} \int d^5x \operatorname{Tr} \left(\frac{1}{4} F^{MN} F_{MN} + \frac{1}{2} D^M X^I D_M X_I - \frac{1}{4} [X^I, X^J] [X_I, X_J] \right), \quad (2.20)$$

with $X^I = (\sigma_1, \sigma_2, X_3, X_4, X_5)$. This is precisely the bosonic part of the 5d MSYM action on \mathbb{R}^5 . (The same result is obtained if W is multiplied by an arbitrary phase factor, but such a phase can be set to one by a $U(1)_V$ rotation.)

There is, however, a problem with the above F-term. In our theory the superpotential W has $q = 0$. If the theory is formulated on $\mathbb{R}^2 \times M$, this simply means that $U(1)_V$ is broken; W must have $q = 2$ for S_W to have $q = 0$. In fact, the lack of $U(1)_V$ was already apparent in our consideration of the twisting, where we saw that the R-symmetry of the twisted theory includes $U(1)_A$ but not $U(1)_V$. On $S^2 \times M$, the consequence is more serious: S_W is not supersymmetric unless W has $q = 2$.

Still, one can find correction terms such that a half of $\mathcal{N} = (2, 2)$ supersymmetry on S^2 are preserved upon adding them to S_W . The supersymmetry variation of S_W is

$$\delta S_W = \frac{1}{2e^2 r} \int_{S^2} \sqrt{h} d^2x \left(-\psi_W \gamma_{\hat{3}} \varepsilon + \bar{\psi}_W \gamma_{\hat{3}} \bar{\varepsilon} \right), \quad (2.21)$$

where ψ_W is the fermion in the superpotential chiral multiplet (W, ψ_W, F_W) . For any $a \in \mathbb{C}^\times$, this can be canceled by the supersymmetry variation of

$$S'_W = \frac{1}{2e^2 r} \int_{S^2} \sqrt{h} d^2x \left(aW - a^{-1} \bar{W} \right), \quad (2.22)$$

provided that one imposes the parameters $(\varepsilon, \bar{\varepsilon})$ to satisfy the relation

$$\varepsilon = -a \gamma_{\hat{3}} \bar{\varepsilon}. \quad (2.23)$$

This relation halves the number of independent conformal Killing spinors. Hence, there is generally a family of F-terms $S_W + S'_W$ that preserve a half of the supersymmetries.

In the present case, we can actually determine the value of a as follows. Suppose that \mathcal{A} is constant on the S^2 . Then S'_W is equal to

$$\frac{2\pi r}{e^2} (a - a^{-1}) \operatorname{Re} W + \frac{2\pi i r}{e^2} (a + a^{-1}) \operatorname{Im} W. \quad (2.24)$$

With W proportional the Chern-Simons functional, for the path integral to be well-defined with such terms the coefficient of $\operatorname{Re} W$ must be equal to an integer k , the level of the Chern-Simons coupling, multiplied by a universal factor. Since we want to keep e and r to be free parameters, the quantization of the level requires $a = \pm 1$ and $k = 0$. The two cases of opposite signs for a are related by orientation reversal of M , so we can take $a = -1$ without loss of generality. Then we have

$$S'_W = -\frac{1}{4e^2 r} \int_{S^2} \sqrt{h} d^2x \int_M (\operatorname{CS}(\mathcal{A}) - \operatorname{CS}(\bar{\mathcal{A}})), \quad (2.25)$$

and $(\varepsilon, \bar{\varepsilon}) = (\varepsilon_\alpha, \bar{\varepsilon}_\alpha)$ satisfy the constraint (2.23).

To summarize, 5d MSYM can be formulated on $S^2 \times M$ preserving the half of $\mathcal{N} = (2, 2)$ supersymmetry on S^2 generated by

$$\overline{Q}_\alpha + Q_\alpha. \quad (2.26)$$

The total action is the sum

$$S = S_V + S_C + S_W + S'_W \quad (2.27)$$

of (2.16), (2.17), (2.19), and (2.25). We remark that the twisted theory is invariant under the rescaling

$$e^2 \rightarrow l e^2, \quad r \rightarrow l r, \quad g_M \rightarrow l^2 g_M, \quad (2.28)$$

as this can be absorbed by a scale transformation of the fields.

This is a good point to pause and discuss a possible alternative formulation of the twisted theory, which we have ignored so far. When we determined the field content of the twisted theory as an $\mathcal{N} = (2, 2)$ theory on S^2 , after we worked out the transformation properties of the fields we jumped to the conclusion that the theory is described by a vector and chiral multiplets. Actually, from the transformation properties alone we cannot tell whether the fields come from a vector and chiral multiplets, or from a twisted vector and twisted chiral multiplets, since we do not know a priori whether $U(1)_R$ corresponds to $U(1)_A$ or $U(1)_V$. $\mathcal{N} = (2, 2)$ twisted multiplets on S^2 are constructed in appendix C.

A twisted superpotential always preserves the full $\mathcal{N} = (2, 2)$ supersymmetry on S^2 , provided that corrections similar to (2.22) are included. However, the twisted vector multiplet action is problematic, being unbounded from below. So if one wants to formulate the theory using twisted multiplets, the best one can hope for is to do that preserving some but not all of the supersymmetries.

The relevant supersymmetries are picked by a constraint imposed on the parameters $(\varepsilon, \bar{\varepsilon})$. The only constraints that are compatible with the conformal Killing spinor conditions, and are invariant under rotations of S^2 , are those of the form (2.23), preserving the linear combinations $-a\overline{Q}_\alpha + Q_\alpha$. Furthermore, $U(1)_V$ must be broken to \mathbb{Z}_2 , or it would imply all the four supersymmetries (unless $a = 0$ or ∞ , but in these cases it is apparently impossible to write down a sensible action). Thus, as far as the symmetries are concerned, a formulation using twisted multiplets can be as good as the formulation using untwisted multiplets, but not better. Besides, preserving two supersymmetries with a twisted vector multiplet is hard, and we do not know how to do this.

3 TQFT in the infinite volume limit

Having formulated the twisted theory on $S^2 \times M$, let us analyze its dependence on the geometry of M . We pick a supercharge Q which we use as the BRST operator. For definiteness we set

$$Q = \overline{Q}_1 + Q_1, \quad (3.1)$$

but the conclusions will be the same for any other linear combinations of the preserved supercharges (2.26). We will be interested in quantities that are computed by path integrals only involving Q -invariant operators and states. Such quantities are invariant under Q -exact shifts in the operators and states involved.

In many examples, twisting a supersymmetric theory leads to a TQFT. What usually happens is that the action becomes independent of the spacetime metric up to Q -exact terms, and therefore varying the metric just inserts Q -exact operators in the path integral, leaving Q -invariant quantities unchanged. Since our theory is partially twisted along M , one may expect that it is topological on M modulo Q -exactness. This is not the case, however.

In our theory the superpotential is topological, so the g_M -dependence comes from S_V and S_C . These can be expressed as

$$\begin{aligned} S_V &= \{\bar{Q}_1, [Q_2, V_V]\} = -\{Q_1, [\bar{Q}_2, V_V]\}, \\ S_C &= \{\bar{Q}_1, [Q_2, V_C]\} = -\{Q_1, [\bar{Q}_2, V_C]\}, \end{aligned} \quad (3.2)$$

with suitable functionals V_V and V_C .³ From these formulas we see that $S_V + S_C$ would be Q -exact if Q were, say, $\bar{Q}_1 + Q_2$ or $\bar{Q}_2 + Q_1$. However, it is not Q -exact with respect to our choice (3.1) or, for that purpose, none of the preserved supercharges. Thus the twisted theory depends on g_M .

Nevertheless, one can argue that the twisted theory does become topological on M in the limit where the coupling e^2 and the radius r of the S^2 are rescaled as

$$e^2 \rightarrow l^{-1}e^2, \quad r \rightarrow l^{-1}r \quad (3.5)$$

and l is sent to infinity, or equivalently by the scale invariance (2.28), in the limit where g_M is rescaled by an infinitely large factor:

$$g_M \rightarrow l^2 g_M, \quad l \rightarrow \infty. \quad (3.6)$$

Therefore, the infinite volume limit of the twisted theory is a TQFT. We now present the argument.

Using the formulas (3.2) we can write

$$\begin{aligned} S_V &= S_V^+ - \{Q, [\bar{Q}_2, V_V]\} = S_V^- + \{Q, [Q_2, V_V]\}, \\ S_C &= S_C^+ - \{Q, [\bar{Q}_2, V_C]\} = S_C^- + \{Q, [Q_2, V_C]\}, \end{aligned} \quad (3.7)$$

where S_V^+ and S_C^+ have $q = +2$, while S_V^- and S_C^- have $q = -2$. Thus a deformation of g_M brings down operators T^\pm of $q = \pm 2$ into the correlation function:

$$\frac{\delta}{\delta g^{mn}} \langle \cdots \rangle = \langle T_{mn}^+ \cdots \rangle = \langle T_{mn}^- \cdots \rangle. \quad (3.8)$$

³Explicitly,

$$\begin{aligned} V_V &= \frac{1}{2e^2} \int \sqrt{g} d^5x \operatorname{Tr} \left(\bar{\lambda} \gamma_3 \lambda - 4i D \sigma_2 - \frac{2i}{r} \sigma_2^2 \right), \\ V_C &= -\frac{1}{2e^2} \int \sqrt{g} d^5x \operatorname{Tr} \left(\bar{\psi}^m \gamma_3 \psi_m - 4D^m X_m \sigma_2 + \frac{2i}{r} X^m X_m \right). \end{aligned} \quad (3.3)$$

To obtain V_C from the corresponding formula in two dimensions, one uses the fact that for an adjoint chiral multiplet scalar ϕ with $q = 0$, the equation

$$\bar{Q}_\alpha Q_\beta \operatorname{Tr} \bar{\phi} \phi = \frac{1}{2} \bar{Q}_\alpha Q_\beta \operatorname{Tr} (2\bar{\phi} \phi - \phi^2 - \bar{\phi}^2) = 2\bar{Q}_\alpha Q_\beta \operatorname{Tr} (\operatorname{Im} \phi)^2 \quad (3.4)$$

holds modulo total derivatives.

Now, the rescaling $g_M \rightarrow l^2 g_M$ is the same as keeping g_M fixed but changing the action to

$$S_l = l^3 S_V + l S_C + S_W + S'_W. \quad (3.9)$$

We see that in the limit $l \rightarrow \infty$, the F-term is negligible compared to the other part of the action, hence $U(1)_V$ is restored. Then

$$\langle T_{mn}^+ \cdots \rangle = 0, \quad (3.10)$$

except possibly when the insertion \cdots has terms with $q = -2$. This possibility is excluded by considering $\langle T_{mn}^- \cdots \rangle$ instead, and we conclude that the theory is independent of g_M in the limit $l \rightarrow \infty$.

We were a bit sloppy when we asserted the equation (3.10) in the above argument. This point deserves a more careful explanation.

In the familiar story of compactification, one expands fields in the eigenmodes of kinetic operators on the small “internal” space, and their eigenvalues appear as masses from the point of view of the large “external” space. Similarly, the nonzero modes of λ , $\bar{\lambda}$ along the S^2 give masses of order l^3 from the point of view of M , and those of ψ_m , $\bar{\psi}_m$ give masses of order l . The kinetic terms describing the dynamics of these modes on M are accompanied by a factor of l . Then natural variables for the fermion integration for large l are obtained by rescaling

$$\lambda_0 \rightarrow l^{-1/2} \lambda_0, \quad \lambda' \rightarrow l^{-3/2} \lambda', \quad \psi_m \rightarrow l^{-1/2} \psi_m, \quad (3.11)$$

where λ_0 and λ' are the zero- and nonzero-mode parts of λ , respectively.⁴ The rescaling absorbs an infinite power of l in the normalization of the path integral measure. On the other hand, the equations of motion set

$$F_m = -\frac{i}{2l} \epsilon_{mnp} \bar{\mathcal{F}}^{np}. \quad (3.12)$$

We write

$$\lambda_0, \psi_m \sim l^{-1/2}, \quad \lambda' \sim l^{-3/2}, \quad F_m \sim l^{-1} \quad (3.13)$$

to indicate the order of these fields according to the power of l .

Since S_V^+ and S_C^+ have $q = 2$, they consist of terms with a single \bar{F}_m or two of $\bar{\lambda}$, $\bar{\psi}_m$ (and terms with four or more of these fields and fields with negative R-charge which we can neglect). At such, they are at most of order l^{-1} :

$$S_V^+ \lesssim l^{-1}, \quad S_C^+ \lesssim l^{-1}. \quad (3.14)$$

Then T_{mn}^+ , derived from $l^3 S_V^+ + l S_C^+$, is at most of order l^2 :

$$T_{mn}^+ \lesssim l^2. \quad (3.15)$$

Meanwhile, $U(1)_V$ is broken at order l^{-1} by S_W . All in all, the counting suggests that $\langle T_{mn}^+ \cdots \rangle$ is at most of order l ,

$$\langle T_{mn}^+ \cdots \rangle \lesssim l, \quad (3.16)$$

⁴For simplicity we assume that there are no fermion zero modes on M . This is generically true on an odd-dimensional compact manifold since the relevant index is zero.

and could grow like l for large l , contrary to our previous assertion. Apparently our estimate was not very good.

So we need a more refined argument. Let us consider a slightly different setup, in which the action is taken to be

$$S_u = l^3 S_V + u S_C + S_W + S'_W, \quad (3.17)$$

with $l \leq u \leq l^3$. First we show that the twisted theory is independent of u in the limit $l \rightarrow \infty$.

With the factor l in front of S_C replaced by u , this time we have

$$\lambda_0, \psi_m \sim u^{-1/2}, \quad \lambda' \sim l^{-3/2}, \quad F_m \sim u^{-1}. \quad (3.18)$$

Following the same logic, we learn

$$S_V^+ \lesssim u^{-1}, \quad S_C^+ \lesssim u^{-1}, \quad (3.19)$$

and $U(1)_V$ is broken at order u^{-1} . Then the change in the correlation function between $u = l$ and $u = l'$ is

$$\langle \cdots \rangle_{u=l'} - \langle \cdots \rangle_{u=l} = - \int_l^{l'} du \langle S_C^+ \cdots \rangle \lesssim l^{-1}. \quad (3.20)$$

This goes to zero as $l \rightarrow \infty$, so the correlator is independent of u in this limit, as promised.

Using the freedom to choose u , we consider deformations of g_M at $u = l^3$. Here we find that T_{mn}^+ is at most of order 1 and

$$\langle T_{mn}^+ \cdots \rangle_{u=l^3} \lesssim l^{-3}. \quad (3.21)$$

This vanishes in the limit $l \rightarrow \infty$, which was what we wanted to show.

Another way of understanding the topological invariance in the infinite volume limit is to note that as a result of the restoration of $U(1)_V$, in this limit the full $\mathcal{N} = (2, 2)$ supersymmetry on S^2 is restored and the g_M -dependent part of the action becomes exact with respect to \overline{Q}_α and Q_α . Then the topological invariance of correlators is manifest for operator insertions invariant under any of these supercharges, and this covers a large class of Q -invariant operators. This perspective also shows clearly that these correlators are independent of u .

Moreover, choosing the BRST operator to be \overline{Q}_α or Q_α allows to construct interesting observables. For example, to a closed loop $L \subset M$ one can associate a Wilson-loop-like operator

$$\text{Tr } P \exp \left(\int_{S^2} \sqrt{h} d^2x \oint_M \mathcal{A} \right), \quad (3.22)$$

which is \overline{Q}_α -invariant. So this is a very attractive possibility, though justifying it surely requires a more careful analysis.

4 Localization to three dimensions

We have seen that the twisted theory on $S^2 \times M$ reduces to a TQFT on M in the limit where M is infinitely large. We now try to determine this TQFT by exploiting one of the salient features of supersymmetric theories: localization of the path integral.

Let us recall how the localization works. Suppose one has a theory with a fermionic charge Q , and wants to compute a correlation function of Q -invariant operators. To simplify the path integral, one picks a Q^2 -invariant functional V such that the bosonic terms in $\{Q, V\}$ are nonnegative. Then one shifts the action as

$$S \rightarrow S + t\{Q, V\}. \quad (4.1)$$

By virtue of the Q -symmetry, the path integral is independent of t . Whereas one recovers at $t = 0$ the original path integral, the path integral in the limit $t \rightarrow \infty$ localizes to the locus where the bosonic terms in $\{Q, V\}$ all vanish. In this limit the path integral can be evaluated by first integrating over fluctuations around a given configuration on the localization locus, and then over all such background configurations. In a typical situation the localization locus is finite-dimensional. Thus, the path integral over an infinite-dimensional field space gets reduced to finite-dimensional integrals.

For the purpose of computing physical quantities in our theory, this localization method does not seem particularly useful; it is rather difficult to find a good candidate for V since Q^2 does not generate a simple bosonic symmetry. This is to be compared with the situation where the action is Q -exact, in which case one can always rescale the action by an infinitely large factor to go to the zero coupling limit, whether $Q^2 = 0$ or not. By contrast, none of the Q -invariant pieces of our action is Q -exact. One might be tempted to rescale the Q -exact terms in the expressions (3.7), but doing so would break the Q -symmetry as these terms are not Q -invariant by themselves.

However, we are interested in the infinite volume limit of M , and the things are better here. As we saw in the previous section, in this limit we can replace the action by the one given in (3.17), so let us define $t = l^3$ and set $u = t$:

$$S_t = t(S_V + S_C) + S_W + S'_W. \quad (4.2)$$

This is just like what one would do to localize the path integral if $S_V + S_C$ were Q -exact, namely, rescale

$$S_V + S_C \rightarrow t(S_V + S_C) \quad (4.3)$$

and take $t \rightarrow \infty$. Then localization should take place and it should be possible to rewrite the $t \rightarrow \infty$ limit of the path integral with respect to S_t as an integral over a smaller space. Hopefully, the expression after the localization can be interpreted as the path integral for some three-dimensional TQFT.

Let us analyze the localization property of this path integral. Integrating out the auxiliary fields F_m , \bar{F}_m leaves the potential

$$\frac{1}{4e^2 t} \int \sqrt{g} d^5 x \operatorname{Tr} \bar{\mathcal{F}}^{mn} \mathcal{F}_{mn}, \quad (4.4)$$

which disappears in the limit $t \rightarrow \infty$. The rest of the bosonic terms in S_V and S_C are nonnegative (except one imaginary term) and multiplied by a factor of t . Thus the path integral localizes to the saddle-point configurations:

$$0 = F_{\hat{1}\hat{2}} + \frac{\sigma_1}{r} = D_\mu \sigma_1 = D_\mu \sigma_2 = [\sigma_1, \sigma_2] = \mathcal{F}_{\mu m} = \mathcal{D}_m \sigma_1 = \mathcal{D}_m \sigma_2 = D^m X_m. \quad (4.5)$$

For these configurations, the gauge field $A_{S^2} = A_\mu dx^\mu$ on S^2 solves the Yang-Mills equation $D^\mu F_{\mu\nu} = 0$. In two dimensions gauge fields have no physical propagating degrees of freedom, so in each topological class there is a unique solution up to gauge transformations. Convenient representatives are

$$A_{S^2}^\pm = B(\pm 1 - \cos \theta) d\varphi, \quad (4.6)$$

where \pm refers to the chart covering the north or south pole, and

$$B = \frac{1}{2\pi} \int_{S^2} F_{\hat{1}\hat{2}} \quad (4.7)$$

is the magnetic flux through the S^2 which can be chosen to be diagonal. Each diagonal entry of B represents the Chern number of a line bundle and must be quantized. The residual gauge symmetry is the group of three-dimensional gauge transformations

$$g: M \rightarrow G_B \quad (4.8)$$

valued in the stabilizer $G_B \subset G$ of B .

For the above choice of A_{S^2} , we have

$$\sigma_1 = \frac{B}{2r}. \quad (4.9)$$

Then $D_\mu \sigma_1 = 0$ is automatically satisfied. On the other hand, $\mathcal{D}_m \sigma_1 = 0$ shows that \mathcal{A} takes values in $\mathfrak{g}_B \otimes \mathbb{C}$, with \mathfrak{g}_B the Lie algebra of G_B . Given this, $\mathcal{F}_{\mu m} = 0$ implies that \mathcal{A} is constant on the S^2 :

$$\mathcal{A}_m = \mathcal{A}_m(x^m) \in \mathfrak{g}_B \otimes \mathbb{C}. \quad (4.10)$$

The imaginary part of \mathcal{A} obeys the D-term equation $\mu(\mathcal{A}) = 0$, where

$$\mu(\mathcal{A}) = D^m X_m. \quad (4.11)$$

Likewise, $D_\mu \sigma_2 = [\sigma_1, \sigma_2] = 0$ implies that σ_2 is constant on the S^2 and valued in \mathfrak{g}_B ,

$$\sigma_2 = \sigma_2(x^m) \in \mathfrak{g}_B, \quad (4.12)$$

and the last equation

$$\mathcal{D}_m \sigma_2 = 0 \quad (4.13)$$

says σ_2 is covariantly constant on M .

Now that we know the saddle-point configurations, we can write down the localized expression of the path integral. If we denote by (B, \mathcal{A}_B, a_B) the saddle-point configuration with a magnetic flux B and $(\mathcal{A}, \sigma_2) = (\mathcal{A}_B, a_B)$, then the partition function is

$$\sum_B \int \mathcal{D}\mathcal{A}_B da_B Z_1(B, \mathcal{A}_B, a_B) \exp(-S_0(\mathcal{A}_B)). \quad (4.14)$$

Here $Z_1(B, \mathcal{A}_B, a_B)$ represents the one-loop determinants for fluctuations around the background (B, \mathcal{A}_B, a_B) , and

$$S_0(\mathcal{A}_B) = -\frac{\pi r}{e^2} \int_M (\text{CS}(\mathcal{A}_B) - \text{CS}(\bar{\mathcal{A}}_B)) \quad (4.15)$$

is the action evaluated at the background. Correlation functions of Q -invariant operators are given by inserting the operators evaluated at the saddle-point configurations in the above path integral.

Suppose we have done the integrations over the a_B . Then, for each B , we are left with the path integral

$$\int \widetilde{\mathcal{D}\mathcal{A}_B} \exp(-S_0(\mathcal{A}_B)), \quad (4.16)$$

with the modified measure

$$\widetilde{\mathcal{D}\mathcal{A}_B} = \mathcal{D}\mathcal{A}_B \int da_B Z_1(B, \mathcal{A}_B, a_B). \quad (4.17)$$

Written in this form, it is clear that each summand of the localized path integral is the path integral for a three-dimensional gauge theory, whose action is independent of g_M . This is in keeping with the expectation that the twisted theory reduces to a three-dimensional TQFT in the infinite volume limit.

There is, however, something not very beautiful about the expression (4.16). The field variable of the path integral is a gauge field \mathcal{A}_B for the complex gauge group $(G_B)_{\mathbb{C}}$. Moreover, the action $S_0(\mathcal{A}_B)$ is invariant under the complex gauge transformations. These facts strongly suggest that we should interpret it as the path integral for a gauge theory with gauge group $(G_B)_{\mathbb{C}}$. And yet, the integration is performed within the space of complex gauge fields modulo *real* gauge transformations, since the gauge group of the original theory is G , not $G_{\mathbb{C}}$. The question is then whether we can convert the gauge group from G_B to $(G_B)_{\mathbb{C}}$.

To phrase this more precisely, let \mathcal{U} be the space of real gauge fields and \mathcal{G} the group of real gauge transformations, and let $\mathcal{U}_{\mathbb{C}}$ and $\mathcal{G}_{\mathbb{C}}$ be the complexifications of \mathcal{U} and \mathcal{G} . (The discussion will be identical for all values of B , so we will not specify it to avoid cluttering the notation.) Then the integration domain is

$$\mu^{-1}(0)/\mathcal{G} \subset \mathcal{U}_{\mathbb{C}}/\mathcal{G}, \quad (4.18)$$

where μ was defined in (4.11). The question is, can we embed this space into the moduli space $\mathcal{U}_{\mathbb{C}}/\mathcal{G}_{\mathbb{C}}$ of complex gauge fields, rather than $\mathcal{U}_{\mathbb{C}}/\mathcal{G}$?

The answer is yes, and indeed it is done naturally. To understand how, recall a standard fact in supersymmetric gauge theories about the moduli space of vacua. This is the space of solutions to the D- and F-term equations (the equations obtained by setting to zero the D- and F-term potentials) modulo gauge transformations. Besides this description, there is another way to describe the same space: one drops the D-term equation and takes the quotient instead by the complexified gauge transformations.

In our situation, the D-term equation is $\mu = 0$, while the F-term equation is absent as the potential (4.4) becomes zero in the limit $t \rightarrow \infty$. Then the two descriptions of the moduli space lead to the identification of $\mu^{-1}(0)/\mathcal{G}$ and $\mathcal{U}_{\mathbb{C}}/\mathcal{G}_{\mathbb{C}}$. In fact, relevant points in $\mathcal{U}_{\mathbb{C}}$ are only those that can be mapped into the locus $\mu = 0$ by a complex gauge transformation; such points are called semistable. Thus we have

$$\mu^{-1}(0)/\mathcal{G} \cong \mathcal{U}_{\mathbb{C}}^{\text{ss}}/\mathcal{G}_{\mathbb{C}}, \quad (4.19)$$

where $\mathcal{U}_{\mathbb{C}}^{\text{ss}}$ is the set of semistable points of $\mathcal{U}_{\mathbb{C}}$. Mathematically, μ is (the dual of) the moment map for the \mathcal{G} -action on $\mathcal{U}_{\mathbb{C}}$ with respect to the Kähler form

$$\omega = \frac{i}{2} \int_M \text{Tr} (\delta \mathcal{A} \wedge \star \delta \bar{\mathcal{A}}), \quad (4.20)$$

and we have an identification of the symplectic and GIT quotients.

Therefore, we arrive at the expression

$$\sum_B \int_{\mathcal{M}_B^{\text{ss}}} \widetilde{\mathcal{D}\mathcal{A}_B} \exp(-S_0(\mathcal{A}_B)), \quad (4.21)$$

where $\mathcal{M}_B^{\text{ss}}$ is the semistable locus in the moduli space \mathcal{M}_B of $(G_B)_{\mathbb{C}}$ gauge fields.

Apart from the modifications in the measure and the integration domain, each summand of the above expression is the path integral for Chern-Simons theory on M with gauge group $(G_B)_{\mathbb{C}}$:

$$\int_{\mathcal{M}_B} \mathcal{D}\mathcal{A}_B \exp(iS_{\text{CS}}(\mathcal{A}_B)). \quad (4.22)$$

The complex Chern-Simons action is given by

$$S_{\text{CS}} = -\frac{k+is}{8\pi} \int_M \text{CS}(\mathcal{A}) - \frac{k-is}{8\pi} \int_M \text{CS}(\bar{\mathcal{A}}), \quad (4.23)$$

with $k \in \mathbb{Z}$ and $s \in \mathbb{R}$. The comparison shows

$$k = 0, \quad s = \frac{8\pi^2 r}{e^2}. \quad (4.24)$$

Notice that the parameter s is invariant under a rescaling of e^2 and r by a common factor. This is consistent with the fact that the twisted theory is invariant under the rescaling (2.28) and becomes topological on M in the infinite volume limit.

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A Spinors on S^2

The metric of a round two-sphere S^2 of radius r is

$$h = r^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad (\text{A.1})$$

with (θ, φ) the spherical coordinates. We set

$$(x^1, x^2) = (\theta, \varphi), \quad (\text{A.2})$$

and introduce the orthonormal vectors

$$e_{\hat{1}} = \frac{1}{r} \frac{\partial}{\partial \theta}, \quad e_{\hat{2}} = \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi}. \quad (\text{A.3})$$

We use the indices μ, ν, \dots to refer to the coordinates, and $\hat{\mu}, \hat{\nu}, \dots$ to refer to the orthonormal frame.

We choose the gamma matrices representing the Clifford algebra $\{\gamma_{\hat{\mu}}, \gamma_{\hat{\nu}}\} = 2\delta_{\hat{\mu}\hat{\nu}}$ to be

$$\gamma_{\hat{\mu}} = \tau_{\hat{\mu}}, \quad (\text{A.4})$$

where $\tau_{\hat{\mu}}$ are the Pauli matrices:

$$\tau_{\hat{1}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_{\hat{2}} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_{\hat{3}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (\text{A.5})$$

The chirality operator is

$$\gamma_{\hat{3}} = \tau_{\hat{3}}. \quad (\text{A.6})$$

The spin connection is denoted by ∇ .

The product of two spinors ψ and χ is defined as

$$\psi\chi = \psi^T C\chi, \quad (\text{A.7})$$

where the charge conjugation operator

$$C = i\tau_{\hat{2}}. \quad (\text{A.8})$$

For anticommuting spinors, we have

$$\psi\chi = \chi\psi, \quad \psi\gamma_{\hat{a}}\chi = -\chi\gamma_{\hat{a}}\psi \quad (\hat{a} = \hat{1}, \hat{2}, \hat{3}). \quad (\text{A.9})$$

More generally,

$$\psi\gamma_{\hat{a}_1} \cdots \gamma_{\hat{a}_n} \chi = (-1)^{n(n+1)/2} \chi\gamma_{\hat{a}_1} \cdots \gamma_{\hat{a}_n} \psi \quad (\text{A.10})$$

for the \hat{a}_i all different.

The following identities are useful in calculating supersymmetry variations:

$$2(\varepsilon\eta)(\alpha\beta) = (\varepsilon\alpha)(\eta\beta) - \sum_{\hat{a}=\hat{1}}^{\hat{3}} (\varepsilon\gamma_{\hat{a}}\alpha)(\eta\gamma_{\hat{a}}\beta), \quad (\text{A.11})$$

$$(\varepsilon\eta)(\alpha\beta) - (\varepsilon\gamma_{\hat{3}}\eta)(\alpha\gamma_{\hat{3}}\beta) = (\varepsilon\alpha)(\eta\beta) - (\varepsilon\gamma_{\hat{3}}\alpha)(\eta\gamma_{\hat{3}}\beta). \quad (\text{A.12})$$

Here ε, η are commuting spinors, and α, β are anticommuting spinors.

B $\mathcal{N} = (2, 2)$ supersymmetry on S^2

In this appendix we review the construction of $\mathcal{N} = (2, 2)$ supersymmetric gauge theories on S^2 . We refer the reader to [17, 18] for more discussions.

The $\mathcal{N} = (2, 2)$ supersymmetry group on S^2 is a subgroup of the $\mathcal{N} = (2, 2)$ superconformal group on S^2 that does not include conformal transformations. The latter group is generated by the rotations and conformal transformations of S^2 , the vector R-symmetry $U(1)_V$, and the axial R-symmetry $U(1)_A$, as well as eight supersymmetries. On fermionic fields Ψ which are Dirac spinors, the R-symmetries act by

$$\begin{aligned} U(1)_V \ni e^{i\alpha} : \Psi &\mapsto \exp(iq_V\alpha)\Psi, \\ U(1)_A \ni e^{i\beta} : \Psi &\mapsto \exp(iq_A\beta\gamma_3)\Psi, \end{aligned} \quad (\text{B.1})$$

where q_V and q_A are the vector and axial R-charges of Ψ . On bosonic fields they both act in the same way, without the γ_3 insertion for $U(1)_A$.

There are two kinds of supercharges, \overline{Q}_ε and $Q_{\bar{\varepsilon}}$, labeled by commuting conformal Killing spinors ε or $\bar{\varepsilon}$ which can be chosen to obey

$$\nabla_\mu \varepsilon = \pm \frac{1}{2r} \gamma_\mu \gamma_3 \varepsilon, \quad \nabla_\mu \bar{\varepsilon} = \pm \frac{1}{2r} \gamma_\mu \gamma_3 \bar{\varepsilon}. \quad (\text{B.2})$$

They satisfy the commutation relations

$$\{\overline{Q}_{\varepsilon_1}, \overline{Q}_{\varepsilon_2}\} = \{Q_{\bar{\varepsilon}_1}, Q_{\bar{\varepsilon}_2}\} = 0 \quad (\text{B.3})$$

and

$$\{\overline{Q}_\varepsilon, Q_{\bar{\varepsilon}}\} = P_\xi + i\alpha F_V + i\beta F_A. \quad (\text{B.4})$$

Here P_ξ generates the rotations and conformal transformations by the conformal Killing vector ξ , and F_V and F_A are the generators of $U(1)_V$ and $U(1)_A$, respectively; the parameters are given by

$$\xi^\mu = i\varepsilon\gamma^\mu\bar{\varepsilon}, \quad \alpha = -\frac{1}{4}(\varepsilon\nabla\bar{\varepsilon} - \bar{\varepsilon}\nabla\varepsilon), \quad \beta = \frac{1}{4}(\varepsilon\gamma_3\nabla\bar{\varepsilon} - \bar{\varepsilon}\gamma_3\nabla\varepsilon). \quad (\text{B.5})$$

Two important types of field representations of the $\mathcal{N} = (2, 2)$ superconformal group are a vector multiplet

$$(A, \sigma, \lambda, \bar{\lambda}, D) \quad (\text{B.6})$$

consisting of a gauge field A , a complex scalar σ , Dirac spinors $\lambda, \bar{\lambda}$, and a real auxiliary field D , and a chiral multiplet

$$(\phi, \psi, F) \quad (\text{B.7})$$

consisting of a complex scalar ϕ , a Dirac spinor ψ , and a complex auxiliary field F . The Weyl weight w and the vector R-charge q of a chiral multiplet are related by

$$q = 2w. \quad (\text{B.8})$$

The following table summarizes the Weyl weight, vector R-charge, and axial R-charge of the component fields:

	A_μ	σ	λ	$\bar{\lambda}$	D	ϕ	ψ	F
w	0	1	3/2	3/2	2	$q/2$	$(q+1)/2$	$(q+2)/2$
F_V	0	0	-1	1	0	q	$q-1$	$q-2$
F_A	0	2	-1	1	0	0	1	0

(B.9)

The supersymmetry transformation of a vector multiplet is

$$\begin{aligned}
 \delta A_\mu &= -\frac{i}{2}(\bar{\varepsilon}\gamma_\mu\lambda + \varepsilon\gamma_\mu\bar{\lambda}), \\
 \delta\sigma &= \bar{\varepsilon}\gamma_-\lambda - \varepsilon\gamma_+\bar{\lambda}, \\
 \delta\bar{\sigma} &= \bar{\varepsilon}\gamma_+\lambda - \varepsilon\gamma_-\bar{\lambda}, \\
 \delta\lambda &= i\left(F_{12}\gamma_3 + \gamma_-\not{D}\sigma + \gamma_+\not{D}\bar{\sigma} + \frac{1}{2}[\sigma, \bar{\sigma}]\gamma_3 + iD\right)\varepsilon + i(\sigma\gamma_- + \bar{\sigma}\gamma_+)\not{V}\varepsilon, \\
 \delta\bar{\lambda} &= i\left(F_{12}\gamma_3 - \gamma_+\not{D}\sigma - \gamma_-\not{D}\bar{\sigma} - \frac{1}{2}[\sigma, \bar{\sigma}]\gamma_3 - iD\right)\bar{\varepsilon} - i(\sigma\gamma_+ + \bar{\sigma}\gamma_-)\not{V}\bar{\varepsilon}, \\
 \delta D &= -\frac{i}{2}\bar{\varepsilon}(\not{D}\lambda + [\sigma, \gamma_+\lambda] + [\bar{\sigma}, \gamma_-\lambda]) - \frac{i}{2}\lambda\not{V}\bar{\varepsilon} \\
 &\quad + \frac{i}{2}\varepsilon(\not{D}\bar{\lambda} - [\sigma, \gamma_-\bar{\lambda}] - [\bar{\sigma}, \gamma_+\bar{\lambda}]) + \frac{i}{2}\bar{\lambda}\not{V}\varepsilon,
 \end{aligned}
 \tag{B.10}$$

where $\gamma_\pm = (1 \pm \gamma_3)/2$ are the projectors to the positive- and negative-chirality subspaces. The supersymmetry transformation of a chiral multiplet coupled to a vector multiplet is

$$\begin{aligned}
 \delta\phi &= \bar{\varepsilon}\psi, \\
 \delta\bar{\phi} &= \varepsilon\bar{\psi}, \\
 \delta\psi &= i(\not{D}\phi + \sigma\phi\gamma_+ + \bar{\sigma}\phi\gamma_-)\varepsilon + i\frac{q}{2}\phi\not{V}\varepsilon + F\bar{\varepsilon}, \\
 \delta\bar{\psi} &= i(\not{D}\bar{\phi} + \bar{\phi}\sigma\gamma_- + \phi\bar{\sigma}\gamma_+)\bar{\varepsilon} + i\frac{q}{2}\bar{\phi}\not{V}\bar{\varepsilon} + \bar{F}\varepsilon, \\
 \delta F &= i\varepsilon(\not{D}\psi - \gamma_-\sigma\psi - \gamma_+\bar{\sigma}\psi - \lambda\phi) + i\frac{q}{2}\psi\not{V}\varepsilon, \\
 \delta\bar{F} &= i\bar{\varepsilon}(\not{D}\bar{\psi} - \gamma_+\bar{\psi}\sigma - \gamma_-\bar{\psi}\bar{\sigma} + \bar{\phi}\bar{\lambda}) + i\frac{q}{2}\bar{\psi}\not{V}\bar{\varepsilon}.
 \end{aligned}
 \tag{B.11}$$

These transformation rules satisfy the commutation relations (B.3), (B.4) modulo gauge transformations; see [17] for details of the calculation.

On S^2 there are four independent conformal Killing spinors, leading to the total of eight supercharges. The supercharges of $\mathcal{N} = (2, 2)$ supersymmetry are obtained by restricting $\varepsilon, \bar{\varepsilon}$ to satisfy

$$\nabla_\mu\varepsilon = +\frac{1}{2r}\gamma_\mu\gamma_3\varepsilon, \tag{B.12}$$

$$\nabla_\mu\bar{\varepsilon} = -\frac{1}{2r}\gamma_\mu\gamma_3\bar{\varepsilon}. \tag{B.13}$$

For this choice of the signs, ξ is a Killing vector and generates an isometry, and $\beta = 0$. Thus the $\mathcal{N} = (2, 2)$ supersymmetry group is generated by rotations, $U(1)_V$, and four supersymmetry transformations.

We pick conformal Killing spinors ε_α ($\alpha = 1, 2$) that satisfy (B.12) and

$$\varepsilon_1 \varepsilon_2 = 1. \quad (\text{B.14})$$

Concretely, we can choose

$$\varepsilon_1 = e^{i\varphi/2} \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix}, \quad \varepsilon_2 = e^{-i\varphi/2} \begin{pmatrix} -\sin(\theta/2) \\ \cos(\theta/2) \end{pmatrix}. \quad (\text{B.15})$$

We set $\bar{\varepsilon}_\alpha = \gamma_3 \varepsilon_\alpha$. Then $\bar{\varepsilon}_\alpha$ satisfy (B.13) and $\bar{\varepsilon}_1 \bar{\varepsilon}_2 = -1$. We write \bar{Q}_α for $\bar{Q}_{\varepsilon_\alpha}$ and Q_α for $Q_{\bar{\varepsilon}_\alpha}$. The corresponding supersymmetry variations are denoted by \bar{Q}_α , Q_α .

The vector multiplet action S_V and the chiral multiplet action S_C are given by

$$(\varepsilon_\alpha \varepsilon_\beta) S_V = \bar{Q}_\alpha Q_\beta \int \sqrt{h} d^2x \text{Tr} \left(\bar{\lambda} \gamma_3 \lambda - 4i D \sigma_2 - \frac{2i}{r} \sigma_2^2 \right), \quad (\text{B.16})$$

$$(\varepsilon_\alpha \varepsilon_\beta) S_C = -\bar{Q}_\alpha Q_\beta \int \sqrt{h} d^2x \text{Tr} \left[\bar{\psi} \gamma_3 \psi - 2\bar{\phi} \left(\sigma_2 + i \frac{q}{2r} \right) \phi + \frac{i}{r} \bar{\phi} \phi \right]. \quad (\text{B.17})$$

These formulas together with the relations $\bar{Q}_\alpha^2 = Q_\alpha^2 = 0$ show that S_V and S_C are invariant under $\mathcal{N} = (2, 2)$ supersymmetry. More explicitly,

$$S_V = \int \sqrt{h} d^2x \text{Tr} \left[\left(F_{12} + \frac{\sigma_1}{r} \right)^2 + D^\mu \bar{\sigma} D_\mu \sigma + \frac{1}{4} ([\sigma, \bar{\sigma}])^2 + i\lambda (\not{D}\lambda + [\sigma, \gamma_+ \lambda] + [\bar{\sigma}, \gamma_- \lambda]) + \left(D + \frac{\sigma_2}{r} \right)^2 \right], \quad (\text{B.18})$$

$$S_C = \int \sqrt{h} d^2x \left[\bar{\phi} \left(-D_\mu^2 + \frac{1}{2} \{ \sigma, \bar{\sigma} \} + iD + i \frac{q}{r} \sigma_2 - \frac{q^2 - 2q}{4r^2} \right) \phi - i\bar{\psi} \left(\not{D} - \sigma \gamma_- - \bar{\sigma} \gamma_+ + \frac{q}{2r} \gamma_3 \right) \psi + i\bar{\psi} \lambda \phi - i\bar{\phi} \bar{\lambda} \psi + \bar{F} F \right], \quad (\text{B.19})$$

where $\sigma = \sigma_1 - i\sigma_2$.

Given a gauge-invariant holomorphic function $W(\phi)$ of chiral multiplet scalars ϕ^i with total vector R-charge $q = 2$, one can construct an F-term S_W that is invariant under $\mathcal{N} = (2, 2)$ supersymmetry. The superpotential W may itself be thought of as the lowest component of a chiral multiplet (W, ψ_W, F_W) , with the auxiliary field

$$F_W = F^i \partial_i W - \frac{1}{2} \psi^i \psi^j \partial_i \partial_j W. \quad (\text{B.20})$$

In terms of this multiplet,

$$S_W = i \int \sqrt{h} d^2x (F_W + \bar{F}_W). \quad (\text{B.21})$$

Note that S_W must be purely imaginary in order to produce a positive F-term potential.

C $\mathcal{N} = (2, 2)$ twisted multiplets on S^2

$\mathcal{N} = (2, 2)$ vector and chiral multiplets have “twisted” cousins of the identical field content but with different R-charge assignments. A twisted vector multiplet

$$(A, \rho, \kappa, \bar{\kappa}, C) \tag{C.1}$$

consists of a gauge field A , a complex scalar ρ , Dirac spinors $\kappa, \bar{\kappa}$, and a real auxiliary field C ; a twisted chiral (plus antichiral) multiplet

$$(v, \chi, \bar{\chi}, E) \tag{C.2}$$

consists of a complex scalar v , Dirac spinors $\chi, \bar{\chi}$, and a complex auxiliary field E . Their Weyl weights and R-charges are as follows:

	A_μ	ρ	κ	$\bar{\kappa}$	C	v	χ	E
w	0	1	3/2	3/2	2	$q/2$	$(q+1)/2$	$(q+2)/2$
F_V	0	-2	-1	1	0	0	-1	0
F_A	0	0	-1	1	0	$-q$	$1-q$	$2-q$

(C.3)

The supersymmetry transformation rules for twisted multiplets can be obtained from those for the corresponding untwisted multiplets by replacing

$$\begin{aligned} \varepsilon &\rightarrow \gamma_- \varepsilon + \gamma_+ \bar{\varepsilon}, \\ \bar{\varepsilon} &\rightarrow \gamma_- \bar{\varepsilon} + \gamma_+ \varepsilon, \end{aligned} \tag{C.4}$$

and mapping the fields properly. The closure of the superconformal algebra is ensured by the fact that the right-hand sides are again conformal Killing spinors. In the case of the flat-space supersymmetry, the parameters can be chosen to be constant spinors of definite chirality, ε_\pm and $\bar{\varepsilon}_\pm$ with $\gamma_3 = \pm 1$, and the replacement is merely the relabeling of the parameters $\varepsilon_+ \rightarrow \bar{\varepsilon}_+$, $\bar{\varepsilon}_+ \rightarrow \varepsilon_+$; the distinction between twisted and untwisted multiplets is therefore a matter of convention. On S^2 , this procedure gives essentially different multiplets since one cannot choose the conformal Killing spinors ε_α (and hence also $\bar{\varepsilon}_\alpha = \gamma_3 \varepsilon_\alpha$) to have definite chirality.

The replacement (C.4) has the effect of exchanging $\alpha \leftrightarrow -\beta$ in the commutation relation (B.4). This is harmless for bosonic fields as $U(1)_V$ and $U(1)_A$ act in the same way on them, so this just switches the charges. However, to maintain the supersymmetry algebra on the fermions, we must map them in a similar fashion to (C.4) so that $U(1)_V$ and $U(1)_A$ are exchanged. We map the vector to twisted vector multiplet fields as

$$\begin{aligned} \sigma &\rightarrow \rho, \\ \lambda &\rightarrow \gamma_- \kappa + \gamma_+ \bar{\kappa}, \\ \bar{\lambda} &\rightarrow \gamma_- \bar{\kappa} + \gamma_+ \kappa, \\ D &\rightarrow C, \end{aligned} \tag{C.5}$$

and the chiral to twisted chiral multiplet fields as

$$\begin{aligned}
 \phi &\rightarrow v, \\
 \psi &\rightarrow \gamma_- \bar{\chi} + \gamma_+ \chi, \\
 \bar{\psi} &\rightarrow \gamma_- \chi + \gamma_+ \bar{\chi}, \\
 F &\rightarrow E.
 \end{aligned} \tag{C.6}$$

The latter mixes the representation R of the multiplets with its complex conjugate \bar{R} , so makes sense only if R is real; implicitly assumed is an isomorphism that exchanges $R \leftrightarrow \bar{R}$ and commutes with gauge transformations.

Applying the above prescription, we find that the supersymmetry transformation of a twisted vector multiplet is

$$\begin{aligned}
 \delta A_\mu &= -\frac{i}{2}(\bar{\varepsilon}\gamma_\mu \kappa + \varepsilon\gamma_\mu \bar{\kappa}), \\
 \delta \rho &= -\varepsilon\gamma_3 \kappa, \\
 \delta \bar{\rho} &= \bar{\varepsilon}\gamma_3 \bar{\kappa}, \\
 \delta \kappa &= i\left(F_{12}\gamma_3 - \frac{1}{2}[\rho, \bar{\rho}] - iC\gamma_3\right)\varepsilon - i\gamma_3 \not{D}\rho \bar{\varepsilon} - i\rho\gamma_3 \not{\nabla}\bar{\varepsilon}, \\
 \delta \bar{\kappa} &= i\left(F_{12}\gamma_3 + \frac{1}{2}[\rho, \bar{\rho}] + iC\gamma_3\right)\bar{\varepsilon} + i\gamma_3 \not{D}\bar{\rho} \varepsilon + i\bar{\rho}\gamma_3 \not{\nabla}\varepsilon, \\
 \delta C &= \frac{i}{2}\varepsilon(\gamma_3 \not{D}\bar{\kappa} - [\bar{\rho}, \kappa]) - \frac{i}{2}\bar{\kappa}\gamma_3 \not{\nabla}\varepsilon - \frac{i}{2}\bar{\varepsilon}(\gamma_3 \not{D}\kappa + [\rho, \bar{\kappa}]) + \frac{i}{2}\kappa\gamma_3 \not{\nabla}\bar{\varepsilon},
 \end{aligned} \tag{C.7}$$

and that of a twisted chiral multiplet is

$$\begin{aligned}
 \delta v &= \bar{\varepsilon}\gamma_+ \chi + \varepsilon\gamma_- \bar{\chi}, \\
 \delta \bar{v} &= \varepsilon\gamma_+ \bar{\chi} + \bar{\varepsilon}\gamma_- \chi, \\
 \delta \chi &= \gamma_+(i\not{D}v\varepsilon + i\rho v\bar{\varepsilon} + E\varepsilon) + \gamma_-(i\not{D}\bar{v}\bar{\varepsilon} - i\rho\bar{v}\bar{\varepsilon} + \bar{E}\bar{\varepsilon}) + i\frac{q}{2}(v\gamma_+ + \bar{v}\gamma_-)\not{\nabla}\varepsilon, \\
 \delta \bar{\chi} &= \gamma_-(i\not{D}v\bar{\varepsilon} + i\bar{\rho}v\varepsilon + E\bar{\varepsilon}) + \gamma_+(i\not{D}\bar{v}\varepsilon - i\bar{\rho}\bar{v}\varepsilon + \bar{E}\varepsilon) + i\frac{q}{2}(v\gamma_- + \bar{v}\gamma_+)\not{\nabla}\bar{\varepsilon}, \\
 \delta E &= i\varepsilon\gamma_+(\not{D}\bar{\chi} - \bar{\rho}\chi - \bar{\kappa}v) + i\bar{\varepsilon}\gamma_-(\not{D}\chi - \rho\bar{\chi} - \kappa v) + i\frac{q}{2}(\bar{\chi}\gamma_+ \not{\nabla}\varepsilon + \chi\gamma_- \not{\nabla}\bar{\varepsilon}), \\
 \delta \bar{E} &= i\bar{\varepsilon}\gamma_+(\not{D}\chi + \rho\bar{\chi} - \kappa\bar{v}) + i\varepsilon\gamma_-(\not{D}\bar{\chi} + \bar{\rho}\chi - \bar{\kappa}\bar{v}) + i\frac{q}{2}(\chi\gamma_+ \not{\nabla}\bar{\varepsilon} + \bar{\chi}\gamma_- \not{\nabla}\varepsilon).
 \end{aligned} \tag{C.8}$$

Notice that $(v, \gamma_+ \chi, \gamma_- \bar{\chi}, E)$ and $(\bar{v}, \gamma_- \chi, \gamma_+ \bar{\chi}, \bar{E})$ transform among themselves.

We face a difficulty when we try to construct an $\mathcal{N} = (2, 2)$ supersymmetric action $S_{\tilde{V}}$ for a twisted vector multiplet. The natural choice

$$(\varepsilon_\alpha \varepsilon_\beta) S_{\tilde{V}} = \bar{\mathcal{Q}}_\alpha \mathcal{Q}_\beta \int \sqrt{h} d^2x \text{Tr} \left(\bar{\kappa} \gamma_3 \kappa + \frac{2i}{r} \rho \bar{\rho} \right), \tag{C.9}$$

which is manifestly invariant under $\mathcal{N} = (2, 2)$ supersymmetry, does not work since its bosonic part

$$\int \sqrt{h} d^2x \text{Tr} \left(F_{12}^2 + D^\mu \bar{\rho} D_\mu \rho - \frac{1}{r^2} \bar{\rho} \rho + \frac{1}{4} [\rho, \bar{\rho}]^2 + C^2 \right) \tag{C.10}$$

is not bounded from below, plagued by the negative potential term $-\text{Tr} \bar{\rho} \rho / r^2$.

A sensible supersymmetric action $S_{\tilde{C}}$ can be constructed for twisted chiral multiplets. It is convenient to shift

$$E \rightarrow E - \frac{iq}{2r}v, \quad (\text{C.11})$$

which absorbs all the q -dependent terms from the supersymmetry transformation (C.8). Then the action can be written as

$$(\varepsilon_\alpha \varepsilon_\beta) S_{\tilde{C}} = \overline{\mathcal{Q}}_\alpha \mathcal{Q}_\beta \int \sqrt{h} d^2x \left(v \overline{E} - \bar{v} E + \frac{i}{r} \bar{v} v \right). \quad (\text{C.12})$$

Computing the supersymmetry variations we get

$$S_{\tilde{C}} = \int \sqrt{h} d^2x \left[\bar{v} \left(-D_\mu^2 + \frac{1}{2} \{ \rho, \bar{\rho} \} + iC \right) v + \overline{E} E \right. \\ \left. - i\chi \not{D} \bar{\chi} + i\chi \left(\gamma_+ \bar{\kappa} v + \gamma_- \bar{\kappa} \bar{v} + \frac{1}{2} \gamma_3 \bar{\rho} \chi \right) + i\bar{\chi} \left(\gamma_- \kappa v + \gamma_+ \kappa \bar{v} + \frac{1}{2} \gamma_3 \rho \bar{\chi} \right) \right]. \quad (\text{C.13})$$

For any gauge-invariant holomorphic function $\widetilde{W}(v)$ of twisted chiral multiplet scalars v^i , the twisted F-term

$$S_{\widetilde{W}} = i \int \sqrt{h} d^2x \left(E_{\widetilde{W}} + \overline{E}_{\widetilde{W}} + \frac{2}{r} \text{Im } \widetilde{W} \right) \quad (\text{C.14})$$

is invariant under $\mathcal{N} = (2, 2)$ supersymmetry. Here $E_{\widetilde{W}}$ is the auxiliary field of the twisted chiral multiplet $(\widetilde{W}, \chi_{\widetilde{W}}, \bar{\chi}_{\widetilde{W}}, E_{\widetilde{W}})$, given by

$$E_{\widetilde{W}} = E^i \partial_i \widetilde{W} - (\bar{\chi}^i \gamma_+ \chi^j) \partial_i \partial_j \widetilde{W}. \quad (\text{C.15})$$

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