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Ω-deformation and quantization

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Abstract: We formulate a deformation of Rozansky-Witten theory analogous to the Ω-deformation. It is applicable when the target space X is hyperkähler and the spacetime is of the form $\mathbb{R}\times\Sigma$, with Σ being a Riemann surface. In the case that Σ is a disk, the Ω-deformed Rozansky-Witten theory quantizes a symplectic submanifold of X, thereby providing a new perspective on quantization. As applications, we elucidate two phenomena in four-dimensional gauge theory from this point of view. One is a correspondence between the Ω-deformation and quantization of integrable systems. The other concerns supersymmetric loop operators and quantization of the algebra of holomorphic functions on a hyperkähler manifold.

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Contents

1 Introduction

The goal of this paper is to develop a new perspective on quantization, from which some intriguing phenomena in four-dimensional gauge theory may be naturally understood.

Specifically, the phenomena that we wish to understand are the following. At low energies, an $\mathcal{N} = 2$ supersymmetric gauge theory compactified on a circle S^1 is described by a three-dimensional $\mathcal{N} = 4$ supersymmetric sigma model [1]. The target space $\mathcal{M}$ of the sigma model is a hyperkähler manifold, which is moreover a complex integrable system in one of the complex structures [2]. On the one hand, it was discovered by Nekrasov and Shatashvili [3] that an Ω-deformation on a two-plane quantizes a real symplectic submanifold of the complex integrable system. On the other hand, it was found by Gaiotto, Moore and Neitzke [4] and Ito, Okuda and Taki [5] that if the spacetime $\mathbb{R}^3 \times S^1$ is replaced with a twisted product of $\mathbb{R}^3$ and S^1, then supersymmetric loop operators form a noncommutative deformation of the algebra of holomorphic functions on $\mathcal{M}$ in other complex structures.

Despite the similarities between the two phenomena, explanations from a unified point of view have been lacking. In this paper we provide such explanations, based on a connection that we establish between a deformation of $\mathcal{N} = 4$ supersymmetric sigma model and quantization of symplectic manifolds.

More precisely, the main result of the paper concerns the topologically twisted version of the sigma model, known as Rozansky-Witten theory [6]. We formulate it on a three-manifold of the form $\mathbb{R} \times \Sigma$, with Σ a Riemann surface, taking the target space to be a hyperkähler manifold X. Given a complex structure on X and a Killing vector field V on Σ, we construct a deformation of the theory analogous to the Ω-deformation in four dimensions. In the case that Σ is a disk D, we show that the Ω-deformed Rozansky-Witten theory is equivalent to a quantum mechanical system whose phase space is a symplectic submanifold of X determined by the boundary condition. The phenomena exhibited by the four-dimensional gauge theory then follow as special cases of this result, applied to low-energy effective sigma models with target space $X = \mathcal{M}$. The two cases differ merely in the choice of complex structure.

While the four-dimensional phenomena are explained with a three-dimensional theory, the construction of this theory is best understood from a two-dimensional point of view. To formulate the Ω-deformation for Rozansky-Witten theory on $\mathbb{R} \times \Sigma$ with target space X, we view the theory as a B-twisted Landau-Ginzburg model [7] on Σ whose target space is the space of maps from $\mathbb{R}$ to X. For this reason we first formulate the Ω-deformation for general B-twisted Landau-Ginzburg models, and then use this formulation to construct the Ω-deformed Rozansky-Witten theory.

The connection between the Ω-deformed Rozansky-Witten theory and quantization involves D-branes of novel type, which can be introduced supersymmetrically in B-twisted Landau-Ginzburg models only in the presence of the Ω-deformation. An amusing fact is that in general these branes are similar to A-branes, rather than B-branes. There is an even more interesting analogy if we specialize to the Ω-deformed Rozansky-Witten theory. In this case, relevant branes are much like (A, B, A)-branes in $\mathcal{N} = (4, 4)$ supersymmetric sigma models: the support of a brane is the space of maps from $\mathbb{R}$ to a submanifold L of X that is Lagrangian with respect to the Kähler forms ω_I and ω_K associated to two complex structures I and K of the hyperkähler structure of X, while holomorphic in the third complex structure J. This implies in particular that L is a symplectic manifold with symplectic form given by the restriction of ω_J.

For $\Sigma = D$ with a brane of this type placed on the boundary, by localization of the path integral we will derive a formula that expresses a correlation function in terms of an integral over the support of the brane. In the context of Rozansky-Witten theory, the localization formula gives an integration over maps from $\mathbb{R}$ to L. This is nothing but the path integral for a quantum mechanical system on L. We therefore conclude that the Ω-deformed Rozansky-Witten theory quantizes the symplectic submanifold (L, ω_J) of X. The algebra of observables is found to be a noncommutative deformation of the algebra of functions on L that are restrictions of holomorphic functions on X.

The appearance of (A, B, A)-like branes in our framework suggests a relation to another approach to quantization, namely the one using A-branes, developed by Gukov and Witten [8]. Indeed, the correspondence between Ω-deformed $\mathcal{N} = 2$ supersymmetric gauge theories and quantum integrable systems was explained by Nekrasov and Witten [9] from this perspective. (For explanations from other perspectives, see [10, 11].) Their argument, however, relies on the fact that the Ω-deformation may be canceled away from the origin

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of the two-plane by a redefinition of fields, and this makes the logic a little involved. Our approach hopefully renders the connection between the Ω-deformation and quantization more transparent.

In this paper we apply our framework to two specific problems in four-dimensional gauge theory. It will be interesting to find further applications. For example, the quantization of Seiberg-Witten curves proposed in [12] may find a natural place in the present framework. Besides, the framework itself may be generalized. One direction in this regard would be to consider a gauged version of Rozansky-Witten theory [13], which is obtained by topological twisting of an $\mathcal{N} = 4$ supersymmetric gauge theory constructed by Gaiotto and Witten [14].

Lastly, the Ω-deformation of B-twisted theories should have a broader range of applications. The construction can be extended to include gauge theories (the details of which will appear elsewhere), and this extension may shed light on the correspondence between $\mathcal{N} = 2$ superconformal theories in three dimensions and analytically continued Chern-Simons theory [15–19] via arguments along the lines of [20] (see also [21, 22]). Purely in two dimensions, it may prove fruitful to study mirror symmetry between Ω-deformed B-twisted theories and Ω-deformed A-twisted theories [15, 23].

The rest of the paper is organized as follows. In section 2, we formulate the Ω-deformation of B-twisted Landau-Ginzburg models and derive the localization formula. In section 3, we construct the Ω-deformed Rozansky-Witten theory and establish its connection to quantization. Section 4 discusses the applications to four-dimensional gauge theory. In the appendix we review the Ω-deformation of twisted $\mathcal{N} = 2$ supersymmetric gauge theories in four dimensions.

2 Ω-deformation of B-twisted Landau-Ginzburg models

In this section we formulate the Ω-deformation of B-twisted Landau-Ginzburg models in two dimensions, based on which the Ω-deformed Rozansky-Witten theory is constructed. Furthermore, we study boundary conditions in the presence of the Ω-deformation, and derive a localization formula for correlation functions on a disk. The results obtained here will be essential in our discussion in the next section.

2.1 Ω-deformed B-twisted Landau-Ginzburg models

Let us recall how the Ω-deformation in four dimensions works [24, 25]. A topologically twisted $\mathcal{N} = 2$ supersymmetric gauge theory [26] has a single scalar supercharge Q, satisfying the relation $Q^2 = 0$ modulo a gauge transformation (as well as a flavor symmetry transformation if hypermultiplet masses are nonzero [27, 28]). This is used as a BRST operator, meaning that physical operators and states are Q-cohomology classes. To introduce an Ω-deformation, one chooses a vector field V generating an isometry of the spacetime four-manifold with respect to a given metric. With this choice understood, the BRST operator of the Ω-deformed theory obeys the deformed relation

$$Q^2 = L_V, \tag{2.1}$$

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where L_V is the conserved charge that acts on fields as the Lie derivative $\mathcal{L}_V$ by V. The construction of the Ω-deformed theory is reviewed in the appendix.

Similarly, a B-twisted Landau-Ginzburg model in two dimensions [7] has a BRST operator Q satisfying $Q^2 = 0$ up to a central charge. In order to formulate an Ω-deformation of this theory, we should therefore pick a Killing vector field V on the worldsheet Σ equipped with a hermitian metric h, and deform the theory so that the modified BRST operator obeys the deformed relation (2.1). This is what we are aiming for.[1]

How can we achieve such a deformation? To get the idea, consider the simplest case $\Sigma = \mathbb{C}$. In this case the theory retains the full $\mathcal{N} = (2,2)$ supersymmetry in the twisted form, generated by two scalar supercharges $\overline{Q}_+$, $\overline{Q}_-$ and a one-form supercharge $G = G_z \mathrm{d}z + G_{\bar{z}} \mathrm{d}\bar{z}$. These satisfy $\{\overline{Q}_-, G_z\} = P_z$ and $\{\overline{Q}_+, G_{\bar{z}}\} = P_{\bar{z}}$, where $P = P_z \mathrm{d}z + P_{\bar{z}} \mathrm{d}\bar{z}$ is the generator of translations; the other commutators either vanish or give central charges. The BRST operator of the undeformed theory is $Q = \overline{Q}_+ + \overline{Q}_-$. If we take a vector field $V = V^z \partial_z + V^{\bar{z}} \partial_{\bar{z}}$ with constant components V^z, $V^{\bar{z}}$ and modify the BRST operator to $Q = \overline{Q}_+ + \overline{Q}_- + \iota_V G$, then we obtain the desired relation $Q^2 = \iota_V P$. We are going to generalize this construction to an arbitrary worldsheet Σ.

The target space of the theory is a Kähler manifold Y. Classically the following construction makes sense for any Kähler target space, but for quantum anomalies to be absent, Y has to be Calabi-Yau. (This is essentially due to the fact that the axial U(1) R-symmetry used in the B-twist is anomalous unless $c_1(Y) = 0$.) We denote the holomorphic and antiholomorphic tangent bundles of Y by TY and $\overline{TY}$, respectively, and their duals with superscript $\vee$. We pick a Kähler metric g on Y.

The bosonic field of the theory is a map $\Phi\colon \Sigma \to Y$. Given local coordinates on Y, we can express Φ locally by a set of functions $(\phi^i, \bar{\phi}^{\bar{\imath}})$. In the standard formulation, the fermionic fields of the B-twisted theory are scalars η with values in $\overline{TY}$ and θ with values in $T^\vee Y$, and a one-form ρ with values in TY. In our construction, we use instead of θ a two-form μ with values in $\overline{TY}$; the two are related by the Hodge duality and the isomorphism between $\overline{TY}$ and $T^\vee Y$ induced by g. Thus the fermionic fields of the theory are

$$\eta \in \Omega^0(\Sigma, \Phi^* \overline{TY}), \quad \rho \in \Omega^1(\Sigma, \Phi^* TY), \quad \mu \in \Omega^2(\Sigma, \Phi^* \overline{TY}). \tag{2.2}$$

We also introduce auxiliary bosonic fields. They are two-forms F with values in TY and $\overline{F}$ with values in $\overline{TY}$:

$$F \in \Omega^2(\Sigma, \Phi^* TY), \quad \overline{F} \in \Omega^2(\Sigma, \Phi^* \overline{TY}). \tag{2.3}$$

Starting from $\mathcal{N} = (2,2)$ supersymmetry transformation laws for B-twisted chiral multiplets [30], it is straightforward to write down the Ω-deformed supersymmetry transformation laws, following the same procedure as in the flat case described above. After

[1] In [29], a supergravity background was found that realizes the Ω-deformation of A-twisted theories on S^2. As mentioned in that paper, one can combine it with a $\mathbb{Z}_2$-action implementing mirror symmetry to obtain an Ω-deformation of B-twisted theories. Our formulation presumably reproduces their results for $\Sigma = S^2$. I would like to thank Stefano Cremonesi for explaining their work.

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shifting $\overline{F}$ to absorb dependence on the worldsheet metric, we have

$$\begin{aligned}
\delta\phi^i &= \iota_V\rho^i, & \delta\bar\phi^{\bar\imath} &= \eta^{\bar\imath},\\
\delta_\nabla\rho^i &= \mathrm{d}\phi^i + \iota_V F^i, & \delta_\nabla\eta^{\bar\imath} &= V(\bar\phi^{\bar\imath}),\\
\delta_\nabla F^i &= \mathrm{d}_\nabla\rho^i + \frac{1}{2}R^i{}_{j\bar kl}\eta^{\bar k}\rho^j\wedge\rho^l, & \delta_\nabla\mu^{\bar\imath} &= \overline{F}^{\bar\imath},\\
& & \delta_\nabla\overline{F}^{\bar\imath} &= \mathrm{d}_\nabla\iota_V\mu^{\bar\imath} + R^{\bar\imath}{}_{\bar\jmath k\bar l}\iota_V\rho^k\eta^{\bar l}\mu^{\bar\jmath}.
\end{aligned} \tag{2.4}$$

Here δ_∇ is the supersymmetry variation coupled to the pullback of the Levi-Civita connection ∇ of g; for example, $\delta_\nabla\rho^i = \delta\rho^i + \delta\phi^k\Gamma^i_{kj}\rho^j$. (More generally, ∇ can be any torsion-free connection on TY whose curvature form R is of type $(1,1)$ and obeys the first Bianchi identity.) Notice the similarity to the supersymmetry transformation laws (A.1) for Ω-deformed theories in four dimensions.

One can verify that the "raw" supersymmetry variation δ satisfies $\delta^2 = \mathcal{L}_V$.[2] As usual, we let Q denote the generator of the supersymmetry variation δ. Then it obeys the deformed relation (2.1), as desired. Strictly speaking, on the right-hand side of this relation may appear an extra conserved charge that commutes with fields and coincides for $V = 0$ with a central charge of the B-twisted $\mathcal{N} = 2$ supersymmetry algebra. Although such an extra term is important when one considers the action of Q on states, it plays no role in our discussion and hence will be ignored.

For the moment we assume that Σ has no boundary; we will discuss boundary effects shortly. Then the Q-invariant action S of the Ω-deformed theory consists of two pieces, $S = S_0 + S_W$. The first piece S_0 is Q-exact and contains kinetic terms. The second piece S_W is constructed from a superpotential $W\colon Y \to \mathbb{C}$, which is a holomorphic function of ϕ^i. Unlike S_0, this one is not Q-exact.

Concretely, we can take S_0 to be

$$S_0 = \delta\int_\Sigma \Big(g_{i\bar\jmath}\rho^i\wedge\star\big(\mathrm{d}\bar\phi^{\bar\jmath} + \iota_V\overline{F}^{\bar\jmath}\big) + g_{i\bar\jmath}F^i\wedge\star\mu^{\bar\jmath}\Big), \tag{2.5}$$

while S_W is given by

$$S_W = i\int_\Sigma\bigg(F^i\partial_i W + \frac{1}{2}\rho^i\wedge\rho^j\nabla_i\partial_j W + \delta\big(\mu^{\bar\imath}\partial_{\bar\imath}\overline{W}\big)\bigg). \tag{2.6}$$

The Q-invariance of the action relies on the assumption that V is a Killing vector field. This ensures that $\mathcal{L}_V$ commutes with $\star$, so if we write the Q-exact part of the Lagrangian as $\delta\mathcal{V}$, then $\delta^2\mathcal{V} = \mathcal{L}_V\mathcal{V} = \mathrm{d}\iota_V\mathcal{V}$ by the formula $\mathcal{L}_V = \mathrm{d}\iota_V + \iota_V\mathrm{d}$.

Computing the supersymmetry variation we find

$$\begin{aligned}
S_0 = \int_\Sigma\bigg(& g_{i\bar\jmath}\big(\mathrm{d}\phi^i + \iota_V F^i\big)\wedge\star\big(\mathrm{d}\bar\phi^{\bar\jmath} + \iota_V\overline{F}^{\bar\jmath}\big) + g_{i\bar\jmath}F^i\wedge\star\overline{F}^{\bar\jmath} - g_{i\bar\jmath}\rho^i\wedge\star\mathrm{d}_\nabla\eta^{\bar\jmath} + g_{i\bar\jmath}\mathrm{d}_\nabla\rho^i\wedge\star\mu^{\bar\jmath}\\
&+\frac{1}{2}R_{\bar\imath j\bar kl}\eta^{\bar k}\rho^j\wedge\rho^l\wedge\star\mu^{\bar\imath} - \rho^i\wedge\star\iota_V\big(g_{i\bar\jmath}\mathrm{d}_\nabla\iota_V\mu^{\bar\jmath} + R_{i\bar\jmath k\bar l}\iota_V\rho^k\eta^{\bar l}\mu^{\bar\jmath}\big)\bigg).
\end{aligned} \tag{2.7}$$

[2] An easy way to see this is to define $G^i = F^i - \frac{1}{2}\Gamma^i_{kj}\rho^k\wedge\rho^j$ and $\overline{G}^{\bar\imath} = \overline{F}^{\bar\imath} - \Gamma^{\bar\imath}_{\bar k\bar\jmath}\eta^{\bar k}\mu^{\bar\jmath}$, in terms of which one has $\delta\rho^i = \mathrm{d}\phi^i + \iota_V G^i$, $\delta G^i = \mathrm{d}\rho^i$ and $\delta\mu^{\bar\imath} = \overline{G}^{\bar\imath}$, $\delta\overline{G}^{\bar\imath} = \mathrm{d}\iota_V\mu^{\bar\imath}$.

For the superpotential terms we get

$$S_W = i\int_\Sigma \left(F^i\partial_i W + \frac{1}{2}\rho^i\wedge\rho^j\nabla_i\partial_j W + \overline{F}^{\bar{\imath}}\partial_{\bar{\imath}}\overline{W} + \eta^{\bar{\imath}}\mu^{\bar{\jmath}}\nabla_{\bar{\imath}}\partial_{\bar{\jmath}}\overline{W}\right). \tag{2.8}$$

Integrating out the auxiliary fields produces the potential

$$\star\,\|\mathrm{d}W\|^2 + \cdots = \star(g^{i\bar{\jmath}}\partial_i W\partial_{\bar{\jmath}}\overline{W}) + \cdots, \tag{2.9}$$

where we have abbreviated the terms involving V. One can add more Q-exact terms to the Lagrangian if one wishes, such as $\delta(g_{\bar{\imath}j}\eta^{\bar{\imath}}V(\phi^j))$ which produces the V-dependent potential $\|V(\phi)\|^2$.

When $V=0$, the supersymmetry transformation and the action constructed above reduce to those of the ordinary B-twisted Landau-Ginzburg model, up to the replacement of θ with μ explained above. This construction therefore defines a deformation of the latter theory for $V\neq 0$.

The worldsheet metric h appears in the action only through the Hodge duality within the Q-exact piece S_0. Hence, the Ω-deformed theory is quasi-topological, in the sense that it is invariant under deformations of the metric as long as V remains to be a Killing vector field. In addition, the theory is invariant under overall rescaling of the target space metric g, as this leaves the supersymmetry transformation invariant and the metric enters the action through S_0.

The observables of the theory are the Q-closed operators that are not Q-exact. At the zeros of V, any local observables of the undeformed theory remain to be observables. There is a class of local observables that are in one-to-one correspondence with the elements of the $\bar\partial$-cohomology $H^{0,\bullet}(Y;\mathbb{C})$. To see this, note that for $V=0$, the action of Q on ϕ^i, $\bar\phi^{\bar{\imath}}$ and η coincides with that of $\bar\partial$ under the identification of $\eta^{\bar{\imath}}$ with $\mathrm{d}\bar\phi^{\bar{\imath}}$. Thus, if $\omega = \omega_{\bar{\imath}_1\ldots\bar{\imath}_q}\mathrm{d}\bar\phi^{\bar{\imath}_1}\wedge\cdots\wedge\mathrm{d}\bar\phi^{\bar{\imath}_q}$ is a $\bar\partial$-closed $(0,q)$-form on Y, then $\omega_{\bar{\imath}_1\ldots\bar{\imath}_q}\eta^{\bar{\imath}_1}\cdots\eta^{\bar{\imath}_q}$ inserted at a zero of V is a Q-closed local operator, and represents a nonzero Q-cohomology class if and only if ω represents a nonzero element in $H^{0,\bullet}(Y;\mathbb{C})$. In particular, holomorphic functions on Y correspond to local observables.

2.2 Incorporating boundaries

So far we have assumed that Σ has no boundary. Now we consider the situation that Σ has a boundary and the isometry generated by V restricts to an isometry of $\partial\Sigma$. In this situation the Q-invariance of the action must be reexamined. Also, we have to ask what sort of boundary conditions are physically sensible. In the following we will express the Q-variation by Q-commutator, reserving δ for arbitrary variation of fields in order to avoid possible confusion.

Let us address the issue of Q-invariance. The Q-exact part of the action remains to be Q-invariant in the presence of boundary. For, if $\mathcal{V}$ is a two-form, then

$$\int_\Sigma [Q,\{Q,\mathcal{V}\}] = \int_\Sigma(\mathrm{d}\iota_V + \iota_V\mathrm{d})\mathcal{V} = \int_{\partial\Sigma}\iota_V\mathcal{V} = 0. \tag{2.10}$$

The last equality follows from the assumption that V generates an isometry of $\partial\Sigma$ and hence is tangent to $\partial\Sigma$. The potential problem therefore comes from the non-Q-exact part. Indeed, its Q-variation gives

$$i\int_{\partial\Sigma}\rho^i\partial_i W, \tag{2.11}$$

breaking the Q-invariance by a boundary contribution.

We must somehow eliminate this boundary contribution to recover the Q-invariance. In the case of ordinary B-twisted Landau-Ginzburg models, one can do this by imposing a B-brane boundary condition. This condition requires that Φ map $\partial\Sigma$ to a submanifold γ of Y,

$$\Phi(\partial\Sigma)\subset\gamma, \tag{2.12}$$

and $\mathrm{d}W|_\gamma=0$, that is, W be locally constant on γ. The boundary contribution vanishes then.

In the Ω-deformed case, there is another way of eliminating the boundary contribution. For simplicity, suppose that there is only one connected boundary component and it is compact. Let φ be a periodic coordinate on $\partial\Sigma$ such that $h_{\varphi\varphi}$ is constant. In this coordinate,

$$V|_{\partial\Sigma}=\varepsilon\partial_\varphi \tag{2.13}$$

for some real constant ε. Assuming that $\varepsilon\neq 0$, we can add to the action the boundary term

$$-\frac{i}{\varepsilon}\int_{\partial\Sigma}\mathrm{d}\varphi\,(W+W_0), \tag{2.14}$$

where W_0 is a locally constant function on γ. The Q-variation of this term cancels the boundary contribution in question, recovering the Q-invariance of the action.

One interpretation of the above boundary term is that it is the action for a theory living on the boundary, with ε being the Planck constant. The undeformed limit $\varepsilon\to 0$ is the classical limit, and in this limit Φ obeys the equation of motion $\mathrm{d}W=0$ on the boundary, which reproduces the ordinary B-brane condition on W.

This mechanism of recovering the Q-invariance is interesting since it is available only when the Ω-deformation is turned on. Moreover, it requires a weaker boundary condition on W compared to the B-brane condition. For the boundary term (2.14) to not spoil the convergence of the path integral, its real part had better be nonnegative. For our purposes it is sufficient to consider the situation that the boundary term is purely imaginary. To ensure this property, we place on the boundary a brane supported on γ, and impose

$$\operatorname{Im}\mathrm{d}W|_\gamma=0. \tag{2.15}$$

Then, the constant imaginary part of W can be absorbed into W_0, and the boundary term can be written as

$$-\frac{i}{\varepsilon}\int_{\partial\Sigma}\mathrm{d}\varphi\,(\operatorname{Re}W+W_0), \tag{2.16}$$

with W_0 now chosen to be real.

We would like to write down a set of boundary conditions that defines this brane. A guiding principle for determining physically sensible conditions is that in a weak coupling

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limit, solutions to equations of motion should be saddle point configurations of the path integral. In other words, when the fields are varied in that limit, boundary terms should not arise in the variation of the action. Recalling that in our case the bulk theory is invariant under rescaling of the target space metric g, we see that there is a natural weak coupling limit, namely the limit in which g is rescaled by a large factor. This is also the limit we will consider in the derivation of the localization formula for correlation functions.[3] We therefore define our brane as the boundary condition obtained by taking variations in this limit. (A similar choice was made in [31] where $\mathcal{N} = (2,2)$ supersymmetric theories on a hemisphere were studied.)

Since S_0 dominates in the limit under consideration, we can ignore the remaining part of the action in our analysis. Setting $\delta S_0 = 0$ and using the equations of motion for F and $\overline{F}$ derived from S_0, we find the constraints

$$g(\delta\Phi, \iota_V \star \mathrm{d}\Phi)|_{\partial\Sigma} = g\big([Q,\delta\Phi], (\iota_V \star \rho, \star\mu)\big)|_{\partial\Sigma} = 0. \tag{2.17}$$

The boundary condition for Φ implies that any variation of Φ is tangent to γ on $\partial\Sigma$, and we require that the same be true for the Q-variation $[Q,\Phi]$; thus we have

$$(\iota_V \rho, \eta) \in T_{\mathbb{C}}\gamma \tag{2.18}$$

at each point on $\partial\Sigma$. Assuming that the variation of Φ is not constrained in any other way, we conclude that

$$\iota_V \star \mathrm{d}\Phi \in N_{\mathbb{R}}\gamma, \quad (\iota_V \star \rho, \star\mu) \in N_{\mathbb{C}}\gamma, \tag{2.19}$$

where $N_{\mathbb{R}}\gamma$ is the normal bundle of γ and $N_{\mathbb{C}}\gamma$ is its complexification. In particular, Φ obeys the Neumann boundary condition in the direction normal to the boundary, just as in the case of ordinary B-branes (with vanishing B-field and Chan-Paton gauge field).

Furthermore, in order for Q to act on the space of allowed field configurations, the boundary condition itself must be invariant under the action of Q (or more precisely, the covariant version of it, coupled to the Levi-Civita connection). This leads to additional constraints generated by repeated action of Q on the constraints described above. An example is the constraint (2.18), which comes from the D-brane constraint $\Phi(\partial\Sigma) \subset \gamma$. This procedure generates only a few new constraints since $Q^2 = L_V$ leaves invariant the space of sections of a vector bundle over $\partial\Sigma$. It turns out that these additional constraints follow from the constraints discussed already if we use the equations of motion derived from the quadratic part of S_0.

The above boundary condition is independent of g, thanks to the limit considered here which decouples dependence on the superpotential. It is also independent of the component of the worldsheet metric h normal to $\partial\Sigma$. (If n is a coordinate in the normal direction, we have $\iota_V \star \mathrm{d}\Phi|_{\partial\Sigma} = \varepsilon\sqrt{h_{\varphi\varphi}/h_{nn}}(\partial_n\phi^i, \partial_n\bar{\phi}^{\bar{\imath}})$ and $(\iota_V \star \rho, \star\mu)|_{\partial\Sigma} = \sqrt{h_{\varphi\varphi}/h_{nn}}(\varepsilon\rho^i_n, h^{\varphi\varphi}\mu^{\bar{\imath}}_{n\varphi})$.) Hence, the invariance of the bulk theory under relevant deformations of the metrics is mostly preserved by the brane, broken only by the explicit dependence on $h_{\varphi\varphi}$.

[3] Actually we will consider a slightly different limit which simplifies the analysis, but the two limits lead to the same boundary condition.

2.3 Localization

As in ordinary A- and B-twisted theories, the path integral for a correlation function of Q-invariant operators in the Ω-deformed B-twisted Landau-Ginzburg model reduces to an integral over a small subspace of the field space. Let us derive a formula for correlation functions in the case that Σ is a disk D, with a brane of the above type placed on the boundary.

We equip D with the metric of the form $h = h_{rr}(r)\mathrm{d}r^2 + h_{\varphi\varphi}(r)\mathrm{d}\varphi^2$, where (r, φ) are polar coordinates. Then $V = \varepsilon\partial_\varphi$, with ε constant. Here ε is real, but it is also possible to make it complex since V only needs to satisfy the Killing equation.

Our theory is invariant under rescaling of the target space metric g. In particular, we can rescale it as $g \to t^2 g$ and take the limit $t \to \infty$. Integrating out the auxiliary fields, we find that in this limit the action diverges away from the locus where

$$\mathrm{d}\phi^i = 0. \tag{2.20}$$

The path integral therefore localizes to the constant maps, that is to say, receives contributions solely from an arbitrarily small neighborhood of the subspace of constant maps in the space of maps from D to Y.

Such a neighborhood may be thought of as a fibration over the space of constant maps. By the boundary condition, the constant maps are required to map into $\gamma \subset Y$, the support of the brane. Thus the base is isomorphic to γ. The fiber consists of the bosonic fluctuation φ around a constant map. We can extend the fiber so that it includes the fermionic fields as well. The path integral is then an integral over the total space of the extended fibration. What we want to do now is to perform the integration over the fiber, and reduce the path integral to an integral over the base.

It may be helpful to recall how the fiber integration is done in a simpler setting where $V = 0$ and Σ has no boundary. We combine η and μ into a single field $\zeta = -\eta + \mu$ which is an even-degree form with values in $\overline{TY}$. To quadratic order, the part of the action relevant for large t can be written as

$$t^2\big(\langle\varphi, \Delta\varphi\rangle + \langle(\mathrm{d}_\nabla + \mathrm{d}_\nabla^*)\rho, \zeta\rangle + \langle\rho, (\mathrm{d}_\nabla + \mathrm{d}_\nabla^*)\zeta\rangle\big). \tag{2.21}$$

Here $\Delta = (\mathrm{d}_\nabla + \mathrm{d}_\nabla^*)^2$, and $\langle\cdot,\cdot\rangle$ is an inner product defined by the metric on Σ and the original metric on Y before the rescaling. We expand φ, ρ and ζ in orthonormal bases of eigenmodes of Δ, and express the quadratic part of the action in terms of the expansion coefficients. The integration over the fiber is integration over these coefficients. We can rescale the coefficients by $1/t$ to absorb the overall t^2 factor in the quadratic part. Provided that there are no fermion zero modes, after doing so the terms of higher order are suppressed by inverse powers of t. In the limit $t \to \infty$, the fiber integration produces the ratio of the bosonic and fermionic one-loop determinants, $\prod_\beta \sqrt{\lambda'_\beta}/\prod_\alpha \lambda_\alpha$, where λ_α are nonzero eigenvalues for φ, and λ'_β are those for ρ and ζ. The nonzero eigenvalues for ρ and ζ agree since their nonzero modes are related by the action of $\mathrm{d}_\nabla + \mathrm{d}_\nabla^*$, which commutes with Δ. Similarly, as Δ and $\star$ commute, the nonzero modes for zero- and two-forms are related by the Hodge duality and have the same eigenvalues. This means that the set

$\{\lambda'_\beta\}$ consists of two copies of the set $\{\lambda_\alpha\}$. Hence, this ratio is equal to 1, and the fiber integration is trivial in this case.

We want to carry out a similar computation in the case at hand, where V generates rotations of $\Sigma = D$. Here the analysis is a bit more complicated.

One complication is that if $V \neq 0$, the action contains additional terms and they modify the quadratic part. To simplify the analysis, we replace the Q-exact piece S_0 of the action with

$$t^2\delta \int_\Sigma \Big(g_{i\bar{\jmath}}\rho^i \wedge \star\big(\mathrm{d}\bar{\phi}^{\bar{\jmath}} + \iota_V \overline{F}^{\bar{\jmath}}\big) + s g_{i\bar{\jmath}} F^i \wedge \star\mu^{\bar{\jmath}}\Big), \tag{2.22}$$

rescale $\mu \to \mu/s$, and take the limit $s \to \infty$. Integrating out the auxiliary fields and performing integration by parts using the boundary condition, we find that the relevant part of the on-shell action now takes the identical form (2.21) as in the case with $V = 0$.

Another complication comes from the presence of boundary, which makes the analysis of mode expansion more difficult. We can deal with this problem as follows. Using the freedom of deforming the worldsheet metric, we can choose D to have the shape of a sphere S^2 with a small hole; in spherical coordinates (ϑ, φ), the metric takes the form $h = R^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\varphi^2)$, with ϑ ranging from 0 to some value $\vartheta_{\partial D}$ where the boundary is located. (Our boundary condition depends on $h_{\varphi\varphi}$. It should be given with respect to the original metric and fixed throughout the deformation.) Then we take the limit $\vartheta_{\partial D} \to \pi$. In this limit D becomes the whole S^2, and the boundary state gets mapped to a Q-invariant local operator inserted at $\vartheta = \pi$. If we now expand the fields in the eigenmodes of Δ on S^2, then in terms of the expansion coefficients the quadratic part of the action has the same expression as before.

The conclusion is therefore that the fiber integration is again exact at one loop and produces a factor similar to the ratio of the bosonic and fermionic determinants, assuming that there are no fermion zero modes. The difference is that this time the path integral receives contributions only from the locus where the expansion coefficients obey various relations, imposed by the boundary state or the operator inserted at $\vartheta = \pi$.

The question is whether this one-loop factor depends on the background constant map Φ_0 around which we are expanding. If it does, the dependence should come from g, W or γ, since these are the only objects defined on the target space that enter our setup. The one-loop computation refers to just the quadratic part of the action in the limit $t \to \infty$, and this is independent of W. It is also independent of g if we use holomorphic normal coordinates centered at Φ_0, in which $g_{i\bar{\jmath}}(\Phi_0) = \delta_{ij}$ and $\partial_k g_{i\bar{\jmath}}(\Phi_0) = \partial_{\bar{k}} g_{i\bar{\jmath}}(\Phi_0) = 0$. In fact, in these coordinates the quadratic part takes exactly the same form as the action for an affine target space $\mathbb{C}^n$ with the standard metric.

This leaves γ as the only possible source for nontrivial dependence on the background. Indeed, the choice of γ may introduce such dependence, since it specifies the boundary condition which in turn determines the relations among the mode expansion coefficients. Put another way, the one-loop factor is independent of the background if we can choose γ in such a way that the boundary condition is the same for all backgrounds. In view of the fact that the boundary condition in a background $\Phi_0 \in \gamma$ is determined by the tangent and normal spaces of γ at Φ_0, a sufficient condition for background independence is that the tangent spaces (and hence also the normal spaces) at any two points of γ can be made identical, when regarded as subspaces of $\mathbb{C}^n$ via a suitable choice of normal coordinate systems centered at these points. Taking into account the freedom in choosing normal coordinates, we see that the two spaces need to be identical up to an action of $\mathrm{U}(n)$.

One way to satisfy this condition is to take γ to be a complex submanifold of Y, as in the case for ordinary B-branes. In this case the superpotential W restricts to a holomorphic function on γ. As we require $\mathrm{Im}\, \mathrm{d}W|_\gamma = 0$, W would then have to be locally constant on γ. This is in fact the ordinary B-brane condition on W. However, it is not desirable for our purposes. We would like to view W as the Lagrangian of a boundary theory, from which the equation $\mathrm{d}W = 0$ follows as a classical equation of motion.

A more interesting possibility is to take γ to be a Lagrangian submanifold of Y with respect to the Kähler form. The condition for background independence is then satisfied since $\mathrm{U}(n)$ acts transitively on the Lagrangian Grassmannian $\mathrm{U}(n)/\mathrm{O}(n)$, the space of Lagrangian subspaces of $\mathbb{R}^{2n}$. From now on we will consider this kind of supports.

Finally, we have to make sure that the assumption of absence of fermion zero modes is actually true. Since the result of the path integral is independent of the size of the S^2, we will show this in the limit where the S^2 is very small. First of all, there are no harmonic one-forms on S^2 and hence no zero modes for ρ. The zero modes of η are constants, while those of μ are their Hodge duals. For these modes, we have to look at the constraints imposed by the boundary condition. In the limit we are considering, the nonzero modes are very massive and decouple, so the fermions can be replaced by their zero mode parts. Then the boundary condition forces $(0, \eta) \in T_{\mathbb{C}}\gamma$ and $(0, \star\mu) \in N_{\mathbb{C}}\gamma$. Since γ is a Lagrangian submanifold of a Kähler manifold, we have $I(T_{\mathbb{R}}\gamma) = N_{\mathbb{R}}\gamma$ and it follows that $\eta = \mu = 0$ on the boundary. Hence, the zero modes of η and μ are actually identically zero in this limit.

We have found that the fiber integration just produces an irrelevant constant. The remaining step in the path integral is to integrate over all background constant maps. For a constant map Φ_0, the action is evaluated as

$$S(\Phi_0) = -\frac{2\pi i}{\varepsilon}\big(\mathrm{Re}\, W + W_0\big)(\Phi_0). \tag{2.23}$$

Altogether, we conclude that the path integral for the Ω-deformed B-twisted Landau-Ginzburg model on a disk reduces to the form

$$\langle \mathcal{O} \rangle = \int_\gamma \mathrm{d}\Phi_0 \exp\left(\frac{2\pi i}{\varepsilon}\big(\mathrm{Re}\, W + W_0\big)(\Phi_0)\right) \mathcal{O}(\Phi_0), \tag{2.24}$$

where the operator insertion on the right-hand side is evaluated for constant maps $\Phi_0 \in \gamma$, with fermions set to zero. In this expression we have renormalized W_0 to absorb the one-loop factor.

3 Quantization via the Ω-deformed Rozansky-Witten theory

Having constructed Ω-deformed B-twisted Landau-Ginzburg models, we now move up one dimension higher and formulate the Ω-deformation of Rozansky-Witten theory in three dimensions. We will then establish, via localization of the path integral, the connection between the Ω-deformed Rozansky-Witten theory and quantization of symplectic submanifolds of the hyperkähler target space.

3.1 Rozansky-Witten theory

To begin, let us review the basic aspects of Rozansky-Witten theory. We refer the reader to the original paper [6] for more details.

Rozansky-Witten theory is a three-dimensional supersymmetric sigma model which can be defined on a general three-manifold M. The target space of the theory is a complex symplectic manifold (X,Ω). It is a complex manifold X equipped with a nondegenerate closed holomorphic two-form Ω, called a holomorphic symplectic form.

Let $\Phi\colon M\to X$ be the bosonic map of the sigma model, and write $(\phi^i,\bar\phi^{\bar\imath})$ for a local expression of Φ. The theory has two fermionic fields, η and χ, and is invariant under the following supersymmetry:

$$\begin{aligned} \delta\phi^i &= 0, & \delta\bar\phi^{\bar\imath} &= \eta^{\bar\imath}, \\ \delta\chi^i &= \mathrm{d}\phi^i, & \delta\eta^{\bar\imath} &= 0. \end{aligned} \tag{3.1}$$

As can be seen from the transformation laws, η is a scalar on M with values in $\overline{TX}$, and χ is a one-form on M with values in TX. The generator Q of the supersymmetry satisfies $Q^2=0$. We use it as a BRST operator, declaring that physical operators and states are Q-cohomology classes.

To construct a Q-invariant action we need to make some choices. We pick a Riemannian metric h on M and a hermitian metric g on X. In addition, we choose a torsion-free connection $\nabla=\mathrm{d}+\Gamma$ on TX, with connection matrices $\Gamma^i_{kj}=\Gamma^i_{jk}$. The $(1,1)$-part of the curvature form of ∇ (given by the matrix elements $R^i{}_{jk\bar l}\mathrm{d}\phi^k\wedge\mathrm{d}\bar\phi^{\bar l}$ with $R^i{}_{j\bar lk}=\partial_{\bar l}\Gamma^i_{kj}$) represents a $\bar\partial$-cohomology class, known as the Atiyah class of X. It is the obstruction to the existence of a holomorphic connection on TX.

The action is the sum of two pieces, $S=S_1+S_2$. The first piece is Q-exact:

$$S_1=\delta\int_M g_{i\bar\jmath}\chi^i\wedge\star\mathrm{d}\bar\phi^{\bar\jmath}=\int_M\bigl(g_{i\bar\jmath}\mathrm{d}\phi^i\wedge\star\mathrm{d}\bar\phi^{\bar\jmath}-g_{i\bar\jmath}\chi^i\wedge\star\mathrm{d}_{\widetilde{\nabla}}\eta^{\bar\jmath}\bigr). \tag{3.2}$$

Here the connection $\widetilde\nabla=\mathrm{d}+\widetilde\Gamma$ is defined by $(\widetilde\Gamma_{\bar k})^{\bar\imath}{}_{\bar\jmath}=g^{\bar\imath l}\partial_{\bar\jmath}g_{l\bar k}$; if g is Kähler, $\widetilde\nabla$ is the Levi-Civita connection of g. The second piece is Q-invariant, but not Q-exact:

$$S_2=-\frac{i}{4}\int_M\left(\Omega_{ij}\chi^i\wedge\mathrm{d}_\nabla\chi^j-\frac13\Omega_{ij}R^j{}_{kl\bar m}\chi^i\wedge\chi^k\wedge\chi^l\eta^{\bar m}+\frac13\nabla_k\Omega_{ij}\mathrm{d}\phi^i\wedge\chi^j\wedge\chi^k\right). \tag{3.3}$$

The particular normalization is chosen for later convenience.

Since the spacetime metric h appears only in the Q-exact part S_1 through the Hodge duality, the theory is topological. Likewise, the target space metric g appears only in S_1, so the theory is independent of the choice of g. It turns out that the theory is also independent of the choice of the connection ∇, different choices leading to the same expression for S_2 modulo Q-exact terms.

The local observables of Rozansky-Witten theory are in one-to-one correspondence with the $\bar\partial$-cohomology classes of X under the identification of $\eta^{\bar\imath}$ with $\mathrm{d}\bar\phi^{\bar\imath}$. There are also

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nonlocal observables. Given a connection A of type $(1,0)$ on any holomorphic G-bundle $E \to X$, we define a Q-invariant connection

$$\mathcal{A} = A_i \mathrm{d}\phi^i + F_{i\bar{j}} \chi^i \eta^{\bar{j}}, \tag{3.4}$$

where F is the curvature of A. Using this connection we can construct a Q-invariant loop operator

$$\mathrm{Tr}\, P \exp\left(\oint_{\mathcal{C}} \mathcal{A}\right), \tag{3.5}$$

by taking the trace of the holonomy of $\mathcal{A}$ along a closed path $\mathcal{C}$ in M.

A special case of interest is when X admits a hyperkähler structure (g, I, J, K). In this case, X has a two-sphere $\mathbb{CP}^1$ of complex structures, and g is Kähler with respect to all of them. Elements of the $\mathbb{CP}^1$ are linear combinations $aI + bJ + cK$, with $a^2 + b^2 + c^2 = 1$ and I, J, K satisfying the quaternion relations

$$I^2 = J^2 = K^2 = IJK = -1. \tag{3.6}$$

If we write ω_J and ω_K for the Kähler forms associated to J and K, respectively, then

$$\Omega_I = \omega_J + i\omega_K \tag{3.7}$$

is a holomorphic symplectic form in complex structure I. Thus we can regard X as a complex symplectic manifold with complex symplectic structure (I, Ω_I), and construct Rozansky-Witten theory, choosing ∇ to be the Levi-Civita connection of g.

Of course, which complex structure to call I is just a matter of convention, and any other complex structure in the $\mathbb{CP}^1$ gives an equally good target space. In other words, there is a family of target spaces parametrized by the $\mathbb{CP}^1$. We can continuously change the theory by moving within this family. Equivalently, we may fix the target space and vary the BRST operator. In the hyperkähler case the theory has a second supercharge $\overline{Q}$ which generates the transformations

$$\begin{aligned} \bar{\delta}\phi^i &= \Omega^{ij} g_{j\bar{k}} \eta^{\bar{k}}, & \bar{\delta}\bar{\phi}^{\bar{\imath}} &= 0, \\ \bar{\delta}\chi^i &= -\Omega^{ij} g_{j\bar{k}} \mathrm{d}\bar{\phi}^{\bar{k}} - \Gamma^i_{kj} \Omega^{kl} g_{l\bar{m}} \eta^{\bar{m}} \chi^j, & \bar{\delta}\eta^{\bar{\imath}} &= 0, \end{aligned} \tag{3.8}$$

where $\Omega^{ij}\Omega_{jk} = \delta^i_k$. As $\overline{Q}$ squares to zero and commutes with Q, any linear combination $Q_\zeta \propto Q + \zeta\overline{Q}$ with $\zeta \in \mathbb{CP}^1$ serves as a BRST operator. (The Q-exact part S_1 of the action is also Q_ζ-exact since $g_{i\bar{j}}\chi^i \wedge \star \mathrm{d}\bar{\phi}^{\bar{j}} = \bar{\delta}(\Omega_{ij}\chi^i \wedge \star\chi^j/2)$ and $\delta\bar{\delta} = (\delta + \zeta\bar{\delta})\bar{\delta}$. Thus, the topological invariance of the theory remains to hold.) We see that for $\zeta \neq 0$, Q_ζ annihilates holomorphic functions on X in a complex structure different from I, so varying the BRST operator indeed amounts to changing the complex structure of the target space.

3.2 Ω-deformation of Rozansky-Witten theory

Now let us formulate the Ω-deformation of Rozansky-Witten theory. Our goal is the following. Consider Rozansky-Witten theory on $M = \mathbb{R} \times \Sigma$, equipped with a product metric $h = h_{\mathbb{R}} \oplus h_\Sigma$. Assume that the target space X is a hyperkähler manifold with hyperkähler metric g. Given a vector field V generating an isometry of Σ, we wish to construct a deformation of this theory such that it has a supercharge Q obeying the deformed relation (2.1).

There are a couple of indications that such a deformation does exist. One is that, as we will explain in section 4.1, Rozansky-Witten theory with hyperkähler target space arises naturally from $\mathcal{N}=2$ supersymmetric gauge theories in four dimensions by compactification on S^1. If we turn on an Ω-deformation in four dimensions, some deformation should be induced in three dimensions as well. Another indication is that reduction of Rozansky-Witten theory on S^1 gives the B-model with the same target space [32]. (More generally, if the target space is not hyperkähler, the dimensional reduction yields a generalization of the B-model.) As we could construct the Ω-deformation in two dimensions, it is natural to expect that there is a corresponding deformation in three dimensions. We take the second observation as a starting point of our construction.

Our strategy is to describe the Ω-deformed Rozansky-Witten theory on $\mathbb{R}\times\Sigma$ with target space X as an Ω-deformed B-twisted Landau-Ginzburg model on Σ with target space $Y=\mathrm{Map}(\mathbb{R},X)$, the space of maps from $\mathbb{R}$ to X. In order to specify the latter theory, we need to pick a complex structure on Y. Such a complex structure is naturally induced from a complex structure chosen on X. This construction therefore singles out a distinguished element in the $\mathbb{CP}^1$ of complex structures on X. We call it I.

Roughly speaking, having $\mathrm{Map}(\mathbb{R},X)$ as the target space means that we should regard a coordinate t of the $\mathbb{R}$ as a continuous index, putting it on the same footing as coordinate indices of X. Hence, the formula for the action (2.5) now contains an integration over t in addition to summation over the other indices:

$$S_0=\delta\int_{\mathbb{R}\times\Sigma}\sqrt{h_{\mathbb{R}}}\mathrm{d}t\wedge\Big(g_{i\bar{j}}\rho^i\wedge\star_\Sigma\big(\mathrm{d}_\Sigma\bar{\phi}^{\bar{j}}+\iota_V\star_\Sigma\overline{F}^{\bar{j}}\big)+g_{i\bar{j}}F^i\wedge\star_\Sigma\mu^{\bar{j}}\Big). \tag{3.9}$$

Here $\star_\Sigma$ and d_Σ denote the Hodge star operator and the exterior derivative (coupled to the Levi-Civita connection) on Σ. The supersymmetry transformation laws take the same form (2.4) as before, the only difference being that the fields have dependence on t.

The above action lacks terms involving t-derivatives, which are crucial for fully three-dimensional dynamics. These missing terms are to be provided by a superpotential W. In the present context, W is a holomorphic function on $\mathrm{Map}(\mathbb{R},X)$ that respects locality, namely a holomorphic functional of a map from $\mathbb{R}$ to X.

To construct a suitable superpotential, we complete the complex structure I into a triple (I,J,K) compatible with the hyperkähler structure, and set $\Omega=\Omega_I$. Since Ω is a closed holomorphic two-form, we can locally write $\Omega=\mathrm{d}\Lambda$ with some holomorphic one-form Λ. Using this form we define W by

$$W(\Phi)=\frac{1}{2}\int_{\mathbb{R}}\Phi^*\Lambda. \tag{3.10}$$

In this formula we have abused the notation and let Φ denote a point on the target space $\mathrm{Map}(\mathbb{R},X)$, as is customary in the finite-dimensional setting. With this choice of W, the superpotential terms (2.6) are given by

$$S_W=\frac{i}{2}\int_{\mathbb{R}\times\Sigma}\Big(\Omega_{ij}F^i\mathrm{d}_{\mathbb{R}}\phi^j-\frac{1}{2}\Omega_{ij}\rho^i\wedge\mathrm{d}_{\mathbb{R}}\rho^j+\overline{\Omega}_{\bar{i}\bar{j}}\overline{F}^{\bar{i}}\mathrm{d}_{\mathbb{R}}\bar{\phi}^{\bar{j}}+\overline{\Omega}_{\bar{i}\bar{j}}\eta^{\bar{i}}\mathrm{d}_{\mathbb{R}}\mu^{\bar{j}}\Big). \tag{3.11}$$

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When $V=0$, the theory described by the action S_0+S_W reduces on-shell to Rozansky-Witten theory. Integrating out the auxiliary fields sets

$$\sqrt{h_{\mathbb{R}}}\star_\Sigma F^i=-\frac{i}{2}g^{i\bar{\jmath}}\overline{\Omega}_{\bar{\jmath}\bar{k}}\partial_t\bar{\phi}^{\bar{k}},\quad \sqrt{h_{\mathbb{R}}}\star_\Sigma \overline{F}^{\bar{\imath}}=-\frac{i}{2}g^{\bar{\imath}j}\Omega_{jk}\partial_t\phi^k. \tag{3.12}$$

From the expression of $\overline{F}^{\bar{\imath}}$, one readily sees that for $V=0$, the supersymmetry transformation laws (2.4) coincide with the formula (3.1) under the identification

$$\rho^i_\mu=\chi^i_\mu,\quad \sqrt{h_{\mathbb{R}}}\star_\Sigma \mu^{\bar{\imath}}=-\frac{i}{2}g^{\bar{\imath}j}\Omega_{jk}\chi^k_t. \tag{3.13}$$

One can also check that the action coincides with the Rozansky-Witten action under this identification.[4] For example, the F-term potential is

$$\|\delta W\|^2=\frac{1}{4}\int_{\mathbb{R}}\mathrm{d}t\sqrt{h_{\mathbb{R}}}h^{tt}g^{i\bar{\jmath}}\Omega_{ik}\partial_t\phi^k\overline{\Omega}_{\bar{\jmath}\bar{l}}\partial_t\bar{\phi}^{\bar{l}}=\int_{\mathbb{R}}\mathrm{d}t\sqrt{h_{\mathbb{R}}}g_{i\bar{\jmath}}h^{tt}\partial_t\phi^i\partial_t\bar{\phi}^{\bar{\jmath}}, \tag{3.14}$$

and this is precisely the kinetic term for the bosonic field along the t-direction. (Matching of the fermionic terms is straightforward.) Thus, the theory constructed above provides a good definition for the Ω-deformation of Rozansky-Witten theory on $\mathbb{R}\times\Sigma$ with target space X and complex symplectic structure (I,Ω_I).

There is a slight generalization of this construction. It is possible to modify the definition of W by terms that vanish for $V=0$. Locality requires that this is done through a deformation of Λ by a locally-defined holomorphic one-form on X, which in turn gives rise to a deformation of Ω by the equation $\Omega=\mathrm{d}\Lambda$. The latter is what really matters as far as the effect on the theory is concerned. After a deformation of this type is included, generically Ω would remain nondegenerate and define a deformed complex symplectic structure. However, it may no longer be a holomorphic symplectic form associated to some complex structure in a hyperkähler structure. This generalization will not be considered in what follows, except that we will briefly discuss its relevance in applications to $\mathcal{N}=2$ supersymmetric gauge theories in four dimensions.

3.3 Reduction to quantum mechanics

Finally we are ready to present the main result of this paper. Suppose that Σ is a disk D and V generates its rotations. In this situation we can apply the localization formula for correlation functions obtained in the previous section. Using the formula, we show that this system is equivalent to a quantum mechanical system on a real symplectic submanifold of X.

First of all, we have to specify the support γ of the brane placed on the boundary of D. We recall that γ is a Lagrangian submanifold of $Y=\mathrm{Map}(\mathbb{R},X)$, and $\operatorname{Im}W$ must be locally constant on γ. The first condition suggests that we should take $\gamma=\mathrm{Map}(\mathbb{R},L)$, with L being a Lagrangian submanifold of X with respect to the Kähler form ω_I. The second condition says that we must have

$$\delta\operatorname{Im}W=\frac{1}{2}\int_{\mathbb{R}}\operatorname{Im}\Omega_{ij}\delta\phi^i\mathrm{d}\phi^j=0 \tag{3.15}$$

[4] In checking this, one uses the identity $g^{i\bar{\jmath}}\Omega_{ik}\overline{\Omega}_{\bar{\jmath}\bar{l}}=4g_{k\bar{l}}$, which follows from the equation $\Omega=\omega_J+i\omega_K=-g(J+iK)$, and the fact that Ω is covariantly constant, which in particular implies $\Omega_{ik}R^k{}_{jl\bar{m}}+\Omega_{kj}R^k{}_{il\bar{m}}=0$.

on the boundary. This is satisfied if L is Lagrangian with respect to $\operatorname{Im}\Omega = \omega_K$. Then, both I and K give an isomorphism between $T_{\mathbb{R}}L$ and $N_{\mathbb{R}}L$, and $J = KI$ is an endomorphism of $T_{\mathbb{R}}L$. Hence, L is a Lagrangian submanifold with respect to ω_I and ω_K, while a complex submanifold in J. In particular, it is a symplectic manifold with symplectic form $\operatorname{Re}\Omega = \omega_J$.

We also need to specify the locally constant function W_0 on γ, which is part of the boundary term. For this one, we introduce a U(1) connection A on a line bundle over L and set

$$W_0 = \frac{1}{2}\int_{\mathbb{R}} \Phi^* A. \tag{3.16}$$

For W_0 to be locally constant, A must be flat.

For simplicity, let us assume for a moment that Ω is an exact form so that a holomorphic one-form Λ satisfying $\Omega = \mathrm{d}\Lambda$ exists globally on X. Then W is given by the integral (3.10), and the localization formula (2.24) reads

$$\langle \mathcal{O} \rangle = \int_{\mathrm{Map}(\mathbb{R},L)} \mathcal{D}\Phi_0 \exp\left(\frac{i}{\hbar} S(\Phi_0)\right) \mathcal{O}(\Phi_0), \tag{3.17}$$

where the action S and the Planck constant $\hbar$ are given by

$$S(\Phi_0) = \int_{\mathbb{R}} \Phi_0^*(\operatorname{Re}\Lambda + A), \quad \hbar = \frac{\varepsilon}{\pi}. \tag{3.18}$$

The right-hand side of the formula is the path integral for a quantum mechanical system with phase space $(L, \operatorname{Re}\Omega)$; in local Darboux coordinates (p_a, q^a), $a = 1, \cdots, \frac{1}{2}\dim L$ such that $\operatorname{Re}\Omega|_L = \mathrm{d}p_a \wedge \mathrm{d}q^a$, we have $\operatorname{Re}\Lambda|_L + A = p_a \mathrm{d}q^a$ up to an exact form, so the Lagrangian is $p_a \dot{q}^a$ up to a total derivative. Therefore, this system quantizes the symplectic manifold $(L, \operatorname{Re}\Omega)$. Notice that the Hamiltonian of the system is zero, as is consistent with the fact that we started with a (quasi-)topological field theory.

Observables of the Ω-deformed Rozansky-Witten theory include local operators that may be regarded as elements of the $\bar\partial$-cohomology $H^{0,\bullet}(X;\mathbb{C})$ in complex structure I, inserted at the origin of D and arbitrary points on the $\mathbb{R}$. Among these observables, those that are nonvanishing after the localization are holomorphic functions on X. The path integral turns these observables into operators in the quantum mechanical system which form a noncommutative algebra. They act on the Hilbert space whose elements are, say, functions of q^a locally. What we have obtained is thus a noncommutative deformation of the algebra of functions on L that are restrictions of holomorphic functions on X, acting on the space of sections of a hermitian line bundle over L.

It is worth noting that the localization formula derived here is very similar to one for an $\mathcal{N} = 4$ supersymmetric sigma model on $\mathbb{R} \times S^2$, constructed from twisted chiral multiplets of $\mathcal{N} = (2,2)$ supersymmetry on S^2 [33]. In that case, the action contains the term

$$\frac{2i}{r}\int_{S^2} \operatorname{Re} W, \tag{3.19}$$

where r is the radius of the S^2 [34]. This term corresponds to our boundary term (2.16), and eventually becomes the action of a quantum mechanical system. The localization again requires the bosonic field to be constant, so the above term is multiplied by the area of the S^2 and the Planck constant is proportional to $1/r$.

Let us discuss what changes have to be made if we remove the assumption that Ω is exact. In this case Λ can exist only locally, so the formula for W is not well-defined. This is not a problem when we work with a spacetime with no boundary, since what enters the action then is not W itself, but the derivative of W, which can be expressed in terms of Ω. However, it does cause a problem in the present setup where W enters the boundary term in the action.

A better definition for W is the following. First, in each homotopy class $\mathcal{P}$ of $\mathrm{Map}(\mathbb{R},X)$ we fix a reference map $\Phi_{\mathcal{P}}$, so that any map $\Phi \in \mathcal{P}$ can be deformed to $\Phi_{\mathcal{P}}$ without altering the behavior at infinity. Next, given a map $\Phi \in \mathcal{P}$, we pick a homotopy $\widehat{\Phi}\colon [0,1] \to \mathrm{Map}(\mathbb{R},X)$ from $\Phi_{\mathcal{P}}$ to Φ; thus $\widehat{\Phi}(0) = \Phi_{\mathcal{P}}$ and $\widehat{\Phi}(1) = \Phi$. Finally, we define

$$W(\Phi) = \frac{1}{2}\int_{[0,1]\times\mathbb{R}} \widehat{\Phi}^*\Omega, \tag{3.20}$$

viewing $\widehat{\Phi}$ as a map from $[0,1]\times\mathbb{R}$ to X. Changing the reference maps $\Phi_{\mathcal{P}}$ shifts W by a locally constant function, but such a shift can be absorbed in the definition of W_0. If Ω is exact, W is given as before by the integral of a holomorphic one-form Λ such that $\Omega = \mathrm{d}\Lambda$. Since the functional derivative of W can be computed locally, and locally we can always write Ω as $\Omega = \mathrm{d}\Lambda$, this definition of W leads to the same superpotential terms (3.11).

The function W so defined is actually not single-valued on $\mathrm{Map}(\mathbb{R},X)$. If we choose a different homotopy $\widehat{\Phi}'$, then the two homotopies combine into a map $\Delta\widehat{\Phi}\colon S^1\times\mathbb{R}\to X$, and W changes by

$$\Delta W = \frac{1}{2}\int_{S^1\times\mathbb{R}} \Delta\widehat{\Phi}^*\Omega. \tag{3.21}$$

By assumption the homotopies leave the behavior at infinity intact, so $\Delta\widehat{\Phi}$ maps each end of the cylinder $S^1\times\mathbb{R}$ to a point. As such, $\Delta\widehat{\Phi}$ may be thought of as really a map from a two-sphere to X. For the path integral with the boundary term (2.16) to be well-defined, the change in the boundary term must be always an integer multiple of $2\pi i$. This requirement places the constraint

$$\frac{1}{2\pi\hbar}[\mathrm{Re}\,\Omega] \in H^2(L;\mathbb{Z}). \tag{3.22}$$

This is nothing but the quantization condition for $(L, \mathrm{Re}\,\Omega)$.

Now we summarize what we have found. Let (X,g,I,J,K) be a hyperkähler manifold. Pick a submanifold L of X that is Lagrangian with respect to ω_I and ω_K, and holomorphic in J. Then, the Ω-deformed Rozansky-Witten theory with target space X in complex symplectic structure (I,Ω_I), formulated on $\mathbb{R}\times D$ with boundary condition specified by a brane supported on L, is equivalent to a quantum mechanical system whose phase space is the symplectic manifold (L,ω_J). The Planck constant is proportional to the Ω-deformation parameter ε.

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3.4 Comparison with the A-model approach

In [8], Gukov and Witten developed a framework for quantization of symplectic manifolds using branes in the A-model. It is illuminating to compare their approach with ours.

In the A-model approach, one first embeds the symplectic manifold (L, ω) that one wants to quantize into a complex symplectic manifold (X, Ω) of twice the dimension such that $\operatorname{Re}\Omega|_L = \omega$ and $\operatorname{Im}\Omega|_L = 0$. (We are using the same symbols as in our approach to emphasize parallels.) One then considers the A-model whose target space is the symplectic manifold $(X, \operatorname{Im}\Omega)$. Taking the worldsheet to be a strip, one places two types of A-branes on its sides. On one side is a Lagrangian A-brane supported on L and endowed with a complex line bundle with a flat U(1) connection A. On the other is a canonical coisotropic A-brane [35] whose support is the whole X, endowed with a line bundle with a connection of curvature $\operatorname{Re}\Omega$. It turns out that this system quantizes (L, ω) just like our system does: the open strings with both ends attached on the canonical coisotropic brane form a noncommutative deformation of the algebra of holomorphic functions on X, whereas the strings stretched between the two branes span a Hilbert space on which the deformed algebra acts.

A particularly nice situation is when X admits a hyperkähler structure such that $\Omega = \Omega_I$ and L is a complex submanifold in complex structure J, since in this case one can study the B-model of complex structure J and describe the Hilbert space explicitly. Since $\omega_K = \operatorname{Im}\Omega$ vanishes on L, so does $\omega_I = -\omega_K J$ then. Thus L is a Lagrangian submanifold with respect to ω_I and ω_K, and a complex submanifold in complex structure J; the branes are of type (A, B, A). Interestingly, this is precisely the property required of the support of a brane used in our approach. In our case we have a single brane on the boundary of D, and it may be thought of as playing the role of a combination of the two branes in the A-model approach. For example, $\operatorname{Re} W$ and W_0 correspond to the gauge fields on the space-filling and middle-dimensional branes, respectively, as can be seen from their expressions (3.10) and (3.16).

In view of these similarities, it may be reasonable to expect that if we equip D with a cigar metric and regard D as an S^1-fibration over an interval, our system reduces to the A-brane system at low energies. Nekrasov and Witten [9] showed that this is indeed the case under certain circumstances. We will discuss this point briefly in section 4.3

4 Applications to four-dimensional gauge theory

In the final section we discuss applications of our framework to $\mathcal{N} = 2$ supersymmetric gauge theories in four dimensions. For each of these theories, there is a class of physical quantities captured by Rozansky-Witten theory with hyperkähler target space. Using this fact and the results obtained in the previous section, we establish connections between the gauge theory and quantization of objects associated with the target space.

4.1 Rozansky-Witten theory from four dimensions

First we clarify the relation between $\mathcal{N} = 2$ supersymmetric gauge theories and Rozansky-Witten theory, and explain some important features of the geometry of the emergent target spaces. For more details, see [1, 36].

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Consider an $\mathcal{N} = 2$ supersymmetric gauge theory, compactified on S^1. On the Coulomb branch, the theory is described at low energies by a three-dimensional abelian gauge theory with $\mathcal{N} = 4$ supersymmetry. Dualizing the gauge fields to periodic scalars, we get a sigma model. Its target space $\mathcal{M}$ is required to be hyperkähler by the $\mathcal{N} = 4$ supersymmetry, and has dimension $4r$, where r is the rank of the gauge group. For a large class of theories obtained by compactification of M5-branes on punctured Riemann surfaces, $\mathcal{M}$ is the Hitchin moduli space of the relevant surface [36, 37].

If we instead start from the topologically twisted version of the same theory, then the resulting sigma model is twisted as well. Placing the ultraviolet theory on $M \times S^1$, we get Rozansky-Witten theory on M with target space $\mathcal{M}$. Recall that when the target space is hyperkähler, Rozansky-Witten theory has two supercharges Q and $\overline{Q}$. The first one exists in the more general case of complex symplectic target spaces, but the second does not. From the four-dimensional viewpoint, Q is the scalar supercharge of the twisted theory, while $\overline{Q}$ is the component of the one-form supercharge along the S^1. Any linear combination $Q_\zeta \propto Q + \zeta\overline{Q}$ with $\zeta \in \mathbb{CP}^1$ may be used as a BRST operator. This corresponds to the fact that $\mathcal{M}$ has a $\mathbb{CP}^1$-worth of complex structures J_ζ in which Rozansky-Witten theory can be formulated. We write Ω_ζ for the holomorphic symplectic form associated to J_ζ.

The above effective description is valid at length scales that are much larger than the radius of the S^1 so that the theory looks effectively three-dimensional, but much smaller compared to the size of M so that the effects of the curvature of M are negligible on the massive modes that are integrated out. This requirement can always be met by rescaling of the spacetime metric which leaves physical quantities unaffected. This is clearly true for $\zeta = 0$, that is when the BRST operator is Q, in which case the twisted theory is well-known to be a topological field theory [26]. It is also true for $\zeta \neq 0$. The reason is that correlation functions of Q_ζ-invariant operators on $M \times S^1$ are supersymmetric indices and protected under deformations of the parameters of the theory.[5]

The geometry of $\mathcal{M}$ is very interesting, in that there is a distinguished complex structure in which $\mathcal{M}$ is the phase space of a complex integrable system [2]. This is the complex structure J_0, and usually called I. In this complex structure, $\mathcal{M}$ is a torus fibration over a complex manifold $\mathcal{B}$ whose fibers are complex Lagrangian submanifolds with respect to the holomorphic symplectic form Ω_I. The base $\mathcal{B}$ is the Coulomb moduli space of the ultraviolet theory placed on $\mathbb{R}^4$; it is topologically an affine space $\mathbb{C}^r$, parametrized by the vacuum expectation values of the gauge-invariant polynomials in the vector multiplet scalar. The torus fibers are parametrized by the holonomies of the infrared abelian gauge fields and their magnetic duals around the S^1.

There are particularly nice coordinates on $\mathcal{M}$ in this context. $\mathcal{N} = 2$ supersymmetry requires that $\mathcal{B}$ admits local holomorphic coordinates a^i, $i = 1, \cdots, r$, and their duals $a_{D,i}$ related through a holomorphic function $\mathcal{F}$ as $a_{D,i} = \partial\mathcal{F}/\partial a^i$. The second derivatives $\tau_{ij} = \partial^2\mathcal{F}/\partial a^i\partial a^j = \partial a_{D,i}/\partial a^j$ encode the complexified gauge couplings of the effective abelian gauge theory. If we write the electric and magnetic holonomies as $\exp(i\theta_e^i)$ and

[5]We assume that the Q_ζ-invariant states form a discrete spectrum, which should be the case if M is compact. Although the choice $M = \mathbb{R} \times D$ that we will consider is not compact, we can replace the $\mathbb{R}$ by a finite interval without altering the conclusions.

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$\exp(i\theta_{m,i})$, then $z_i = \theta_{m,i} - \tau_{ij}\theta_e^j$ are holomorphic coordinates on the torus fiber of $\mathcal{M}$. Moreover, (a^i, z_i) are complex Darboux coordinates:[6]

$$\Omega_I = \mathrm{d}a^i \wedge \mathrm{d}z_i. \tag{4.1}$$

The a^i are conserved charges generating translations in the fiber directions, and commute with one another with respect to the Poisson bracket derived from Ω_I. There are r such charges in the phase space $\mathcal{M}$ of complex dimension $2r$, reflecting the fact that the system is completely integrable in the complex sense.

4.2 Quantization by twisting of the spacetime

Now we wish to modify the ultraviolet theory in such a way that the effective theory undergoes an Ω-deformation. For $\zeta \neq 0, \infty$, we can achieve this by replacing the spacetime $M \times S^1$ with a twisted product between M and S^1, which is a nontrivial M-fibration over S^1. More specifically, we take the trivial fibration $M \times [0,1]$, and identify the fibers at the two ends of the interval $[0,1]$ with the action of an isometry of M. Writing this isometry as $\exp(V)$ with V a Killing vector field, we denote the resulting fibration by $M \times_V S^1$.

Since Q_ζ are scalars on M, it commutes with isometries on M and correlation functions of Q_ζ-invariant operators on $M \times_V S^1$ are still protected indices. As such, they may be computed by the effective sigma model. We have $\{Q, \overline{Q}\} \propto P_4$ and hence $Q_\zeta^2 \propto P_4$ for $\zeta \neq 0, \infty$ (up to a central charge), where P_4 acts on fields by ∂_4. Due to the isometry twist, at low energies P_4 is replaced by L_V,[7] leading to the deformed relation $Q_\zeta^2 \propto L_V$ in three dimensions. We thus identify the effective theory for $\zeta \neq 0, \infty$ with Rozansky-Witten theory subject to the Ω-deformation by a (complex) Killing vector field proportional to V.

Taking $M = \mathbb{R} \times D$ and V to generate rotations of the disk D, we conclude that a twisted $\mathcal{N} = 2$ supersymmetric gauge theory on the corresponding fibration quantizes a symplectic submanifold $(L, \mathrm{Re}\,\Omega_\zeta)$ of $\mathcal{M}$, where L is the support of the brane on the boundary of D.

It is interesting to consider Q_ζ-invariant operators. In three dimensions, relevant operators are holomorphic functions on $\mathcal{M}$ in complex structure J_ζ, inserted at the center of D and points on the $\mathbb{R}$. They form a noncommutative deformation of the algebra of holomorphic functions on $\mathcal{M}$. In four dimensions, these local observables may be represented by line operators wrapped on the S^1. This explains the observation made in [4, 5] that supersymmetric loop operators realize a deformation quantization of the algebra of holomorphic functions on $\mathcal{M}$.

We remark that due to the twisting of the spacetime, there may be corrections to the holomorphic symplectic form Ω_ζ when it appears in the Ω-deformed Rozansky-Witten theory and hence in the quantum mechanical system; see the comment at the end of section 3.2 for this point.

[6] Holomorphic objects in complex structure I, especially the structure of complex integrable system, do not receive instanton corrections coming from BPS particles circling around the S^1. This is because the action for such particles is not holomorphic, but rather the absolute value of a holomorphic function [1].

[7] To see this, one can use coordinates $(y^\mu, y^4) = (\exp(x^4 V)x^\mu, x^4)$, where x^μ and x^4 are coordinates on M and S^1, respectively. In these coordinates the fibration is "untwisted," $(y^\mu, 0) \sim (y^\mu, 1)$, and at low energies we simply have $\partial/\partial y^4 = \partial/\partial x^4 - V = 0$ on functions.

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4.3 Quantization by the Ω-deformation in four dimensions

The above argument does not apply when $\zeta = 0$ or ∞. For $\zeta = 0$, there is a more obvious way to induce an Ω-deformation in the effective Rozansky-Witten theory. That is to turn on an Ω-deformation in the ultraviolet.

What we find in this case is that a twisted $\mathcal{N} = 2$ supersymmetric gauge theory on $\mathbb{R} \times D \times S^1$, subject to the Ω-deformation by a rotation generator of D, quantizes a real symplectic submanifold $(L, \operatorname{Re}\Omega_I)$ of the complex integrable system $(\mathcal{M}, \Omega_I)$. The commuting Hamiltonians of the quantum integrable system are the operators corresponding to the special coordinates a^i. In the ultraviolet theory, they are realized by the gauge-invariant polynomials in the vector multiplet scalar inserted at the origin of D.

Since the twisted theory is topological and its Hamiltonian is identically zero, the states of the quantum mechanical system correspond to the vacua of the gauge theory. Let us find where the vacua are located. To be concrete, we take L to be the locus given locally by the equations

$$\operatorname{Im} a_{D,i} = \theta_{m,i} = 0. \tag{4.2}$$

This is a good choice; we may choose a hyperkähler metric such that $\omega_I|_L = 0$ (for example, the semiflat metric obtained by dimensional reduction of the effective abelian gauge theory), while $\Omega = \mathrm{d}a^i \wedge \mathrm{d}\theta_{m,i} - \mathrm{d}a_{D,i} \wedge \mathrm{d}\theta_e^i$ and hence $\operatorname{Im}\Omega_K|_L = 0$, showing that L is a Lagrangian submanifold with respect to ω_I and $\omega_K = \operatorname{Im}\Omega$ as required. L is a real integrable system over the real submanifold of $\mathcal{B}$ parametrized by $\operatorname{Re} a_{D,i}$, with the torus fiber parametrized by θ_e^i. The Lagrangian of the quantum mechanical system is $-\operatorname{Re} a_{D,i}\mathrm{d}\theta_e^i$. Integrating over the periodic scalars θ_e^i imposes the constraints $\operatorname{Re} a_{D,i}/\hbar \in \mathbb{Z}$. Combining these constraints with the equations $\operatorname{Im} a_{D,i} = 0$, we obtain

$$\exp\left(\frac{2\pi i}{\hbar} a_{D,i}\right) = 1. \tag{4.3}$$

These are the equations that determine the locations of the vacua.

The above equations are to be identified with the Bethe equations in the integrable system [3]. Let $\mathcal{F}$ be the (ε-corrected) prepotential of the gauge theory, and $\widetilde{W} = 2\pi i\mathcal{F}/\hbar$. Then we can rewrite the equations as

$$\exp\left(\frac{\partial \widetilde{W}}{\partial a^i}\right) = 1. \tag{4.4}$$

If the radius of S^1 is much larger than $1/\varepsilon$, then there is a low-energy regime in which the Ω-deformed theory is effectively described by a two-dimensional theory with $\mathcal{N} = (2,2)$ supersymmetry. The function $\widetilde{W}$ may be interpreted as the twisted superpotential for this effective theory on $\mathbb{R} \times S^1$. In the quantum integrable system, it is interpreted as the Yang-Yang function.

In the work of Nekrasov and Witten [9], the quantization of the integrable system was explained in the A-brane framework disscussed in section 3.4. The starting point of their approach is the same as ours, namely the twisted theory on $\mathbb{R} \times D \times S^1$, subject to the Ω-deformation on D. One puts a cigar metric on D and thinks of it as an S^1-fibration over an interval. One then reduces the theory to a two-dimensional theory on a strip. It turns out that away from the tip of the cigar, the effect of the Ω-deformation can be canceled by a redefinition of fields. One makes use of this observation and deduces that the two-dimensional theory is an $\mathcal{N} = (4,4)$ supersymmetric sigma model with target space $\mathcal{M}$, with a space-filling (A, B, A)-brane placed on one side of the strip and a middle-dimensional (A, B, A)-brane placed on the other. This configuration fits in the A-brane framework, and one concludes that it quantizes a real integrable system which is the support of the middle-dimensional brane. Here we have presented an alternative derivation based on the framework developed in this paper. The relation between the two approaches should be clear from the discussion in section 3.4.

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A Ω-deformation of twisted $\mathcal{N} = 2$ supersymmetric gauge theories

In this appendix we review the Ω-deformation of topologically twisted $\mathcal{N} = 2$ supersymmetric gauge theories in four dimensions [24, 25]. We only consider the case of pure super Yang-Mills theory, constructed from a vector multiplet. The bosonic fields of the twisted theory are a gauge field A, a complex scalar ϕ, and an auxiliary self-dual two-form H. The fermionic fields are a zero-form η, a one-form ψ, and a self-dual two-form χ. The spacetime can be any four-manifold M that admits an isometry.

To introduce an Ω-deformation, one lifts the theory to a six-dimensional gauge theory, formulated on a nontrivial M-fibration over a two-torus T^2 such that the fiber is acted upon by isometries as one goes around cycles in the base. Killing vector fields generating the isometries are assumed to commute with each other. One then dimensionally reduces the lifted theory down to four dimensions. Formally this procedure has the effect of replacing ϕ and its hermitian conjugate $\bar{\phi}$ by differential operators as $\phi \to \phi + V^\mu D_\mu$ and $\bar{\phi} \to \bar{\phi} + \overline{V}^\mu D_\mu$, where V is a linear combination of the Killing vector fields and $\overline{V}$ is its complex conjugate.

The Ω-deformed supersymmetry transformation laws are

$$\begin{aligned}
\delta A &= \psi, \\
\delta\phi &= \iota_V \psi, \\
\delta\bar{\phi} &= \eta + \iota_{\overline{V}}\psi, \\
\delta\eta &= i[\phi, \bar{\phi}] - \iota_{\overline{V}} \mathrm{d}_A \phi + \iota_V \mathrm{d}_A \bar{\phi} + \iota_V \iota_{\overline{V}} F_A, \\
\delta\psi &= \mathrm{d}_A \phi + \iota_V F_A, \\
\delta\chi &= iH, \\
\delta H &= [\phi, \chi] - i\mathcal{L}_V \chi.
\end{aligned} \tag{A.1}$$

Here $\mathrm{d}_A = \mathrm{d} - iA$ is the exterior derivative coupled to A, and F_A is the curvature of A. The above transformation preserves the self-duality of H since the (gauge-covariant) Lie derivative $\mathcal{L}_V = \mathrm{d}_A\iota_V + \iota_V\mathrm{d}_A$ commutes with the Hodge duality if V is a Killing vector field. The supersymmetry algebra closes provided that V and $\overline{V}$ commute. The generator Q of the supersymmetry satisfies $Q^2 = L_V$ modulo a gauge transformation, where L_V is the conserved charge that acts on fields as $\mathcal{L}_V$.

The action of the Ω-deformed theory is

$$S = \frac{\operatorname{Im}\tau}{4\pi}\delta\int_M \operatorname{Tr}\left(\frac{1}{2}(-\delta\chi + 4F_A^+)\wedge\star\chi + \overline{\delta\psi}\wedge\star\psi + \frac{1}{2}\overline{\delta\eta}\wedge\star\eta\right) + \frac{i\tau}{4\pi}\int_M \operatorname{Tr} F_A\wedge F_A, \quad \text{(A.2)}$$

where $\tau = \theta/2\pi + 4\pi i/e^2$ is the complexified gauge coupling and F_A^+ is the self-dual part of F_A. Integrating out the auxiliary field, we find that the bosonic part of the action is given by

$$\frac{\operatorname{Im}\tau}{4\pi}\int_M \operatorname{Tr}\Big(F_A\wedge\star F_A + (\mathrm{d}_A\bar\phi + \iota_{\overline{V}}F_A)\wedge\star(\mathrm{d}_A\phi + \iota_V F_A)$$
$$+\frac{1}{2}\big([\phi,\bar\phi] + i\iota_{\overline{V}}\mathrm{d}_A\phi - i\iota_V\mathrm{d}_A\bar\phi - i\iota_V\iota_{\overline{V}}F_A\big)^2\Big) + \frac{i\operatorname{Re}\tau}{4\pi}\int_M \operatorname{Tr} F_A\wedge F_A. \quad \text{(A.3)}$$

When $V = 0$, this reduces to the bosonic part of the standard $\mathcal{N} = 2$ super Yang-Mills action.

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