

Branes and integrable lattice models

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Published 29 December 2016

This is a brief review of my work on the correspondence between four-dimensional $\mathcal{N} = 1$ supersymmetric field theories realized by brane tilings and two-dimensional integrable lattice models. I explain how to construct integrable lattice models from extended operators in partially topological quantum field theories, and elucidate the correspondence as an application of this construction.

Keywords: Branes; integrable lattice models; topological quantum field theories.

PACS Nos.: 11.25.-w, 02.30.Ik, 05.50.+q

1. Introduction

In supersymmetric field theories, exact computations are often possible for a limited class of physical quantities. Supersymmetric indices are primary examples of such quantities, and have been extensively studied in connection with gauge theory dualities, holography and other interesting phenomena.

Around 2010, it was discovered that supersymmetric indices of certain four-dimensional $\mathcal{N} = 1$ supersymmetric gauge theories coincide with the partition functions of two-dimensional integrable lattice models in statistical mechanics.^{1–3} As was later recognized,⁴ these gauge theories are realized by particular configurations of branes in string theory, called brane tilings.^{5,6} The lattice models in question are known as the Bazhanov–Sergeev models^{1,3} and have continuous spin variables. The Yang–Baxter equations that guarantee the integrability of the models are integral identities obeyed by the elliptic gamma function. On the gauge theory side, they translate to the invariance of the indices under Seiberg duality.⁷

This paper provides a concise review of the main results of Ref. 8 where the above correspondence was elucidated from the perspective of topological quantum field theories (TQFTs). Also discussed is the role played by surface defects, which was partly understood in Ref. 9. I hope that this review will serve as an introduction to the beautiful yet largely unexplored connections between branes, supersymmetric field theories, TQFTs and integrable lattice models.

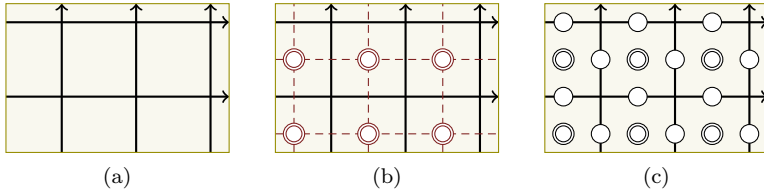


Fig. 1. Construction of a lattice model from line operators. (a) A lattice of line operators on a torus. (b) The torus with holes obtained by gluing of pieces. (c) The corresponding spin model.

2. TQFTs with Extra Dimensions and Integrable Lattice Models

Underlying the correspondence between brane tilings and integrable lattice models is a general method to construct such models from extended operators in partially topological quantum field theories. To begin with, I present this construction at a formal level. The essential idea of it was developed by Costello.^{10,11}

2.1. Lattice models from line operators in two-dimensional TQFTs

Suppose that we have a two-dimensional TQFT \mathbb{T} equipped with line operators. Place this theory on a torus T^2 , and wrap line operators \mathcal{L}_i , $i = 1, \dots, l$ around closed curves C_i in such a way that they form an $m \times n$ lattice. Figure 1(a) illustrates the case $(m, n) = (2, 3)$. We wish to compute the correlation function for this configuration of line operators on the torus.

Our strategy is to break up the torus into small pieces and first perform the path integral piecewise. Then we combine the results from these pieces and reconstruct the original correlation function.^a

Consider the following piece of surface containing an intersection of two line operators \mathcal{L}_i and \mathcal{L}_j :



This picture represents the worldsheet of two scattering open strings, each carrying a particle whose worldline is one of the line operators. The end points of the strings sweep out the double-lined arcs. The strings are attached to D-branes there, and subject to boundary conditions which are specified by labels a, b, c, d (“Chan–Paton factors”). We denote the set of boundary conditions by B .

The path integral for the above surface produces a linear map

$$\tilde{R}_{ij} \begin{pmatrix} a & d \\ b & c \end{pmatrix} : V_{ab,i} \otimes V_{bc,j} \rightarrow V_{ad,j} \otimes V_{dc,i}, \quad (2)$$

^aIn the axiomatic language, we will compute the correlation function by embedding the closed TQFT with line operators into an open/closed TQFT with line operators.

where $V_{ab,i}$ is the space of states on an interval intersected by \mathcal{L}_i , with the boundary conditions on the left and the right ends being a and b , respectively. We call this map the *R-matrix* (or *R-operator*) associated with this decorated surface.

To reconstruct the lattice on the whole torus, we glue pieces similar to the above together. Gluing amounts to composing the corresponding *R*-matrices. For example, gluing two pieces horizontally gives

$$\begin{array}{c} a \quad d \quad e \\ \leftarrow \quad \rightarrow \quad \rightarrow \\ i \quad \quad \quad j \quad k \quad f \\ b \quad c \quad f \end{array} = \check{R}_{ik} \begin{pmatrix} d & e \\ c & f \end{pmatrix} \circ_{V_{ac,i}} \check{R}_{ij} \begin{pmatrix} a & d \\ b & c \end{pmatrix}. \quad (3)$$

The torus thus obtained by gluing, however, has holes in it and looks as in Fig. 1(b). On the boundaries of these holes are imposed various boundary conditions, specified by labels a , b , etc. We must fill these holes.

This is achieved by summation over the boundary conditions. The path integral on a finite-length cylinder, with boundary condition a imposed on one end, defines on the other end a closed string state $|a\rangle$, called a boundary state. Similarly, the path integral on a disk with no insertion of operators defines a state $|1\rangle$ on the boundary. Assume that we have chosen the set B to be sufficiently large so that $|1\rangle$ can be written as a superposition of boundary states:

$$|1\rangle = \sum_{a \in B} c_a |a\rangle. \quad (4)$$

Then, summing over the boundary conditions we get $|1\rangle$ on the boundary of each hole, which may in turn be replaced with a disk:

$$\sum_{a \in B} c_a \left(a \text{ (hole) } \right) = \sum_{a \in B} c_a \left(|a\rangle \text{ (hole) } \right) = |1\rangle \text{ (hole) } = \text{ (disk) }. \quad (5)$$

The holes are filled and disappear from the torus, as desired.

Having understood how to reconstruct the correlation function on the torus, let us interpret this procedure as an operation in statistical mechanics. To this end, choose a basis for the open string state space $V_{ab,i}$ for each a , b and i . According to what we have just found, the procedure for computing the correlation function consists of three steps: First, pick a basis state for every side of the pieces comprising the torus and a boundary condition for every hole in the torus. Second, take the product of the corresponding *R*-matrix elements from all pieces as well as the coefficients c_a associated with the boundary conditions from all holes. Finally, sum over all possible assignments of basis states and boundary conditions.

The first step may be alternatively thought of as assigning basis states to the circles \bigcirc and boundary conditions to the double-lined circles \odot on the torus shown in Fig. 1(c). Rephrased in this way, it is clear that the above procedure defines the partition function of a spin model. The model has spins located at two kinds of

sites, \circ and \odot . A spin at \circ takes values in the chosen basis for the relevant open string state space, while that at \odot is valued in B . The Boltzmann weights for their interactions are determined by the R -matrix elements and the coefficients c_a .

Thus, we conclude that the correlation function for a lattice of line operators coincides with the partition function of a spin model defined on the same lattice:

$$\left\langle \prod_{i=1}^l \mathcal{L}_i(C_i) \right\rangle_{\mathbb{T}, T^2} = Z_{\mathbb{L}(\mathbb{T}), \{\mathcal{L}_i(C_i)\}}. \quad (6)$$

Here, $\mathbb{L}(\mathbb{T})$ denotes the lattice model arising from the TQFT \mathbb{T} by this construction and $\{\mathcal{L}_i(C_i)\}$ is the lattice formed by line operators \mathcal{L}_i wrapped around C_i .

If B consists of a single boundary condition a , we simply write V_i for $V_{aa,i}$ and represent the R -matrix $\check{R}_{ij}: V_i \otimes V_j \rightarrow V_j \otimes V_i$ by a crossing of two lines:

$$\check{R}_{ij} = i \begin{array}{c} \uparrow \\ \text{---} \text{---} \text{---} \\ \downarrow \\ j \end{array}. \quad (7)$$

In this case, we can ignore the spins at \odot since there is no summation for them. This means that $\mathbb{L}(\mathbb{T})$ is a *vertex model*: the spins live on the edges of the lattice and interact at the vertices. We may think of V_i as a vector space carried by \mathcal{L}_i .

If $\dim V_{ab,i} = 1$ for all a, b and i , we only sum over the boundary conditions instead. In this case, the spins at \circ can be ignored and $\mathbb{L}(\mathbb{T})$ is an *interaction-round-a-face model* (or *IRF model* for short): the spins are placed on the faces and the interaction takes place among four spins surrounding a vertex.

Formally, we can always recast our lattice model into a vertex model by setting $V_i = \bigoplus_{a,b \in B} V_{ab,i}$ and declaring that all newly introduced R -matrix elements, which correspond to scattering processes with inconsistent Chan–Paton factors, vanish. We can also absorb the coefficients c_a into the R -matrix elements by appropriate rescaling. In what follows this reformulation is implicitly performed.

2.2. Integrability from extra dimensions

A remarkable aspect of this construction of lattice models is that it allows us to understand integrability from a higher-dimensional point of view. This is the crucial observation by Costello.^{10,11}

In our lattice model, consider a row where a horizontal line operator \mathcal{L}_i intersects the vertical line operators \mathcal{L}_j , $j = 1, \dots, n$. Concatenating the R -matrices in this row, we get the row-to-row *transfer matrix*

$$T_i = i \begin{array}{c} \uparrow \quad \uparrow \quad \dots \quad \uparrow \\ \text{---} \text{---} \text{---} \text{---} \text{---} \\ \downarrow \quad \downarrow \quad \dots \quad \downarrow \\ 1 \quad 2 \quad \dots \quad n \end{array} = \text{Tr}_{V_i}(\check{R}_{in} \circ_{V_i} \dots \circ_{V_i} \check{R}_{i1}). \quad (8)$$

(The hooks on the horizontal line are to remind us that the periodic boundary condition is imposed.) This object is an endomorphism of $\bigotimes_{j=1}^n V_j$ which maps

a state just below \mathcal{L}_i to another state just above it. In terms of transfer matrices the partition function is written as a trace:

$$Z_{\mathbf{L}(\mathbf{T}), \{\mathcal{L}_i(C_i)\}} = \text{Tr}(T_{n+m} \cdots T_{n+1}). \quad (9)$$

In the TQFT context, T_i may be regarded as a time-evolution operator induced by \mathcal{L}_i , acting on the Hilbert space $\bigotimes_{j=1}^n V_j$. Since the theory is topological, a state evolves trivially unless it hits something — line operators in the present case.

Now, suppose that each line operator depends on a parameter which is an element of some set S . This parameter is called the *spectral parameter* of the lattice model. We denote the spectral parameter of \mathcal{L}_i by u_i . Thus, \check{R}_{ij} is a function of two parameters u_i, u_j , whereas T_i carries $n+1$ parameters u_1, \dots, u_n and u_i . To avoid clutter, we fix u_1, \dots, u_n and suppress them below.

A vertex model is said to be *integrable* if $T_i(u_i)$ is a smooth function of u_i (hence S is a smooth manifold), and moreover the relation

$$\begin{array}{c} j \\ \updownarrow \\ \updownarrow \\ \updownarrow \\ \dots \\ \updownarrow \\ \updownarrow \\ i \end{array} = \begin{array}{c} i \\ \updownarrow \\ \updownarrow \\ \updownarrow \\ \dots \\ \updownarrow \\ \updownarrow \\ j \end{array} \iff [T_i(u_i), T_j(u_j)] = 0 \quad (10)$$

holds for $u_i \neq u_j$. When the model is integrable, we can find a series of mutually commuting operators on $\bigotimes_{j=1}^n V_j$ from the Taylor expansions of transfer matrices.

These conditions for integrability are naturally satisfied if the TQFT has “extra dimensions.” In this scenario, we really start with a higher-dimensional theory \mathbf{T} formulated on $S \times T^2$ that is topological on T^2 but not on S . We wrap line operators \mathcal{L}_i around closed curves $u_i \times C_i$, where u_i are points in S . They may or may not have parameters.

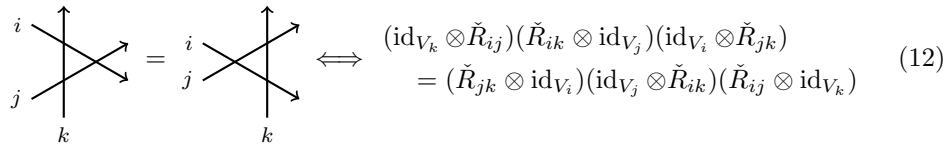
To someone unaware of the presence of the extra dimensions S , the theory appears as a two-dimensional TQFT, which we call $\mathbf{T}[S]$.^b This observer finds that line operators $\mathcal{L}_i[u_i]$ carrying continuous parameters u_i are wrapped around C_i in the seemingly two-dimensional spacetime T^2 , and the correlation function for this configuration is given by the partition function of a lattice model $\mathbf{L}(\mathbf{T}[S])$ defined on the lattice $\{\mathcal{L}_i[u_i](C_i)\}$. For a generic choice of the points u_i , the transfer matrices of $\mathbf{L}(\mathbf{T}[S])$ commute since the two horizontal line operators in (10) may move freely and interchange their positions due to the topological invariance along T^2 ; no phase transition occurs when they pass each other as they do not meet in the full spacetime $S \times T^2$. Thus, the integrability follows from the existence of extra dimensions, whose coordinates provide continuous spectral parameters.

In fact, we can say more. By the same logic, we deduce that the *unitarity relation*

$$\begin{array}{c} i \\ \searrow \\ \swarrow \\ j \end{array} = \begin{array}{c} i \\ \longrightarrow \\ j \end{array} \iff \check{R}_{ji} \check{R}_{ij} = \text{id}_{V_i \otimes V_j} \quad (11)$$

^bNote that $\mathbf{T}[S]$ is not the dimensional reduction of \mathbf{T} on S . Here we are keeping all Kaluza–Klein modes and therefore $\mathbf{T}[S]$ captures the full content of \mathbf{T} .

and the *Yang–Baxter equation*


$$\begin{array}{c} i \\ \diagdown \\ j \end{array} \begin{array}{c} \diagup \\ k \end{array} = \begin{array}{c} i \\ \diagup \\ j \end{array} \begin{array}{c} \diagdown \\ k \end{array} \iff (\text{id}_{V_k} \otimes \check{R}_{ij})(\check{R}_{ik} \otimes \text{id}_{V_j})(\text{id}_{V_i} \otimes \check{R}_{jk}) \\ = (\check{R}_{jk} \otimes \text{id}_{V_i})(\text{id}_{V_j} \otimes \check{R}_{ik})(\check{R}_{ij} \otimes \text{id}_{V_k}) \quad (12)$$

also hold. (For brevity the spectral parameters are omitted.) These relations imply the commutativity of transfer matrices and hence the integrability of the model.

2.3. Correspondence with extended operators in extra dimensions

The above argument generalizes in a couple of ways. First of all, we can formulate the higher-dimensional theory on a manifold of the form $S \times \Sigma$, with Σ being any surface, and put line operators along various curves $u_i \times C_i$ in such a way that no three curves intersect at a point on Σ . In this situation, we get a spin model placed on the lattice drawn on Σ by the curves C_i .^c The model is integrable in the sense that its R -matrix satisfies the Yang–Baxter equation with spectral parameter.

The surface Σ may have a boundary. If it does, we make “dents” on the boundary between line operators and impose boundary conditions there:


$$\text{---} \rightarrow \text{---} \quad (13)$$

This process assigns spins to the faces touching the boundary. These spins, together with those assigned to the edges intersecting the boundary, provide the data defining the TQFT states living on the boundary (with dents).

Second, the line operators may descend from extended operators of dimension greater than one. Consider a theory T formulated on $S \times M \times \Sigma$, where M is some manifold. Suppose that it is topological on Σ and has extended operators \mathcal{E}_i whose codimension is greater than $\dim S$. Place \mathcal{E}_i on submanifolds of the form $u_i \times N_i \times C_i$. Since $T[S \times M]$ — the theory T “compactified” on $S \times M$ and regarded as a two-dimensional theory, though neither S nor M needs to be compact — is a TQFT, the correlation function of this configuration still equals the partition function of an integrable lattice model $L(T[S \times M])$. The model is placed on the lattice constructed from the line operators $\mathcal{E}_i[u_i \times N_i]$, the images of \mathcal{E}_i in $T[S \times M]$.

In the previous paragraph, we regarded our theory as a TQFT on Σ , but we may also view it as a theory $T[\Sigma]$ on $S \times M$. In the latter theory, \mathcal{E}_i appear as extended operators $\mathcal{E}_i[C_i]$ supported on $u_i \times N_i$, and we have

$$\left\langle \prod_{i=1}^l \mathcal{E}_i[C_i](u_i \times N_i) \right\rangle_{T[\Sigma], S \times M} = Z_{L(T[S \times M]), \{\mathcal{E}_i[u_i \times N_i](C_i)\}} \quad (14)$$

^cIf necessary, we introduce “invisible” line operators so that a lattice is formed. They are the unit of the algebra of line operators and have no effect on the correlation function.

Thus we have arrived at a correspondence between the theory on the extra dimensions $S \times M$ and the integrable lattice model on Σ . If Σ has a boundary, this is an equality between linear functionals on the Hilbert space of states.

2.4. Higher-dimensional lattice models

Our construction can also be extended to higher-dimensional lattice models. In a d -dimensional TQFT, a generic configuration of $(d-1)$ -dimensional extended operators makes a lattice. The correlation function for this configuration gives the partition function of a d -dimensional lattice model. If the theory has extra dimensions and there is enough room there for these operators to avoid one another, the model is integrable and satisfies a d -dimensional analog of the Yang–Baxter equation. For $d=3$, the relevant equation is Zamolodchikov’s tetrahedron equation.^{12,13} Our argument shows that the partition function equals a correlation function of extended operators in a theory formulated on the extra dimensions.

3. Branes and Integrable Lattice Models

We have seen above that a lattice model is realized by a lattice of line operators in a two-dimensional TQFT, and it is integrable if the TQFT is embedded in higher dimensions and the line operators come from extended operators localized in some directions of the extra dimensions. Now I explain how to get such structures of TQFTs with extra dimensions using branes in string theory.

Consider a stack of N NS5-branes supported on $\mathbb{R}^{3,1} \times \Sigma \times 0$ in type II string theory in the spacetime $\mathbb{R}^{3,1} \times T^*\Sigma \times \mathbb{R}^2$. Here Σ is a surface (without boundary, for simplicity) embedded in $T^*\Sigma$ as the zero section. To this configuration, we introduce Dp -branes Dp_i ending on the NS5-branes along curves C_i on Σ , as in Fig. 2. Let their worldvolumes be $\mathbb{R}^{p-2,1} \times \Sigma_i \times 0$, where $\mathbb{R}^{p-2,1}$ is a subspace of $\mathbb{R}^{3,1}$ and Σ_i are surfaces in $T^*\Sigma$ such that $\partial\Sigma_i \cap \Sigma = C_i$. Provided that Σ_i are suitably chosen, this brane system preserves four supercharges.

The low-energy dynamics of the NS5-branes is governed by a six-dimensional theory T_{NS5} , which is either $\mathcal{N} = (2,0)$ superconformal field theory of type A_{N-1} or $\mathcal{N} = (1,1)$ super Yang–Mills theory with gauge group $SU(N)$, depending on whether we are considering type IIA or IIB string theory. The theory T_{NS5} is formulated on $\mathbb{R}^{3,1} \times \Sigma$, with topological twist along Σ which breaks half of the sixteen supercharges. In this twisted theory, Dp_i create p -dimensional defects \mathcal{E}_{Dp_i} on $\mathbb{R}^{p-2,1} \times C_i$, reducing the number of unbroken supercharges to four. From the

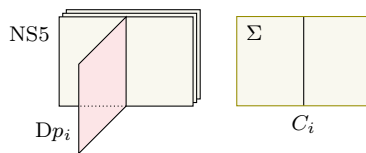


Fig. 2. The Dp -brane Dp_i ending on the NS-branes creates a defect \mathcal{E}_{Dp_i} along C_i .

point of view of a four-dimensional observer, this brane configuration gives half-BPS defects $\mathcal{E}_{Dp_i}[C_i]$ in an $\mathcal{N} = 2$ theory $T_{NS5}[\Sigma]$. The total system is invariant under a $U(1)$ R -symmetry originating from the rotational symmetry on the \mathbb{R}^2 factor of the ten-dimensional spacetime.

Let us take a three-manifold M and $(p-2)$ -submanifolds N_i of M , and modify the above construction so that the worldvolumes of the NS5-branes and the Dp -branes become $S^1 \times M \times \Sigma$ and $S^1 \times N_i \times \Sigma_i$, respectively. At low energies, we get the same theory T_{NS5} formulated on $S^1 \times M \times \Sigma$, with \mathcal{E}_{Dp_i} located on $S^1 \times N_i \times C_i$. In general, this modification completely breaks supersymmetry. However, for certain choices of M and N_i , there is a string background in which a fraction of supersymmetry is still preserved. In such a background, the path integral computes the *supersymmetric index* of T_{NS5} , defined with respect to the Hilbert space on $M \times \Sigma$ in the presence of the defects \mathcal{E}_{Dp_i} inserted on $N_i \times C_i$.

A salient feature of supersymmetric indices is that they are protected against continuous changes of various parameters of the theory. This is because a supersymmetric index is the trace of $(-1)^F$ over the space of states annihilated by a set of supercharges, usually refined by gradings with respect to some conserved charges commuting with those supercharges. Under variations of continuous parameters, such states are created or annihilated in boson-fermion pairs and there is no net change in the index.

For the same reason, the index of our theory is invariant under deformations of the geometric data of Σ and C_i , namely the metric on Σ and the shapes of C_i . In other words, T_{NS5} on $S^1 \times M \times \Sigma$ is topological on Σ , as far as the computation of the index is concerned.

To connect the present setup to the one considered in the previous section, we apply T -duality along S^1 . It turns Dp_i into $D(p-1)$ -branes $D(p-1)_i$ localized at points u_i on the dual circle \tilde{S}^1 , while sending the NS5-branes to those in the other type II string theory.^d The new NS5-branes produce the dual six-dimensional theory \tilde{T}_{NS5} on $\tilde{S}^1 \times M \times \Sigma$, and in this theory $D(p-1)_i$ create $(p-1)$ -dimensional defects $\mathcal{E}_{D(p-1)_i}$ on $u_i \times N_i \times C_i$.

Furthermore, we know that \tilde{T}_{NS5} is topological on Σ if we restrict the allowed operators to a subset which includes these defects. Thus we are in the situation studied in the last section, and the correlation function for this configuration coincides with the partition function of an integrable lattice model:

$$\left\langle \prod_{i=1}^l \mathcal{E}_{Dp_i}[C_i](S^1 \times N_i) \right\rangle_{T_{NS5}[\Sigma], S^1 \times M} = Z_{L(\tilde{T}_{NS5}[\tilde{S}^1 \times M]), \{\mathcal{E}_{D(p-1)_i}[u_i \times N_i](C_i)\}} \cdot \quad (15)$$

Here the correlation function is expressed in the original frame; it implicitly depends on each spectral parameter u_i through the holonomy $\exp(2\pi i u_i)$ around S^1 of the

^dMore precisely, we obtain branes in an exotic variant of type II string theory with Euclidean D-branes, as we have applied time-like T -duality.¹⁴

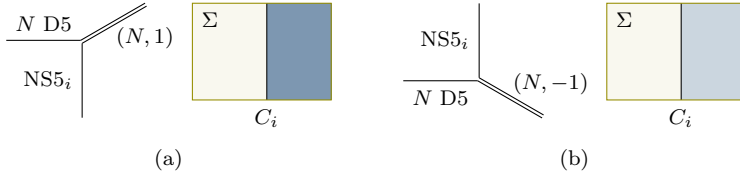


Fig. 3. An NS5-brane combines with a stack of N D5-branes, forming (a) an $(N, 1)$ 5-brane or (b) an $(N, -1)$ 5-brane. The 5-brane junction is a domain wall in T_{D5} . The shaded regions shown above support a nonzero NS5-brane charge $q = \pm 1$.

gauge field for the flavor symmetry $U(1)_i$ supported on Dp_i . The holonomy appears in the index as a refinement parameter, or *fugacity*, associated with $U(1)_i$. In that context, u_i is a *chemical potential* for $U(1)_i$.

4. Integrable Lattice Models from Brane Tilings

Finally, we apply the framework developed in the previous sections to the main theme of this article: the integrable lattice models arising from brane tilings.

4.1. Brane tilings

Let us consider the case $p = 5$ in the brane construction described in the last section. To conform with the standard convention, we go to the S -dual frame where the D5-branes and the NS5-branes are interchanged. Thus, we have a stack of N D5-branes wrapping $S^1 \times M \times \Sigma$, together with NS5-branes $NS5_i$ supported on $S^1 \times N_i \times \Sigma_i$ creating defects \mathcal{E}_{NS5_i} on $S^1 \times N_i \times C_i$ in the theory T_{D5} on the D5-branes. In addition, we allow Σ to have a boundary where the 5-branes end on 7-branes.

The first thing to notice is that we necessarily have $N_i = M$, i.e. \mathcal{E}_{NS5_i} wrap the whole M . Accordingly, the half-BPS defects $\mathcal{E}_{NS5_i}[C_i]$ in the four-dimensional $\mathcal{N} = 2$ theory $T_{D5}[\Sigma]$ cover the entire spacetime $S^1 \times M$, and may be thought of as changing $T_{D5}[\Sigma]$ to a different theory with $\mathcal{N} = 1$ supersymmetry.

Another peculiarity is that the NS5-branes cannot just end on the D5-branes. Rather, when an NS5-brane meets N D5-branes, they combine to form a bound state. In the language of (p, q) 5-branes, this bound state is either an $(N, 1)$ 5-brane or an $(N, -1)$ 5-brane, depending on the relative positions of the branes; see Fig. 3. Therefore, \mathcal{E}_{NS5_i} are domain walls in T_{D5} partitioning the spacetime into regions with different values of the NS5-brane charge q . (In this sense, T_{D5} is not supported solely on D5-branes.) The curves C_i along which these domain walls are located are known as *zigzag paths*. Across a zigzag path the value of q jumps by one.

Conversely, given a configuration of curves C_i on Σ and a 5-brane charge assignment consistent with it, we can construct a 5-brane system whose zigzag paths are C_i : we take NS5-branes approaching the D5-branes from transverse directions, and let them meet along C_i and form bound states over regions with $q \neq 0$. Such a 5-brane system is called a *brane tiling*^{5,6} on Σ . The reader is referred to Refs. 16 and 17 for reviews of brane tilings.

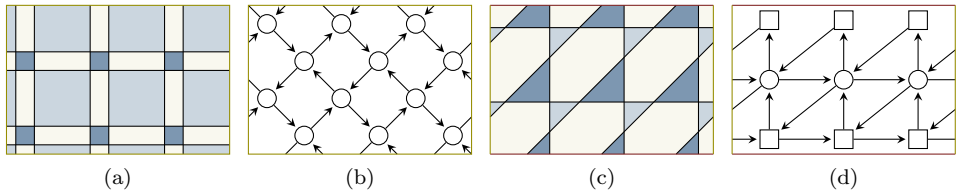


Fig. 4. (a) A brane tiling on a torus. (b) The periodic quiver associated with (a). (c) A brane tiling on a finite-length cylinder. The horizontal direction is periodic. (d) The quiver for (c).

As we just explained, a brane tiling gives rise to a four-dimensional $\mathcal{N} = 1$ theory. A concrete description of this theory is known for the subset of brane tilings that involve only $(N, 0)$ 5-branes (i.e. N coincident D5-branes) and $(N, \pm 1)$ 5-branes.⁶ Given a brane tiling in this subset, we indicate $(N, 1)$ and $(N, -1)$ 5-brane regions by dark and light shading, respectively, while leaving $(N, 0)$ regions unshaded. After the shading, we get a checkerboard-like pattern on Σ where shaded faces adjoin unshaded ones and two shaded faces sharing a vertex are of different types.^e Examples are shown in Fig. 4.

Each unshaded region supports N D5-branes, hence an $SU(N)$ vector multiplet lives there. If the region contains part of the boundary, the multiplet is frozen by boundary conditions and the associated symmetry is an $SU(N)$ flavor symmetry; otherwise it is dynamical and we have an $SU(N)$ gauge symmetry. Adopting the quiver notation, we represent a dynamical vector multiplet by a gauge node \bigcirc and a nondynamical one by a flavor node \square .

From open strings stretched between two unshaded regions sharing a vertex, we get a chiral multiplet that transforms in the fundamental representation under one of the associated gauge or flavor groups and in the antifundamental representation under the other. We represent it by an arrow between the two nodes:

$$\begin{array}{c} \text{Diagram of a square with a diagonal arrow from bottom-left to top-right, labeled } i \text{ and } j. \end{array} \rightsquigarrow \begin{array}{c} \square \\ \uparrow \\ \square \end{array} . \quad (16)$$

The arrow points from the antifundamental side to the fundamental side. See Fig. 4 for examples of quivers obtained from brane tilings.

Moreover, for every set of zigzag paths bounding a shaded region, we have a loop of arrows and worldsheet instantons generate a superpotential term given by the trace of the product of the bifundamental chiral multiplets in the loop. The coefficient of this term is positive or negative depending on whether the

^eIt is more common to represent such a brane tiling by a bipartite graph, placing a white node in each $(N, 1)$ region and a black node in each $(N, -1)$ region, and connecting every pair of black and white nodes placed in two shaded regions sharing a vertex. The term “zigzag paths” originated in this context; we can draw them as lines running zigzag to avoid these nodes.

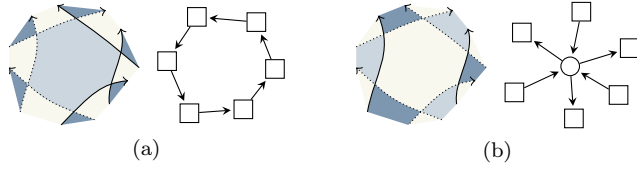


Fig. 5. Zigzag paths bounding (a) a shaded region and (b) an unshaded region. In either case, the R -charges of two of the arrows are different from those of the rest.

direction of the loop is clockwise or counterclockwise. Thus, the four-dimensional theory realized by a brane tiling in the subset under consideration is an $\mathcal{N} = 1$ supersymmetric gauge theory described by a quiver with potential drawn on Σ .

Each NS5_i supports a $\text{U}(1)$ flavor symmetry $\text{U}(1)_i$. An arrow is charged under $\text{U}(1)_i$ if it is crossed by C_i . The charge F_i of $\text{U}(1)_i$ can be normalized in such a way that the arrow in (16) has $F_i = -1$ and $F_j = +1$. The diagonal combination of all $\text{U}(1)_i$ acts on the theory trivially since every arrow is crossed by exactly two zigzag paths from the opposite sides.

The theory also has an R -symmetry $\text{U}(1)_R$. Its definition is not unique as the R -charge R can be shifted by a linear combination of $\text{U}(1)$ flavor charges. However, the R -charge assignment is constrained by two conditions. The first is that $\text{U}(1)_R$ must be unbroken by the superpotential and therefore the R -charges of the chiral multiplets contained in each superpotential term must add up to two. The second is that $\text{U}(1)_R$ must be free of anomaly. This requires that for every gauge node, the sum of the R -charges of the arrows starting from or ending at that node must equal the number of the arrows minus two.

To fix the R -charge assignment, let us assume that we can orient the zigzag paths (and deform them if necessary) and bound every shaded or unshaded region with zigzag paths all heading upward, for some choice of the “vertical” direction in the neighborhood of that region. This is the case for the examples in Fig. 4. The zigzag paths thus oriented fall into two groups; when a zigzag path goes upward and we cross it from the left to the right, q increases or decreases by one. We distinguish the latter case from the former by drawing the zigzag path with a dotted line. Then, we give an arrow $R = 0$ if it originates from a crossing of two zigzag paths of the same type, and $R = 1$ otherwise. With this R -charge assignment the two conditions described above are satisfied; see Fig. 5 for illustration.

The rule for reading off the quiver from zigzag paths is summarized in Fig. 6.

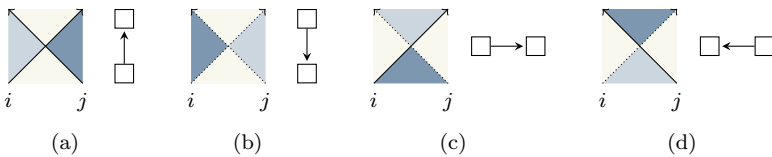


Fig. 6. The rule for assigning a quiver to a brane tiling diagram. The arrows in (a) and (b) have $(R, F_i, F_j) = (0, -1, 1)$. Those in (c) and (d) have $(R, F_i, F_j) = (1, 1, -1)$.

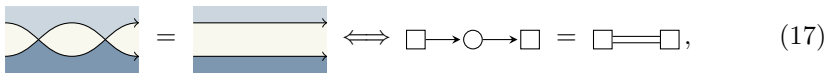
4.2. Integrable lattice models arising from brane tilings

From the supersymmetric index of the four-dimensional $\mathcal{N} = 1$ theory realized by a brane tiling, we obtain an integrable lattice model defined on the lattice $\{C_i\}$ consisting of the zigzag paths. Each C_i carries a spectral parameter u_i . S -duality followed by T-duality on S^1 turns NS5 $_i$ into a D4-brane, and its coordinate on the dual circle \tilde{S}^1 is u_i . Instead, we can apply T-duality on S^1 and lift NS5 $_i$ to an M5-brane; then u_i is the coordinate on the M-theory circle. Either way, u_i is determined by the holonomy of the U(1) gauge field on NS5 $_i$ around S^1 .

If the theory is described by a quiver, translation between the gauge theory and the lattice model goes as follows.²

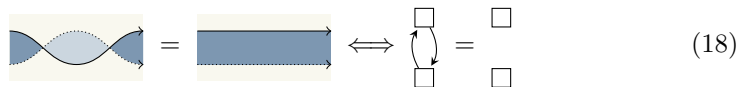
Nodes are interpreted as spin sites. For each flavor node, we can turn on a holonomy of the associated gauge field. The index depends on the conjugacy class of the holonomy, which is uniquely represented by a diagonal matrix $\text{diag}(z_1, \dots, z_N)$ up to permutations of the entries. The index is therefore a symmetric function of the U(1)-valued variables (z_1, \dots, z_N) obeying the constraint $z_1 \cdots z_N = 1$. These variables are fugacities for the SU(N) flavor symmetry and parametrize the value of the spin at this node; thus spins take values in the maximal torus $U(1)^{N-1}$ of SU(N). For a gauge node, integration is performed over the fugacities since its gauge field is a path integral variable. This is the summation over the values of a spin placed on an internal face. Finally, arrows represent interactions between spins.

The unitarity relations are satisfied if the contributions to the index from arrows with $R = 0$ are properly normalized. For example, consider the relation^f



$$\text{wavy arrow} = \text{straight arrow} \iff \square \rightarrow \bigcirc \rightarrow \square = \square \equiv \square, \quad (17)$$

where the right-hand side is a “delta function” that equates two flavor nodes when one of them is gauged. The theory on the left-hand side is SQCD with N colors and N flavors. It exhibits confinement and has a vacuum in which the mesons take nonzero expectation values and the flavor symmetry $SU(N) \times SU(N)$ is broken to the diagonal subgroup.¹⁸ The index computed in this vacuum is given by the right-hand side, provided that we cancel the contributions from the surviving baryon and antibaryon. Another relation



$$\text{loop} = \text{straight arrow} \iff \square \rightarrow \bigcirc \rightarrow \square = \square \equiv \square \quad (18)$$

holds since the two arrows on the left-hand side form a loop and generates a mass term in the superpotential. We can send the mass to infinity so that these arrows decouple from the theory, and are left with the right-hand side.

^fBy an equality of two quivers, we mean that the supersymmetric indices of the theories described by those quivers are equal.

The Yang–Baxter equation with three zigzag paths is harder to understand, as it always involves an (N, q) region with $|q| > 1$ and a quiver description is not available. The problem stems from the fact that our defects are domain walls across which q changes. To circumvent the difficulty, we take a pair of zigzag paths of different types and think of it as a single line:

$$\longrightarrow = \begin{array}{c} \longrightarrow \\ \cdots \longrightarrow \end{array} . \quad (19)$$

This line does not alter the value of q . Taking two copies of this line and placing them in an $(N, -1)$ background, we can make the R -matrix

$$\begin{array}{c} \uparrow \\ \hline \longrightarrow \end{array} = \begin{array}{c} \uparrow \\ \hline \begin{array}{c} \longrightarrow \\ \cdots \longrightarrow \end{array} \end{array} = \begin{array}{c} \square \\ \nearrow \quad \searrow \\ \square \quad \square \\ \nwarrow \quad \nearrow \\ \square \quad \square \\ \nwarrow \quad \nearrow \\ \square \end{array} . \quad (20)$$

A lattice model constructed from this R -matrix is a vertex model whose quiver consists of diamonds of arrows; see Fig. 4(b). The vector space carried by a line is the space of symmetric functions of fugacities (z_1, \dots, z_N) .

Alternatively, we can place these lines in an $(N, 0)$ background and force them to exchange their constituent zigzag paths as they cross:

$$\begin{array}{c} \uparrow \\ \hline \longrightarrow \end{array} = \begin{array}{c} \uparrow \\ \hline \begin{array}{c} \longrightarrow \\ \cdots \longrightarrow \end{array} \end{array} = \begin{array}{c} \square \\ \nearrow \quad \searrow \\ \square \quad \square \\ \nwarrow \quad \nearrow \\ \square \quad \square \\ \nwarrow \quad \nearrow \\ \square \end{array} . \quad (21)$$

This R -matrix leads to an IRF model described by a quiver with triangles of arrows, as shown in Fig. 4(d). The corresponding Yang–Baxter equation, after cancellation of some factors with the help of the unitarity relation (18), reads

$$\begin{array}{c} \square \\ \nearrow \quad \searrow \\ \square \quad \square \\ \nwarrow \quad \nearrow \\ \square \quad \square \\ \nwarrow \quad \nearrow \\ \square \end{array} = \begin{array}{c} \square \\ \nearrow \quad \searrow \\ \square \quad \square \\ \nwarrow \quad \nearrow \\ \square \quad \square \\ \nwarrow \quad \nearrow \\ \square \end{array} . \quad (22)$$

The two sides are related by Seiberg duality⁷ for SQCD with N colors and $2N$ flavors, so their indices are indeed equal. The Yang–Baxter equation for the R -matrix (20), though more complicated, also follows from this equality. The relation between the Yang–Baxter move and Seiberg duality was pointed out in Ref. 15.

4.3. Surface defects as transfer matrices

The brane tiling construction of integrable lattice models can be enriched by introduction of surface defects.⁹ Consider a brane tiling configuration, and add to it a D3-brane that creates a defect \mathcal{E}_{D3} in \mathbb{T}_{D5} . Let the support of \mathcal{E}_{D3} be $S^1 \times N \times C$, where N is a curve in M and C is a closed curve on Σ . In the four-dimensional $\mathcal{N} = 1$ theory, \mathcal{E}_{D3} becomes a half-BPS surface defect $\mathcal{E}_{D3}[C]$ on $S^1 \times N$.

In the lattice model, \mathcal{E}_{D3} appears as a new oriented line, which we represent by a dashed line. Now that we have two kinds of lines, we can define three R -matrices:

$$\check{R} = \begin{array}{c} \uparrow \\ | \\ \text{---} \rightarrow \end{array}, \quad \check{L} = \begin{array}{c} \uparrow \\ | \\ \text{---} \leftarrow \end{array}, \quad \check{\mathcal{R}} = \begin{array}{c} \uparrow \\ | \\ \text{---} \rightarrow \\ | \\ \text{---} \end{array}. \quad (23)$$

The middle one is called the L -operator. Correspondingly, we have four Yang–Baxter equations, involving zero to three dashed lines. Those that mix different R -matrices,

$$\begin{array}{c} \uparrow \\ | \\ \text{---} \rightarrow \\ | \\ \text{---} \end{array} = \begin{array}{c} \uparrow \\ | \\ \text{---} \leftarrow \\ | \\ \text{---} \end{array} \quad \text{and} \quad \begin{array}{c} \uparrow \\ | \\ \text{---} \rightarrow \\ | \\ \text{---} \end{array} = \begin{array}{c} \uparrow \\ | \\ \text{---} \leftarrow \\ | \\ \text{---} \end{array}, \quad (24)$$

are called *RLL relations*.

The effect of the surface defect on the lattice model can be phrased compactly in terms of the L -operator. The neighborhood of the dashed line looks like

$$\leftarrow \begin{array}{c} \uparrow \\ | \\ \text{---} \rightarrow \end{array} \begin{array}{c} \uparrow \\ | \\ \text{---} \leftarrow \end{array} \dots \begin{array}{c} \uparrow \\ | \\ \text{---} \rightarrow \end{array} \rightarrow. \quad (25)$$

This picture shows that the surface defect acts on the lattice model by a transfer matrix constructed from L -operators.

In fact, \mathbb{T}_{D5} has a whole family of half-BPS defects. Each of them corresponds to an irreducible representation of $SU(N)$ and is constructed with D3-branes that stretch between the D5-branes and extra NS5-branes. The defect \mathcal{E}_{D3} described above is the simplest member of this family, corresponding to the fundamental representation. Thus, the four-dimensional theory has a family of surface defects parametrized by the irreducible representations of $SU(N)$. The insertion of a surface defect is mapped in the lattice model to the action of a transfer matrix constructed from L -operators, which contains a dashed line labeled with a representation in addition to the spectral parameter and the curve N . The vector space carried by the dashed line is the representation space.

4.4. The three-sphere case

To conclude our discussion, we describe the integrable lattice models arising from brane tilings concretely for $M = S^3$, equipped with the round metric of radius 1. Other cases are possible and interesting; the case where M is a lens space $L(r, 1)$ was considered in Ref. 19.

Parametrize S^3 by two complex variables (ζ_p, ζ_q) satisfying $|\zeta_p|^2 + |\zeta_q|^2 = 1$, and denote the isometry groups acting on ζ_p and ζ_q by $U(1)_p$ and $U(1)_q$, respectively. We take $S^1 \times M$ to be a twisted product; we prepare a trivial S^3 -fibration over an interval $[0, \beta]$ and identify the fibers at the ends of the base using an isometry $(e^{i\theta_p}, e^{i\theta_q}) \in U(1)_p \times U(1)_q$. On this spacetime, the partition function of the quiver gauge theory realized by a brane tiling gives the supersymmetric index refined by the isometries and the flavor symmetries.^{20–22}

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the L -operator gives a representation of Felder's elliptic quantum group for \mathfrak{sl}_N .^{30–32} The details will be presented elsewhere.

Acknowledgments

I would like to thank Kazunobu Maruyoshi and Kevin Costello for illuminating discussions, and Petr Vaško for careful reading of the manuscript. This work is supported by the ERC Starting Grant No. 335739 “Quantum fields and knot homologies” funded by the European Research Council under the European Union's Seventh Framework Programme.

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