




Poisson vertex algebras in supersymmetric field theories

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Abstract

A large class of supersymmetric quantum field theories, including all theories with $\mathcal{N} = 2$ supersymmetry in three dimensions and theories with $\mathcal{N} = 2$ supersymmetry in four dimensions, possess topological–holomorphic sectors. We formulate Poisson vertex algebras in such topological–holomorphic sectors and discuss some examples. For a four-dimensional $\mathcal{N} = 2$ superconformal field theory, the associated Poisson vertex algebra is the classical limit of a vertex algebra generated by a subset of local operators of the theory.

Keywords Supersymmetric quantum field theory · Poisson vertex algebras · Superconformal field theory

Mathematics Subject Classification 81T60 · 17B69

1 Introduction

In this work, we study algebraic structures in topological–holomorphic sectors of supersymmetric quantum field theories, with one or more topological directions and a single holomorphic direction. These structures, known as *Poisson vertex algebras* [10,16,17], exist in $\mathcal{N} = 2$ supersymmetric field theories in three dimensions and $\mathcal{N} = 2$ supersymmetric field theories in four dimensions, among others.

Our motivation for studying the Poisson vertex algebras comes from two main sources.

One is the fact that local operators in topological quantum field theories (TQFTs) of cohomological type [32,33] form Poisson algebras, some aspects of which were eluci-

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dated in the recent paper [5]. TQFTs being special cases of topological–holomorphic theories, it is natural to ask what the analogs of these Poisson algebras are in the topological–holomorphic setting. An answer is Poisson vertex algebras.

Another source of motivation is the presence of vertex algebras in four-dimensional $\mathcal{N} = 2$ superconformal field theories, which was uncovered in [6] and has been a subject of intensive research for the past several years. The classical limits of these vertex algebras are Poisson vertex algebras.

The construction of Poisson vertex algebras we discuss in this paper indeed provides a bridge between the two lines of developments: whereas the Poisson vertex algebra for a four-dimensional unitary $\mathcal{N} = 2$ superconformal field theory has a canonical deformation to the vertex operator algebra introduced in [6], dimensional reduction turns it into a Poisson algebra associated with a TQFT in one dimension less. We emphasize, however, that the construction itself requires no conformal invariance and applies to a broader class of theories.

In Sect. 2, we describe the structures that define a topological–holomorphic sector within a supersymmetric field theory and demonstrate that local operators in this sector comprise a Poisson vertex algebra. Essential for the definition of a Poisson vertex algebra is the binary operation called the λ -bracket [21], which plays a role similar to the Poisson bracket in Poisson algebras. The λ -bracket is constructed via a topological–holomorphic analog of topological descent [32], much as the Poisson bracket between local operators in a TQFT is constructed via the descent procedure. This construction may be seen as providing a concrete physical realization of a special instance of the Poisson additivity theorem, proved by Nick Rozenblyum and also independently in [28].¹

In Sect. 3, we identify topological–holomorphic sectors in theories with $\mathcal{N} = 2$ supersymmetry in three dimensions and determine the Poisson vertex algebras for chiral multiplets and vector multiplets. We content ourselves mostly with free theories here; to go beyond that one needs to incorporate quantum corrections, both perturbative and nonperturbative.

In Sect. 4, we consider Poisson vertex algebras for $\mathcal{N} = 2$ supersymmetric field theories in four dimensions. We show that if a theory is conformal, the associated Poisson vertex algebra is the classical limit of a vertex algebra, which is isomorphic to the vertex operator algebra of [6] in the unitary case. For free hypermultiplets, we confirm this relation by explicitly computing the Poisson vertex algebra and comparing it with the vertex operator algebra. For gauge theories constructed from vector multiplets and hypermultiplets, the relation to the vertex algebras allows us to propose a description of the associated Poisson vertex algebras.

There are many directions to explore in connection with Poisson vertex algebras. To conclude this introduction, we briefly describe some general ideas.

Theories related by dualities, such as mirror symmetry and S-dualities, should have isomorphic Poisson vertex algebras. Dualities may thus lead to interesting isomorphisms between Poisson vertex algebras. Conversely, isomorphisms established between Poisson vertex algebras associated with apparently different theories may serve as evidence that those theories are actually equivalent.

¹ We are grateful to Dylan Butson for explaining this point to us.

The structures considered in this paper can be further enriched by introduction of topological–holomorphic boundaries and defects, which themselves support Poisson vertex algebras (or vertex algebras if they are two-dimensional and have no topological direction). The interplay between the Poisson vertex algebras associated with the bulk theory and the boundaries and defects would be a fascinating topic. In three dimensions, relations between bulk Poisson vertex algebras and boundary vertex algebras are discussed in [13].

It would also be interesting to study the Poisson vertex algebras for theories originating from compactification of six-dimensional $\mathcal{N} = (2, 0)$ superconformal field theories on Riemann surfaces [18, 19] and 3-manifolds [14, 15, 29, 30]. These Poisson vertex algebras should be geometric invariants.

Intriguing observations made in [12] suggest that the Poisson vertex algebras for four-dimensional $\mathcal{N} = 2$ supersymmetric field theories are related to wall-crossing phenomena. Understanding this relation may be an outstanding problem.

Lastly, it would be fruitful to investigate connections to integrable hierarchies of soliton equations. This is the context in which the theory of Poisson vertex algebras was originally developed [1, 10, 16, 17].

2 Poisson vertex algebras in topological–holomorphic sectors

In this paper, we consider supersymmetric field theories that contain sectors that are topological in $d \geq 1$ directions and holomorphic in one complex direction. The topological directions form a d -dimensional manifold M , parametrized by real coordinates $y = (y^i)_{i=1}^d$; the holomorphic direction is a Riemann surface C with complex coordinate z . We also use x^μ , $\mu = i, z, \bar{z}$, for the coordinates of $M \times C$, with $x^i = y^i$, $x^z = z$ and $x^{\bar{z}} = \bar{z}$. Typically, we take $M = \mathbb{R}^d$ and $C = \mathbb{C}$ or a cylinder.

In this section, we explain how such a topological–holomorphic sector arises from a supersymmetric field theory and construct a (d -shifted) Poisson vertex algebra in this sector. The construction is similar to that of the Poisson algebras for TQFTs, treated in detail in [5]. Then, we discuss some of the basic properties of the Poisson vertex algebra thus obtained.

2.1 Topological–holomorphic sector

Suppose that a quantum field theory on $M \times C$ is invariant under translations shifting x^μ and has symmetries generated by a fermionic conserved charge Q and a fermionic one-form conserved charge

$$Q = Q_i dy^i + Q_{\bar{z}} d\bar{z}, \quad (1)$$

satisfying the relations

$$Q^2 = 0, \quad (2)$$

$$[Q, P_\mu] = 0 \quad (3)$$

$$[Q, P_\mu] = 0, \quad (4)$$

$$[Q, Q] = iP_i dy^i + iP_{\bar{z}} d\bar{z}. \quad (5)$$

Here, P_μ is the generator of translations in the x^μ -direction, acting on a local operator \mathcal{O} by²

$$iP_\mu \cdot \mathcal{O} = \partial_\mu \mathcal{O}. \quad (6)$$

The graded commutator $[\cdot, \cdot]$ is defined by $[a, b] = ab - (-1)^{F(a)F(b)}ba$, where $(-1)^{F(a)} = +1$ if a is bosonic and -1 if fermionic.

Since Q squares to zero, we can use it to define a cohomology in the space of operators. The commutation relations (3) and (5) show that P_μ act on the Q -cohomology, and P_i and $P_{\bar{z}}$ act trivially. Therefore, the Q -cohomology class $\llbracket \mathcal{O}(z) \rrbracket$ of a Q -closed local operator $\mathcal{O}(y, z, \bar{z})$ is independent of its position in M and varies holomorphically on C . The same is true for correlation functions of Q -closed local operators, which depend only on the Q -cohomology classes and not the choices of representatives.

Given multiple Q -closed local operators $\mathcal{O}_a(x_a)$, $a = 1, \dots, n$, located at $x_a = (y_a, z_a, \bar{z}_a)$, we define the product of their Q -cohomology classes $\llbracket \mathcal{O}_a(z_a) \rrbracket$ by

$$\llbracket \mathcal{O}_1(z_1) \rrbracket \cdots \llbracket \mathcal{O}_n(z_n) \rrbracket = \llbracket \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rrbracket, \quad x_a \neq x_b \text{ if } a \neq b, \quad (7)$$

displacing the operators in M if necessary. (Note that this definition requires $d \geq 1$.) The condition that no two operators are located at the same point in $M \times C$ is important because a product of coincident operators is generally singular in a quantum field theory.

As long as no two operators are located at the same point in M , we may set all z_a equal in the above definition of product. Pointwise multiplication on C furnishes the Q -cohomology of local operators with the structure of an algebra, which is furthermore equipped with a derivation $P_{\bar{z}}$. We denote this algebra by \mathcal{V} .

While the space of operators is always \mathbb{Z}_2 -graded by the fermion parity $(-1)^F$, in many examples this grading is refined by a \mathbb{Z} -grading by a weight F of a $U(1)$ symmetry such that

$$F(Q) = -F(Q) = 1, \quad F(P_\mu) = 0. \quad (8)$$

In that case, \mathcal{V} is \mathbb{Z} -graded by F . It may also happen that only a \mathbb{Z}_{2n} subgroup of such a $U(1)$ symmetry is unbroken and \mathcal{V} is \mathbb{Z}_{2n} -graded. There may be additional $U(1)$ symmetries providing further gradings.

We will refer to the collection of various structures associated with the Q -cohomology described above as the *topological–holomorphic sector* of the theory.

² In the path integral formulation, the action of a conserved charge X on a local operator \mathcal{O} located at a point x is implemented by the integration of the Hodge dual of the associated conserved current over a simply-connected codimension-1 cycle surrounding x . If we take this cycle to be the difference of two time slices sandwiching x , we obtain the familiar formula $X \cdot \mathcal{O} = [X, \mathcal{O}]$ as an operator acting on the Hilbert space of states. The cycle can also be infinitesimally small, so the operation $\mathcal{O} \mapsto X \cdot \mathcal{O}$ is local.

This definition is somewhat weaker than the usual notion of a topological–holomorphic field theory of cohomological type, in which one requires that the components T_{ij} and $T_{\mu\bar{z}} = T_{\bar{z}\mu}$ of the stress–energy tensor are Q -exact.

2.2 Topological–holomorphic descent

In cohomological TQFTs, there is a procedure called topological descent [32] which produces Q -cohomology classes of nonlocal operators starting from a Q -cohomology class of local operator. This construction can be adapted in the topological–holomorphic setting.

For a local operator \mathcal{O} , let us define its k th descendant $\mathcal{O}^{(k)}$ by

$$\mathcal{O}^{(k)} = \frac{1}{k!} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_k} (\mathbf{Q}_{\mu_1} \cdots \mathbf{Q}_{\mu_k}) \cdot \mathcal{O}. \quad (9)$$

Here, we have set $\mathbf{Q}_z = 0$. We denote the total descendant of \mathcal{O} by \mathcal{O}^* :

$$\mathcal{O}^* = \sum_{k=0}^{d+1} \mathcal{O}^{(k)} = \exp(\mathbf{Q}) \cdot \mathcal{O}. \quad (10)$$

Similarly, we introduce

$$\mathcal{O}^{-*} = \exp(-\mathbf{Q}) \cdot \mathcal{O}. \quad (11)$$

These descendant operators satisfy

$$Q \cdot \mathcal{O}^{\pm*} = \pm d' \mathcal{O}^{\pm*} + (Q \cdot \mathcal{O})^{\mp*}, \quad (12)$$

where $\mathcal{O}^{+*} = \mathcal{O}^*$ and

$$d' = dy^i \partial_i + d\bar{z} \partial_{\bar{z}}. \quad (13)$$

Now suppose that \mathcal{O} is a Q -closed local operator. Then, for any holomorphic one-form $\omega(z) = \omega_z(z) dz$ on C , we have

$$Q \cdot (\omega \wedge \mathcal{O}^*) = -d(\omega \wedge \mathcal{O}^*). \quad (14)$$

Therefore, for any cycle $\Gamma \subset M \times C$, the integral

$$\int_{\Gamma} \omega \wedge \mathcal{O}^* \quad (15)$$

is a Q -closed operator. The Q -cohomology class of this operator depends on Γ only through the homology class $[\Gamma] \in H_*(M \times C)$ since $d(\omega \wedge \mathcal{O}^*) = 0$ in the Q -cohomology. It also depends on \mathcal{O} only through the Q -cohomology class $[\mathcal{O}]$ because shifting \mathcal{O} by $Q \cdot \tilde{\mathcal{O}}$ just changes the integral by $Q \cdot \int_{\Gamma} \omega \wedge \tilde{\mathcal{O}}^{-*}$. We call this procedure of constructing nonlocal Q -cohomology classes from local ones *topological–holomorphic descent*.

More generally, for a product $\mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n)$ of n \mathcal{Q} -closed local operators, subject to the condition that $x_a \neq x_b$ if $a \neq b$, we define its descendant as a differential form on the configuration space of n points on $M \times C$,

$$\text{Conf}_n(M \times C) = \{(x_1, \dots, x_n) \in (M \times C)^n \mid x_a \neq x_b \text{ if } a \neq b\}. \quad (16)$$

Let $\pi_a: \text{Conf}_n(M \times C) \rightarrow M \times C$ be the projection to the a th factor, and for differential forms $\alpha_1, \dots, \alpha_n \in \Omega^\bullet(M \times C)$ let

$$\alpha_1 \boxtimes \cdots \boxtimes \alpha_n = \pi_1^* \alpha_1 \wedge \cdots \wedge \pi_n^* \alpha_n \in \Omega^\bullet(\text{Conf}_n(M \times C)). \quad (17)$$

In this language, we may think of the product of local operators at n distinct points as the evaluation of a local operator at a single point in $\text{Conf}_n(M \times C)$:

$$\mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) = (\mathcal{O}_1 \boxtimes \cdots \boxtimes \mathcal{O}_n)(x_1, \dots, x_n). \quad (18)$$

We define the descendant of $\mathcal{O}_1 \boxtimes \cdots \boxtimes \mathcal{O}_n$ to be

$$(\mathcal{O}_1 \boxtimes \cdots \boxtimes \mathcal{O}_n)^* = \mathcal{O}_1^* \boxtimes \sigma^{F_1} \mathcal{O}_2^* \boxtimes \cdots \boxtimes \sigma^{F_1 + \cdots + F_{n-1}} \mathcal{O}_n^*, \quad (19)$$

where $F_a = F(\mathcal{O}_a)$ and the operator σ multiplies even forms by $+1$ and odd forms by -1 . The descendant satisfies

$$\mathcal{Q} \cdot (\mathcal{O}_1 \boxtimes \cdots \boxtimes \mathcal{O}_n)^* = d'(\mathcal{O}_1 \boxtimes \cdots \boxtimes \mathcal{O}_n)^*, \quad (20)$$

with d' acting on $\Omega^\bullet(\text{Conf}_n(M \times C))$ in the obvious manner. The role of σ in the definition (19) is to supply, via the relation $\sigma d' = -d' \sigma$, a factor of $(-1)^{F_1 + \cdots + F_{a-1}}$ when d' acts on \mathcal{O}_a^* .

Given a holomorphic top-form ω on C^n (or more precisely, its pullback to $\text{Conf}_n(M \times C)$) and a cycle $\Gamma \subset \text{Conf}_n(M \times C)$, the integral

$$\int_{\Gamma} \omega \wedge (\mathcal{O}_1 \boxtimes \cdots \boxtimes \mathcal{O}_n)^* \quad (21)$$

is a \mathcal{Q} -closed operator whose \mathcal{Q} -cohomology class depends only on ω , the homology class $[\Gamma]$ and the \mathcal{Q} -cohomology classes $[\mathcal{O}_a]$.

2.3 Secondary products

The algebra \mathcal{V} of \mathcal{Q} -cohomology of local operators possesses *secondary products*, in addition to the ordinary (or primary) product given by pointwise multiplication on C . These products make use of topological–holomorphic descent.

Choose a translation invariant holomorphic two-form $\kappa(z_1, z_2) = \kappa_{z_1 z_2}(z_1, z_2) dz_1 \wedge dz_2$ on $C \times C$. By translation invariance, we mean

$$\mathcal{L}_{\partial_{z_1} + \partial_{z_2}} \kappa = 0, \quad (22)$$

or $(\partial_{z_1} + \partial_{z_2})\kappa_{z_1 z_2}(z_1, z_2) = 0$, where \mathcal{L}_V denotes the Lie derivative with respect to V . For two Q -closed local operators \mathcal{O}_1 and \mathcal{O}_2 , we define a new Q -closed operator $\mathcal{O}_1 \star \mathcal{O}_2$ by

$$(\mathcal{O}_1 \star \mathcal{O}_2)(x_2) = \left(\int_{S_{x_2}^{d+1}} \iota_{\partial_{z_2}} \kappa(z_1, z_2) \wedge \mathcal{O}_1^{(d)}(x_1) \right) \mathcal{O}_2(x_2), \quad (23)$$

where $S_{x_2}^{d+1}$ is a $(d+1)$ -sphere centered at the point x_2 at which \mathcal{O}_2 is placed, with the radius taken to be sufficiently small. On this operator iP_μ acts as ∂_μ since infinitesimally displacing \mathcal{O}_1 in the integrand is equivalent to shifting the center of $S_{x_2}^{d+1}$, which does not change the homology class, plus shifting the argument z_2 in $\iota_{\partial_{z_2}} \kappa(z_1, z_2)$. In particular, the relation (22) is crucial for the topological–holomorphic descent to hold for $\mathcal{O} = \mathcal{O}_1 \star \mathcal{O}_2$.

The Q -cohomology class $[\![\mathcal{O}_1 \star \mathcal{O}_2]\!]$ depends only on $[\![\mathcal{O}_1]\!]$ and $[\![\mathcal{O}_2]\!]$. Hence, \star_κ defines a secondary product for \mathcal{V} . It has cohomological degree $-d$ since $\mathcal{O}_1 \star \mathcal{O}_2$ uses the d th descendant of \mathcal{O}_1 which has fermion number $F_1 - d$.

The secondary product \star_κ acts on the primary product as a derivation: it satisfies the *Leibniz rule*

$$\begin{aligned} & [\![\mathcal{O}_1] \star_\kappa ([\![\mathcal{O}_2]\!] [\![\mathcal{O}_3]\!]) \\ &= ([\![\mathcal{O}_1] \star_\kappa [\![\mathcal{O}_2]\!]) [\![\mathcal{O}_3]\!] + (-1)^{(F_1+d)F_2} [\![\mathcal{O}_2]\!] ([\![\mathcal{O}_1] \star_\kappa [\![\mathcal{O}_3]\!]) . \end{aligned} \quad (24)$$

This identity simply says that S^{d+1} enclosing both \mathcal{O}_2 and \mathcal{O}_3 can be divided into two S^{d+1} , one containing \mathcal{O}_2 and the other containing \mathcal{O}_3 . Such a decomposition is possible because \mathcal{O}_2 and \mathcal{O}_3 in the product $[\![\mathcal{O}_2]\!] [\![\mathcal{O}_3]\!]$ are by definition separated in M .

2.4 λ -bracket

Secondary products are defined locally; we can make $S_{x_2}^{d+1}$ as small as we wish, for changing the radius does not affect its homology class. Then, by the locality of quantum field theory (expressed, for example, as the gluing axiom in the Atiyah–Segal formulation), any secondary product can be computed purely based on the knowledge of the behavior of the theory in a neighborhood $U \times V \subset M \times C$ of the point at which we are taking the product, where $U \times V \simeq \mathbb{R}^d \times \mathbb{C}$ topologically.

On V , we can expand $\kappa(z_1, z_2)$ around z_2 :

$$\kappa(z_1, z_2) = \sum_{k=0}^{\infty} \frac{1}{k!} (z_1 - z_2)^k \partial_{z_1}^k \kappa_{z_1 z_2}(z_2, z_2) dz_1 \wedge dz_2. \quad (25)$$

Since \star_κ is linear in κ , all information about secondary products in the neighborhood $U \times V$ is encoded in the cases when $\kappa(z_1, z_2) = (z_1 - z_2)^k dz_1 \wedge dz_2$. The generating function of such forms is $e^{\lambda(z_1 - z_2)} dz_1 \wedge dz_2$, where λ is a formal variable.

This consideration motivates us to introduce the λ -bracket [21]:

$$\{\llbracket \mathcal{O}_1 \rrbracket_\lambda \llbracket \mathcal{O}_2 \rrbracket\}(z_2) = (-1)^{F_1 d} \left[\left(\int_{S_{x_2}^{d+1}} e^{\lambda(z_1 - z_2)} dz_1 \wedge \mathcal{O}_1^{(d)}(x_1) \right) \mathcal{O}_2(x_2) \right]. \quad (26)$$

The space-time is here taken to have the topology of $\mathbb{R}^d \times \mathbb{C}$, but not necessarily given the standard metric. In general, the λ -bracket may depend on the choice of metric and other geometric structures.

The λ -bracket satisfies a few important identities, besides the Leibniz rule (24). First, it has *sesquilinearity*:

$$\begin{aligned} \{\partial_z \llbracket \mathcal{O}_1 \rrbracket_\lambda \llbracket \mathcal{O}_2 \rrbracket\} &= -\lambda \{\llbracket \mathcal{O}_1 \rrbracket_\lambda \llbracket \mathcal{O}_2 \rrbracket\}, \\ \{\llbracket \mathcal{O}_1 \rrbracket_\lambda \partial_z \llbracket \mathcal{O}_2 \rrbracket\} &= (\lambda + \partial_z) \{\llbracket \mathcal{O}_1 \rrbracket_\lambda \llbracket \mathcal{O}_2 \rrbracket\}. \end{aligned} \quad (27)$$

Second, it has the following *symmetry* under exchange of the two arguments:

$$\{\llbracket \mathcal{O}_1 \rrbracket_\lambda \llbracket \mathcal{O}_2 \rrbracket\} = -(-1)^{(F_1+d)(F_2+d)} \{\llbracket \mathcal{O}_2 \rrbracket_{-\lambda-\partial_z} \llbracket \mathcal{O}_1 \rrbracket\}. \quad (28)$$

The λ -bracket with λ being an operator Λ is to be understood as a power series in Λ acting on the local operators in the bracket:

$$\{\llbracket \mathcal{O}_2 \rrbracket_\Lambda \llbracket \mathcal{O}_1 \rrbracket\} = \left(\exp(\Lambda \partial_\lambda) \{\llbracket \mathcal{O}_2 \rrbracket_\lambda \llbracket \mathcal{O}_1 \rrbracket\} \right) \Big|_{\lambda=0}. \quad (29)$$

Finally, the λ -bracket satisfies the *Jacobi identity*

$$\begin{aligned} &\{\llbracket \mathcal{O}_1 \rrbracket_\lambda \{\llbracket \mathcal{O}_2 \rrbracket_\mu \llbracket \mathcal{O}_3 \rrbracket\}\} \\ &= \{\{\llbracket \mathcal{O}_1 \rrbracket_\lambda \llbracket \mathcal{O}_2 \rrbracket\}_{\lambda+\mu} \llbracket \mathcal{O}_3 \rrbracket\} + (-1)^{(F_1+d)(F_2+d)} \{\llbracket \mathcal{O}_2 \rrbracket_\mu \{\llbracket \mathcal{O}_1 \rrbracket_\lambda \llbracket \mathcal{O}_3 \rrbracket\}\}. \end{aligned} \quad (30)$$

Let us prove the above identities one by one. Since these are equalities between holomorphic functions on \mathbb{C} , it suffices to demonstrate that they hold on $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$. Thus, we replace the space-time with $\mathbb{R}^d \times \mathbb{C}^\times$ during the derivation.

The sesquilinearity (27) is straightforward to show. The first equation follows from the fact that to the integral appearing in the definition (26) of the λ -bracket, an infinitesimal displacement of \mathcal{O}_1 in the z -direction has the same effect as shifting the parameter z_2 in $S_{x_2}^{d+1}$ and $e^{\lambda(z_1 - z_2)}$. To prove the second equation, one may just take a derivative of the λ -bracket with respect to z_2 .

To establish the symmetry (28), we show

$$\begin{aligned} &\int_{S_0^1} f(z_2) dz_2 \{\llbracket \mathcal{O}_1 \rrbracket_\lambda \llbracket \mathcal{O}_2 \rrbracket\}(z_2) \\ &= -(-1)^{(F_1+d)(F_2+d)} \int_{S_0^1} f(z_1) dz_1 \{\llbracket \mathcal{O}_2 \rrbracket_{-\lambda-\partial_z} \llbracket \mathcal{O}_1 \rrbracket\}(z_1) \end{aligned} \quad (31)$$

for any holomorphic function f on \mathbb{C}^\times , where S_0^1 is a circle around $z = 0$. To this end, we adopt the point of view of the configuration space. As an integral in $\text{Conf}_2(\mathbb{R}^d \times$

\mathbb{C}^\times), the left-hand side is given by

$$(-1)^{F_1 d} \left[\int_{S_{x_2}^{d+1} \times S_0^1} f(z_2) dz_2 \wedge e^{\lambda z_{12}} dz_{12} \wedge (\mathcal{O}_1 \boxtimes \mathcal{O}_2)^* \right]. \quad (32)$$

Here, we have defined $x_{12}^\mu = x_1^\mu - x_2^\mu$ and think of (x_{12}^μ, x_2^μ) as coordinates on $\text{Conf}_2(\mathbb{R}^d \times \mathbb{C}^\times)$. The cycle $S_{x_2}^{d+1} \times S_0^1 \subset \text{Conf}_2(\mathbb{R}^d \times \mathbb{C}^\times)$ is homologous to $(-1)^d S_0^1 \times S_{x_1}^{d+1}$,³ so we can rewrite this integral as

$$- (-1)^{F_1 d + d} \left[\int_{S_0^1 \times S_{x_1}^{d+1}} f(z_2) dz_1 \wedge e^{-\lambda z_{21}} dz_{21} \wedge (\mathcal{O}_1 \boxtimes \mathcal{O}_2)^* \right]. \quad (34)$$

We expand $f(z_2)$ around z_1 and replace the powers of z_{21} in the series with powers of $-\partial_\lambda$ to get

$$\begin{aligned} & - (-1)^{F_1 d + d} \sum_{k=0}^{\infty} \frac{1}{k!} (-\partial_\lambda)^k \left[\int_{S_0^1 \times S_{x_1}^{d+1}} \partial_z^k f(z_1) dz_1 \wedge e^{-\lambda z_{21}} dz_{21} \wedge (\mathcal{O}_1 \boxtimes \mathcal{O}_2)^* \right] \\ & = - (-1)^{F_1 d + d} \sum_{k=0}^{\infty} \frac{1}{k!} (i P_z \partial_\lambda)^k \left[\int_{S_0^1 \times S_{x_1}^{d+1}} f(z_1) dz_1 \wedge e^{-\lambda z_{21}} dz_{21} \wedge (\mathcal{O}_1 \boxtimes \mathcal{O}_2)^* \right]. \end{aligned} \quad (35)$$

In the equality, we have used the invariance of the Q -cohomology class of the integral under a shift of the cycle in the z -direction. The last expression is equal to

$$- (-1)^{F_1 d + d + F_1 d + F_1(F_2 + d) + F_2 d} \int_{S_0^1} f(z_1) dz_1 \exp(i P_z \partial_\lambda) \{ [\![\mathcal{O}_2]\!]_{-\lambda} [\![\mathcal{O}_1]\!] \} (z_1), \quad (36)$$

where a factor of $(-1)^{F_1 d}$ comes from σ^{F_1} in the definition (19) of $(\mathcal{O}_1 \boxtimes \mathcal{O}_2)^*$, the factor $(-1)^{F_1(F_2 + d)}$ is due to an interchange of $\mathcal{O}_1^{(0)}$ and $\mathcal{O}_2^{(d)}$, and a factor of $(-1)^{F_2 d}$ comes from σ^{F_2} in $(\mathcal{O}_2 \boxtimes \mathcal{O}_1)^*$. This is what we wanted.

The Jacobi identity (30), like the Leibniz rule, is based on the decomposition of a large S^{d+1} surrounding both \mathcal{O}_2 and \mathcal{O}_3 into a small S^{d+1} around \mathcal{O}_2 and another

³ Since $H_{d+2}(\text{Conf}_2(\mathbb{R}^d \times \mathbb{C}^\times)) \cong \mathbb{Z}$, to establish this relation we just need to evaluate a suitable cohomology class on these cycles. We can take this class to be $d(\theta_1 + \theta_2) \wedge d^{-1}(\bigwedge_\mu \delta(x_{12}^\mu) dx_{12}^\mu)$, where $\theta = \arg z$ and $\delta(x)$ is the delta function. To compute the homology group, let $U = \text{Conf}_2(\mathbb{R}^d \times \mathbb{C}^\times)$ and $V \simeq \mathbb{R}^{2d+3} \times S^1$ be a normal neighborhood of the diagonal Δ of $(\mathbb{R}^d \times \mathbb{C}^\times)^2$. Then, $U \cup V = (\mathbb{R}^d \times \mathbb{C}^\times)^2 \simeq \mathbb{R}^{2d+2} \times S^1 \times S^1$ and $U \cap V = V \setminus \Delta \simeq \mathbb{R}^{d+2} \times S^1 \times S^{d+1}$. From the Mayer-Vietoris sequence

$$\cdots \rightarrow H_{d+3}(U \cup V) \rightarrow H_{d+2}(U \cap V) \rightarrow H_{d+2}(U) \oplus H_{d+2}(V) \rightarrow H_{d+2}(U \cup V) \rightarrow \cdots, \quad (33)$$

we get $H_{d+2}(U) \cong H_{d+2}(S^1 \times S^{d+1}) \cong \mathbb{Z}$.

small S^{d+1} around \mathcal{O}_3 . The latter small S^{d+1} gives the second term on the right-hand side.

Obtaining the first term is a little tricky. The shift in μ is easy to understand. What is tricky is that this term involves $(\mathcal{O}_1^* \mathcal{O}_2^*)^*$, while the left-hand side only contains $\mathcal{O}_1^* \mathcal{O}_2^*$, which differs by terms involving \mathbf{Q}_μ acting on \mathcal{O}_1^* . To avoid getting these extra terms, we can use the symmetry (28).

Let us switch to the configuration space viewpoint, as we did in the proof of the symmetry. The relevant cycle in $\text{Conf}_3(\mathbb{R}^d \times \mathbb{C}^\times)$ is one that represents the situation in which \mathcal{O}_1^* is integrated over S^{d+1} containing \mathcal{O}_2^* , while \mathcal{O}_2^* is integrated over S^{d+1} around \mathcal{O}_3^* , and finally \mathcal{O}_3^* is integrated over S^1 around the origin of \mathbb{C}^\times . Up to a sign, this cycle is homologous to one in which \mathcal{O}_2^* is surrounded by \mathcal{O}_1^* , and by \mathcal{O}_3^* farther back, and itself circles around the origin of \mathbb{C}^\times .⁴ Again up to a sign, the second cycle gives $\{\llbracket \mathcal{O}_3 \rrbracket - \lambda - \mu - \partial_z \{\llbracket \mathcal{O}_1 \rrbracket_\lambda \llbracket \mathcal{O}_2 \rrbracket\}\}$ and equals the term in question in the Jacobi identity. To determine the sign, we can go back to the original viewpoint and consider the special case when \mathbf{Q}_μ annihilate $\int_{S_{x_2}^{d+1}} e^{\lambda(z_1 - z_2)} dz_1 \wedge \mathcal{O}_1^*$.

2.5 Poisson vertex algebra

The Q -cohomology of local operators \mathcal{V} is a graded $\mathbb{C}[P_z]$ -module endowed with the λ -bracket $\{\lambda\}: \mathcal{V} \otimes \mathcal{V} \rightarrow \mathcal{V}[\lambda]$ with fermion number

$$F(\{\lambda\}) = -d, \quad (37)$$

satisfying the sesquilinearity (27), the symmetry (28) and the Jacobi identity (30). These properties make \mathcal{V} a *d-shifted Lie conformal algebra* [21].

In addition, the λ -bracket satisfies the Leibniz rule (24). A unital commutative associative *d-shifted Lie conformal algebra*, with the Leibniz rule obeyed, is called a *d-shifted Poisson vertex algebra*. Thus, we have shown that the algebra of local operators in the topological–holomorphic sector of a quantum field theory on $M \times C$ possesses the structure of a *d-shifted Poisson vertex algebra*. We refer the reader to [22] for an introduction to Poisson vertex algebras.

Suppose that the theory has rotation symmetry on C , and let J be its generator. We normalize J in such a way that P_z has $J = 1$. If Q has a definite spin $J(Q)$ (as opposed to being a linear combination of operators with different spins), then \mathcal{V} , as an algebra, is also graded by J . In this case, we can choose \mathbf{Q} to have spin

$$J(\mathbf{Q}) = -J(Q). \quad (38)$$

As it is, the λ -bracket is not compatible with this grading. The problem is that for computation of the λ -bracket, one of the local operators needs to be displaced from the center of rotations x_2 , and while $\llbracket S_{x_2}^{d+1} \rrbracket$ used in the λ -bracket is invariant under rotations, the factor $e^{\lambda(z_1 - z_2)}$ is not. This means that the λ -bracket, when expanded in

⁴ Think of \mathcal{O}_1^* , \mathcal{O}_2^* and \mathcal{O}_3^* as the moon, the earth and the sun, respectively, and consider the heliocentric versus geocentric descriptions of their relative motion. The origin of \mathbb{C}^\times can be the center of our galaxy.

λ , consists of infinitely many parts carrying different values of J . We can, however, remedy this problem if we simultaneously shift the phase of λ . In other words, the λ -bracket respects the J -grading if we assign

$$J(\lambda) = 1. \quad (39)$$

Since its definition contains the one-form dz_1 and d copies of \mathbf{Q} , the λ -bracket changes the J -grading by

$$J(\{\lambda\}) = -J(Q)d - 1. \quad (40)$$

2.6 Dimensional reduction to the Poisson algebra of a TQFT

Poisson vertex algebras are generalizations of Poisson algebras, which are commutative associative algebras endowed with a Lie bracket obeying the Leibniz rule. Indeed, the quotient of \mathcal{V} by the ideal $(\partial_z \mathcal{V})\mathcal{V}$ is a d -shifted Poisson algebra with respect to the Poisson bracket induced from the λ^0 -order part $\{\lambda\}_{|\lambda=0}$ of the λ -bracket.

This Poisson algebra may be interpreted as a structure associated with the $(d+1)$ -dimensional theory that one obtains by taking C to be a cylinder and performing dimensional reduction on the circumferential direction. Since local Q -cohomology classes vary holomorphically on C , after the reduction they become independent of their positions in the longitudinal direction. Thus, the topological–holomorphic sector on $M \times C$ is turned into a topological sector on $M \times \mathbb{R}$. The latter carries a Poisson bracket of degree $-d$ on the algebra of local operators [5], and in favorable situations, $\mathcal{V}/(\partial_z \mathcal{V})\mathcal{V}$ coincides with this Poisson algebra.

In general, the relation between the two Poisson algebras can be more complicated. For instance, the theory on $M \times \mathbb{R}$ may have local operators that come from line operators in $M \times C$ extending in the reduced direction. If these local operators are Q -closed, they may represent Q -cohomology classes that are not present in $\mathcal{V}/(\partial_z \mathcal{V})\mathcal{V}$. If they are not Q -closed, then some elements of $\mathcal{V}/(\partial_z \mathcal{V})\mathcal{V}$ may be paired up with them and get annihilated from the Q -cohomology.

2.7 Lie algebra of zero modes

As a vector space, we can also consider the quotient $\overline{\mathcal{V}} = \mathcal{V}/\partial_z \mathcal{V}$. This is the space of zero modes of local Q -cohomology classes. The quotient map $\mathcal{V} \rightarrow \overline{\mathcal{V}}$ is denoted by \int .

The space $\overline{\mathcal{V}}$ is actually a d -shifted Lie algebra, with the Lie bracket $\{\cdot, \cdot\}$ of degree $-d$ given by

$$\{\int \llbracket \mathcal{O}_1 \rrbracket, \int \llbracket \mathcal{O}_2 \rrbracket\} = \int \{\llbracket \mathcal{O}_1 \rrbracket_\lambda \llbracket \mathcal{O}_2 \rrbracket\}_{|\lambda=0}. \quad (41)$$

Moreover, the Lie algebra $\overline{\mathcal{V}}$ acts on \mathcal{V} by

$$\{\int \llbracket \mathcal{O}_1 \rrbracket, \llbracket \mathcal{O}_2 \rrbracket\} = \{\llbracket \mathcal{O}_1 \rrbracket_\lambda \llbracket \mathcal{O}_2 \rrbracket\}_{|\lambda=0}. \quad (42)$$

As such, $\bar{\mathcal{V}}$ may be regarded as a Lie algebra generating a continuous symmetry of \mathcal{V} . A proof of these statements is straightforward [22].

3 $\mathcal{N} = 2$ supersymmetric field theories in three dimensions

The lowest dimensionality in which the structures of topological–holomorphic sectors arise is three. In this section, we define Poisson vertex algebras for three-dimensional $\mathcal{N} = 2$ supersymmetric field theories and determine them in basic examples. We will take

$$M \times C = \mathbb{R} \times \mathbb{C} \quad (43)$$

and denote the coordinate on \mathbb{R} by t .

3.1 Topological–holomorphic sector

The $\mathcal{N} = 2$ supersymmetry algebra in $2 + 1$ dimensions has four supercharges Q_α , \bar{Q}_α , $\alpha = \pm$, satisfying the commutation relations⁵

$$[Q_\pm, \bar{Q}_\pm] = -P_0 \pm P_2, \quad (44)$$

$$[Q_\pm, \bar{Q}_\mp] = P_1 \mp iZ, \quad (45)$$

$$[Q_\alpha, Q_\beta] = [\bar{Q}_\alpha, \bar{Q}_\beta] = 0, \quad (46)$$

where Z is a real central charge. Performing the Wick rotation $x^0 \rightarrow -ix^3$ and introducing the complex coordinate $z = (x^2 + ix^3)/2$, the commutation relations (44) become

$$[Q_+, \bar{Q}_+] = P_z, \quad [Q_-, \bar{Q}_-] = -P_{\bar{z}}. \quad (47)$$

Taking the x^1 -direction as the topological direction $M = \mathbb{R}$ and the z -plane as the holomorphic direction $C = \mathbb{C}$, we wish to find a supercharge Q such that $Q^2 = 0$ and P_t , $P_{\bar{z}}$ are Q -exact. Moreover, $P_{\bar{z}}$ should not appear in any commutators between Q and other generators so that it is unconstrained in the Q -cohomology. These conditions require Q to take the form

$$Q = aQ_- + d\bar{Q}_-, \quad (48)$$

with a, d being complex numbers. We have $Q^2 = -adP_{\bar{z}}$, so we must set either a or d to zero. Without loss of generality, we can take

$$Q = \bar{Q}_-. \quad (49)$$

From the commutation relations (44)–(46), we see that in general P_t cannot be genuinely Q -exact; rather, the combination $P_t - iZ$ is Q -exact. An obvious way to

⁵ The $\mathcal{N} = 2$ supersymmetry algebra in $2 + 1$ dimensions can be obtained from the $\mathcal{N} = 1$ supersymmetry algebra in $3 + 1$ dimensions by dimensional reduction. For the latter algebra, we follow the conventions of [31], except that we rescale the supercharges by a factor of $1/\sqrt{2}$. We have chosen to perform the reduction along the x^2 -direction and subsequently renamed x^3 to x^2 .

construct a Poisson vertex algebra in such a situation is to take the Q -cohomology in the subalgebra of local operators that have $Z = 0$.

Instead, we may simply regard $P_t - iZ$ as generating a modified translation symmetry in the t -direction, which is “twisted” by the central charge. Given a local operator \mathcal{O} , we define its twisted-translated counterpart ${}^Z\mathcal{O}$ by

$${}^Z\mathcal{O}(t, z, \bar{z}) = \exp(i(P_t - iZ)t) \cdot \mathcal{O}(0, z, \bar{z}) = \exp(Zt) \cdot \mathcal{O}(t, z, \bar{z}). \quad (50)$$

Then, the Q -exact operator $i(P_t - iZ)$ acts on ${}^Z\mathcal{O}$ as ∂_t . We can thus construct a Poisson vertex algebra from local operators translated by $P_t - iZ$.

In many cases, there is a $U(1)$ R-symmetry under which Q_α has charge $R = -1$ and \bar{Q}_α has $R = 1$. Requiring $R(Q) = -R(\mathbf{Q}) = 1$ fixes the one-form supercharge:

$$\mathbf{Q} = iQ_+ dt - iQ_- d\bar{z}. \quad (51)$$

We have $J(Q) = -J(\mathbf{Q}) = -1/2$, so the λ -bracket has $(R, J) = (-1, -1/2)$.

It is often useful to twist the theory by regarding

$$J' = J + \frac{R}{2} \quad (52)$$

as a generator of rotations on C . Under the twisted rotations, $J'(Q) = J'(\mathbf{Q}) = 0$, which means that Q is a scalar and \mathbf{Q} transforms correctly as a differential form. We have

$$J'(\{\lambda\}) = -1. \quad (53)$$

3.2 Free chiral multiplets

Let us determine the Poisson vertex algebra for a free theory of chiral multiplets. This is arguably the simplest $\mathcal{N} = 2$ supersymmetric field theory.

A chiral multiplet Φ consists of a complex bosonic scalar ϕ , a pair of fermionic spinors ψ_\pm and a complex bosonic scalar F . Under the action of the linear combination $\epsilon_- Q_+ - \epsilon_+ Q_- - \bar{\epsilon}_- \bar{Q}_+ + \bar{\epsilon}_+ \bar{Q}_-$ of the supercharges, these fields transform as

$$\begin{aligned} \delta\phi &= \epsilon_- \psi_+ - \epsilon_+ \psi_- , \\ \delta\psi_+ &= i\bar{\epsilon}_- \partial_z \phi - i\bar{\epsilon}_+ (\partial_t + m)\phi + \epsilon_+ F , \\ \delta\psi_- &= i\bar{\epsilon}_- (\partial_t - m)\phi + i\bar{\epsilon}_+ \partial_z \phi + \epsilon_- F , \\ \delta F &= -i\bar{\epsilon}_+ (\partial_z \psi_+ + (\partial_t + m)\psi_-) - i\bar{\epsilon}_- ((\partial_t - m)\psi_+ - \partial_z \psi_-) , \end{aligned} \quad (54)$$

where m is a real mass parameter called the twisted mass. The supersymmetry transformations for the conjugate antichiral multiplet $\bar{\Phi} = (\bar{\phi}, \bar{\psi}_\pm, \bar{F})$ can be obtained by Hermitian conjugation, which exchanges unbarred fields to the corresponding barred fields and vice versa. (Note that $\partial_z^\dagger = \partial_{\bar{z}}$ and $\partial_{\bar{z}}^\dagger = \partial_z$ because the Hermitian conjugation is defined with respect to Minkowski space-time.) The central charges of Φ and $\bar{\Phi}$ are given by their masses:

$$Z(\Phi) = m, \quad Z(\bar{\Phi}) = -m. \quad (55)$$

We consider a theory described by n chiral multiplets $\Phi^a = (\phi^a, \psi_{\pm}^a, F^a)$, $a = 1, \dots, n$, and their conjugates $\bar{\Phi}^{\bar{a}} = (\bar{\phi}^{\bar{a}}, \bar{\psi}_{\pm}^{\bar{a}}, \bar{F}^{\bar{a}})$, with no superpotential. The action of the theory is

$$\begin{aligned} & \int_{\mathbb{R} \times \mathbb{C}} d^3x (\bar{Q}_- Q_- Q_+ \bar{Q}_+) \cdot (g_{a\bar{b}} \phi^a \bar{\phi}^{\bar{b}}) \\ &= \int_{\mathbb{R} \times \mathbb{C}} d^3x g_{a\bar{b}} \left(\partial_{\bar{z}} \phi^a \partial_z \bar{\phi}^{\bar{b}} + (\partial_t + m_a) \phi^a (\partial_t + m_b) \bar{\phi}^{\bar{b}} - F^a \bar{F}^{\bar{b}} \right. \\ & \quad \left. + i(\partial_{\bar{z}} \psi_+^a + (\partial_t + m_a) \psi_-^a) \bar{\psi}_+^{\bar{b}} + i\psi_-^a \partial_z \bar{\psi}_-^{\bar{b}} - i\psi_+^a (\partial_t + m_b) \bar{\psi}_-^{\bar{b}} \right), \quad (56) \end{aligned}$$

where $g_{a\bar{b}} = \delta_{ab}$. We assign $(R, J) = (0, 0)$ to Φ^a . Then, ϕ^a , ψ_{\pm}^a and F^a have $(R, J) = (0, 0)$, $(-1, \pm 1/2)$ and $(-2, 0)$, respectively, and $(-1)^F = (-1)^R$.

To deal with the Q -cohomology of this theory, it is better to switch to a notation suitable for twisted theories. Under the twisted rotations on \mathbb{C} , all fields transform like components of differential forms, which we can make manifest by renaming them as follows:

$$\begin{aligned} \chi^a &= i\psi_+^a dt - i\psi_-^a d\bar{z}, & G^a &= F^a dt \wedge d\bar{z}, \\ \bar{\eta}^{\bar{a}} &= \bar{\psi}_-^{\bar{a}}, & \bar{\xi}_a &= g_{a\bar{b}} \bar{\psi}_+^{\bar{b}} dz, & \bar{G}_a &= g_{a\bar{b}} \bar{F}^{\bar{b}} dz. \end{aligned} \quad (57)$$

The supercharge $Q = \bar{Q}_-$ acts on the fields as

$$\begin{aligned} Q \cdot \phi^a &= 0, & Q \cdot \bar{\phi}^{\bar{a}} &= \bar{\eta}^{\bar{a}}, \\ Q \cdot \chi^a &= d'_{m_a} \phi^a, & Q \cdot \bar{\eta}^{\bar{a}} &= 0, \\ Q \cdot G^a &= d'_{m_a} \chi^a, & Q \cdot \bar{\xi}_a &= \bar{G}_a, \\ & & Q \cdot \bar{G}_a &= 0. \end{aligned} \quad (58)$$

Here,

$$d'_m = dt(\partial_t + m) + d\bar{z}\partial_{\bar{z}} \quad (59)$$

is the d' -operator twisted by mass m . The action (56) can be written as

$$\int_{\mathbb{R} \times \mathbb{C}} Q \cdot \left(g_{a\bar{b}} \chi^a \wedge \star (dt(\partial_t + m_b) + d\bar{z}\partial_{\bar{z}}) \bar{\phi}^{\bar{b}} + 2iG^a \wedge \bar{\xi}_a \right). \quad (60)$$

The Poisson vertex algebra \mathcal{V} is the Q -cohomology taken in the space of twisted-translated local operators. In terms of these operators the twisted masses disappear from the Q -action:

$$Q \cdot {}^Z \chi^a = d'^Z \phi^a, \quad Q \cdot {}^Z G^a = d'^Z \chi^a. \quad (61)$$

It follows that the Q -cohomology is independent of the twisted masses. (The dependence of the action on the twisted masses is Q -exact.) This is a consequence of the locality of the Poisson vertex algebra: the twisted masses may be thought of as vacuum

expectation values of scalars in nondynamical vector multiplets for flavor symmetries and, as such, are determined by the boundary condition at infinity, on which \mathcal{V} does not depend. We will set $m_a = 0$ below to simplify our argument.

Not all local operators are relevant for the computation of \mathcal{V} . Suppose that we rewrite the action in such a way that it couples to the product metric on $\mathbb{R} \times \mathbb{C}$ in a manner that is invariant under diffeomorphisms on \mathbb{R} and those on \mathbb{C} . (This is achieved by introduction of the topological term $\int_{\mathbb{R} \times \mathbb{C}} dt \wedge \phi^* \omega_{\mathbb{C}^n}$ to the action, where $\omega_{\mathbb{C}^n} = i g_{a\bar{b}} d\phi^a \wedge d\bar{\phi}^{\bar{b}}$ is the Kähler form on \mathbb{C}^n .) If V is a vector field generating such diffeomorphisms, $J_V = V^\mu T_{\mu\nu} dx^\nu$ is a current such that an integral of $\star J_V$ acts on local operators by V . The components T_{tt} and $T_{\bar{z}\bar{z}}$ of the stress–energy tensor are Q -exact because the action is Q -exact and variations of the corresponding components of the metric commute with the Q -action, while $T_{t\bar{z}} = T_{\bar{z}t} = 0$ as the metric is of a product form and $T_{z\bar{z}} = 0$ by conformal invariance on \mathbb{C} . As a result, the diffeomorphisms on \mathbb{R} and antiholomorphic reparametrizations on \mathbb{C} act on the Q -cohomology trivially. In particular, Q -closed operators that have nonzero scaling dimension on \mathbb{R} or nonzero antiholomorphic scaling dimension on \mathbb{C} are Q -exact.

The relevant operators are therefore local operators constructed from ϕ^a , $\bar{\phi}^{\bar{a}}$, $\bar{\eta}^{\bar{a}}$, $\bar{\xi}_a$ and \bar{G}_a , as well as their z -derivatives. Among these, \bar{G}_a are auxiliary fields and vanish by equations of motion. Since $g_{a\bar{b}} \partial_z \bar{\eta}^{\bar{b}} = \partial_t \bar{\xi}_{a\bar{z}}$ by an equation of motion, $\partial_z^k \bar{\eta}^{\bar{a}}$ with $k \geq 1$ can be dropped. In the absence of $\partial_z^k \bar{\eta}^{\bar{a}}$ and $\partial_t \bar{\xi}_{a\bar{z}}$, operators that contain $\partial_z^k \bar{\phi}^{\bar{a}}$ cannot be Q -closed. Thus, we can also drop $\partial_z^k \bar{\phi}^{\bar{a}}$.

We may think of $(\phi^a, \bar{\phi}^{\bar{a}})$ as coordinates on the target space

$$X = \mathbb{C}^n \quad (62)$$

and identify $\bar{\eta}^{\bar{a}}$ with $d\bar{\phi}^{\bar{a}}$. Furthermore, for each $k \geq 1$, we may regard $\partial_z^k \phi^a$ as a section of the holomorphic cotangent bundle T_X^\vee , and $\partial_z^{k-1} \bar{\xi}_{a\bar{z}}$ as a section of the tangent bundle T_X . Under this identification, the relevant operators with $J' = j'$ are sections of $\Omega_X^{0,\bullet} \otimes E_X^{j'}$, where $E_X^{j'}$ is a holomorphic vector bundle given by the formal series

$$\bigoplus_{J'=0}^{\infty} u^{J'} E_X^{J'} = \bigotimes_{k=1}^{\infty} \left(\left(\bigoplus_{l=0}^{\infty} u^{lk} S^l T_X^\vee \right) \otimes \left(\bigoplus_{m=0}^{\infty} u^{mk} \Lambda^m T_X \right) \right). \quad (63)$$

For each k , the factor of the l th symmetric tensor power $S^l T_X^\vee$ accounts for operators that take the form $f(\phi, \bar{\phi})_{a_1 \dots a_l} \partial_z^k \phi^{a_1} \dots \partial_z^k \phi^{a_l}$, and the m th exterior power $\Lambda^m T_X$ accounts for those of the form $f(\phi, \bar{\phi})^{a_1 \dots a_m} \partial_z^{k-1} \bar{\xi}_{a_1 \bar{z}} \dots \partial_z^{k-1} \bar{\xi}_{a_m \bar{z}}$.

Using the equations of motion $\bar{G}_a = 0$, we see that Q acts on these sections as the Dolbeault operator $\bar{\partial}$, increasing R by 1 but keeping J' unchanged. We conclude that the Q -cohomology of local operator is isomorphic to the Dolbeault cohomology of this bundle:

$$\mathcal{V} = \bigoplus_{J'=0}^{\infty} \mathcal{V}^{J'}, \quad \mathcal{V}^{J'} \cong H_{\bar{\partial}}^\bullet(X; E_X^{J'}). \quad (64)$$

For $X = \mathbb{C}^n$, we have $H_{\bar{\partial}}^q(X; E_X^{J'}) = 0$ for $q \geq 1$ by the $\bar{\partial}$ -Poincaré lemma. Therefore, \mathcal{V} is isomorphic as an algebra to the algebra of holomorphic sections of E_X^\bullet .

The λ -bracket has $(R, J') = (-1, -1)$. There are three combinations of local operators for which we need to compute the λ -bracket, namely $\{\llbracket \phi^a \rrbracket_\lambda \llbracket \phi^b \rrbracket\}$, $\{\llbracket \tilde{\xi}_{az} \rrbracket_\lambda \llbracket \tilde{\xi}_{bz} \rrbracket\}$ and $\{\llbracket \tilde{\xi}_{az} \rrbracket_\lambda \llbracket \phi^b \rrbracket\}$; the other combinations can be obtained from these by the Leibniz rule, sesquilinearity and symmetry of the λ -bracket.

Since there are no Q -cohomology classes with $R < 0$, we immediately find

$$\{\llbracket \phi^a \rrbracket_\lambda \llbracket \phi^b \rrbracket\} = 0. \quad (65)$$

In general, $\{\llbracket \mathcal{O}_1 \rrbracket_\lambda \llbracket \mathcal{O}_2 \rrbracket\}$ vanishes unless the d th descendant of \mathcal{O}_1 produces a singularity when placed at the same point as \mathcal{O}_2 , for otherwise the integration cycle $S_{x_2}^{d+1}$ can be shrunk to a point. The theory under consideration is a free theory, so the product of a bosonic field and a fermionic one, such as those that appear in $\{\llbracket \phi^a \rrbracket_\lambda \llbracket \phi^b \rrbracket\}$, cannot be singular. For the same reason, we have

$$\{\llbracket \tilde{\xi}_{az} \rrbracket_\lambda \llbracket \tilde{\xi}_{bz} \rrbracket\} = 0. \quad (66)$$

The remaining combination $\{\llbracket \tilde{\xi}_{az} \rrbracket_\lambda \llbracket \phi^b \rrbracket\}$ is a polynomial in λ with $(R, J') = (0, 0)$. Since $J'(\lambda) = 1$ and there are no local Q -cohomology classes with $J' < 0$, it must be actually independent of λ and equal to a Q -cohomology class with $(R, J') = (0, 0)$, that is, a holomorphic function of ϕ^a . The theory has a $U(n)$ global symmetry under which ϕ^a transform in the vector representation and $\tilde{\xi}_a$ in the dual representation, and the λ -bracket must be invariant under this symmetry. These constraints leave the only possibility to be that $\{\llbracket \tilde{\xi}_{az} \rrbracket_\lambda \llbracket \phi^b \rrbracket\}$ is proportional to δ_a^b .

Let us calculate this λ -bracket explicitly. For this calculation, we restore the twisted masses to demonstrate that they do not affect the result.

The first descendant of $\tilde{\xi}_{az}$ is given by

$$\mathbf{Q} \cdot \tilde{\xi}_{az} = g_{a\bar{b}}(dt \partial_z - d\bar{z}(\partial_t + m_a))\bar{\phi}^{\bar{b}}, \quad (67)$$

so we have

$$\begin{aligned} & \{\llbracket \tilde{\xi}_{az} \rrbracket_\lambda \llbracket \phi^b \rrbracket\}(0) \\ &= -\left[\left(\int_{S_0^2} e^{\lambda z} dz \wedge e^{-m_a t} g_{a\bar{c}}(dt \partial_z - d\bar{z}(\partial_t + m_a))\bar{\phi}^{\bar{c}}(x) \right) \phi^b(0) \right] \end{aligned} \quad (68)$$

where the factor of $e^{-m_a t}$ comes from twisted translation. Using the Stokes theorem, we can convert this integral to one over the 3-ball B_0^3 that bounds S_0^2 :

$$-\frac{i}{2} \left[\left(\int_{B_0^3} d^3 x e^{\lambda z - m_a t} g_{a\bar{c}}(\partial_t^2 + \partial_z \partial_{\bar{z}} - m_a^2)\bar{\phi}^{\bar{c}}(x) \right) \phi^b(0) \right]. \quad (69)$$

By integration by part in the path integral, we obtain the identity

$$\frac{\delta \phi^b(0)}{\delta \phi^a(x)} = \frac{\delta S}{\delta \phi^a(x)} \phi^b(0), \quad (70)$$

which holds inside correlation functions as long as no other operators are present at x . The left-hand side is δ_a^b times the delta function supported at $x = 0$, whereas the right-hand side is $-g_{a\bar{c}}(\partial_t^2 + \partial_z \partial_{\bar{z}} - m_a^2)\bar{\phi}^{\bar{c}}(x)\phi^b(0)$. Plugging this relation into the integral (69), we find

$$\{\llbracket \bar{\xi}_{a\bar{z}} \rrbracket_\lambda \llbracket \phi^b \rrbracket\}(0) = \frac{i}{2} \delta_a^b. \quad (71)$$

As expected, the result is proportional to δ_a^b , with the proportionality constant independent of the twisted masses.

Finally, let us consider the Poisson algebra $\mathcal{V}/(\partial_z \mathcal{V})\mathcal{V}$ associated with the dimensional reduction of the theory. The elements of $\mathcal{V}/(\partial_z \mathcal{V})\mathcal{V}$ are in one-to-one correspondence with the local operators constructed from ϕ^a and $\bar{\xi}_{a\bar{z}}$ but not their z -derivatives, that is, the holomorphic sections of $\Lambda^\bullet T_X$. The Poisson bracket on $\mathcal{V}/(\partial_z \mathcal{V})\mathcal{V}$ coincides with the Schouten–Nijenhuis bracket. This Poisson algebra is the one for the B-model with target space X [5].

3.3 Sigma models

The construction of the Poisson vertex algebra discussed above can be generalized to the case when the target space X is not \mathbb{C}^n but any other Kähler manifold. Thus, we get a map

$$X \mapsto \mathcal{V}(X) \quad (72)$$

which assigns to a Kähler manifold X a Poisson vertex algebra $\mathcal{V}(X)$. Classically, $\mathcal{V}(X)$ is still described by the Dolbeault cohomology of X with values in the holomorphic vector bundles (63). Quantum corrections may alter this description, however.

3.4 Free vector multiplets

A theory of free vector multiples provides an example of a sigma model with $X \neq \mathbb{C}^n$. In three dimensions, an abelian gauge field A can be dualized to a periodic scalar γ through the relation $d\gamma = i \star dA$ (in Euclidean signature). Under this dualization process, a vector multiplet is mapped to a chiral multiplet whose scalar field is $\phi = \sigma + i\gamma$, where σ is the adjoint scalar in the vector multiplet. Hence, a theory of n abelian vector multiplets is equivalent to a sigma model with target $X = (\mathbb{R} \times S^1)^n$.

The target space metric enters the action through Q -exact terms, so we can actually take

$$X = (\mathbb{C}^\times)^n, \quad (73)$$

with the standard flat metric on it. The target being flat, the theory is free and there are no quantum corrections to the description of the Poisson vertex algebra as the Dolbeault cohomology.

Compared to the case of $X = \mathbb{C}^n$, we have more ingredients to build Q -cohomology classes from: holomorphic sections of the bundles E_X^\bullet can have poles at $\phi^a = 0$, and $\bar{\eta}^{\bar{a}}/\bar{\phi}^{\bar{a}}$ are Q -closed but not Q -exact. Accordingly, we have more combinations to consider for the λ -bracket.

Specifically, we have the cases when either operator in the λ -bracket involves $\llbracket \bar{\eta}^{\bar{a}}/\bar{\phi}^{\bar{a}} \rrbracket$ and need to evaluate $\{\llbracket \phi^a \rrbracket_\lambda \llbracket \bar{\eta}^{\bar{b}}/\bar{\phi}^{\bar{b}} \rrbracket\}$, $\{\llbracket \bar{\eta}^{\bar{a}}/\bar{\phi}^{\bar{a}} \rrbracket_\lambda \llbracket \bar{\eta}^{\bar{b}}/\bar{\phi}^{\bar{b}} \rrbracket\}$ and $\{\llbracket \bar{\xi}_{az} \rrbracket_\lambda \llbracket \bar{\eta}^{\bar{b}}/\bar{\phi}^{\bar{b}} \rrbracket\}$. However, these additional combinations all vanish. For the first two, this is because there are simply no Q -cohomology classes with $J' = -1$. The last one has $(R, J') = (1, 0)$ and may be proportional to $\llbracket \bar{\eta}^{\bar{a}}/\bar{\phi}^{\bar{a}} \rrbracket$. This vanishes since neither $\bar{\eta}^{\bar{a}}$ nor $\bar{\phi}^{\bar{a}}$ produces a singularity when multiplied by $\bar{\xi}_{az}^{(1)}$.

3.5 $\mathcal{N} = 2$ superconformal field theories

Although the Poisson vertex algebra of an interacting $\mathcal{N} = 2$ supersymmetric field theory is difficult to determine, some general statements can be made about it if the theory is unitary and has conformal symmetry.

In that case, by conformal symmetry the Q -cohomology of local operators is isomorphic as a vector space to the Q -cohomology of states in radial quantization, and by unitarity the latter is isomorphic to the space of Q -harmonic states on which $\{Q, Q^\dagger\} = 0$. In radial quantization, the Hermitian conjugate of Q is a conformal supercharge and satisfies

$$[Q, Q^\dagger] = D - R - J, \quad (74)$$

where D is the dilatation operator. Hence, a local Q -cohomology class is represented uniquely by a local operator with $D - R - J = 0$. If we assign $D(\lambda) = 1$, the λ -bracket has $(D, R, J) = (-3/2, -1, -1/2)$ and preserves $D - R - J$.

There are many $\mathcal{N} = 2$ superconformal multiplets that contain such local operators; see [11] for a comprehensive list. Especially important are those with conserved currents.

For example, there is a flavor current multiplet which contains a fermionic operator Ψ_0 with $(D, R, J) = (3/2, 1, 1/2)$. The first descendant $\Psi_0^{(1)}$ of this operator is given by $d\bar{z} \wedge \Psi_0^{(1)} = \star(j - j_{\bar{z}}d\bar{z})$, where j is the conserved current for a flavor symmetry. The component $j_{\bar{z}}$ is Q -exact, so the zero mode $\int \llbracket \Psi_0 \rrbracket$ acts on \mathcal{V} by an infinitesimal flavor transformation. For the theory of free chiral multiplets, $(\Psi_0)^a_b = \bar{\xi}_{bz}\phi^a$.

Similarly, for each integer $n \geq 1$, there is a multiplet that contains conserved currents and a local operator Ψ_n with $(D, R, J) = (n/2 + 3/2, 1, n/2 + 1/2)$ representing a Q -cohomology class. For $n = 1$, the first descendant $\Psi_1^{(1)}$ is part of a spin-3/2 current, and $\int \llbracket \Psi_1 \rrbracket$ acts by a supersymmetry transformation. The multiplet with $n = 2$ contains a stress-energy tensor, and $\int \llbracket \Psi_2 \rrbracket$ acts by the holomorphic derivative ∂_z . For free chiral multiplets, $(\Psi_2)^a_b = \bar{\xi}_{bz}\partial_z\phi^a$. The multiplets with $n \geq 3$ contain higher spin currents.

4 $\mathcal{N} = 2$ supersymmetric field theories in four dimensions

Next, we turn to Poisson vertex algebras for four-dimensional supersymmetric field theories. In four dimensions, we must have twice as many topological directions as in three dimensions. It turns out that we also need twice as many supercharges, namely eight supercharges, generating $\mathcal{N} = 2$ supersymmetry. We will take

$$M \times C = \mathbb{R}^2 \times \mathbb{C}. \quad (75)$$

4.1 Topological–holomorphic sector

The supercharges $Q_\alpha^A, \bar{Q}_{A\dot{\alpha}}, A = 1, 2, \alpha = \pm$, of the $\mathcal{N} = 2$ supersymmetry algebra in $3 + 1$ dimensions satisfy the commutation relations

$$[Q_\alpha^A, \bar{Q}_{B\dot{\alpha}}] = \sigma_{\alpha\dot{\alpha}}^\mu P_\mu \delta_B^A, \quad (76)$$

$$[Q_\alpha^A, Q_\beta^B] = \epsilon_{\alpha\beta} \epsilon^{AB} Z, \quad (77)$$

$$[\bar{Q}_{A\dot{\alpha}}, \bar{Q}_{B\dot{\beta}}] = -\epsilon_{\dot{\alpha}\dot{\beta}} \epsilon_{AB} \bar{Z}, \quad (78)$$

where $\bar{Z} = Z^\dagger$ is a complex central charge. Our convention is such that

$$\sigma^0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (79)$$

and $\epsilon^{12} = -\epsilon_{12} = -\epsilon_{+-} = -\epsilon_{-+} = 1$. We raise and lower indices as $\bar{Q}_{\dot{\alpha}}^A = \epsilon^{AB} \bar{Q}_{B\dot{\alpha}}, \bar{Q}_{A\dot{\alpha}} = \epsilon_{AB} \bar{Q}_{\dot{\alpha}}^B$.

We perform the Wick rotation $x^0 \rightarrow -ix^4$ and introduce the complex coordinates $w = (x^1 + ix^2)/2$ and $z = (x^3 + ix^4)/2$. The generators $P_{\alpha\dot{\alpha}} = \sigma_{\alpha\dot{\alpha}}^\mu P_\mu$ of translations are then given by

$$\begin{pmatrix} P_{++} & P_{+-} \\ P_{-+} & P_{--} \end{pmatrix} = \begin{pmatrix} P_z & P_w \\ P_{\bar{w}} & -P_{\bar{z}} \end{pmatrix}. \quad (80)$$

We choose the w -plane for $M = \mathbb{R}^2$ and the z -plane for $C = \mathbb{C}$.

We need a supercharge Q such that $Q^2 = 0$ and P_{+-}, P_{-+}, P_{--} are Q -exact. Since P_z should not enter any Q -commutators, Q must be of the form

$$Q = aQ_-^1 + bQ_-^2 - c\bar{Q}_{2\dot{-}} + d\bar{Q}_{1\dot{-}}. \quad (81)$$

We have

$$Q^2 = -\det(A)P_{\bar{z}}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \quad (82)$$

For $Q^2 = 0$, the two rows of the matrix A must be proportional to each other. On the other hand, the Q -commutators of supercharges that contain P_{+-} and P_{-+} can be written as

$$\left[Q, \begin{pmatrix} Q_+^2 & -Q_+^1 \\ \bar{Q}_{1\dot{+}} & \bar{Q}_{2\dot{+}} \end{pmatrix} \right] = \begin{pmatrix} Z & -P_w \\ P_{\bar{w}} & \bar{Z} \end{pmatrix} A. \quad (83)$$

(The commutator on the left-hand side is to be understood as acting on each entry of the matrix.) For P_w and $P_{\bar{w}}$ to both appear on the right-hand side, neither row of A should be zero. Thus, we find

$$Q = a(Q_-^1 + t\overline{Q}_{2\dot{-}}) + b(Q_-^2 - t\overline{Q}_{1\dot{-}}) \quad (84)$$

for some $t \neq 0$.

From the commutators (83), we see that the linear combinations

$$P_w + t^{-1}Z, \quad P_{\bar{w}} - t\overline{Z} \quad (85)$$

are Q -exact. As in the case of three-dimensional $\mathcal{N} = 2$ supersymmetry, we define twisted-translated local operators by

$$\begin{aligned} {}^{t,Z}\mathcal{O}(w, \bar{w}, z, \bar{z}) &= \exp(i(P_w + t^{-1}Z)w + i(P_{\bar{w}} - t\overline{Z})\bar{w}) \cdot \mathcal{O}(0, 0, z, \bar{z}) \\ &= \exp(it^{-1}Zw - it\overline{Z}\bar{w}) \cdot \mathcal{O}(w, \bar{w}, z, \bar{z}) \end{aligned} \quad (86)$$

so that $i(P_w + t^{-1}Z)$ and $i(P_{\bar{w}} - t\overline{Z})$ act on ${}^{t,Z}\mathcal{O}$ as ∂_w and $\partial_{\bar{w}}$, respectively. Twisted-translated local Q -cohomology classes form a Poisson vertex algebra \mathcal{V} .⁶

Let us suppose that the theory has an R-symmetry $SU(2)_R$ under which (Q_α^1, Q_α^2) and $(\overline{Q}_\alpha^1, \overline{Q}_\alpha^2)$ transform as doublets. Then, the structure of \mathcal{V} does not depend on the parameters a, b even though they appear in the formula (84) for Q . Under the action of the complexification $SL(2, \mathbb{C})_R$ of the R-symmetry group, \mathcal{V} gets mapped to an isomorphic algebra. Since the linear combinations of supercharges in the parentheses in the expression (84) form a doublet of $SU(2)_R$, by the transformation by

$$\begin{pmatrix} 0 & -b \\ b^{-1} & a \end{pmatrix} \in SL(2, \mathbb{C})_R \quad (87)$$

we can set

$$Q = Q_-^1 + t\overline{Q}_{2\dot{-}} \quad (88)$$

(assuming $b \neq 0$ without loss of generality). With this choice, Q has charge $R = 1$ under the diagonal subgroup $U(1)_R$ of $SU(2)_R$. Under the rotation symmetry on \mathbb{C} , it has spin $J(Q) = -1/2$.

Also, \mathcal{V} is independent of t if the theory has rotation symmetry on \mathbb{R}^2 . Since Q_-^1 and $\overline{Q}_{2\dot{-}}$ have spin $\tilde{J} = -1/2$ and $1/2$ under the rotation symmetry, we can change the value of t using complexified rotations; in effect, t transforms as if it has weight 1 under this \mathbb{C}^\times -action, just as ∂_w does. Since rotations act on the space of local operators at $w = 0$, it follows that the Q -cohomology of local operators, as a vector space, is independent of t at $w = 0$, hence anywhere on \mathbb{R}^2 . Moreover, by the action of the rotation symmetry, we can also show that the algebra structure and the λ -bracket remain the same under phase rotation of t .⁷ These structures depend holomorphically on t , so they are actually entirely independent of t . Thus, we can take

⁶ The fact that the Q -cohomology of local operators is a Poisson vertex algebra was mentioned in [2–4].

⁷ Here, we cannot use the \mathbb{C}^\times -action because we need to place local operators away from $w = 0$ in order to define these structures. The action by $\alpha \in \mathbb{C}^\times$ transforms ${}^{t,Z}\mathcal{O}(w, \bar{w}, z, \bar{z})$ to ${}^{\alpha t, Z}\mathcal{O}(\alpha w, \alpha^{-1}\bar{w}, z, \bar{z})$ (assuming, for simplicity, that \mathcal{O} is a scalar operator). For αw and $\alpha^{-1}\bar{w}$ to be complex conjugate to each other, we must have $\alpha \in U(1)$ or $w = 0$.

$$t = 1 \quad (89)$$

and

$$Q = Q_-^1 + \overline{Q}_{2-} \cdot \quad (90)$$

We will write ${}^Z\mathcal{O}$ for ${}^{1,Z}\mathcal{O}$:

$${}^Z\mathcal{O}(w, \bar{w}, z, \bar{z}) = \exp(iZw - i\bar{Z}\bar{w}) \cdot \mathcal{O}(w, \bar{w}, z, \bar{z}). \quad (91)$$

The one-form supercharge is not uniquely determined. The general form of \mathbf{Q} that has $(R, J) = (-1, 1/2)$ is

$$\mathbf{Q} = iQ_+^2 dw + i\overline{Q}_{1+} d\bar{w} + i((u-1)Q_-^2 - u\overline{Q}_{1-}) d\bar{z}, \quad (92)$$

with u being an arbitrary complex number. The λ -bracket does not depend on u , however. To see this, pick a boundary condition such that all local Q -cohomology classes vanish at the infinity of \mathbb{R}^2 . Furthermore, we choose to represent the homology class $[[S_{x_2}^3]]$ in the definition (26) of the λ -bracket by a “cylinder” whose “side” is $\mathbb{R}^2 \times S_{z_2}^1 \subset \mathbb{R}^2 \times \mathbb{C}$, where $S_{z_2}^1$ has a fixed radius everywhere except at the infinity of \mathbb{R}^2 where it shrinks to a point; by the Poincaré conjecture, this cylinder is homeomorphic to $S_{x_2}^3$. Then, $\mathbf{Q}_{\bar{z}}$ drops out of the computation of the λ -bracket because the only contributions come from the region in which the pullback of $dz \wedge d\bar{z}$ vanishes.

4.2 Vertex algebras for $\mathcal{N} = 2$ superconformal field theories

If the theory has not only $\mathcal{N} = 2$ supersymmetry but also conformal symmetry, \mathcal{V} can be deformed to a family of vertex algebras V^{\hbar} parametrized by $\hbar \in \mathbb{C}$. This is essentially quantization of \mathcal{V} by Ω -deformation [20, 26], and related via dimensional reduction to the quantization of a Poisson algebra associated with an $\mathcal{N} = 4$ superconformal field theory in three dimensions [5, 8].

An $\mathcal{N} = 2$ superconformal field theory has eight conformal supercharges $S_A^\alpha, \overline{S}_{\dot{\alpha}}^A$, in addition to the Poincaré supercharges $Q_\alpha^A, \overline{Q}_{A\dot{\alpha}}$. In radial quantization, the two sets of supercharges are related by Hermitian conjugation:

$$(Q_\alpha^A)^\dagger = S_A^\alpha, \quad (\overline{Q}_{A\dot{\alpha}})^\dagger = \overline{S}^{A\dot{\alpha}}. \quad (93)$$

Furthermore, the theory has an extra R-symmetry $U(1)_r$, under which $Q_\alpha^A, \overline{S}_{\dot{\alpha}}^A$ have charge $r = 1/2$ and $\overline{Q}_{A\dot{\alpha}}, S_A^\alpha$ have $r = -1/2$. The central charge Z necessarily vanishes because of $U(1)_r$.

Let us introduce the following deformations of P , Q and \mathbf{Q} :

$$P^{\hbar} = P_w dw + P_{\bar{w}} d\bar{w} + P_z dz + (P_{\bar{z}} - \hbar R^2_1) d\bar{z}, \quad (94)$$

$$Q^{\hbar} = Q_-^1 + \overline{Q}_{2-} + \hbar(\overline{S}^{2-} - S_1^-), \quad (95)$$

$$\mathbf{Q}^{\hbar} = \mathbf{Q}. \quad (96)$$

Here, R^A_B are the generators of $SU(2)_R \times U(1)_r \cong U(2)$, in terms of which $R = R^1_1 - R^2_2$ and $r = R^1_1 + R^2_2$. These deformed generators satisfy the relations

$$(Q^{\hbar})^2 = \hbar(\tilde{J} + r), \quad (97)$$

$$[Q^{\hbar}, P^{\hbar}_{\mu}] = -i\hbar\partial_{\mu}\tilde{J}^{\nu}Q^{\hbar}_{\nu}, \quad (98)$$

$$[Q^{\hbar}, P^{\hbar}_{\mu}] = 0, \quad (99)$$

$$[Q^{\hbar}, Q^{\hbar}] = iP^{\hbar}_i dy^i + iP^{\hbar}_{\bar{z}} d\bar{z}. \quad (100)$$

Recall that \tilde{J} is the generator of rotations on \mathbb{R}^2 ; it is normalized in such a way that $[\tilde{J}, P_w] = P_w$ and $[\tilde{J}, P_{\bar{w}}] = -P_{\bar{w}}$. We have also denoted the corresponding vector field by the same symbol:

$$\tilde{J} = w\partial_w - \bar{w}\partial_{\bar{w}}. \quad (101)$$

We regard P^{\hbar} as a twisted translation generator and define twisted-translated local operators by

$$\mathcal{O}^{\hbar}(w, \bar{w}, z, \bar{z}) = \exp(izP^{\hbar}_z + i\bar{z}P^{\hbar}_{\bar{z}}) \cdot \mathcal{O}(w, \bar{w}, 0, 0) \quad (102)$$

so that iP^{\hbar}_{μ} act on \mathcal{O}^{\hbar} as ∂_{μ} .

The deformed supercharge Q^{\hbar} does not square to zero, but instead to the generator

$$\tilde{J}' = \tilde{J} + r \quad (103)$$

of twisted rotations on \mathbb{R}^2 . (In this sense, Q^{\hbar} defines an Ω -deformation [24,25] and induces quantization [34].) We can still consider the Q^{\hbar} -cohomology in the space of operators that are annihilated by \tilde{J}' . The vertex algebra V^{\hbar} is the Q^{\hbar} -cohomology of twisted-translated local operators, placed at $w = 0$ and annihilated by \tilde{J}' . Since $P^{\hbar}_{\bar{z}}$ is Q^{\hbar} -exact, classes in V^{\hbar} vary holomorphically on \mathbb{C} , just as in the undeformed case. Unlike the undeformed case, however, the product $\llbracket \mathcal{O}^{\hbar}_1(z_1) \rrbracket \llbracket \mathcal{O}^{\hbar}_2(z_2) \rrbracket$ of two local Q -cohomology classes can be singular at $z_1 = z_2$ because operators representing these classes are located at the same point $w = 0$ on \mathbb{R}^2 .

The structure of V^{\hbar} is most naturally described in the Kapustin twist [23], in which the rotation generators (\tilde{J}, J) are replaced by (\tilde{J}', J') , where $J' = J + R/2$ as in the case of three-dimensional $\mathcal{N} = 2$ supersymmetry. In the Kapustin twist, Q transforms under rotations as a scalar and P^{\hbar}, Q as one-forms. The R -grading is well defined in V^{\hbar} if we assign

$$R(\hbar) = 2, \quad (104)$$

for then Q^{\hbar} has a definite charge $R(Q^{\hbar}) = 1$, while $R(P^{\hbar}) = 0$ and the twisted translation preserves R . The operator product expansion (OPE) takes the form

$$\llbracket \mathcal{O}^{\hbar}_a(z_1) \rrbracket \llbracket \mathcal{O}^{\hbar}_b(z_2) \rrbracket = \sum_c \frac{C_{ab}{}^c(\hbar) \llbracket \mathcal{O}^{\hbar}_c(z_2) \rrbracket}{(z_1 - z_2)^{h_a + h_b - h_c}}, \quad (105)$$

where $h_a = J'(\mathcal{O}_a)$ and $C_{ab}^c(\hbar)$ are analytic functions of \hbar . Having a singular OPE structure, V^\hbar is a vertex algebra, rather than a Poisson vertex algebra.

For a unitary $\mathcal{N} = 2$ superconformal field theory, it was shown in [26] that V^\hbar is isomorphic to the vertex operator algebra introduced in [6]. The latter is the cohomology of twisted-translated local operators with respect to the supercharge $Q_-^1 + \hbar \bar{S}^{2-}$, which is “half” of Q^\hbar , so to speak. This supercharge has $r = 1/2$, so V^\hbar has an additional grading by r in the unitary case.

We can encode the singular part of the OPE in the λ -bracket $[\]_\lambda: V^\hbar \otimes V^\hbar \rightarrow V^\hbar[\lambda]$ on V^\hbar , defined by

$$\begin{aligned} [[\mathcal{O}_1^\hbar]_\lambda [\mathcal{O}_2^\hbar]](z_2) &= \int_{S^1_{z_2}} \frac{dz_1}{2\pi i} e^{\lambda(z_1 - z_2)} [[\mathcal{O}_1^\hbar](z_1)] [[\mathcal{O}_2^\hbar](z_2)] \\ &= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \sum_c \delta_{h_1 + h_2 - h_c, n+1} C_{12}^c(\hbar) [[\mathcal{O}_c^\hbar](z_2)]. \end{aligned} \quad (106)$$

The λ -bracket may be considered for any vertex algebra, but that of V^\hbar has a special property. In the limit $\hbar \rightarrow 0$, nontrivial Q^\hbar -cohomology classes reduce to nontrivial Q -cohomology classes; the \hbar -correction to Q can destroy but not create cohomology classes, as we will argue shortly. Since the OPE between Q -cohomology classes is regular, the OPE coefficients $C_{ab}^c(\hbar)$ for $h_a + h_b - h_c > 0$ contain only terms of positive order when expanded in powers of \hbar . In particular, the λ -bracket is at least of order \hbar .⁸

It is known that in such a situation, the family of vertex algebras V^\hbar reduces in the limit $\hbar \rightarrow 0$ to a Poisson vertex algebra \mathcal{V}' ; see [22] for a proof. The λ -bracket of \mathcal{V}' is given by

$$[[\mathcal{O}_1]_\lambda [\mathcal{O}_2]] = \lim_{\hbar \rightarrow 0} \frac{1}{\hbar} [[\mathcal{O}_1^\hbar]_\lambda [\mathcal{O}_2^\hbar]]. \quad (107)$$

The appearance of Poisson vertex algebras in this context was previously noted in [7, 9].

Now we show that \mathcal{V}' is a subalgebra of the Poisson vertex algebra \mathcal{V} associated with the topological–holomorphic sector of the theory.

First of all, we note that it makes sense to compare V^\hbar and \mathcal{V} as vector spaces. Since \tilde{J}' is Q -exact, for the computation of the Q -cohomology of local operators we can restrict Q to the kernel of \tilde{J}' . We can also compute it anywhere on \mathbb{R}^2 . Thus, we can compute \mathcal{V} , as a vector space, as the Q -cohomology in the space of local operators that have $\tilde{J}' = 0$ and are located at $w = 0$. This is the space in which the Q^\hbar -cohomology of local operators is defined.

An injection $V^\hbar \rightarrow \mathcal{V}$ is constructed as follows. Let $[[\mathcal{O}^\hbar]]$ be a nontrivial Q^\hbar -cohomology class. We may assume that $[[\mathcal{O}^\hbar]]$ neither vanishes nor diverges in the

⁸ In [26], the construction of vertex algebras was extended to $\mathcal{N} = 2$ supersymmetric gauge theories which are not necessarily conformal. For a nonconformal theory, the associated vertex algebra is anomalous, but the anomaly can be canceled if an $\mathcal{N} = (0, 2)$ supersymmetric surface defect carrying an appropriate vertex algebra is inserted at $w = 0$. Although the combined vertex algebra is well defined, it may not have a classical limit since the vertex algebra on the surface defect may not have one. What goes wrong in the above argument is the assertion that the OPE between Q -cohomology classes is regular.

limit $\hbar \rightarrow 0$; we can multiply $[\![\mathcal{O}^{\hbar}]\!]$ by an appropriate power of \hbar if needed. Thus, we have $\mathcal{O}^{\hbar} = \mathcal{O} + \hbar\mathcal{O}_1$, with \mathcal{O} containing no \hbar and \mathcal{O}_1 nonnegative powers of \hbar . If we write $Q^{\hbar} = Q + \hbar Q_1$, then the relation $(Q^{\hbar})^2 = 0$ (which holds in the space of operators under consideration) implies $[Q, Q_1] = Q_1^2 = 0$, while $Q^{\hbar} \cdot \mathcal{O}^{\hbar} = 0$ implies $Q \cdot \mathcal{O} = Q_1 \cdot \mathcal{O} + Q \cdot \mathcal{O}_1 + \hbar Q_1 \cdot \mathcal{O}_1 = 0$. Hence, \mathcal{O} represents a Q -cohomology class. This class is nontrivial because if $\mathcal{O} = Q \cdot \mathcal{O}'$ for some local operator \mathcal{O}' , then $Q^{\hbar} \cdot (\mathcal{O}_1 - Q_1 \cdot \mathcal{O}') = 0$ and $\mathcal{O}^{\hbar} = Q^{\hbar} \cdot \mathcal{O}' + \hbar(\mathcal{O}_1 - Q_1 \cdot \mathcal{O}')$, in contradiction with the assumption.

To show that the λ -bracket of V^{\hbar} reduces to that of \mathcal{V} in the limit $\hbar \rightarrow 0$, we introduce an equivariant analog of topological–holomorphic descent. The key Eq. (12) is deformed to

$$\begin{aligned} Q^{\hbar} \cdot (\mathcal{O}^{\hbar}(w, \bar{w}, z, \bar{z}))^{\pm*} &= \pm(d' + \hbar\iota\tilde{\mathcal{J}})(\mathcal{O}^{\hbar}(w, \bar{w}, z, \bar{z}))^{\pm*} \\ &+ (\exp(iwP_w^{\hbar} + i\bar{w}P_{\bar{w}}^{\hbar}) \cdot (Q^{\hbar} \cdot \mathcal{O}(0, 0, z, \bar{z})))^{\mp*}. \end{aligned} \quad (108)$$

For a local operator \mathcal{O} that is Q^{\hbar} -closed at $w = 0$, we have

$$Q^{\hbar} \cdot (\omega \wedge (\mathcal{O}^{\hbar})^*) = -d^{\hbar}(\omega \wedge (\mathcal{O}^{\hbar})^*) \quad (109)$$

for any holomorphic one-form ω on C , where

$$d^{\hbar} = d + \hbar\iota\tilde{\mathcal{J}}. \quad (110)$$

Compared to the relation (14), the exterior derivative d is deformed to its equivariant version d^{\hbar} , which squares to the Lie derivative by $\hbar\tilde{\mathcal{J}}$.

Accordingly, we can integrate $\omega \wedge (\mathcal{O}^{\hbar})^*$ over an equivariant cycle Γ^{\hbar} to obtain a Q^{\hbar} -closed operator, and the Q^{\hbar} -cohomology class of the resulting operator depends only on $[\![\mathcal{O}^{\hbar}]\!] \in V^{\hbar}$ and the equivariant homology class $[\![\Gamma^{\hbar}]\!]$. The equivariant homology is the homology with respect to the boundary operator

$$\partial^{\hbar} = \partial + \hbar\mathcal{J}, \quad (111)$$

where \mathcal{J} is the dual of $\iota\tilde{\mathcal{J}}$. Acting on a k -chain located at some fixed value of $\varphi = \arg w$, the operator \mathcal{J} sweeps it around in the φ -direction to produce a $(k+1)$ -chain that is invariant under the action of $\tilde{\mathcal{J}} = -i\partial_{\varphi}$.

Let us consider the Q^{\hbar} -cohomology class

$$\left[\left(\int_{S_{x_2}^3} e^{\lambda(z_1 - z_2)} dz_1 \wedge (\mathcal{O}_1^{\hbar})^*(x_1) \right) \mathcal{O}_2^{\hbar}(x_2) \right]. \quad (112)$$

There is a 2-disk $D_{x_2}^2$ such that $\partial^{\hbar} D_{x_2}^2 = S_{z_2}^1 + \hbar S_{x_2}^3 / 2\pi i$, with $S_{z_2}^1$ being a circle lying in \mathbb{C} , centered at z_2 and located at $w = 0$.⁹ Hence, we have

$$[\![S_{x_2}^3]\!] = -\frac{2\pi i}{\hbar} [\![S_{z_2}^1]\!] \quad (114)$$

in the equivariant homology, and the above Q^{\hbar} -cohomology class is equal, up to an overall numerical factor, to $[\![\mathcal{O}_1^{\hbar}]\!]_{\lambda} [\![\mathcal{O}_2^{\hbar}]\!](z_2)/\hbar$. In the limit $\hbar \rightarrow 0$, the expression (112) reduces to the λ -bracket $\{[\![\mathcal{O}_1]\!]_{\lambda} [\![\mathcal{O}_2]\!]\}(z_2)$, leading to the desired relation between the two λ -brackets.

If the theory is unitary, the limit $\hbar \rightarrow 0$ of V^{\hbar} equals \mathcal{V} and not a proper subalgebra thereof. In this case, the elements of V^{\hbar} are in one-to-one correspondence with the harmonic states of Q^{\hbar} , that is, those states satisfying the condition

$$[Q^{\hbar}, (Q^{\hbar})^{\dagger}] = (1 + \hbar^2)(D - J - R) = 0, \quad (115)$$

where D is the dilatation operator. This condition is independent of \hbar , hence equivalent to the harmonic condition $[Q, Q^{\dagger}] = 0$ characterizing the elements of \mathcal{V} .

The vacuum character of V^{\hbar} computes the Schur limit of the superconformal index of the theory [6]. Since the character is independent of \hbar , the character of \mathcal{V} also equals the Schur index. The Poisson vertex algebra can be defined for nonconformal theories, and its character may be taken as a definition of the Schur index for those theories.

4.3 Free hypermultiplets

Let us determine the Poisson vertex algebra for the theory of a free hypermultiplet. To this purpose, it is convenient to adopt a two-dimensional point of view.

The supercharges Q_{-}^1, \bar{Q}_{2-}^1 and Q_{+}^2, \bar{Q}_{1+}^2 which comprise Q and (part of) \mathbf{Q} form a subalgebra isomorphic to the $\mathcal{N} = (2, 2)$ supersymmetry algebra on \mathbb{R}^2 . The latter may be obtained by dimensional reduction from the $\mathcal{N} = 1$ supersymmetry algebra in four dimensions or the $\mathcal{N} = 2$ supersymmetry algebra in three dimensions. It is generated by four supercharges Q_{\pm}, \bar{Q}_{\pm} , satisfying

$$\begin{aligned} [Q_{+}, \bar{Q}_{+}] &= P_w, & [Q_{-}, \bar{Q}_{-}] &= -P_{\bar{w}}. \\ [Q_{+}, \bar{Q}_{-}] &= Z, & [Q_{-}, \bar{Q}_{+}] &= \bar{Z}, \\ [Q_{+}, Q_{-}] &= 0, & [\bar{Q}_{+}, \bar{Q}_{-}] &= 0, \end{aligned} \quad (116)$$

⁹ In terms of the spherical coordinates $(\psi, \theta, \varphi) \in [0, \pi] \times [0, \pi] \times [0, 2\pi)$, defined by

$$\begin{aligned} x_1^1 - x_2^1 &= r \sin \psi \sin \theta \cos \varphi, \\ x_1^2 - x_2^2 &= r \sin \psi \sin \theta \sin \varphi, \\ x_1^3 - x_2^3 &= r \cos \psi, \\ x_1^4 - x_2^4 &= r \sin \psi \cos \theta, \end{aligned} \quad (113)$$

the 2-disk $D_{x_2}^2$ is located at $\varphi = 0$ and has the boundary $S_{z_2}^1$ at $\theta = 0, \pi$. One can easily show that for an equivariantly closed form, the integral over $S_{x_2}^3$ reduces to an integral over $S_{z_2}^1$.

where Z is a complex central charge. We have the identification

$$Q_-^1 = \bar{Q}_-, \quad \bar{Q}_{2-} = \bar{Q}_+, \quad Q_+^2 = Q_+, \quad \bar{Q}_{1+} = -Q_-, \quad (117)$$

and Z coincides with the central charge in the four-dimensional $\mathcal{N} = 2$ supersymmetry algebra. (In general, the $\mathcal{N} = (2, 2)$ supersymmetry algebra has an additional central charge, which is zero in the present case.)

We may therefore think of any $\mathcal{N} = 2$ supersymmetric field theory on $\mathbb{R}^2 \times \mathbb{C}$ as an $\mathcal{N} = (2, 2)$ supersymmetric field theory on \mathbb{R}^2 . From this two-dimensional point of view, a hypermultiplet consists of a pair of $\mathcal{N} = (2, 2)$ chiral multiplets, which have “continuous indices” (z, \bar{z}) , namely their coordinates on \mathbb{C} .

A chiral multiplet Φ of $\mathcal{N} = (2, 2)$ supersymmetry is the dimensional reduction of a chiral multiplet of $\mathcal{N} = 2$ supersymmetry in three dimensions. Under the action of $\epsilon_- Q_+ - \epsilon_+ Q_- - \bar{\epsilon}_- \bar{Q}_+ + \bar{\epsilon}_+ \bar{Q}_-$, the fields of Φ transform as

$$\begin{aligned} \delta\phi &= \epsilon_- \psi_+ - \epsilon_+ \psi_-, \\ \delta\psi_+ &= i\bar{\epsilon}_- \partial_w \phi + \bar{\epsilon}_+ m\phi + \epsilon_+ F, \\ \delta\psi_- &= -\bar{\epsilon}_- \bar{m}\phi + i\bar{\epsilon}_+ \partial_{\bar{w}} \phi + \epsilon_- F, \\ \delta F &= -i\bar{\epsilon}_+ (\partial_{\bar{w}} \psi_+ + im\psi_-) - i\bar{\epsilon}_- (i\bar{m}\psi_+ - \partial_w \psi_-), \end{aligned} \quad (118)$$

with m a complex mass parameter, the twisted mass of Φ . The central charge is given by

$$Z(\Phi) = m. \quad (119)$$

Let us rename the fields of Φ and its conjugate $\bar{\Phi}$ as

$$\begin{aligned} \chi &= i\psi_+ dw - i\psi_- d\bar{w}, & G &= F dw \wedge d\bar{w}, \\ \bar{\eta} &= \bar{\psi}_+ + \bar{\psi}_-, & \bar{\mu} &= \frac{1}{2}(\bar{\psi}_+ - \bar{\psi}_-). \end{aligned} \quad (120)$$

Then, the action of $Q = \bar{Q}_- + \bar{Q}_+$ on the fields can be written as

$$\begin{aligned} Q \cdot \phi &= 0, & Q \cdot \bar{\phi} &= \bar{\eta}, \\ Q \cdot \chi &= d_m \phi, & Q \cdot \bar{\eta} &= 0, \\ Q \cdot G &= d_m \chi, & Q \cdot \bar{\mu} &= \bar{F}, \\ & & Q \cdot \bar{F} &= 0, \end{aligned} \quad (121)$$

where

$$d_m = dw(\partial_w + im) + d\bar{w}(\partial_{\bar{w}} - i\bar{m}) \quad (122)$$

is the exterior derivative twisted by the twisted mass m . As in the case of $\mathcal{N} = 2$ supersymmetry in three dimensions, in terms of the twisted-translated fields the twisted mass disappears from the formula.

We are interested in a theory consisting of multiple chiral multiplets $\Phi^a = (\phi^a, \psi_{\pm}^a, F^a)$, coupled through a superpotential W , which is a holomorphic function

of ϕ^a . Integrating out the auxiliary fields sets

$$F^a = -g^{a\bar{b}} \frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{b}}}, \quad \bar{F}^{\bar{a}} = -g^{\bar{a}b} \frac{\partial W}{\partial \phi^b}, \quad (123)$$

with $g^{a\bar{b}} = g^{\bar{a}b} = \delta_{ab}$. For a generic superpotential, the fields of the antichiral multiplets $\bar{\Phi}^{\bar{a}}$ do not contribute to the Q -cohomology. Then, the Q -cohomology is spanned (at $w = 0$) by holomorphic functions of ϕ^a , but there is the relation

$$dW = 0 \quad (124)$$

coming from $Q \cdot \bar{\mu}^{\bar{a}}$. Modulo $d\bar{z}$, \mathbf{Q} acts on Φ^a as

$$\mathbf{Q} \cdot \phi^a = \chi^a, \quad \frac{1}{2} \mathbf{Q} \cdot \chi^a = G^a, \quad (125)$$

from which it follows

$$(\phi^a)^* = \phi^a + \chi^a + G^a. \quad (126)$$

A hypermultiplet on $\mathbb{R}^2 \times \mathbb{C}$ consists of a pair of chiral multiplets with opposite masses m and $-m$, whose scalar fields $q^a(z, \bar{z})$, $a = 1, 2$, have $(R, J') = (1, 1/2)$ and are coupled by the superpotential

$$W \propto \int_C dz \wedge d\bar{z} \epsilon_{ab} q^a \partial_{\bar{z}} q^b. \quad (127)$$

The Q -cohomology of local operators is the algebra of holomorphic functions of q^a , and by the relation (124) its classes vary holomorphically on \mathbb{C} , as they should.

The λ -bracket has $(R, J') = (-2, -1)$, hence $\{\llbracket q^a \rrbracket_\lambda \llbracket q^b \rrbracket\}$ is a Q -cohomology class of $(R, J') = (0, 0)$. Furthermore, it should be invariant under the $SU(2)$ symmetry rotating (q^1, q^2) as a doublet. The only possibility is a multiple of ϵ^{ab} . Let us show this explicitly.

As explained near the end of Sect. 4.1, we can compute the λ -bracket by taking the integration cycle to be $\mathbb{R}^2 \times S^1_{z_2}$:

$$\{\llbracket q^a \rrbracket_\lambda \llbracket q^b \rrbracket\}(z_2) = \left[\left(\int_{\mathbb{R}^2 \times S^1_{z_2}} e^{\lambda(z_1 - z_2)} dz_1 \wedge ({}^Z q^a)^*(x_1) \right) q^b(z_2) \right]. \quad (128)$$

We have

$$\begin{aligned} & \left(\int_{\mathbb{R}^2} ({}^Z q^a)^*(x_1) \right) q^b(0) \\ & \propto \left(\int_{\mathbb{R}^2} e^{imw_1 - i\bar{m}\bar{w}_1} g^{a\bar{c}} \epsilon_{\bar{c}\bar{d}} \partial_z \bar{q}^{\bar{d}}(x_1) dw_1 \wedge d\bar{w}_1 \right) q^b(x_2), \end{aligned} \quad (129)$$

which, using the propagator, we can rewrite as

$$\int_{\mathbb{R}^2} e^{imw_1 - i\bar{m}\bar{w}_1} \epsilon^{ab} \partial_{z_1} \left(\int_{\mathbb{R}^4} d^4 p \frac{e^{-ip_\mu(x_1^\mu - x_2^\mu)}}{p^\mu p_\mu + |m|^2} \right) dw_1 \wedge d\bar{w}_1. \quad (130)$$

Integrating over w_1, \bar{w}_1 yields delta functions which set $p_w = m$ and $p_{\bar{w}} = -\bar{m}$, leaving

$$\epsilon^{ab} \partial_{z_1} \left(\int_{\mathbb{R}^2} d^2 p \frac{e^{-ip_z(z_1 - z_2) - i\bar{p}_{\bar{z}}(\bar{z}_1 - \bar{z}_2)}}{p_z p_{\bar{z}}} \right) \propto \frac{\epsilon^{ab}}{z_1 - z_2}. \quad (131)$$

The last proportionality can be seen from the fact that acted on with $\partial_{\bar{z}_1}$, both sides become a delta function supported at $z_1 = z_2$. Performing the integral over $S^1_{z_2}$, we find

$$\{\llbracket q^a \rrbracket_\lambda \llbracket q^b \rrbracket\} \propto \epsilon^{ab}, \quad (132)$$

as expected. The result is also independent of m due to the locality of \mathcal{V} , as explained in Sect. 3.2.

For $m = 0$, the hypermultiplet is conformal. The associated vertex algebra is known to be the algebra of symplectic bosons [6], characterized by the OPE

$$\llbracket (q^a)^{\hbar}(z_1) \rrbracket \llbracket (q^b)^{\hbar}(z_2) \rrbracket \sim \hbar \frac{\epsilon^{ab}}{z_1 - z_2}. \quad (133)$$

In the classical limit $\hbar \rightarrow 0$, this vertex algebra indeed reduces to the Poisson vertex algebra just found.

The Poisson algebra $\mathcal{V}/(\partial_z \mathcal{V})\mathcal{V}$ is isomorphic to that of holomorphic functions on \mathbb{C}^2 with respect to the holomorphic symplectic form $\epsilon_{ab} dq^a \wedge dq^b$. This is the Poisson algebra associated with the Rozansky–Witten twist [27] of a three-dimensional $\mathcal{N} = 4$ hypermultiplet [5].

4.4 Gauge theories

Finally, let us determine the Poisson vertex algebra for an $\mathcal{N} = 2$ supersymmetric gauge theory, constructed from a vector multiplet for a gauge group G and a hypermultiplet in a representation ρ of G .

If ρ is chosen appropriately and the hypermultiplet masses are zero, the theory is conformal. In this case, the associated vertex algebra can be defined, and it is known to be the algebra of gauged symplectic bosons [6, 20, 26]. This algebra is constructed as follows.

Let γ and β be bosonic fields of conformal weight $J' = 1/2$, valued in ρ and its dual ρ^\vee , and b and c be fermionic fields with $J' = 1$ and 0 , valued in the adjoint representation of G . The ghost field c is constrained to have no zero mode. The dynamics of these fields are governed by the action

$$\frac{1}{\hbar} \int_{\mathbb{C}} dz \wedge d\bar{z} (b \partial_{\bar{z}} c + \beta \partial_{\bar{z}} \gamma), \quad (134)$$

which lead to the OPEs

$$\gamma(z_1)\beta(z_2) \sim \frac{\hbar \text{id}_\rho}{z_1 - z_2}, \quad b(z_1)c(z_2) \sim \frac{\hbar C_2(\mathfrak{g})}{z_1 - z_2}. \quad (135)$$

Here, id_ρ is the identity operator on the representation space of ρ , and $C_2(\mathfrak{g})$ is the quadratic Casimir element of the Lie algebra \mathfrak{g} of G . Let $V_{bc-\beta\gamma}^\hbar$ be the vertex algebra generated by β, γ, b, c .

The vertex algebra $V_{bc-\beta\gamma}^\hbar$ has a fermionic symmetry, called the BRST symmetry, whose conserved current is given by

$$J_{\text{BRST}} = \text{Tr}(bcc) - \beta c \gamma. \quad (136)$$

The corresponding charge Q_{BRST} acts on an element $\mathcal{O} \in V_{bc-\beta\gamma}^\hbar$ by

$$Q_{\text{BRST}} \cdot \mathcal{O}(z) = \frac{1}{2\pi i \hbar} \left(\int_{S^1_z} J_{\text{BRST}} \right) \mathcal{O}(z) \quad (137)$$

and satisfies $Q_{\text{BRST}}^2 = 0$. The vertex algebra of the gauge theory is the Q_{BRST} -cohomology of $V_{bc-\beta\gamma}^\hbar$. The vertex algebra for a free massless hypermultiplet corresponds to the case when G is trivial and $(\llbracket q_1 \rrbracket, \llbracket q_2 \rrbracket) = (\gamma, \beta)$.

The Poisson vertex algebra associated with the gauge theory is the classical limit of the BRST cohomology and can be described as follows. The vertex algebra $V_{bc-\beta\gamma}^\hbar$ reduces to a Poisson vertex algebra $\mathcal{V}_{bc-\beta\gamma}$ whose λ -bracket is given by

$$\{\gamma \lambda \beta\} \propto \text{id}_\rho, \quad \{b \lambda c\} \propto C_2(\mathfrak{g}). \quad (138)$$

The action (137) of Q_{BRST} on $V_{bc-\beta\gamma}^\hbar$ is given by $1/\hbar$ times $[J_{\text{BRST}} \lambda]_{|\lambda=0}$. In the limit $\hbar \rightarrow 0$, this becomes the action of the zero mode $\int J_{\text{BRST}}$ on $\mathcal{V}_{bc-\beta\gamma}$. Therefore, the Poisson vertex algebra for the $\mathcal{N} = 2$ superconformal gauge theory is the classical BRST cohomology of $\mathcal{V}_{bc-\beta\gamma}$, whose differential is given by $\{\int J_{\text{BRST}}, \cdot\}$.

The classical BRST cohomology of $\mathcal{V}_{bc-\beta\gamma}$ actually makes sense for any choice of ρ , not necessarily one for which the theory is conformal. This is in contrast to its quantum counterpart: Q_{BRST} squares to zero if and only if the one-loop beta function vanishes. The anomaly in the BRST symmetry arises from double contractions in the OPE between two J_{BRST} , which is of order \hbar^2 and discarded in the classical limit.

Based on this observation, we propose that the Poisson vertex algebra for an $\mathcal{N} = 2$ supersymmetric gauge theory, whether conformal or not, is given by the same classical BRST cohomology.

The character of this Poisson vertex algebra can be calculated by a matrix integral formula. In [12], this formula was employed as a definition of the Schur index for nonconformal theories with Lagrangian descriptions and shown to coincide with a certain wall-crossing invariant in a number of examples.

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References

1. Barakat, A., De Sole, A., Kac, V.G.: Poisson vertex algebras in the theory of Hamiltonian equations. *Jpn. J. Math.* **4**(2), 141–252 (2009). <https://doi.org/10.1007/s11537-009-0932-y>
2. Beem, C.: $4d \mathcal{N} = 2$ SCFTs and VOAs. Talk at Pollica Summer Workshop “Mathematical and Geometric Tools for Conformal Field Theories,” June 3–21 (2019)
3. Beem, C.: Building VOAs out of Higgs branches. *Talk String Math* **2019**(July), 1–5 (2019)
4. Beem, C.: Comments on vertex algebras for $\mathcal{N} = 2$ SCFTs. *Talk String Math* **2017**(July), 24–28 (2017)
5. Beem, C., Ben-Zvi, D., Bullimore, M., Dimofte, T., Neitzke, A.: Secondary products in supersymmetric field theory. [arXiv:1809.00009](https://arxiv.org/abs/1809.00009) [hep-th]
6. Beem, C., Lemos, M., Liendo, P., Peelaers, W., Rastelli, L., van Rees, B.C.: Infinite chiral symmetry in four dimensions. *Comm. Math. Phys.* **336**(3), 1359–1433 (2015). <https://doi.org/10.1007/s00220-014-2272-x>
7. Beem, C., Meneghelli, C., Rastelli, L.: Free field realizations from the Higgs branch. [arXiv:1903.07624](https://arxiv.org/abs/1903.07624) [hep-th]
8. Beem, C., Peelaers, W., Rastelli, L.: Deformation quantization and superconformal symmetry in three dimensions. *Commun. Math. Phys.* **354**, 345–392 (2017). <https://doi.org/10.1007/s00220-017-2845-6>
9. Beem, C., Rastelli, L.: Vertex operator algebras, Higgs branches, and modular differential equations. *JHEP* **08**, 114 (2018). [https://doi.org/10.1007/JHEP08\(2018\)114](https://doi.org/10.1007/JHEP08(2018)114)
10. Beilinson, A., Drinfeld, V.: *Chiral Algebras*. American Mathematical Society Colloquium Publications, vol. 51. American Mathematical Society, Providence (2004)
11. Cordova, C., Dumitrescu, T.T., Intriligator, K.: Multiplets of superconformal symmetry in diverse dimensions. *JHEP* **03**, 163 (2019). [https://doi.org/10.1007/JHEP03\(2019\)163](https://doi.org/10.1007/JHEP03(2019)163)
12. Córdova, C., Shao, S.H.: Schur indices, BPS particles, and Argyres–Douglas theories. *JHEP* **01**, 040 (2016). [https://doi.org/10.1007/JHEP01\(2016\)040](https://doi.org/10.1007/JHEP01(2016)040)
13. Costello, K., Dimofte, T., Gaiotto, D.: Boundary chiral algebras and holomorphic twists. [arXiv:2005.00083](https://arxiv.org/abs/2005.00083)
14. Dimofte, T., Gaiotto, D., Gukov, S.: 3-manifolds and 3d indices. *Adv. Theor. Math. Phys.* **17**, 975 (2013). <https://doi.org/10.4310/ATMP.2013.v17.n5.a3>
15. Dimofte, T., Gaiotto, D., Gukov, S.: Gauge theories labelled by three-manifolds. *Commun. Math. Phys.* **325**, 367 (2014). <https://doi.org/10.1007/s00220-013-1863-2>
16. Enriquez, B., Frenkel, E.: Geometric interpretation of the Poisson structure in affine Toda field theories. *Duke Math. J.* **92**(3), 459–495 (1998). <https://doi.org/10.1215/S0012-7094-98-09214-6>
17. Frenkel, E., Ben-Zvi, D.: *Vertex Algebras and Algebraic Curves*. Mathematical Surveys and Monographs, vol. 88, 2nd edn. American Mathematical Society, Providence (2004). <https://doi.org/10.1090/surv/088>
18. Gaiotto, D.: $N = 2$ dualities. *JHEP* **08**, 034 (2012). [https://doi.org/10.1007/JHEP08\(2012\)034](https://doi.org/10.1007/JHEP08(2012)034)
19. Gaiotto, D., Moore, G.W., Neitzke, A.: Wall-crossing, Hitchin systems, and the WKB approximation. *Adv. Math.* **234**, 239–403 (2013). <https://doi.org/10.1016/j.aim.2012.09.027>

20. Jeong, S.: SCFT/VOA correspondence via Ω -deformation. *JHEP* **10**, 171 (2019). [https://doi.org/10.1007/JHEP10\(2019\)171](https://doi.org/10.1007/JHEP10(2019)171)
21. Kac, V.: *Vertex Algebras for Beginners*. University Lecture Series, vol. 10, 2nd edn. American Mathematical Society, Providence (1998). <https://doi.org/10.1090/ulect/010>
22. Kac, V.: Introduction to vertex algebras, Poisson vertex algebras, and integrable Hamiltonian PDE. In: Callegaro, F., Carnovale, G., Caselli, F., De Concini, C., De Sole, A. (eds.) *A Perspectives in Lie Theory*. Springer INdAM Series, vol. 19, pp. 3–72. Springer, Cham (2017)
23. Kapustin, A.: Holomorphic reduction of $\mathcal{N} = 2$ gauge theories, Wilson–’t Hooft operators, and S-duality. [arXiv:hep-th/0612119](https://arxiv.org/abs/hep-th/0612119)
24. Nekrasov, N.A.: Seiberg–Witten prepotential from instanton counting. *Adv. Theor. Math. Phys.* **7**(5), 831–864 (2003). <https://doi.org/10.4310/ATMP.2003.v7.n5.a4>
25. Nekrasov, N.A., Okounkov, A.: Seiberg–Witten theory and random partitions. In: Etingof, P., Retakh, V., Singer, I.M. (eds.) *Singer the Unity of Mathematics*. Progress in Mathematics, vol. 244, p. 525. Birkhäuser, Boston (2006). https://doi.org/10.1007/0-8176-4467-9_15
26. Oh, J., Yagi, J.: Chiral algebras from Ω -deformation. *JHEP* **08**, 143 (2019). [https://doi.org/10.1007/JHEP08\(2019\)143](https://doi.org/10.1007/JHEP08(2019)143)
27. Rozansky, L., Witten, E.: Hyper-Kähler geometry and invariants of three-manifolds. *Selecta Math.* (N.S.) **3**(3), 401 (1997). <https://doi.org/10.1007/s000290050016>
28. Safronov, P.: Braces and Poisson additivity. *Compos. Math.* **154**(8), 1698–1745 (2018). <https://doi.org/10.1112/s0010437x18007212>
29. Terashima, Y., Yamazaki, M.: $SL(2, \mathbb{R})$ Chern-Simons, Liouville, and gauge theory on duality walls. *JHEP* **08**, 135 (2011). [https://doi.org/10.1007/JHEP08\(2011\)135](https://doi.org/10.1007/JHEP08(2011)135)
30. Terashima, Y., Yamazaki, M.: Semiclassical analysis of the 3d/3d relation. *Phys. Rev. D* **88**(2), 026011 (2013). <https://doi.org/10.1103/PhysRevD.88.026011>
31. Wess, J., Bagger, J.: *Supersymmetry and Supergravity*. Princeton Series in Physics, 2nd edn. Princeton University Press, Princeton (1992)
32. Witten, E.: Topological quantum field theory. *Commun. Math. Phys.* **117**(3), 353–386 (1988). <https://doi.org/10.1007/BF01223371>
33. Witten, E.: Topological sigma models. *Commun. Math. Phys.* **118**(3), 411–449 (1988). <https://doi.org/10.1007/BF01466725>
34. Yagi, J.: Ω -deformation and quantization. *JHEP* **08**, 112 (2014). [https://doi.org/10.1007/JHEP08\(2014\)112](https://doi.org/10.1007/JHEP08(2014)112)