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# Chiral Algebras of (0, 2) Models: Beyond Perturbation Theory

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**Abstract.** We show that the chiral algebras of $\mathcal{N} = (0, 2)$ sigma models with no left-moving fermions are totally trivialized by worldsheet instantons for flag manifold target spaces. Consequently, supersymmetry is spontaneously broken in these models. Our results affirm Stolz's idea (Stolz in Math Ann 304(4):785–800, 1996) that there are no harmonic spinors on the loop spaces of flag manifolds. Moreover, they also imply that the kernels of certain twisted Dirac operators on these target spaces will be empty under a quantum deformation of their geometries.

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## 1. Introduction

Twisted $\mathcal{N} = (0, 2)$ models possess a nilpotent global fermionic symmetry $Q$. This leads to the construction of two kinds of $Q$-cohomology groups, namely, the cohomology of operators and the cohomology of states. Let us focus on the cohomology of *local* operators. This is an infinite-dimensional space graded by the right-moving $R$ charge, and is equipped with a natural ring structure defined by the OPE. Unlike the $\mathcal{N} = (2, 2)$ case, twisted $(0, 2)$ models are not topological due to the absence of the left-moving fermionic symmetry. Rather, the cohomology class of an observable varies *holomorphically* in the insertion point (i.e., $\partial_{\bar{z}}\mathcal{O}$ is $Q$-exact if $\mathcal{O}$ is $Q$-closed). Such $Q$-cohomology of local operators will be called the *chiral algebra* of a $(0, 2)$ model, which we will denote by $\mathcal{A}$.

In this paper, we will consider twisted $(0, 2)$ sigma models with no left-moving fermions [2]. At the perturbative level, the chiral algebras of these models were studied by Witten [3]. One of the main results of [3] is that the chiral algebras can be reconstructed, to all orders in perturbation theory, by gluing *free $\beta\gamma$ CFTs* over the target space. More precisely, the perturbative chiral algebra can be formulated as the cohomology of the sheaf of chiral differential operators, which was developed previously by Malikov et al. [4]. From this point of view, the moduli of the models arise as different ways to glue free theories globally over the target space, while the obstruction to doing so consistently is encoded in the sigma model anomalies [5]. The description of chiral algebras by free CFTs naturally extends to include left-moving fermions [6].

Nonperturbatively, worldsheet instantons can change the picture radically [7,8]. A particularly striking example is the model with the target space being the complete flag manifold $G/T$ of a compact semisimple Lie group $G$, where $T$ is the maximal torus. This includes the manifold $SU(3)/U(1)\times U(1)$, which was recently proposed by Tomasiello [9] to be a novel flux vacuum of string theory. The perturbative chiral algebra in this case possesses a structure of a $\hat{\mathfrak{g}}$-module at the critical level $k=-h^\vee$ [4,10,11]. As we will see, however, the chiral algebra becomes trivial (i.e. identically zero) in the presence of instantons. The mechanism of this trivialization is somewhat analogous to the lifting of perturbative ground states in the Witten complex [12] of supersymmetric quantum mechanics. A similar result, albeit derived via a different approach, has also been obtained by Frenkel et al. [13].

In the following we will explain how this phenomenon actually takes place. But before going into the details, let us point out some of its implications. From the fact that the chiral algebra is trivial, it follows that the $Q$-cohomology of states is also trivial. Supersymmetric states, which are harmonic states of $Q$, therefore do not exist; in other word, supersymmetry is spontaneously broken. On the other hand, the cohomology of states can also be considered as the cohomology of the Dirac operator on $LX$, whose index computes the Witten genus of $X$ [14,15]. Our result thus indicates that the kernel of the Dirac operator on $LX$ is empty for $X=G/T$. This is consistent with Stolz's idea [1] that if $X$ has positive Ricci curvature – such is the case for flag manifolds – then the scalar curvature of $LX$ is positive and hence there are no harmonic spinors.

## 2. Generalities

To begin, let us briefly review the relevant features of the models that will be studied. We refer the reader to [3,7] for more details.

### 2.1. THE MODEL

Let $\Sigma$ be a Riemann surface and $X$ a Kähler manifold. Our $(0,2)$ model with worldsheet $\Sigma$ and target space $X$ then has a bosonic field $\phi\colon \Sigma\to X$, and right-moving fermionic fields

$$\rho\in\Gamma(\overline{K}\otimes\phi^*TX),\qquad \alpha\in\Gamma(\phi^*\overline{TX}),\tag{1}$$

where $\overline{K}$ is the anticanonical line bundle of $\Sigma$. The dynamics of the model is governed by the action

$$S = \int_\Sigma d^2z\, g_{i\bar{j}} \left( \partial_{\bar{z}}\phi^i \partial_z \phi^{\bar{j}} + \rho^i_{\bar{z}} D_z \alpha^{\bar{j}} \right) + \int_\Sigma \phi^* \omega. \tag{2}$$

Here, $D_z\alpha^{\bar{j}} = \partial_z \alpha^{\bar{j}} + \partial_z \phi^{\bar{j}} \Gamma^{\bar{i}}_{\bar{j}\bar{k}} \alpha^{\bar{k}}$ is the covariant derivative and $\omega = i g_{i\bar{j}} d\phi^i \wedge d\phi^{\bar{j}}$ is the Kähler form of $X$.

The fermionic symmetry $Q$ acts on operators via the supercommutator, $\delta A = \{Q, A\}$, with transformation laws given by

$$\begin{aligned} &\delta\phi^i = 0, && \delta\phi^{\bar{i}} = \alpha^{\bar{i}}, \\ &\delta\rho^i_{\bar{z}} = -\partial_{\bar{z}}\phi^i, && \delta\alpha^{\bar{i}} = 0. \end{aligned} \tag{3}$$

The cohomological nature of the theory arises from the fact that the action can be written as a $Q$-exact term $\int_\Sigma d^2z \{Q, -g_{i\bar{j}} \rho^i_{\bar{z}} \partial_z \phi^{\bar{j}}\}$ plus a topological invariant. Under the right-moving $R$ symmetry, $\alpha$ has charge 1 and $\rho$ has charge $-1$. Note that $Q$ has charge 1 and is a worldsheet scalar. We will denote the $Q$-cohomology group of charge $q$ by $\mathscr{A}^q$. The chiral algebra is then $\mathscr{A} = \oplus_q \mathscr{A}^q$.

## 2.2. CONFORMAL INVARIANCE

Classically, the theory is conformally invariant and the stress–energy tensor commutes with $Q$. Indeed, by coupling the action with the worldsheet metric $h_{ab}$ and taking a variation $\delta h_{ab}$, one finds $T_{z\bar{z}} = T_{\bar{z}z} = 0$ and

$$T_{zz} = g_{i\bar{j}} \partial_z \phi^i \partial_z \phi^{\bar{j}}, \quad T_{\bar{z}\bar{z}} = \left\{ Q, g_{i\bar{j}} \rho^i_{\bar{z}} \partial_{\bar{z}} \phi^{\bar{j}} \right\}. \tag{4}$$

Being $Q$-exact $T_{\bar{z}\bar{z}}$ is manifestly $Q$-closed, while $T_{zz}$ commutes with $Q$ upon using the equation of motion $D_z\alpha^{\bar{i}} = 0$. Thus, the classical chiral algebra is invariant under conformal transformations. Moreover, it is nontrivial only for the subspace of local operators in which $\overline{L}_n = \oint d\bar{z}\, \bar{z}^{n+1} T_{\bar{z}\bar{z}}$ are zero for all $n$. In particular, a $Q$-closed operator with nonzero antiholomorphic scaling dimension $\overline{L}_0 = \bar{h}$ is $Q$-exact. From this follows the holomorphy of the chiral algebra. Also, since operators of negative $R$ charge must contain at least one $\rho$ field, they have $\bar{h} > 0$ and do not contribute to the chiral algebra.

Quantum mechanically, conformal invariance is in general broken. Still, one can show that $T_{z\bar{z}}$ and $T_{\bar{z}\bar{z}}$ commute with $Q$; in fact, they are both $Q$-exact. However, $T_{zz}$ may no longer be $Q$-closed. In such a case, the quantum chiral algebra lacks the invariance under holomorphic reparametrizations. When $c_1(X) \neq 0$, there exists [7] a perturbative cohomology class $\theta \in \mathscr{A}^1$ which satisfies

$$[Q, T_{zz}] = \partial_z \theta. \tag{5}$$

Explicitly, $\theta$ is given at the lowest order in perturbation theory by

$$\theta = R_{i\bar{j}} \partial_z \phi^i \alpha^{\bar{j}}. \tag{6}$$

Thus, perturbatively $T_{zz}$ is lifted out of the chiral algebra by quantum corrections, and is "connected" to $\partial_z\theta$. This is a reflection of the one-loop beta function which is proportional to the Ricci curvature of $X$ [16,17]. Notice that perturbative corrections can only connect a pair of local operators of the same scaling dimensions. Instanton corrections, however, need not do so.

### 2.3. INSTANTONS

The $Q$-cohomology is invariant under a smooth deformation of the target metric. We can in particular take a large volume limit, in which the path integral for correlation function localizes to the $Q$-invariant bosonic field configurations, called instantons. In the present case, instantons $\phi_0$ satisfy

$$\partial_{\bar{z}}\phi_0^i = 0, \tag{7}$$

so they are holomorphic maps from $\Sigma$ to $X$. We will label the space of instantons by their degree, defined by the Kähler form $\omega$ of $X$ normalized so that

$$\int_\Sigma \phi_0^*\omega = kt \tag{8}$$

for integers $k \geq 0$. The $k$-instanton moduli space $\mathscr{M}_k$ is then the space of instantons of degree $k$ (or $k$-instantons). Zero-instantons are constant maps, and their moduli space can be identified with the target space: $\mathscr{M}_0 \cong X$. The correlation function now decomposes into different instanton sectors and takes the form

$$\langle \ldots \rangle = \sum_{k=0}^{\infty} \mathrm{e}^{-kt} \langle \ldots \rangle_k \,. \tag{9}$$

The $k$-instanton correlation function $\langle \ldots \rangle_k$ involves an integration over $\mathscr{M}_k$. In particular, the perturbative correlation function $\langle \ldots \rangle_0$ contains an integration over $X$.

Corresponding to the expansion (9) of the correlation function, one can also expand an operator $A$ in the instanton weight:

$$A = \sum_{k=0}^{\infty} \mathrm{e}^{-kt} A_k. \tag{10}$$

This can be justified by considering the matrix elements between arbitrary states. To obtain the matrix element $\langle a|A|b\rangle$, one quantizes the theory on a cylinder of infinitesimal length and compute the corresponding path integral with the initial state $|b\rangle$ and the final state $|a\rangle$. If the theory is conformally invariant, one can rescale the cylinder to make it infinitely long, and then compactify to a sphere by adding points at infinity. The matrix element then reduces to a three-point function with vertex operators inserted at $z=0$ and $\infty$:

$$\langle a|A|b\rangle = \langle \mathscr{V}_a(\infty) A \mathscr{V}_b(0)\rangle \,. \tag{11}$$

Together with (9), this makes it clear that one can expand $A$ to obtain (10) as an operator relation. The form of the expansion (10) remains unchanged even when conformal invariance is broken by quantum corrections as in the case for $c_1(X) \neq 0$. This is because at the lowest order in the perturbation theory around instantons, the theory can be treated as conformally invariant; the explicit computations of correlation functions will be carried out with respect to a free field action that is quadratic in the fluctuating fields.[1] Then, the matrix elements can still be computed from three-point functions. However, $A_k$ will now receive perturbative corrections.

A subtlety in the above argument is that the twisted model which we have been discussing so far is anomalous on the sphere if $c_1(X) \neq 0$. However, note that on the cylinder or the sphere with two points removed, the canonical line bundle is trivial. Consequently, the twisting by $\overline{K}^{1/2}$ does nothing upon choosing the trivial spin structure, and the physical and twisted models are equivalent. As such, one may as well work with the physical model, which is free of the above anomaly anyway.

## 2.4. PATH INTEGRAL MEASURE AND ANOMALIES

To actually perform the path integral computation in the $k$-instanton sector, one must define the fermionic path integral measure in the neighborhood of the instanton moduli space $\mathscr{M} = \bigcup_k \mathscr{M}_k$ in the field space. To this end, let us introduce unitary eigenmodes of the laplacians $\Delta_F = h^{z\bar{z}} D_z D_{\bar{z}}$ and $\Delta_F^\dagger = h^{z\bar{z}} D_{\bar{z}} D_z$:

$$\begin{aligned} \Delta_F(\phi) u_{0,r}(z,\bar{z};\phi) &= 0, \quad \Delta_F(\phi) u_n(z,\bar{z};\phi) = \lambda_n(\phi) u_n(z,\bar{z};\phi), \\ \Delta_F^\dagger(\phi) v_{0,s}(z,\bar{z};\phi) &= 0, \quad \Delta_F^\dagger(\phi) v_n(z,\bar{z};\phi) = \lambda_n(\phi) v_n(z,\bar{z};\phi), \end{aligned} \tag{12}$$

One can then expand the fermions as

$$\begin{aligned} \rho(z,\bar{z};\phi) &= \sum_s b_0^s v_{0,s}(z,\bar{z};\phi) + \sum_n b^n v_n(z,\bar{z};\phi), \\ \alpha(z,\bar{z};\phi) &= \sum_r c_0^r \bar{u}_{0,r}(z,\bar{z};\phi) + \sum_n c^n \bar{u}_n(z,\bar{z};\phi), \end{aligned} \tag{13}$$

where $b_0^s$, $c_0^r$, $b^n$, $c^n$ are grassmannian coefficients. The fermionic path integral measure is then defined by the formal product

$$\prod_{r,s,n} \mathrm{d}b_0^s \mathrm{d}c_0^r \mathrm{d}b^n \mathrm{d}c^n. \tag{14}$$

[1] Actually, the path integral measure may transform nontrivially under a conformal transformation, in which case conformal invariance is broken even at the lowest order. However, in proving a relation such as (10), one can redefine the $k$-instanton measure by multiplying a quantity $C_k$ that cancels the conformal anomaly. The theory with the modified measure is now conformally invariant. To be consistent, one then has to multiply the operator by $C_k^{-1}$. This just changes each matrix element by some factor, and will not alter the general form (10).

Of course, this expression is defined assuming that the eigenmodes (12) vary smoothly over a specific local patch in the field space. For each local patch one can define a measure of the above form, and they must glue consistently over the entire field space. This is only possible if the sigma model anomalies are absent [5], which in our case are the familiar $p_1(X)/2$ anomaly and the additional $c_1(\Sigma)c_1(X)$ anomaly introduced by twisting [3].

Let us make several observations that will be important when we consider the trivialization of the chiral algebra. First, we see from (14) that the fermionic measure has $R$ charge violation equal to the difference in the numbers of $\alpha$ zero modes and the $\rho$ zero modes. On a compact Riemann surface $\Sigma$ of genus $g$, this is given by the index theorem as

$$(1-g)\cdot\dim_{\mathbb{C}} X + \int_{\Sigma} \phi^* c_1(X). \tag{15}$$

The first term is absent in the untwisted (or physical) models, where the fermions are worldsheet spinors. Second, the scaling dimension is in general also violated, since scale transformations may act nontrivially on the measure. However, this is only relevant when one considers relations involving two or more instanton sectors with different scaling properties, since an overall factor can be dropped by taking normalized correlation functions. Finally, notice that the eigenmodes (12) in terms of which the fermions are expanded are themselves functions of fluctuating quantum fields. In particular, even the zero mode part of the fermionic fields can produce short-distance singularities.

## 3. The Mechanism of Trivialization

Let us now discuss a general mechanism that renders the chiral algebra trivial. The starting point is the following observation: The chiral algebra is trivial if and only if there exists a local operator $\Theta$ which satisfies

$$\{Q,\Theta\}=1. \tag{16}$$

This is true, for if $\mathscr{O}$ is a $Q$-closed operator, then $\mathscr{O}=\{Q,\Theta\mathscr{O}\}$; conversely, if the $Q$-cohomology is trivial, then the constant operator 1 must be $Q$-exact since it is $Q$-closed. A trivial chiral algebra characterized by (16) implies that the $Q$-cohomology of states is also trivial: a $Q$-closed state $|\Psi\rangle$ can be written as $|\Psi\rangle = Q(\Theta|\Psi\rangle)$. Clearly, it suffices for our purpose to find an operator $V$ which gives

$$\{Q,V\}=W, \tag{17}$$

where $W$ is an *invertible* local operator – by the nilpotency of the supersymmetry transformation, $W$ must be $Q$-closed, and thus $\{Q,W^{-1}V\}=1$.

The relation (16) cannot be induced by perturbative effects. To see this, note that $\Theta$ must have charge $-1$ for (16) to hold at the perturbative level, since the $R$ symmetry is not violated perturbatively. The local operator $\Theta$ must thus have $\bar{h}>0$, but since the scaling dimension is not violated either and $Q$ is dimensionless, $\{Q, \Theta\}$ cannot be equal to 1. Hence, the trivialization of the chiral algebra can only be a purely nonperturbative phenomenon, induced by worldsheet instantons.

Suppose that the trivialization occurs at the $(l+1)$-instanton level with $l \geq 0$, so that up to the $l$-instanton level, the chiral algebra is nontrivial. Then, there exist operators $V$ and $W$ which satisfy

$$\{Q, V\} = W = \mathrm{e}^{-(l+1)t} W_{l+1} + \cdots, \tag{18}$$

where $W$ is expanded in accordance with (10). This equation implies that $V$ is $Q$-closed up to the $l$-instanton level. Let us see if $V$ can be $Q$-exact at the $m$-instanton level for any $m \leq l$. If not, then $V$ will represent a cohomology class up to the $l$-instanton level. If $V$ were to be $Q$-exact at the $m$-instanton level, then it would be written as $V = \{Q, U\} + \mathrm{e}^{-(m+1)t} V'$ for some $U$ and $V'$. Plugging this into (18), one finds

$$\{Q, V'\} = W' = \mathrm{e}^{-(l-m)t} W'_{l-m} + \cdots \tag{19}$$

with $W'_{l-m} = W_{l+1}$. Notice that $W'$ is invertible if and only if $W'_{l-m}$ is invertible, for small higher order corrections cannot affect the invertibility. Now, we know that $W$ is invertible. Applying the same argument, we conclude that its lowest order term $W_{l+1} = W'_{l-m}$ is also invertible, and hence so is $W'$. It then follows from (19) that the chiral algebra is trivialized at the $(l-m)$-instanton level. But since we assumed that the chiral algebra was nontrivial up to the $l$-instanton level, this is a contradiction. Hence, $V$ must represent a cohomology class up to the $l$-instanton level.

The above argument about the invertibility of $W$ also shows that the chiral algebra will remain trivial through all higher instanton levels if it is trivialized at the $(l+1)$-instanton level. In this paper, we will present examples where this occurs at $l = 0$, i.e., the operator $V$ is a perturbative $Q$-cohomology class. Essentially, the trivialization will be a consequence of the fact that the grading by the $R$ charge is anomalously broken to $\mathbb{Z}_2$ in all these examples, in a specific way that will be made clear later.

## 4. The $\mathbb{CP}^1$ Model

The simplest model which exhibits the trivialization of the chiral algebra is when the target space $X = \mathbb{CP}^1$. This is an important example – it serves as a basis for the analysis of more interesting models that will be studied in the next section.

The perturbative chiral algebra of the model with $c_1(X) \neq 0$ lacks conformal invariance due to the lifting of $T_{zz}$. From (5), we see a similarity between $\theta$ and the constant operator 1; namely, both of these operators have vanishing derivatives in the $Q$-cohomology. In the case $X = \mathbb{CP}^1$, there is an even deeper connection between them. Reflecting the geometry of the target space $\mathbb{CP}^1 \simeq SL_2/B$, the model possesses $\widehat{sl}_2$ currents in $\mathscr{A}^0$. The level of the affine algebra is necessarily

$-2$, which is the critical level, for otherwise there will be a stress–energy tensor by the Sugawara construction. The action of these currents by the OPE makes $\mathscr{A}^0$ and $\mathscr{A}^1$ naturally an $\widehat{sl}_2$-module. In fact, they are isomorphic $\widehat{sl}_2$-modules, where the isomorphism is given by $\mathscr{O} \mapsto \mathscr{O}\theta$ [4]. The perturbative chiral algebra of the $\mathbb{CP}^1$ model is therefore constructed by acting "creation operators" $\mathscr{O} \in \mathscr{A}^0$ on the "ground states" $1 \in \mathscr{A}^0$ and $\theta \in \mathscr{A}^1$. This is analogous to the Ramond spectrum of strings.

Being a cohomology class in the zero-instanton sector, $\theta$ satisfies the requirement to be an operator responsible for the trivialization of the chiral algebra. Since $c_1(\mathbb{CP}^1) = 2x$, where $x$ is the generator of $H^2(\mathbb{CP}^1, \mathbb{Z})$, the charge violation is 2 in the one-instanton sector and the grading by $R$ charge is broken to $\mathbb{Z}_2$. Hence, one may expect that there will be a relation

$$\{Q, \theta\} \sim \mathrm{e}^{-t} \tag{20}$$

in the presence of one-instantons. If such a relation exists, it means that the inverse map of $\mathscr{O} \mapsto \mathscr{O}\theta$ is nothing other than the supercharge $Q$.

Although the counting of charge violation works out, there is still a somewhat mysterious property of (20) we have to account for. Plugging (20) into the correlation function (9), we find

$$\langle \{Q, \theta\} \cdots \rangle_1 \sim \langle 1 \cdots \rangle_0 \, . \tag{21}$$

The left-hand side of this equation involves an integration over $\mathscr{M}_1$. The right-hand side, on the other hand, involves an integration over $\mathscr{M}_0 \cong X$. Therefore, $\{Q, \theta\}$ must somehow transform the measure of $\mathscr{M}_1$ into that of $\mathscr{M}_0$. How can this happen?

Suppose that we wish to compute the matrix elements of $\{Q, \theta\}$. We will restrict here to the lowest order in the perturbation theory around instantons, so we may freely exploit conformal field theory arguments. As discussed in the previous section, an arbitrary matrix element can then be obtained from a three-point function on the sphere:

$$\langle a | \{Q, \theta\}(z, \bar{z}) | b \rangle = \langle \mathscr{V}_a(\infty) \{Q, \theta\}(z, \bar{z}) \mathscr{V}_b(0) \rangle \, . \tag{22}$$

Let us use a Möbius transformation on the worldsheet to rearrange the locations of the operators, so that the three-point function (22) becomes

$$\left\langle \{Q', \theta'\}(1) \mathscr{V}'_a(\varepsilon, \bar{\varepsilon}) \mathscr{V}'_b(0) \right\rangle \tag{23}$$

for some $|\varepsilon| \ll 1$, where the prime indicates that the operators are expressed in this new frame. We can expand the operators as $\mathscr{V}'_a = f_a{}^i(\varepsilon, \bar{\varepsilon}) \mathscr{A}_i$ using a complete set of local operators, where the coefficients $f_a{}^i$ depend on the choice of $\varepsilon$. We then have

$$f_a{}^i f_b{}^j(\varepsilon, \bar{\varepsilon}) \left\langle \{Q', \theta'\}(1) \mathscr{A}_i(\varepsilon, \bar{\varepsilon}) \mathscr{A}_j(0) \right\rangle . \tag{24}$$

At this point, we use the OPE

$$\mathscr{A}_i(\varepsilon,\bar{\varepsilon})\mathscr{A}_j(0)=\sum_k c^k{}_{ij}(\varepsilon,\bar{\varepsilon})\mathscr{A}_k(0). \tag{25}$$

The OPEs can of course produce short-distance singularities as $\varepsilon\to 0$. Thus if we started with a finite matrix element, $f_a{}^i f_b{}^j(\varepsilon,\bar{\varepsilon})$ in (24) must contain a factor $\epsilon^m\bar{\varepsilon}^n$ of appropriate powers $m$, $n$ to cancel the most singular terms that appear in the OPEs. These most singular terms will then be the only terms in the OPEs that contribute to the matrix element in the limit $\varepsilon\to 0$. In particular, these terms arise from the complete contraction of fermionic fields (and their derivatives) and the complete contraction of the derivatives of the bosonic field. Therefore, the computation ultimately boils down to two-point functions of the form

$$\langle\{Q,\theta\}(1)\mathscr{V}(0)\rangle\,, \tag{26}$$

where $\mathscr{V}$ is a *function* of the bosonic field.

We can now understand qualitatively how the transmutation of the instanton measure occurs. We wish to compute the two-point function (26) in the one-instanton sector. In the present case, a one-instanton $\phi_0$ is a biholomorphic map from the Riemann sphere to $X=\mathbb{CP}^1$, namely, a Möbius transformation. It can be described by three parameters, which we will conveniently take to be the points in $X$ that the points $z=0$, 1, and $\infty$ on the worldsheet are mapped to. The one-instanton computation thus involves integrations over $\phi_0(0)$, $\phi_0(1)$, and $\phi_0(\infty)$. Schematically, the $\phi_0(\infty)$ integration will give a constant, the $\phi_0(1)$ integration will give an integration of $\{Q,\theta\}$ which will again be a constant. Finally, the $\phi_0(0)$ integration will become an integration of the function $\mathscr{V}$ over the target space. But this is just the one-point function $\langle\mathscr{V}(0)\rangle_0$ in the zero-instanton sector. Consequently, we will have a relation

$$\langle\{Q,\theta\}(1)\mathscr{V}(0)\rangle_1\sim\langle\mathscr{V}(0)\rangle_0\,, \tag{27}$$

which is equivalent to (21).

In the rest of this section, we will make the above argument more precise and explicit. Let us first find the number of fermionic zero modes. Recall from the discussion in the previous section that here we are dealing with the physical model, so the fermions are spinors. The $\alpha$ and $\rho$ zero modes obey

$$\partial_z\alpha^{\bar{\imath}}=\partial_z\rho_{\bar{\imath}}=0. \tag{28}$$

Hence, after taking the complex conjugate, they are respectively holomorphic sections of $\mathscr{O}(-1)\otimes\phi_0^*TX$ and $\mathscr{O}(-1)\otimes\phi_0^*T^*X$, where $\mathscr{O}(-1)$ is the spinor bundle. For one-instantons, $\phi_0^*TX=\mathscr{O}(2)$ and $\phi_0^*T^*X=\mathscr{O}(-2)$. Since $h^0(\mathscr{O}(1))=2$ and $h^0(\mathscr{O}(-3))=0$, we have two $\alpha$ zero modes, and no $\rho$ zero modes. We see that

$\{Q, \theta\}$ indeed contains just the right number of $\alpha$ fields to soak up the fermionic zero modes.[2]

In computing $\{Q, \theta\}$, one must look for antiholomorphic single poles in the OPE of $J(\bar{z}) \cdot \theta(w, \bar{w})$, where $J = g_{i\bar{j}} \partial_{\bar{z}} \phi^i \alpha^{\bar{j}}$ is the supercurrent and $Q = \oint J \mathrm{d}\bar{z}$. This OPE contains two $\alpha$ fields. Notice that the Taylor expansion of $\alpha(\bar{z})\alpha(\bar{w})$ around $\bar{w}$ starts with the first order in $\bar{z} - \bar{w}$, since the zeroth order vanishes by the grassmannian nature of $\alpha$. Thus, in order to get a single pole in the overall computation, we must look for double poles in the contractions of the rest of the fields in $J$ and $\theta$. In view of the expression $\theta = R_{i\bar{j}} \partial_z \phi^i \alpha^{\bar{j}}$, it is not at all obvious how such poles arise.

At this point, one must recall that the fermionic fields carry within themselves fluctuating bosonic field. We must therefore examine more carefully their dependence on the bosonic field. The field configuration space $\mathscr{C} = \mathrm{Map}(\Sigma, X)$ inherits the complex structure of $X$ via the identification

$$\Gamma(\phi^* TX) \cong T_\phi \mathscr{C}. \tag{29}$$

Let $\{\zeta^r\}$ be holomorphic coordinates of $\mathscr{M}$, and $\{\xi^n\}$ be (real) coordinates which parametrize the directions away from $\mathscr{M}$ so that $\xi^n = 0$ on $\mathscr{M}$. Then, a point $\phi$ in the neighborhood of $\mathscr{M}$ is described by coordinates $\{\zeta^r, \bar{\zeta}^r, \xi^n\}$. We will write the full dependence of the bosonic field as $\phi(z, \bar{z}; \zeta, \bar{\zeta}, \xi)$. By construction, the dependence of an instanton is $\phi_0(z; \zeta)$. Since $\alpha$ at $\phi \in \mathscr{C}$ is an odd vector in $\overline{T_\phi \mathscr{C}}$, it can be expanded as

$$\alpha^{\bar{\imath}} = c^r \partial_r \phi^{\bar{\imath}} + c^{\bar{r}} \partial_{\bar{r}} \phi^{\bar{\imath}} + c^n \partial_n \phi^{\bar{\imath}} \tag{30}$$

with grassmanian coefficients $c^r$, $c^{\bar{r}}$, and $c^n$. On $\mathscr{M}$, one has $\partial_r \phi^{\bar{\imath}} = 0$ and $\partial_{\bar{r}} \phi^{\bar{\imath}} = \bar{u}_{0,r}$, where $u_{0,r}$ are the zero modes of the laplacian $\Delta_F$. Then, one can choose $\{\xi^n\}$ for which the expansion (30) coincides with (13).

Given $\zeta \in \mathscr{M}$, a one-instanton $\phi_0$ maps a point on the worldsheet $\Sigma = \mathbb{CP}^1$ to a point in the target space $X = \mathbb{CP}^1$ in a one-to-one manner. The bosonic field can thus be expressed as

$$\phi(z, \bar{z}; \zeta, \bar{\zeta}, \xi) = \hat{\phi}\left(\phi_0(z; \zeta), \bar{\phi}_0(\bar{z}; \bar{\zeta}); \zeta, \bar{\zeta}, \xi\right). \tag{31}$$

From the above and $\partial_{\bar{z}} \phi^{\bar{\imath}} = \partial_{\bar{z}} \phi_0^{\bar{j}} (\partial \hat{\phi}^{\bar{\imath}} / \partial \phi_0^{\bar{j}})$, we see that

$$\partial_{\bar{r}} \phi^{\bar{\imath}} = \frac{\partial \hat{\phi}^{\bar{\imath}}}{\partial \phi_0^{\bar{j}}} \partial_{\bar{r}} \phi_0^{\bar{j}} + \partial_{\bar{r}} \hat{\phi}^{\bar{\imath}} = \frac{\partial_{\bar{z}} \phi^{\bar{\imath}}}{\partial_{\bar{z}} \phi_0^{\bar{j}}} \partial_{\bar{r}} \phi_0^{\bar{j}} + \partial_{\bar{r}} \hat{\phi}^{\bar{\imath}}. \tag{32}$$

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[2]If there exists more fermionic zero modes, we must bring down a term proportional to the Riemann curvature from the action. This contains one $\alpha$ and one $\rho$ zero mode, and contributes to the lowest order in the large volume limit.

Plugging (30) and (32) into (6), we find that $\theta$ in the one-instanton sector contains an operator $\theta'$, where

$$\theta' = R_{i\bar{\jmath}}(\phi_0, \bar{\phi}_0)\partial_z\phi_0^i \frac{\partial_{\bar{z}}\phi^{\bar{\jmath}}}{\partial_{\bar{z}}\phi_0^{\bar{k}}}\alpha_0^{\bar{k}}. \tag{33}$$

Here, $\alpha_0$ is the zero mode part of $\alpha$ evaluated at $\phi_0$. The operator $\theta'$ contains $\partial_{\bar{z}}\phi^{\bar{\imath}}$ that can contract with $\partial_{\bar{z}}\phi^i$ in $J$ to produce an antiholomorphic double pole. Performing the contour integration, this yields

$$\{Q, \theta'\} = R_{i\bar{\jmath}}(\phi_0, \bar{\phi}_0)\frac{\partial_z\phi_0^i}{\partial_{\bar{z}}\phi_0^{\bar{k}}}\partial_{\bar{z}}\alpha_0^{\bar{\jmath}}\alpha_0^{\bar{k}}, \tag{34}$$

up to irrelevant terms that are of higher orders in the perturbation theory around instantons. The other terms in $\theta$ contribute to $\{Q, \theta\}$ obviously as $Q$-exact terms. Hence, they can be ignored.

To complete the computation, we need the explicit forms of the fermionic zero modes. Since a one-instanton $\phi_0$ is a biholomorphic map from $\Sigma = \mathbb{CP}^1$ to $X = \mathbb{CP}^1$, it is given by a Möbius transformation

$$\phi_0(z) = \frac{az+b}{cz+d}; \quad ad - bc = 1. \tag{35}$$

Then, $\alpha_0$ can be expanded as

$$\alpha_0^{\bar{\imath}} = c_0^1\bar{u}_{0,1} + c_0^2\bar{u}_{0,2}, \tag{36}$$

where $\bar{u}_{0,1}$, $\bar{u}_{0,2}$ are [18]

$$\bar{u}_{0,1}(\bar{z}) = \frac{1}{\bar{c}\bar{z}+\bar{d}}, \quad \bar{u}_{0,2}(\bar{z}) = \frac{1}{\bar{c}(\bar{c}\bar{z}+\bar{d})^2}. \tag{37}$$

After the fermionic zero mode integration, $\{Q, \theta'\}$ becomes the pullback of the Kähler form:

$$\int \mathrm{d}c_0^1\mathrm{d}c_0^2\,\{Q, \theta'\} = R_{i\bar{\jmath}}\partial_z\phi_0^i\partial_{\bar{z}}\phi_0^{\bar{\jmath}}. \tag{38}$$

This must be plugged into the two-point function (26), and then integrated over the one-instanton sector. In the region $d \neq 0$, one can set $d = 1$ by an overall rescaling. The conformally invariant measure on $\mathscr{M}_1$ is, up to an overall constant, given by [19]

$$d\mathscr{M}_1 = |ad - bc|^{-4}\, d^2a\, d^2b\, d^2c. \tag{39}$$

Recalling that $\mathscr{V}$ is a function, the two-point function (26) can be evaluated as

$$\int \frac{\mathrm{d}^2Y}{|Y|^4}\int \mathrm{d}^2X_1\, R_{1\bar{1}}(X_1, \overline{X}_1)\int \mathrm{d}^2X_0\,\mathscr{V}(X_0, \overline{X}_0). \tag{40}$$

Here, $X_0 = \phi_0(0) = b$, $X_1 = \phi_0(1) = (a+b)/(c+1)$, $Y = a - bc$. The first divergent integral in $Y$ reflects the noncompactness of the Möbius group. The second integral comes from $\{Q, \theta\}$, and is the integration of $c_1(X)$. And the third integral is an integration of $\mathscr{V}$ over the target space, which will give the zero-instanton one-point function $\langle \mathscr{V}(0) \rangle_0$.

More precisely, let $d\mathscr{M}_0 = \Omega(X_0)\mathrm{d}^2 X_0$ be the volume form of the zero-instanton sector. The nowhere vanishing function $\Omega$ defines an invertible operator in the zero-instanton sector through its matrix elements $\Omega(X_0)$ (expressed in the basis where states are localized in the target space). The computation above demonstrates that we have the operator relation

$$\{Q, \theta\} \sim \mathrm{e}^{-t}\Omega^{-1}. \tag{41}$$

We have found a relation of the form (18), which via (17) implies that we have the relation $\{Q, \Theta\} = 1$. Therefore, the chiral algebra is trivialized in the $\mathbb{CP}^1$ model.

## 5. Flag Manifold Models

The key property of the $\mathbb{CP}^1$ model that was crucial for the trivialization of its chiral algebra is that there are precisely two $\alpha$ zero modes and no $\rho$ zero modes in the one-instanton sector. In this case a perturbative cohomology class $\theta$ exists (since $c_1(\mathbb{CP}^1) \neq 0$), and $\{Q, \theta\}$ contains the right number of fermionic zero modes. Hence, any one-instanton correlation function with $\{Q, \theta\}$ inside reduces to a zero-instanton one, and this establishes the relation (41). Let us see if $\{Q, \theta\}$ leads to a similar relation in the case of other nonanomalous target spaces that have $p_1(x) = 0$ but nonvanishing $c_1(X)$.

Let $X$ be a Kähler manifold of complex dimension $d$, and $\phi_0$ a one-instanton wrapping a rational curve $L \subset X$. We can decompose the tangent bundle as $TX = TL \oplus NL$, where $NL$ is the normal bundle of $L$ in $X$. The pullback bundle $\phi_0^* TL = \mathscr{O}(2)$ for one-instantons, and $\phi_0^* NL$ further splits into the direct sum of line bundles. If $\int_\Sigma \phi_0^* c_1(X) = k$, then we can write

$$\phi_0^* TX \cong \mathscr{O}(2) \oplus \mathscr{O}(p_1) \oplus \cdots \oplus \mathscr{O}(p_{d-1}), \tag{42}$$

where $p_1 + \cdots + p_{d-1} = k - 2$. The number of $\alpha$ or $\rho$ zero modes can be found from the splitting type (42). According to the formula

$$h^0(\mathscr{O}(n)) = \begin{cases} n+1 & \text{for } n \geq 0; \\ 0 & \text{for } n < 0, \end{cases} \tag{43}$$

each $\mathscr{O}(n)$ with $n > 0$ in (42) contributes $n$ of $\alpha$ zero modes. We also know from the index theorem that there are $k$ more $\alpha$ zero modes than $\rho$ zero modes. In order to have exactly two $\alpha$ and no $\rho$ zero modes, it must be that $k = 2$ and the splitting type is

$$\mathscr{O}(2) \oplus \mathscr{O}(0) \oplus \cdots \oplus \mathscr{O}(0). \tag{44}$$

Notice then that the two $\alpha$ zero modes should come solely from $TL$. This means that only the field components tangent to $L$ contribute to $\{Q, \theta\}$. Consequently, our computation will be the same as that in the $\mathbb{CP}^1$ case, and the fermionic zero mode integration turns $\{Q, \theta\}$ into the pullback of the Kähler form (38), but this time restricted to $L$. The integration over the parameters of the instanton then becomes an integration over $L$.

However, this is not quite the end of the story, since $L$ is not a rigid instanton if (44) is true. An infinitesimal deformation of $L$ is given by a holomorphic section of $\phi_0^* NL$. In the case of the splitting type (44), we have $d-1$ independent deformations, one for each normal direction. Intuitively, we therefore expect that the instanton can be infinitesimally translated in every possible direction in the target space. This generates a family of instantons with $d-1$ complex parameters, over which we still have to integrate after the integration over $L$ is done. If the instanton sweeps the whole target space, then we will obtain an integration over the target space.

This can happen for homogeneous spaces $G/H$ equipped with a $G$-invariant Kähler structure. Assuming that the model with target space $G/H$ has a number of topologically distinct one-instantons, pick one and call it $\phi_0$. Then, the transitive $G$-action can map $\phi_0$ to another instanton located anywhere else in $G/H$. This means that there is at least one deformation in every normal direction, i.e., all the $p_i$s in the splitting type (42) must be nonnegative. If any of the $p_i$s is positive, then the instanton has charge violation greater than 2, and hence, does not contribute to $\{Q, \theta\}$. Thus, in order for the chiral algebra to be trivialized, it suffices that there exists at least a single one-instanton such that the tangent bundle has splitting type (44).

This is indeed the case for flag manifolds $G/T$ of compact semisimple Lie groups $G$! It is well-known [20] that $c_1(G/T) = 2(x_1 + \cdots + x_r)$, where $r$ is the rank of $G$. Furthermore, there are $r$ rational curves that are dual to the $x_i$s. Each of the rational curves (i.e. instantons) has the splitting type (44) by the above argument. Therefore, we conclude that the chiral algebras for complete flag manifolds are zero nonperturbatively.

## 6. Supersymmetry Breaking and Loop Space Geometry

We have seen that the chiral algebra is trivial nonperturbatively in the models with flag manifold target spaces, hence so is the cohomology of states. On the other hand, note that supersymmetric states must be annihilated by $Q$ and $Q^\dagger$, i.e., they are harmonic states of $Q$. Since the harmonic space is isomorphic to the cohomology, supersymmetry is spontaneously broken in these models. Although in the present case this phenomenon is a consequence of the trivialization of the chiral algebra, the physics of supersymmetry breaking is interesting in its own right. As we now explain, it is intimately related to the geometry of loop space.

To unravel the connection between supersymmetry and loop space geometry, let us consider the model with target space $X$ and the worldsheet being the cylinder $S^1 \times \mathbb{R}$ with coordinates $(\sigma, \tau)$, $\sigma \sim \sigma + 2\pi$. This may be viewed as supersymmetric quantum mechanics on the loop space $LX = \mathrm{Map}(S^1, X)$, which we can canonically quantize. The fermionic fields then obey the anticommutation relation

$$\{\rho_{\bar{a}}(\sigma, \tau), \alpha^{\bar{b}}(\sigma', \tau)\} = \delta_{\bar{a}}^{\bar{b}} \delta(\sigma - \sigma') \tag{45}$$

expressed in a given local unitary frame $\{e_a\}$ on $TX$. This is the loop-space analog of the Clifford algebra. On the other hand, the supercharge is identified as

$$Q = \int_{S^1} \mathrm{d}\sigma \left( \alpha^{\bar{\imath}} \frac{\mathrm{D}}{\mathrm{D}\phi^{\bar{\imath}}} - i g_{\bar{\imath} j} \alpha^{\bar{\imath}} \partial_\sigma \phi^j \right), \tag{46}$$

where $\mathrm{D}/\mathrm{D}\phi^{\bar{\imath}}$ is the covariant functional derivative on $LX$. Notice that the first term is just (a half of) the Dirac operator $Q_0$ on $LX$. In fact, $Q$ is related to $Q_0$ via a conjugation by a Bott–Morse–Novikov function on $LX$. Given a loop $\gamma \in LX$, let $\gamma_u \colon [0, 1] \to LX$ be a homotopy connecting a reference loop $\gamma_0$ and $\gamma_1 = \gamma$. Such a homotopy defines a map $\hat{\gamma}$ from the annulus $A = [0, 1] \times S^1$ to $X$. If we define a function $h$ by $h(\gamma) = \int_A \hat{\gamma}^* \omega$ and $Q_s = \mathrm{e}^{sh} Q_0 \mathrm{e}^{-sh}$ for $s \in \mathbb{R}$, then [21]

$$Q_s = \int_{S^1} \mathrm{d}\sigma \left( \alpha^{\bar{\imath}} \frac{\mathrm{D}}{\mathrm{D}\phi^{\bar{\imath}}} - i s g_{\bar{\imath} j} \alpha^{\bar{\imath}} \partial_\sigma \phi^j \right) \tag{47}$$

and $Q = Q_1$. The cohomology remains unchanged under such a similarity transformation. Therefore, the $Q$-cohomology of states is also the spinor cohomology on $LX$.

Now, the crucial observation is that the laplacian $\{Q_s, Q_s^\dagger\}$ contains a term

$$s^2 \int_{S^1} \mathrm{d}\sigma \, \|\partial_\sigma \phi\|^2. \tag{48}$$

Thus, harmonic states of $Q_s$ must localize in the limit $s \to \infty$ around constant maps given by $\partial_\sigma \phi = 0$ (or linear combinations thereof). Intuitively, this can also be understood as follows. From (47), we see that $Q_s$ is the supercharge of the theory defined on the cylinder with metric $\mathrm{d}\sigma_s^2 + \mathrm{d}\tau^2$, where $\sigma_s = \sigma/s$. The circumference of the cylinder is $2\pi/s$. As $s$ gets larger and larger, the cylinder becomes narrower and narrower. It then takes an increasing amount of energy to "stretch" the loop to make a large circle in the target space.

Since tiny loops cannot "feel" the curvature of the target space, in the large $s$ limit the harmonic space may be approximated by the Fock space of free closed strings. One can then systematically construct the harmonic space order by order in $1/s$. Looking back at the expression (47), we notice that the parameter $s$ appears in $Q_s$ only in the combination $s g_{i\bar{j}}$. Thus, as far as the determination of the supersymmetric spectrum is concerned, the large $s$ limit is equivalent to the large volume limit – the widths of localized supersymmetric states are controlled by $1/s$, and by increasing $s$ we are effectively inflating the target space. It is now clear that a supersymmetric state is a linear combination of states of the form

$$f_{i_1\cdots i_k}{}^{j_1\cdots j_l} c^{i_1}_{-m_1} \cdots c^{i_k}_{-m_k} c_{-n_1, j_1} \cdots c_{-n_l, j_l} |\psi\rangle, \tag{49}$$

where $|\psi\rangle$ is a spinor ground state, and $c^i_{-m}$, $c_{-n,i}$ are left-moving bosonic creation operators. The right-moving excitations are suppressed since supersymmetric states must have $\{Q, Q^\dagger\} = H - P = 0$. The state (49) can be viewed as a section of the spinor bundle twisted by tensor products of $TX$ and $T^*X$. More precisely, it is a section of $S \otimes R_m \otimes R^*_n$, where $m = m_1 + \cdots + m_k$, $n = n_1 + \cdots + n_l$, and

$$\sum_{k=0}^{\infty} q^k R_k = \bigotimes_{k=0}^{\infty} \bigoplus_{l=0}^{\infty} \mathrm{Sym}^l (q^k \cdot T^*X). \tag{50}$$

The supercharge $Q$, when acting on these states, reduces [7] to the Dirac operator on $X$ twisted by the relevant bundle given by (50).

So far, we have considered supersymmetric states localized around the space of constant loops $X \subset LX$. We can actually go further: By conjugating $Q_s$ with a Morse function $f$ on $X$, supersymmetric states can be localized around the critical points of $f$. This will lead us to holomorphic Morse theory [22] on loop space [23], in which the lifting of approximate supersymmetric states by instantons can be captured by the Witten complex [12]. For example, in the $\mathbb{CP}^1$ model [7] one can localize states of charge 0 at the south pole $S$ and those of charge 1 at the north pole $N$, so that one has two isomorphic Fock spaces at $S$ and $N$. Instantons going from $S$ to $N$ are worldline instantons, which propagate essentially as particles. These capture the classical geometry of the target space, connecting states which do not enter the harmonic spaces of the aforementioned twisted Dirac operators. On the other hand, instantons going from $N$ to $S$ are worldsheet instantons. They sweep a nontrivial two-cycle representing the fundamental class of $\mathbb{CP}^1$, and are responsible for the lifting of the rest of the states in the Fock spaces. In this sense, they capture the "quantum" geometry of the $\mathbb{CP}^1$ model. In principle, one should observe the same phenomena in any model which exhibits supersymmetry breaking, although a direct analysis may not always be tractable.

When combined with the results from the previous section, what we have seen above points to the following. From the loop space viewpoint, the spinor cohomology on $LX$ is zero for $X = G/T$, i.e., complete flag manifolds of compact semisimple Lie groups. Equivalently, this means that the kernel of the Dirac operator on $LX$ is empty. This is consistent with Stolz's idea [1] that if $X$ has positive Ricci curvature, then $LX$ has positive scalar curvature with no harmonic spinors – flag manifolds indeed admit positive Ricci curvature [24]. From the target space viewpoint, the kernels of the Dirac operators twisted by (50) become empty nonperturbatively. This can be interpreted as an effect arising from a "quantum" deformation of the target space geometry by worldsheet instantons.

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