Regret-Optimal Supervisory Control of Partially-Known Discrete-Event Systems

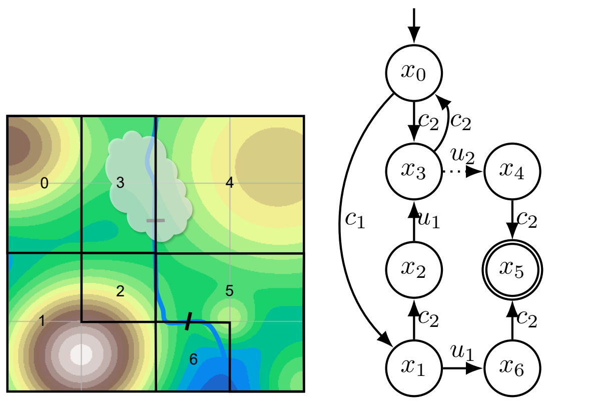
Jianing Zhao[†], Bohan Cui[†], Dimos V. Dimarogonas, Rupak Majumdar and Xiang Yin

***Abstract*— This paper addresses a novel optimal supervisory control problem for reachability tasks in partially-known discrete-event systems (DES). We consider a setting where the supervisor lacks prior knowledge of feasible events in certain states and must discover this information by visiting them. To assess performance in this context, we study regret as a metric that quantifies the difference between the actual cost incurred and the optimal cost achievable with full knowledge. We formalize this problem and propose the algorithm to compute an optimal supervisor that guarantees reachability while minimizing regret. Our results demonstrate that regret serves as a meaningful performance measure for supervisory control in partially-known DES, and our method is both correct and effective in practice.**

I. Introduction

Discrete-Event Systems (DES) are an important class of systems with discrete state spaces and event-triggered dynamics [1]. They play an essential role in modeling, analysis and control of high-level behaviors for engineering cyber-physical systems such as manufacturing systems, embedded software, and communication networks. In the context of DES, one of the central problems is how to enforce the closed-loop properties of the system. Supervisory Control Theory (SCT), initiated by Ramadge and Wonham [2], is a powerful formal framework widely used for the synthesis of DES controllers under some desired specifications, such as safety, liveness and nonblockingness; see, e.g., the textbook [3] and some recent works [4]–[10].

One important branch in SCT is the synthesis of optimal supervisors with respect to some performance measures. This problem, known as the optimal supervisory control problem, has been widely investigated in the literature; see, e.g., [11]–[15]. Particularly, in [14], the authors proposed an optimal supervisory control framework considering both the occurrence cost and disablement cost. The supervisor's objective is to reach marked states optimally in terms of the worst-case total accumulated cost. Following this framework, in [11], a liveness task and the worst-case average cost are considered. In [16], the authors further extended the



(a) Robot Workspace (b) PK-DES

Fig. 1. A motivating example, where a robot needs to reach region 6 from region 0 with partially-known environment information.

specification to the cyclic task in which the supervisor is required to visit marked states infinitely.

In most of existing works in SCT, uncertainties are modeled as uncontrollable events. These models, while suitable for describing disturbances, fall short in capturing the scenarios where the system structure is inherently unknown and the supervisor is required to actively explore the system to uncover the actual configuration. Relying solely on the worst-case analysis may lead to a highly conservative supervisor. To be more specific, let us consider a simple motivating example where a robot moves in a partially-known workspace as shown in Fig 1(a), the corresponding transition model is as shown in Fig 1(b). We assume that the robot knows the terrain and the bridge between regions 5 and 6, but is uncertain about the bridge between regions 3 and 4 due to a cloud. The goal is to reach region 5 from region 0 with minimal energy cost. However, traversing mountainous terrain incurs significantly higher energy consumption compared to the plains. A worst-case strategy follows path $0 \to 1 \to 6 \to 5$ to prevent unnecessary backtracking. However, this may lead to a heavy regret if a bridge does exist at region 3, as the robot would have missed a shortcut. A more adaptive approach is to first check region 3. If a bridge is present, the robot takes the shortcut and saves energy; otherwise, it backtracks to 0. While this strategy may have a slightly higher worst-case cost, it renders much less regret and takes the potential advantage of exploring the unknown regions.

The above example illustrates that even in a fully-controllable setting, worst-case performance may not always be suitable for evaluating a planning strategy. A systematic approach is needed to quantitatively balance the trade-off between exploration and conservatism. Motivated by this,

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J. Zhao, B. Cui and X. Yin are with the School of Automation and Intelligent Sensing, Shanghai Jiao Tong University, Shanghai 200240, China. E-mail: {jnzhao,bohan_cui,yinxiang}@sjtu.edu.cn.

D. V. Dimarogonas is with the School of Electrical Engineering and Computer Science, KTH Royal Institute of Technology, Stockholm SE 10044, Sweden. E-mail: dimos@kth.se.

R. Majumdar is with the Max-Planck Institute for Software Systems, Kaiserslautern 67663, Germany. E-mail: rupak@mpi-sws.org.

† indicates the equal contribution.

we formulate and solve a novel optimal supervisory control problem by considering scenarios where the supervisor operates in a *partially-known* DES: the supervisor knows all system states but lacks prior information about feasible transitions from certain states unless explored. Different from nondeterministic DES, where outcomes are random and may vary even under identical conditions, a partially-known DES involves initial uncertainty about the system's structure but remains deterministic.

Building upon the above setting, the main results of this paper are summarized as follows: i) We propose a formal model for partially-known DES that encompasses all possible actual environments. Building on this model, we define the evolution of the supervisor's knowledge within the partially-known DES and formulate the regret-optimal supervisory control problem for which both the reachability specification and the regret evaluation are introduced; ii) We define a novel tripartite transition system to capture all potential knowledge evolution paths under different control decisions and actual environments; iii) We develop a game-theoretical approach to solve the problem, which is divided into two parts: the first focuses on deriving the optimal micro-strategies for *one layer of exploration*, while the second designs a macro-strategy across *different layers of exploration*. By integrating these micro- and macro-strategies, we obtain an optimal solution that minimizes regret.

Our work is closely related to graph games with quantitative objectives [17], [18], particularly the regret minimization problem [19]–[22]. However, their setting assumes the environment-player's strategy is unrestricted, allowing it to change decisions freely each time it revisits the same state. This assumption does not align with the partially-known environment scenario considered in our work, where the environment is fixed and the environment-player must act consistently when revisiting the same state. In particular, one of our previous works [22] started the study of partially-known environment for temporal logic specification. However, only the path planning problem is solved in [22] and the effect of uncontrollable events, which makes the solution algorithm more complicated, has not been investigated.

II. Supervisory Control of Fully-Known DES

A. System Model

Let Σ be a finite set of events. A *string* is a finite sequence of events and we denote by Σ^* the set of all strings over Σ including the empty string ϵ. For integer n, we denote by Σ^n the set of strings with length n. For string $s \in \Sigma^*$, the length of s is denoted by $|s|$ with $|\epsilon| = 0$. A language $L \subseteq \Sigma^*$ is a set of strings. The prefix-closure of L is denote by $\bar{L}$, i.e., $\bar{L} = \{u \in \Sigma^* : \exists v \in \Sigma^* \text{ s.t. } uv \in L\}$.

We consider a *fully-known* discrete-event system modeled by a deterministic finite-state automaton (DFA)

$$G = (X, \Sigma, \delta_G, x_0, X_m)$$

where X is the finite set of states, Σ is the finite set of events, $\delta_G : X \times \Sigma \to X$ is the partial transition function, $x_0 \in X$ is the initial state, and $X_m \subseteq X$ is the set of marked states. The transition function δ_G can also be extended to $\delta_G : X \times \Sigma^* \to X$ in the usual manner. The language generated by G from state x is defined by $\mathcal{L}(G, x) = \{s \in \Sigma^* : \delta_G(x, s)!\}$, where "!" means "is defined". We also define $\mathcal{L}(G, Q) := \bigcup_{x \in Q} \mathcal{L}(G, x)$ as the language generated from a set of states $Q \subseteq X$, define the language generated by G is $\mathcal{L}(G) := \mathcal{L}(G, x_0)$, and the language marked by G is $\mathcal{L}_m(G) = \{s \in \mathcal{L}(G) : \delta_G(s) \in X_m\}$. For any $s \in \mathcal{L}(G)$, we write $\delta_G(x_0, s)$ simply as $\delta_G(s)$. For any $x \in X$, we define $\Lambda_G(x) = \{\sigma \in \Sigma : \delta_G(x, \sigma)!\}$ as the set of active events at x. For any $s \in \mathcal{L}(G)$, we also write $\Lambda_G(\delta_G(s))$ as $\Lambda_G(s)$ for simplicity.

In the supervisory control framework, a supervisor can restrict the behavior of the system G by dynamically disabling/enabling some system events. In this setting, the event set Σ is partitioned as $\Sigma = \Sigma_c \dot{\cup} \Sigma_{uc}$, where Σ_c is the set of controllable events and Σ_{uc} is the set of uncontrollable events. A control decision $\gamma \in 2^\Sigma$ is said to be admissible if $\Sigma_{uc} \subseteq \gamma$, namely, uncontrollable events can never be disabled. Particularly, in this paper, we consider the control decisions in which *at most one controllable event* is included. This setting is without loss of generality since it does not influence the controllability of the system. We define

$$\Gamma = \left\{\gamma \in 2^\Sigma : \Sigma_{uc} \subseteq \gamma \wedge |\gamma \setminus \Sigma_{uc}| \leq 1\right\}$$

as the set of admissible control decisions. Then, a supervisor is a mapping

$$S : \mathcal{L}(G) \to \Gamma \tag{1}$$

We denote by $\Psi(G)$ the set of all supervisors in G. Moreover, we use the notation S/G to represent the controlled system and the language generated by S/G, denoted by $\mathcal{L}(S/G)$, is defined recursively in the following manners:

i) $\epsilon \in \mathcal{L}(S/G)$; and
ii) for any $s \in \Sigma^*$, $\sigma \in \Sigma$ we have $s\sigma \in \mathcal{L}(S/G)$ iff $s\sigma \in \mathcal{L}(G)$, $s \in \mathcal{L}(S/G)$, and $\sigma \in S(s)$.

The language marked by S/G is denoted by $\mathcal{L}_m(S/G) = \mathcal{L}(S/G) \cap \mathcal{L}_m(G)$.

In this paper, we aim to synthesis a supervisor enforcing *reachability objective*, which requires that the controlled DES will eventually visit the marked states, i.e.,

$$\begin{aligned} &\forall s \in \mathcal{L}(S/G), \exists n \in \mathbb{N}, \forall t \in \mathcal{L}(S/G)/s : \\ &|t| \geq n \Rightarrow \overline{\{st\}} \cap \mathcal{L}_m(G) \neq \emptyset \end{aligned} \tag{2}$$

We define $\Psi_W(G) = \{S \in \Psi(G) : S \text{ satisfies (2)}\}$ as the set of all winning supervisors for the reachability.

In this paper, we consider the optimal supervisory control in a quantitative manner. To this end, for each event, we define the cost function $w : \Sigma \to \mathbb{R}_{\geq 0}$ describing the cost incurred whenever the DES generates an event. Given a finite string $t = \sigma_1 \cdots \sigma_n \in \mathcal{L}(G)$, we define $\mathsf{cost}(t) = \sum_{i=1}^n w(\sigma_i)$ as the total cost along the string s. For a string $s \in \mathcal{L}_m(G)$, we define its cost as the total cost when it first reaches the set of marked states X_m, i.e.,

$$\begin{gathered} \mathsf{cost}(s) = \mathsf{cost}(t), \\ t \in \overline{\{s\}} \cap \mathcal{L}_m(G) \wedge t' \notin \mathcal{L}_m(G), \forall t' \in \overline{\{t\}} \backslash \{t\} \end{gathered} \tag{3}$$

Algorithm 1: Min-Max Game (`SolveMinMax`)

```
Input: A BTS $\mathcal{B}$ with the set of marked states $X_m$
       and a cost function $w_{\mathcal{B}}$
Output: Optimal strategy $\pi^\star : Q_X \to Q_Y$ and the
        optimal minmax value $\mathsf{minmax}$
1  foreach $x \in Q_X$ do
2      if $x \in X_m$ then
3          $\mathsf{Val}^{(0)}(x) \leftarrow 0$ and $\pi^{(0)}(x) \leftarrow \Lambda_G(x) \cup \Sigma_{uc}$
4      else
5          $\mathsf{Val}^{(0)}(x) \leftarrow \infty$
6  repeat
7      foreach $y \in Q_Y$ do
8          $\mathsf{Val}^{(k+1)}(y) \leftarrow \max_{x' \in \mathrm{Succ}(y)} \Big(\mathsf{Val}^{(k)}(x') + w_{\mathcal{B}}(y, x')\Big)$
9      foreach $x \in Q_X$ do
10         $\mathsf{Val}^{(k+1)}(x) \leftarrow \min_{y \in \mathrm{Succ}(x)} \Big(\mathsf{Val}^{(k)}(y) + w_{\mathcal{B}}(x, y)\Big)$
11         $\pi^{(k+1)}(x) \leftarrow \arg\min_{y \in \mathrm{Succ}(x)} \Big(\mathsf{Val}^{(k)}(y) + w_{\mathcal{B}}(x, y)\Big)$
12     $k \leftarrow k + 1$
13 until $\forall z \in Q_X \cup Q_Y : \mathsf{Val}^{(k+1)}(z) = \mathsf{Val}^{(k)}(z)$;
14 return $\pi^{(k)}, \mathsf{Val}^{(k)}(x_0)$
```

Given a winning supervisor $S \in \Psi_W(G)$, we define the cost of the controlled system S/G as

$$\mathsf{cost}(S/G) = \max_{s \in \mathcal{L}_m(S/G)} \mathsf{cost}(s) \tag{4}$$

Then, we aim to synthesize an optimal supervisor $S^\star$ that minimizes the cost with the reachability objective, i.e.,

$$\mathsf{cost}(S^\star/G) \leq \mathsf{cost}(S/G),\ \forall S \in \Psi_W(G) \tag{5}$$

B. Controller Synthesis

In this subsection, we review the optimal supervisory control synthesis algorithm for reachability based on the following structure of bipartite transition system [23].

Definition 1 (Bipartite Transition System): Given system G, we construct its bipartite transition system (BTS) as

$$\mathcal{B} = (Q_X, Q_Y, x_0, h_{XY}, h_{YX}, \Sigma, \Gamma)$$

where

- $Q_X = X \cup \{d\}$ is the set of X-states, where d is the deadlocked state;
- $Q_Y = X \times \Gamma$ is the set of Y-states;
- $x_0 \in Q_Y$ is the initial X-state;
- $h_{XY} : Q_X \times \Gamma \to Q_Y$ is the transition function from X-states to Y-states define by: for any $x \in Q_X$, $\gamma \in \Gamma$ and $y = (x, \gamma) \in Q_Y$, $y = h_{XY}(x, \gamma)$;
- $h_{YX} : Q_Y \times (\Sigma \cup \{\epsilon\}) \to Q_X$ is the transition function from Y-states to X-states define by: for any $y = (x, \gamma) \in Q_Y$, we have
  - if $\Lambda_G(x) \cap \gamma \neq \emptyset$, we define
  $$h_{YX}(y, \sigma) = \delta(x, \sigma),\ \forall \sigma \in \Lambda_G(x) \cap \gamma \tag{6}$$
  - if $\Lambda_G(x) \cap \gamma = \emptyset$, we define
  $$h_{YX}(y, \epsilon) = d \tag{7}$$
- Σ is the set of events in G;
- Γ is the set of admissible control decisions in G.

Here we note that the marked states in G are implicitly included in X-states of $\mathcal{B}$, i.e., $X_m \subset Q_X$. For the sake of writing convenience, in this paper, we use $\mathrm{Succ}(\cdot)$ to denote the successor-states of a given state in the graph, i.e.,

- for each $x \in Q_X$, we have
$$\mathrm{Succ}(x) = \{h_{XY}(x, \gamma) : \gamma \in \Gamma\}$$
- for each $x \in Q_Y$, we have
$$\mathrm{Succ}(x) = \{h_{YX}(y, \sigma) : \sigma \in \Sigma \cup \{\epsilon\} \text{ s.t. } h_{YX}(y, \sigma)!\}$$

Hereafter, the detailed definition of $\mathrm{Succ}(\cdot)$ for other kinds of graphs will be omitted.

To capture the optimality, we define a cost function for $\mathcal{B}$ as follows: for each $x \in Q_X \setminus \{d\}$ and each $y \in h_{XY}(x)$, we define $w_{\mathcal{B}}(x, y) = 0$; for each $y \in Q_Y$ and each $x' = h_{YX}(y, \sigma) \in h_{YX}(y)$, we define $w_{\mathcal{B}}(y, x') = w(\sigma)$.

Then the optimal supervisor can be obtained by solving a standard *min-max reachability game* [24], i.e., Algorithm 1.

Given the optimal strategy $\pi^\star$ returned by Algorithm 1, in execution, the optimal supervisor $S^\star$ can work as follows: at each instant, $S^\star$ remembers the current X-state x and pick the decision γ such that $h_{XY}(x, \gamma) = \pi^\star(x)$. Then we update the current to Y-state according to γ and wait for the next event to occur. After that, we update the current state again to X-state and so forth.

III. Partially-Known DES

Although uncertainties in environments can be captured by the uncontrollable events in the full-known DES, there are also scenarios where there are initially unknown states in the system that could become known after being visited by system trajectories. We capture such kind of *information uncertainty* by the following partially-known DES.

Definition 2 (Partially-Known DES): A partially-known DES (PK-DES) is a 6-tuple

$$\mathbb{G} = (X, \Sigma, \delta, \Delta, x_0, X_m)$$

where X is the set of states; Σ is the set of events; $\delta : X \times \Sigma \to X$ is the transition function; x_0 is the initial state; X_m is the set of marked states; different from the DFA,

$$\Delta : X \to 2^{2^\Sigma}$$

is the *event-pattern function* that assigns each state a family of activated events.

The intuition of the PK-DES $\mathbb{G}$ is explained as follows. Essentially, PK-DES is used to describe the *possible world* for the system. That is, the system has some prior information regarding the possible events of each unknown state but does not know which ones are activated before the system trajectory actually visits it. Therefore, in the PK-DES $\mathbb{G}$, for each state $x \in X$, we have $\Delta(x) = \{o_1, o_2, \ldots, o_{|\Delta(s)|}\}$ where each $o_i \in 2^\Sigma$ is called an *event-pattern* representing a possible set of activated events at state x. For convenience, we refer each $o_i \in \Delta(x)$ to as an *observation* at state x, since the system "observes" the activated events when visiting state x. Therefore, for each $x \in X$, we say x is a *known state* if

$|\Delta(x)| = 1$; and *unknown state* if $|\Delta(x)| > 1$. Accordingly, we partition the state space as $X = X_k \dot\cup X_{uk}$ where X_k is the set of known states and X_{uk} is the set of unknown states. We refer the visit to an unknown state to as an *exploration*.

Definition 3 (Compatible DES): Given $\mathbb{G}$, a DFA $G = (X, \Sigma, \delta_G, x_0, X_m)$ is a compatible DES with $\mathbb{G}$, denoted by $G \in \mathbb{G}$, if for any $x \in X$, we have i) $\Lambda_G(x) \in \Delta(x)$; and ii) for each $\sigma \in \Lambda_G(x)$, we have $\delta_G(x, \sigma) = \delta(x, \sigma)$.

In the partially-known setting, a supervisor cannot be synthesized only based on the finite string generated by the system. In addition, the observed successor-pattern at each state should also be taken into consideration. To be specific, if the system trajectory visits a known state, then there would not be any useful information about the partially-known DES, since $\Delta(x)$ is already a singleton. Only when the system trajectory visits unknown state, it will gain new information and successor-pattern at this state will become known from then on.

To capture the result of an exploration, we call a tuple $\kappa = (x(\kappa), o(\kappa)) \in X \times 2^\Sigma$ as a *knowledge state*, where $o \in \Delta(x)$ is the set of activated events of x that becomes known to the agent after exploring x. We denote by $\mathsf{Kw} = \{\kappa \in X \times 2^\Sigma : o(\kappa) \in \Delta(x(\kappa))\}$ the set of all possible knowledge states. A *history* in $\mathbb{G}$ is a finite sequence

$$\hbar = (x_0, o_0)\sigma_0(x_1, o_1)\sigma_1 \cdots (x_n, o_n) \in (\mathsf{Kw} \cdot \Sigma)^* \mathsf{Kw}$$

such that i) for any $i = 0, 1, \ldots, n-1$, we have $\sigma_i \in o_i$ and $x_{i+1} = \delta(x_i, \sigma_i)$; ii) for any $i, j = 1, \ldots, n$, we have $x_i = x_j \Rightarrow o_i = o_j$. For such history $\hbar$, we call $\sigma_1\sigma_2 \cdots \sigma_{n-1} \in X^*$ its string. We denote by $\mathcal{H}(\mathbb{G})$ and $\mathcal{L}(\mathbb{G})$ the set of all finite histories and paths in PK-DES $\mathbb{G}$, respectively.

Given knowledge on the PK-DES will be accumulated along a history, which is captured by the following concept of *knowledge set*. A knowledge set $\mathcal{K} = \langle \kappa_1, \ldots, \kappa_{|\mathcal{K}|} \rangle$ where $\kappa_i \in \mathcal{K}$ is an *ordered set* of knowledge states such that

$$\forall \kappa, \kappa' \in \mathcal{K} : x(\kappa) = x(\kappa') \Rightarrow o(\kappa) = o(\kappa').$$

We denote by $\mathbb{KW}$ the set of all knowledge sets. With a slight abuse of notation, for each $x \in X$, we write $x \in \mathcal{K}$ if $(x, o) \in \mathcal{K}$ for some observation $o \in \Delta(x)$. We denote by $o_{\mathcal{K}}(x) \in \Delta(x)$ the unique observation such that $(x, o_{\mathcal{K}}(x)) \in \mathcal{K}$.

During the system execution, the system maintains a knowledge set to record its exploration history. Once a new unknown state is explored, the knowledge set is updated according to the following function. Given a knowledge set $\mathcal{K} \in \mathbb{KW}$ and a knowledge state $\kappa \in \mathsf{Kw}$, we have

$$\texttt{update}(\mathcal{K}, \kappa) = \begin{cases} \mathcal{K}, & \text{if } x(\kappa) \in \mathcal{K} \\ \langle \kappa_1, \ldots, \kappa_{|\mathcal{K}|}, \kappa \rangle, & \text{otherwise} \end{cases} \tag{8}$$

Finally, given a knowledge set, the system could refine the PK-DES by eliminating uncertainties that have been explored. Given a PK-DES $\mathbb{G}$ and a knowledge set $\mathcal{K} \in \mathbb{KW}$, the refined PK-DES is a new PK-DES

$$\mathbb{G}_{\mathcal{K}} = (X, \Sigma, \delta, \Delta', x_0, X_m) \tag{9}$$

such that for any $x \in X$, we have

$$\Delta'(x) = \begin{cases} \{o_{\mathcal{K}}(x)\}, & \text{if } x \in \mathcal{K} \\ \Delta(x), & \text{if } x \notin \mathcal{K} \end{cases}$$

Therefore, under the setting of partially-known DES, it is no longer sufficient to synthesize a supervisor in (1), since the supervisor should decide the control decision based on both of what the system has visited and what it has known. Therefore, we define the notion of *strategic supervisor*.

Definition 4 (Strategic Supervisor): A strategic supervisor is a function

$$\mathbb{S} : \mathcal{H}(\mathbb{G}) \to \Gamma$$

such that for any $\hbar = \kappa_0\sigma_1\kappa_1\sigma_2 \cdots \kappa_n$ where $\kappa_i = (x_i, o_i)$, we have $\mathbb{S}(\hbar) = \{\sigma_c\} \cup \Sigma_{uc}$, $\sigma_c \in o_n \cap \Sigma_c$, i.e., the supervisor makes a control decision based on the observed set of activated events at x_n.

We denote by $\Phi(\mathbb{G})$ the set of all strategic supervisors in $\mathbb{G}$.

Given a strategic supervisor $\mathbb{S} \in \Phi(\mathbb{G})$ and an actual DES $G \in \mathbb{G}$, the controlled system, denoted by $\mathbb{S}/G$, whose language $\mathcal{L}(\mathbb{S}/G)$ is recursively defined as:

i) $\epsilon \in \mathcal{L}(\mathbb{S}/G)$;
ii) for any $\hbar = \kappa_0\sigma_1\kappa_1\sigma_2 \cdots \kappa_n$, we have $\sigma_1\sigma_2 \cdots \sigma_n \in \mathcal{L}(\mathbb{S}/G)$ iff $\sigma_1\sigma_2 \cdots \sigma_{n-1} \in \mathcal{L}(\mathbb{S}/G)$ and $\sigma_n \in \mathbb{S}(\hbar)$.

In the partially-known setting, since the actual DES $G \in \mathbb{G}$ is unknown *a priori*, we aim to synthesize a strategic supervisor $\mathbb{S} \in \Phi(\mathbb{G})$ such that

$$\forall G \in \mathbb{G} : \mathcal{L}(\mathbb{S}/G) \text{ satisfies (2)} \tag{10}$$

We denote by $\Phi_W(\mathbb{G}) \subseteq \Phi(\mathbb{G})$ the set of all winning strategic supervisors for the reachability objective.

To evaluate the performance of a strategic supervisor $\mathbb{S}$, a natural approach is to still consider the *worst-case cost* similar to the definition of (4) among all possible actual DES, i.e.,

$$\mathsf{Cost}_{\text{worst}}(\mathbb{S}) := \max_{G \in \mathbb{G}} \mathsf{cost}(\mathbb{S}/G) \tag{11}$$

However, this metric cannot capture the potential benefit obtained from exploring unknown states. To address this issue, we propose to use *regret* to evaluate the performance of a strategic supervisor, which is defined as follows.

Definition 5 (Regret): Given a partially-known DES $\mathbb{G}$, the regret of a supervisor $\mathbb{S} \in \Phi(\mathbb{G})$ is

$$\mathsf{Reg}(\mathbb{S}) = \max_{G \in \mathbb{G}} \left(\mathsf{cost}(\mathbb{S}/G) - \min_{\mathbb{S}' \in \Phi_W(G)} \mathsf{cost}(\mathbb{S}'/G) \right)$$

The intuition is explained as follows. For each strategic supervisor $\mathbb{S} \in \Phi(\mathbb{G})$ and each possible actual DES $G \in \mathbb{G}$, $\mathsf{cost}(\mathbb{S}/G)$ is the actual cost incurred when applying $\mathbb{S}$ to this specific DES, while $\min_{\mathbb{S}' \in \Phi_W(G)} \mathsf{cost}(\mathbb{S}'/G)$ is the *best-response cost* if G had been known to the system with hindsight. Therefore, their difference is the regret of applying $\mathbb{S}$ in G. Since the system does not know which $G \in \mathbb{G}$ the actual DES is, the regret of $\mathbb{S}$ is defined as the worst-case regret among all $G \in \mathbb{G}$.

Finally, we formally formulate the problem that we solve in this paper as follows.

Problem 1 (Regret-Optimal Supervisory Control): Given a partially-known DES $\mathbb{G}$, synthesize a strategic supervisor $\mathbb{S} \in \Phi_W(\mathbb{G})$ such that

$$\forall \mathbb{S}' \in \Phi_W(\mathbb{G}) : \mathsf{Reg}(\mathbb{S}) \leq \mathsf{Reg}(\mathbb{S}'). \tag{12}$$

IV. Main Results

In this section, we first show that the regret-optimal supervisory control problem can be reduced to a quantitative *three-player graph game* by incorporating knowledge into the state space. Then, we propose the algorithm to obtain the regret-optimal supervisor.

A. Tripartite Transition System

We aim to incorporate the knowledge set into the state space of the game arena and explicitly split the movements of the control decision, the system execution, and the non-determinism of the environment, which leads to the following notion of *tripartite transition system*.

Definition 6 (Tripartite Transition System): Given a PK-DES $\mathbb{G}$, its tripartite transition system (TTS) is a tuple

$$\mathcal{G} = (V_X, V_Y, V_Z, v_0, f_{XZ}, f_{ZY}, f_{YX}, \Sigma, \Gamma)$$

where

- $V_X = X \times \mathbb{KW}$ is the set of X-states;
- $V_Y = X \times \mathbb{KW} \times \Gamma$ is the set of Y-states;
- $V_Z = X \times \mathbb{KW} \times X \cup \{d\}$ is the set of Z-states;
- $v_0 = (x_0, \mathcal{K}_0) \in Q_X$ is the initial state with $\mathcal{K}_0$ being the initial knowledge set;
- $f_{XY} : V_X \times \Gamma \to V_Y$ is the transition function from X-states to Y-states defined by: for any $v_x = (x, \mathcal{K}) \in V_X$, $\gamma \in \Gamma$, and $v_y = (x, \mathcal{K}, \gamma) \in V_F$, we have
$$v_y = f_{XY}(v_x, \gamma) \tag{13}$$
- $f_{YZ} : V_Y \times (\Sigma \cup \{\epsilon\}) \to V_Z$ is the transition function from Y-states to Z-states defined by: for any $v_y = (x, \mathcal{K}, \gamma) \in V_Y$, we have
  - if $\gamma \cap o_{\mathcal{K}}(x) \neq \emptyset$, we define
$$f_{YZ}(v_y, \sigma) = (x, \mathcal{K}, \delta(x, \sigma)),\ \forall \sigma \in \gamma \cap o_{\mathcal{K}}(x) \tag{14}$$
  - if $\gamma \cap o_{\mathcal{K}}(x) = \emptyset$, we define
$$f_{YZ}(v_x, \epsilon) = d \tag{15}$$
- $f_{ZX} : V_Z \times 2^{\Sigma} \to V_X$ is the transition function from Z-states to X-states defined by: for any $v_z = (x, \mathcal{K}, x') \in V_Z \setminus \{d\}$ and any $o \in \Delta(x')$, we define
$$v_x = f_{ZX}(v_z, o) = (x', \mathcal{K}') \tag{16}$$
with $\mathcal{K}' = \texttt{update}(\mathcal{K}, (x', o))$;
- Σ is the set of events in G;
- Γ is the set of admissible control decisions in G.

The intuition of the TTS is explained as follows. The graph is tripartite with three types of states: the X-states from which the system chooses a feasible control decision; the Y-states from which the system is executed according the control decision chosen before; and the Z-states from which the actual event-pattern is decided in the possible world.

Note that the knowledge sets are update *monotonely* since they are ordered sets. Consider the Z-states with at least two successors, i.e., $v_z \in V_Z$ satisfying $\text{Succ}(v_z) \geq 2$, which we call as "Z-states with decisions". Essentially, the knowledge sets in $\mathcal{G}$ will be updated after each Z-state with decisions. Then we build the following property for $\mathcal{G}$.

Proposition 1: In the TTS $\mathcal{G}$, there are no cycles encompassing two different Z-states with decisions.

Proof: The proof is given in Appendix I. ■

B. Plays and Strategies

We denote by $V = V_X \cup V_Y \cup V_Z$ the state space of $\mathcal{G}$. Furthermore, we define the set of marked states in $\mathcal{G}$ as $V_m = \{(x, \mathcal{K}) \in V_X : x \in X_m\}$. Given $\mathcal{G}$, we call a finite sequence

$$\rho = v_x^0 v_y^0 v_z^0 v_x^1 v_y^1 v_z^1 \cdots v_x^n \in (V_X \cdot V_Y \cdot V_Z)^* V_X$$

a *play* if $v_x^0 = v_0$ and there are $\gamma_i \in \Gamma$, $\sigma_i \in \Sigma \cup \{\epsilon\}$ and $o_i \in 2^{\Sigma}$ such that $v_y^i = f_{XY}(v_x^i, \gamma_i)$, $v_z^i = f_{YZ}(v_y^i, \sigma_i)$, and $v_x^{i+1} = f_{ZX}(v_z^i, o_i)$ hold for each $i = 0, 1, \ldots, n-1$ and $v_x^i = v_x^n$ for $i = n$. It is obvious that each play in $\mathcal{G}$ induces a history in $\mathbb{G}$, i.e.,

$$\hbar_\rho = (x_0, o_0)\sigma_0(x_1, o_1)\sigma_1 \cdots (x_n, o_n) \in \mathcal{H}(\mathbb{G}).$$

For convenience, we still use ρ to denote the partial play that does not necessarily end with a X-state.

Since only edges from V_z to V_x represent actual movements, we define a weight function for $\mathcal{G}$ as $w_{\mathcal{G}} : V \times V \to \mathbb{R}_{\geq 0}$ where for any $v_z = (x, \mathcal{K}, x')$ and $v_x = (x', \mathcal{K}')$, we have $w_{\mathcal{G}}(v_z, v_x) = w(x, x')$, $w_{\mathcal{G}}(v_x, v_y) = w_{\mathcal{G}}(v_y, v_z) = 0$. The cost of a play $\rho = v_0 v_1 \cdots v_n \in V^*$ is defined as $\mathsf{cost}_{\mathcal{G}}(\rho) = \sum_{i=0}^{n-1} w_{\mathcal{G}}(v_i, v_{i+1})$.

Given the above TTS $\mathcal{G}$, the strategies are functions

$$\pi_x : V^* V_X \to V_Y \cup \{\text{Null}\},\ \pi_y : V^* V_Y \to V_Z,\ \pi_z : V^* V_Z \to V_X$$

for X-player, Y-player and Z-player, respectively. We denote by $\mathfrak{S}_X(\mathcal{G})$, $\mathfrak{S}_Y(\mathcal{G})$, $\mathfrak{S}_Z(\mathcal{G})$ the sets of all X-strategies, Y-strategies and Z-strategies, respectively. In particular, we say a strategy π is *positional* if $\forall \rho, \rho' : \mathsf{last}(\rho) = \mathsf{last}(\rho') \Rightarrow \pi(\rho) = \pi(\rho')$, where $\mathsf{last}(\cdot)$ denotes the last state of a sequence. We denote by $\mathfrak{S}_j^1(\mathcal{G})$ the set of all positional strategies for j-player, where $j = X, Y, Z$. Given strategies $\pi_x \in \mathfrak{S}_X, \pi_y \in \mathfrak{S}_Y, \pi_z \in \mathfrak{S}_Z$, the outcome play $\rho_{\pi_x, \pi_y, \pi_z}$ is the unique sequence $v_0 v_1 \cdots v_n \in V^* V_X$ such that

- $\forall i < n : v_i \in V_X \Rightarrow \pi_x(v_0 v_1 \cdots v_i) = v_{i+1}$;
- $\forall i < n : v_i \in V_Y \Rightarrow \pi_y(v_0 v_1 \cdots v_i) = v_{i+1}$;
- $\forall i < n : v_i \in V_Z \Rightarrow \pi_z(v_0 v_1 \cdots v_i) = v_{i+1}$;
- $\pi_x(v_0 v_1 \cdots v_n) = \text{Null}$.

Here we remark that, since the objective of the supervisor is to satisfy the reachability w.r.t. the set V_m, once the system trajectory visits V_m, it is unncessary to restrict the behaviors of the system. To this end, in the TTS $\mathcal{G}$, for a play ρ satisfying $\mathsf{last}(\rho) \in \text{Null}$, we set $\pi_x(\rho) = \text{Null}$.

Next, we aim to characterize the sets of strategies for the X, Y, Z-players that are sufficient to solve the problem.

For the Y-player, since there is no restriction for the uncontrollable events and no causal relation between two different events, it is sufficient to consider $\Pi_Y = \mathfrak{S}_Y^1(\mathcal{G})$ for all the possible behaviors generated by Y-player.

For the Z-player, it cannot play arbitrarily since the actual DES is fixed. Therefore, the Z-player must commit to a specific event-pattern it chooses at each unknown state when the game begins. To this end, we define the following notion of *strongly positional strategy*. We say a Z-strategy $\pi_z \in \mathfrak{S}_Z(\mathcal{G})$ is strongly positional if for any two plays $\rho, \rho' \in V^* V_Z$ where $\pi_z(\rho) = (x, \mathcal{K})$ and $\pi_z(\rho') = (x', \mathcal{K}')$, we have i) π_z is positional; and ii) $x = x' \Rightarrow o_{\mathcal{K}}(x) = o_{\mathcal{K}'}(x')$.

$$\mathsf{Reg}_{\mathcal{G}}(\pi_x) = \max_{\pi_z \in \Pi_Z} \left(\max_{\pi_y \in \Pi_Y} \mathsf{cost}_{\mathcal{G}}(\rho_{\pi_x,\pi_y,\pi_z}) - \min_{\pi_x' \in \Pi_X} \max_{\pi_y \in \Pi_Y} \mathsf{cost}_{\mathcal{G}}(\rho_{\pi_x',\pi_y,\pi_z}) \right) \tag{18}$$

We denote by $\Pi_Z \subseteq \mathfrak{S}_Z(\mathcal{G})$ the set of all stronly positional strategies for the Z-player.

Finally, we aim to find a winning X-strategy that ensures the outcome play visits the state in V_m. Therefore, we define

$$\Pi_X = \left\{ \pi_x \in \mathfrak{S}_X(\mathcal{G}) : \begin{array}{c} \mathsf{last}(\rho_{\pi_x,\pi_y,\pi_z}) \in V_m, \\ \forall \pi_y \in \Pi_Y, \forall \pi_z \in \Pi_Z \end{array} \right\} \tag{17}$$

Now, similarly to Definition 5, we define the regret of an X-strategy $\pi_x \in \Pi_X$ in $\mathcal{G}$ as (18). It suffices to synthesize an X-strategy $\pi_X \in \Pi_X$ that minimizes $\mathsf{Reg}_{\mathcal{G}}(\pi_x)$. In what follows, we present the solution by introducing two algorithms for micro and macro-strategy synthesis.

C. Strategy Synthesis

Due to the possible existence of cycles in $\mathcal{G}$, there is generally an infinite number of winning strategies for X-player, which makes enumerating all of them infeasible. To tackle this issue, we propose to compute *micro-strategies* to capture the strategies that are sufficient to obtain the regret-optimal X-strategy. Based on Proposition 1, we know that, the evolution of Z-states is monotone and for any plays in $\mathcal{G}$ that end with the same state, they must visit the same Z-states with decisions in the same order. Therefore, it suffices to compute the X-strategies between different *layers of Z-states*. To formally capture this, we define the set of initial states for all knowledge sets as

$$V_I = \{v_0\} \cup \{\mathrm{Succ}(v_z) \in V_X : \mathrm{Succ}(v_z) \geq 2, \forall v_z \in V_Z\} \tag{19}$$

For each $v_I \in V_I$, we define the set of all end states as

$$\begin{aligned} V_F(v_I) = \{v_x \in V_m : \mathcal{K}(v_x) = \mathcal{K}(v_I)\} \quad (20) \\ \cup \left\{ v_x \in V_X : \begin{array}{c} \exists v_z \in V_Z \text{ s.t. } \mathcal{K}(v_z) = \mathcal{K}(v_I) \wedge \\ v_x \in \mathrm{Succ}(v_z) \wedge \mathcal{K}(v_x) \neq \mathcal{K}(v_z) \end{array} \right\} \end{aligned}$$

We denote by $V_F = \bigcup_{v_I \in V_I} V_F(v_I)$ the set of of end nodes in TTS $\mathcal{G}$.

For convenience, in what follows, we use a *tree* in the TTS $\mathcal{G}$ to represent a strategy and we denote by $\mathcal{T}$ the set of all trees in $\mathcal{G}$. For each $\mathcal{T} \in \mathbb{T}$, we denote by $V_{\mathcal{T}}$ and $E_{\mathcal{T}}$ its sets of states and edges. We define the cost of $\mathcal{T}$, denoted by $\mathsf{cost}_{\mathbb{T}}(\mathcal{T})$, as the maximum cost of its plays from the root to the leaves. We denote by $\rho_{\mathcal{T}}$ the play with the maximum cost, i.e., $\mathsf{cost}_{\mathcal{G}}(\rho_{\mathcal{T}}) = \mathsf{cost}_{\mathbb{T}}(\mathcal{T})$. Furthermore, for each $\mathcal{T} \in \mathbb{T}$, it is a map $\mathcal{T} : V_{\mathcal{T}} \to \mathbb{T}$ that assigns each state in $\mathcal{T}$ a subtree. Specifically, we have $\mathcal{T}(v_I) = \mathcal{T}$, where v_I is the root of $\mathcal{T}$.

Definition 7 (Micro-Strategy): Given the TTS $\mathcal{G}$ and for each $v_I \in V_I$, a micro-strategy $\mathcal{T}_{v_I} \in \mathbb{T}$ for X-player is a winning strategy (strategy-tree) such that

i) the root of $\mathcal{T}_{v_I}$ is v_I;
ii) for each $v_x \in V_{\mathcal{T}_{v_I}} \cap V_X$, there is only one successor $v_y \in \mathrm{Succ}(v_x)$ in $\mathcal{G}$ such that $(v_x, v_y) \in E_{\mathcal{T}_{v_I}}$;
iii) for each $v_y \in V_{\mathcal{T}_{v_I}} \cap V_Y$, for any successor $v_z \in \mathrm{Succ}(v_y)$ in $\mathcal{G}$, we have $(v_y, v_z) \in E_{\mathcal{T}_{v_I}}$;
iv) for each $v_z \in V_{\mathcal{T}_{v_I}} \cap V_Z$, all of the successors $v_x \in \mathrm{Succ}(v_z)$ in $\mathcal{G}$ are the leaves;
v) All $v_x \in V_{\mathcal{T}_{v_I}} \cap V_m$ are leaves.

For convenience, for each micro-strategy $\mathcal{T} \in \mathbb{T}$, we denote by $Acc(\mathcal{T})$ the set of all leaves of $\mathcal{T}$.

Intuitively, each micro-strategy captures the partial strategy for X-player in $\mathcal{G}$ that ensures for each $v_I \in V_I$ either the reachability of V_m or a new exploration. For any two micro-strategies $\mathcal{T}_{v_I}$ and $\mathcal{T}'_{v_I}$, we say they are similar if they have the same leaves. For each $v_I \in V_I$, we call the micro-strategy with the minimum cost as the *critical micro-strategy*.

Now, we present the algorithm for finding all critical micro-strategies as Algorithm 2, which is explained in detail as follows. In Line 2, we define the set of all micro-strategies as Π_{v_I} which assigns each feasible combination Acc of end nodes with a micro-strategy and the cost of the strategy. With a slight abuse of notation, we denote $Acc \in \Pi_{v_I}$ if there is a micro-strategy for Acc in Π_{v_I}. We aim to use a depth-first search procedure to find all the strategies. For `DFS`, we use v, V_{vis}, and C to denote the current state, the states that have been searched and the minimum cost of the path from v_I to v. For each v, we aim to build a subtree $\mathcal{T}_v$ recursively (Line 7). If the current state v is an end state, then we compute its visited end nodes as Acc (Line 9) and update the optimal micro-strategy for Acc in Lines 10–15. If the current state v is an X-state, then we use $\tilde{\mathsf{C}}$, $\tilde{v}_y$ and $\tilde{T}$ to record the total cost for its optimal subtree, the root of its optimal subtree, and the optimal subtree, respectively (Lines 18–20). We find the unique optimal subtree for v by Lines 21–31. Specifically, we use Lines 23–25 to ensure that there is no cycles in the searched subtree. If the current state v is a Y-state or Z-state, we keep the subtrees starting from all the successors of v. We summarize the correctness as follows.

Proposition 2: For each $v_I \in V_I$, all its the critical micro-strategies are returned by Algorithm 2.

Proof: The proof is given in Appendix II. ■

With the critical micro-strategies for *one layer of Z-states with decisions*, we next introduce the notion of *macro-strategy* that connects the *different layers of Z-states with decisions* by the micro-strategies.

Definition 8 (Macro-Strategy): Given $\mathcal{G}$, a macro-strategy is a map $\xi : V_I \to \{\Pi_{v_I} : v_I \in V_I\}$ such that for each $v_I \in V_I$, we have $\xi(v_I) = \mathcal{T}_{v_I}$ for some micro-strategy $\mathcal{T}_{v_I} \in \Pi_{v_I}$.

Given a partial macro-strategy ξ, we denote by $\mathsf{dom}(\xi) \subseteq V_I$ the domain of ξ. For each ξ, we define

$$\Upsilon(\xi) = \left(\{v_0\} \cup \bigcup_{v' \in \mathsf{dom}(\xi)} (Acc(\xi(v')) \cap V_X) \right) \backslash \mathsf{dom}(\xi) \tag{21}$$

the set of frontier nodes of ξ, representing the X-states for which the micro-strategies need to be assigned next. Based on this, we say ξ is a *complete strategy* if $\Upsilon(\xi) = \emptyset$.

Algorithm 2: Find All Critical Micro-Strategies (FACMiS)

```
Input: The TTS $\mathcal{G}$ and an initial state $v_I$
Output: All micro-strategies from $v_I$ to $V_F(v_I)$
1  Compute $V_F(v_I)$ as in (20);
2  Define $\Pi_{v_I} : 2^{V_F(v_I)} \to \mathbb{T} \times \mathbb{R}_{\geq 0}$;
3  Call DFS$(v_I, \{v_I\}, 0)$;
4  return $\Pi_{v_I}$
5  procedure DFS$(v, V_{vis}, \mathsf{C})$
6  Define subtree $\mathcal{T}_v : Succ(v) \to \mathbb{T}$;
7  $\mathcal{T}_v \leftarrow$ Null;
8  if $v \in V_F(v_I)$ then
9      $Acc \leftarrow V_{vis} \cap V_F(v_I)$;
10     if $Acc \notin \Pi_{v_I}$ then
11         $\Pi_{v_I}(Acc) \leftarrow (\mathcal{T}_v, \mathsf{C})$;
12     else
13         $(\mathcal{T}, c) \leftarrow \Pi_{v_I}(Acc)$;
14         if $C < c$ then
15             $\Pi_{v_I}(Acc) \leftarrow (\mathcal{T}_v, \mathsf{C})$;
16     return Null
17 if $v \in V_X$ then
18     Define $\tilde{\mathsf{C}} \leftarrow \infty$, $\tilde{v}_y \leftarrow$ Null, $\tilde{T} \leftarrow$ Null;
19     foreach $v_y \in Succ(v)$ do
20         if $v_y \notin V_{vis}$ then
21             $V_{vis} \leftarrow V_{vis} \cup \{v_y\}$;
22             $T \leftarrow$ DFS$(v_y, V_{vis}, \mathsf{C} + w_{\mathcal{G}}(v, v_y))$;
23             $V_{vis} \leftarrow V_{vis} \backslash \{v_y\}$;
24             if $C + w_{\mathcal{G}}(v, v_y) + cost_{\mathbb{T}}(T) < \tilde{C}$ then
25                 $\tilde{\mathsf{C}} \leftarrow \mathsf{C} + w_{\mathcal{G}}(v, v_y) + \mathsf{cost}_{\mathbb{T}}(T)$;
26                 $\tilde{v}_y \leftarrow v_y$;
27                 $\tilde{T} \leftarrow T$;
28     if $\tilde{v}_y \neq Null$ then
29         $\mathcal{T}_v(\tilde{v}_y) \leftarrow \tilde{T}$;
30 if $v \in V_Y \cup V_Z$ then
31     foreach $v' \in Succ(v)$ do
32         if $v' \notin V_{vis}$ then
33             $V_{vis} \leftarrow V_{vis} \cup \{v'\}$;
34             $T \leftarrow$ DFS$(v', V_{vis}, \mathsf{C} + w_{\mathcal{G}}(v, v'))$;
35             $V_{vis} \leftarrow V_{vis} \backslash \{v'\}$;
36             $\mathcal{T}_v(v') \leftarrow T$;
37 return $\mathcal{T}_v$
```

To compute regret of a given complete macro-strategy ξ, we first characterize the set of all Z-strategies Π_Z as follows. Given the unknown states $x_1, \dots, x_n \in X_{uk}$, we construct

$$\Theta = \{(x_1, o_1) : o_1 \in \Delta(x_1)\} \times \cdots \times \{(x_n, o_n) : o_n \in \Delta(x_n)\} \quad (22)$$

For each $\theta = \langle (x_1, o_1), \cdots, (x_n, o_n) \rangle \in \Theta$, we define a Z-strategy π_z^θ as: for any $v_z = (x, \mathcal{K}, x_i)$ where $x_i \in X_{uk}$, we have

$$\pi_z^\theta(v_z) = (x_i, \mathcal{K}'),\ \mathcal{K}' = \mathtt{update}(\mathcal{K}, (x_i, o_i)) \quad (23)$$

For $v_z = (x, \mathcal{K}, x')$ where $x' \in X_k$, we have $\pi_z^\theta(v_z) = (x', \mathcal{K})$.

Algorithm 3: Find Regret-Optimal Macro-Strategy

```
Input: The TTS $\mathcal{G}$ and an initial state $v_0$
Output: The regret-optimal macro-strategy $\xi^\star$ and
        the minimized regret $\mathsf{Reg}^\star$
1  Construct an empty macro-strategy $\xi : V_I \to \mathbb{T}$ with
   $\mathsf{dom}(\xi) = \emptyset$;
2  $\mathsf{Reg}^\star \leftarrow \infty$ and $\xi^\star \leftarrow \xi$;
3  Call DFS2$(v_0, \xi)$;
4  return $\xi^\star$ and $\mathsf{Reg}^\star$
5  procedure DFS2$(v, \xi)$
6  if $v \notin dom(\xi)$ then
7      foreach $\mathcal{T} \in \Pi_v$ do
8          $\xi(v) \leftarrow \mathcal{T}$;
9          ADVANCE$(\xi)$;
10         $\xi(v) \leftarrow \emptyset$;
11 else
12     ADVANCE$(\xi)$;
13 procedure ADVANCE$(\xi)$
14 if $\Upsilon(\xi) = \emptyset$ then
15     $r \leftarrow$ REGRET$(\xi)$;
16     if $r < Reg^\star$ then
17         $\mathsf{Reg}^\star \leftarrow r$ and $\xi^\star \leftarrow \xi$;
18 else
19     Choose any $v' \in \Upsilon(\xi)$;
20     Call DFS2$(v', \xi)$;
```

By th above construction, we have $\Pi_Z = \{\pi_z^\theta : \theta \in \Theta\}$.

Given a Z-strategy π_z^θ, the TTS $\mathcal{G}$ is induced to a tree $\mathcal{G}_{\pi_z^\theta}$ which is essentially a game between X and Y. Therefore, we call Algorithm 1 to compute the best-response cost against π_z^θ, i.e., $\mathsf{minmax}(\mathcal{G}_{\pi_z^\theta}) \leftarrow \mathtt{SolveMinMax}(\mathcal{G}_{\pi_z^\theta}, V_m, w_{\mathcal{G}})$. Given a complete macro-strategy ξ and a Z-strategy π_z^θ, the TTS $\mathcal{G}$ is induced to a tree $\mathcal{G}_{\xi, \pi_z^\theta}$. We denote by $\rho(\mathcal{G}_{\xi, \pi_z^\theta})$ the play in tree $\mathcal{G}_{\xi, \pi_z^\theta}$ with the maximum cost $\mathsf{cost}(\rho(\mathcal{G}_{\xi, \pi_z^\theta}))$, which can be directly computed by a tree dynamic programming. Then, we define

$$\mathtt{REGRET}(\xi) = \max_{\pi_z^\theta \in \Pi_Z} \left(\mathsf{cost}(\rho(\mathcal{G}_{\xi, \pi_z^\theta})) - \mathsf{minmax}(\mathcal{G}_{\pi_z^\theta}) \right)$$

where we use $Acc(\mathcal{G}_{\xi, \pi_z^\theta})$ to denote all the leaves of $\mathcal{G}_{\xi, \pi_z^\theta}$.

Due to the fact that $V_F(v_I) \subseteq V_I$ for each $v_I \in V_I$, it is sufficient to form a *complete X-strategy* in the TTS $\mathcal{G}$ with one macro-strategy with the corresponding micro-strategies, each of which starts from either the initial state v_0 or one leave of the last micro-strategy.

Now, we present the algorithm for finding the regret-optimal macro-strategy as Algorithm 3, which in explained in detail as follows. We first construct an empty macro-strategy ξ and initiate the best regret $\mathsf{Reg}^\star$ and the current best macro-strategy $\xi^\star$ in lines 1–2. Then a depth-first-search procedure DFS2 is started from the initial state v_0 in line 3. Procedure DFS2(v, ξ) is used to explore a complete macro-strategy from state v given the current partial macro-strategy ξ. In this procedure, if v is not defined yet in the domain of ξ, then we enumerate all micro-strategies $\mathcal{T} \in \Pi_v$. For each $\mathcal{T}$,

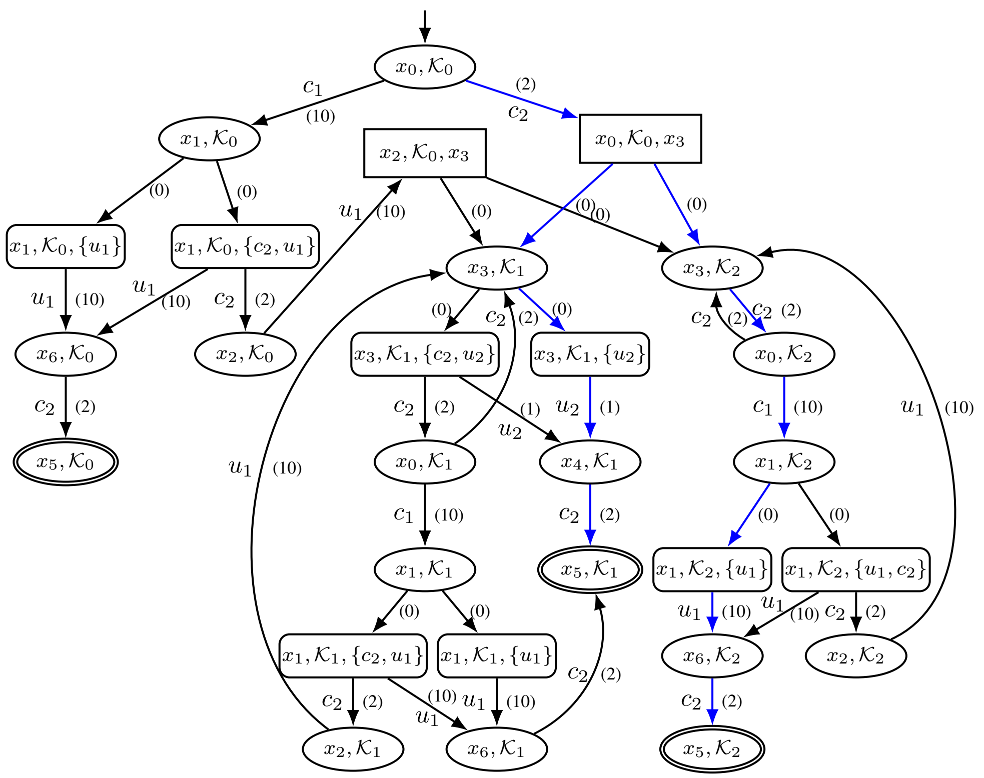

Fig. 2. Tripartite transition system of PK-DES in Fig 1(b).

we temporarily assign it as the current best micro-strategy (line 9) and call $\texttt{ADVANCE}(\xi)$ to push forward the partial macro-strategy ξ. After that, we undo the assignment in line 9 to continue with the next $\mathcal{T}$. Obviously, if v is already defined in ξ, then we directly call $\texttt{ADVANCE}(\xi)$ to proceed to the next frontier node. The procedure $\texttt{ADVANCE}(\xi)$ is used to either detect that ξ is complete and evaluate its regret or choose a new frontier node and recurs. If $\Upsilon(\xi) = \emptyset$ showing that ξ is complete, then we compute its regret and update the incumbent in lines 14–17. Otherwise, we select any $v' \in \Upsilon(\xi)$ and recursively call $\texttt{DFS2}(v', \xi)$ to continue with the search from a new undecided X-state. The correctness is summarized as follows.

Theorem 1: Given the TTS $\mathcal{G}$, the X-strategy formed by the macro-strategy returned by Algorithm 3 and the micro-strategies returned by Algorithm 2 minimizes the regret for X-player, i.e., Problem 1 is solved with minimized $\mathsf{Reg}^\star$.

Proof: The proof is given in Appendix III. ■

Example 1: Let us still consider the PK-DES as shown in Fig 1(b) where $\Sigma_c = \{c_1, c_2\}$, $w(c_2) = 2$, $w(c_1) = w(u_1) = 10$ and $w(u_2) = 1$. The corresponding TTS $\mathcal{G}$ is as shown in Fig 2. For simplicity, we omit Z-states when the corresponding state is already known as well as the controllable transitions that can not satisfy the reachability specification. Since there is only one unknown state in Fig 1(b), the corresponding Z-state in $\mathcal{G}$ is $(x_0, \mathcal{K}_0, x_3)$ from which there are two different X-states with the different updated knowledge set, i.e., $(x_3, \mathcal{K}_1)$ and $(x_3, \mathcal{K}_2)$. We apply the solution synthesis approach provided by Algorithm 2 and Algorithm 3 to obtain the regret-optimal strategic supervisor as the blue transitions in Fig 2 with the minimized $\mathsf{Reg}^\star = 4$.

V. Conclusion

In this paper, we formulate and solve the regret-optimal supervisory control problem for partially-known discrete-event systems. To model the system's interaction with uncertain environments, we introduce a tripartite transition structure that captures all possible actual environments. Then by decomposing the problem into a two-stage reachability game, we develop the solution algorithm. Our results demonstrate that regret serves as a more meaningful performance metric than conventional approaches for handling information that is deterministic yet initially unknown. This result extends optimal supervisory control theory beyond traditional worst-case analysis. For the future work, we plan to extend the partially-known setting to broader specifications including cyclic tasks and mean-payoff cost metrics.

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Appendix I
Proof of Proposition 1

We prove this result by contradiction. Suppose that there exists a cycle that encompasses two different Z-states with decisions. That is, there are $v_z, v'_z \in V_Z$ satisfying

i) $v_z \neq v'_z$;

ii) there is a sequence starting from v_z to v'_z; and

iii) there is also a sequence starting from v'_z to v_z.

Then, based on the construction of $\mathcal{G}$, we know that, after each X-state with decisions, the knowledge sets will be updated to a different one, that is,

$$\forall v_z \in V_Z, \forall v_x \in \text{Succ}(v_z) \geq 2 \Rightarrow \mathcal{K}(v_x) \neq \mathcal{K}(v_z) \quad (24)$$

where $\mathcal{K}(\cdot)$ denotes the knowledge set of a state in $\mathcal{G}$. From ii), we know that, $\mathcal{K}(v'_z)$ is updated from $\mathcal{K}(v_z)$; and from iii), we know that, $\mathcal{K}(v_z)$ is updated from $\mathcal{K}(v'_z)$. Since knowledge sets are ordered sets, to satisfy ii) and iii), we have $\mathcal{K}(v_z) = \mathcal{K}(v'_z)$, which contradicts with (24). The proof is thus completed.

Appendix II
Proof of Proposition 2

We first prove the soundness, i.e., all strategies returned by Algorithm 2 are micro-strategies for each $v_I \in V_I$ according to Definition 7. It is obvious that i) is satisfied since the `DFS` starts from v_I. For each X-state of each returned strategy, there is only one successor based on Lines 17–29, which satisfies ii). Similarly, Lines 30–36 ensure that all the successors of each Y-state and each Z-state in $\mathcal{G}$ are added to the returned strategy tree. Specifically, for each Z-state, since all its successors are included in $V_F(v_I)$, Lines 8–16 ensure that all theses successors are leaves of each returned strategy tree, which satisfies iv). Finally, since the marked states which have the same knowledge set with v_I are also included in $V_F(v_I)$, Lines 8–16 also ensures that these marked states are the leaves of each returned strategy, which satisfies v). Therefore, each returned strategy in Π_{v_I} is a micro-strategy for each $v_I \in V_I$.

Then, we show the completeness, i.e., for each $v_I \in V_I$, for any its micro-strategy in $\mathcal{G}$, there is a corresponding similar micro-strategy in Π_{v_I} returned by Algorithm 2. In the algorithm, we use Acc to record the leaves of each strategy tree. Specifically, by Line 9, we know that, once a leaf state is searched, Acc will be updated. Since `DFS` guarantees the traverse among all the end nodes in $V_F(v_I)$, each possible combination of the end nodes will be captured by different Acc. By Lines 10-11, we know that, for each Acc, there is a corresponding micro-strategy updated, i.e., $\Pi_{v_I}(Acc)$. Therefore, for any micro-strategies with different leaves, there are corresponding micro-strategies in Π_{v_I}.

Finally, we show the optimality, i.e., all the strategies returned by Algorithm 2 are critical strategies. By Lines 20–29, we ensure that, all the micro-strategies will not include any cycles, since only one successor $\tilde{v}_y$ with the corresponding optimal subtree $\tilde{T}$ is added to $\mathcal{T}_v$ and each candidate v_y is updated only when it was not searched, i.e., $v_y \notin V_{vis}$. We use C to record the minimum cost of the path from v_I to the current searched state v. For each possible Acc, we update its optimal micro-strategy by Lines 13–15. Therefore, all the returned strategies are critical micro-strategies. The proof is thus completed.

Appendix III
Proof of Theorem 1

By a similar analysis to Appendix II, we know that, all the complete macro-strategies can be traversed by the depth-first-search. By the construction of $\Pi_Z = \{\pi_z^\theta : \theta \in \Theta\}$, we have $\texttt{REGRET}(\xi) = \mathsf{Reg}_{\mathcal{G}}(\xi)$. Therefore, the returned macro-strategy has the minimal regret value. The proof is thus completed.