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TikhonovFenichelReductions.jl: A systematic approach to geometric singular perturbation theory

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Summary

Singular perturbation theory is a mathematical toolbox that allows dimensionality reduction of ODE systems whose components evolve on different time scales emanating from the presence of a small parameter ε. More precisely, we obtain the *reduced system* as the limit

$$\begin{aligned} \dot{u} &= g(u,v) \\ \varepsilon\dot{v} &= h(u,v) \end{aligned} \quad \xrightarrow[\varepsilon\to 0]{} \quad \begin{aligned} \dot{u} &= g(u,v) \\ 0 &= h(u,v) \end{aligned} \tag{1}$$

as stated in Tikhonov's theorem (1952; Theorem 1.1 in Verhulst, 2007). For autonomous systems, Fenichel (1979) established a geometric singular perturbation theory (GSPT) in coordinate-free settings (see Wechselberger, 2020). Convergence properties then follow from the slow manifold $M_0 = \{(u,v) \,|\, h(u,v) = 0\}$, on which the reduced system is defined.

The algebraic approach to GSPT recently developed by Goeke and Walcher (and colleagues) allows to systematically find all critical parameters admitting a reduction for polynomial or rational ODE systems of the form

$$\dot{x} = f(x,\pi) = f^{(0)}(x) + \varepsilon f^{(1)}(x) + \mathcal{O}(\varepsilon^2), \quad x \in \mathbb{R}^n, \pi \in \mathbb{R}^m, \tag{2}$$

i.e. we obtain all reductions for a system with a slow-fast separation of processes instead of components as in the standard form (1), which renders this a coordinate-free approach. This can be achieved by evaluating necessary conditions for the existence of a reduction for a system as in (2) (Goeke, 2013; Goeke et al., 2015; Goeke & Walcher, 2013, 2014).

`TikhonovFenichelReductions.jl` is a `Julia` (Bezanson et al., 2017) package implementing this approach for polynomial ODE systems. Apelt & Liebscher (2025) provide a showcasing example and more detailed explanations.

Statement of need

The ad-hoc approach to singular perturbation theory requires prior knowledge about a suitable time scale separation and substantial mathematical effort to compute the reduction. The algebraic approach yields algorithmically accessible conditions for the existence of a reduction, which allows us to find all reductions of a given polynomial ODE system using methods from computational algebra, and simplifies the computation of reduced systems (Goeke et al., 2015). `TikhonovFenichelReductions.jl` makes the required computations easily accessible (even for non-expert users) by utilizing `Oscar.jl` (Decker et al., 2025; The OSCAR Team, 2025).

The author is not aware of any publicly available implementation of the theory by Goeke and Walcher, but there exist an implementation for a related approach for computing invariant manifolds by Roberts (n.d., 1997).

Features

The main features provided by `TikhonovFenichelReductions.jl` are the search for critical parameters admitting a reduction, so-called *Tikhonov-Fenichel Parameter Values (TFPVs)*, and the computation of the corresponding reduced systems. Detailed explanations and examples are provided by Apelt & Liebscher (2025) and in the documentation.

Finding TFPVs

The package provides two methods for finding TFPVs admitting a reduction onto an s-dimensional slow manifold.

Restricting the search for TFPVs to *slow-fast separations of rates*, i.e. TFPVs with some parameters set to zero, is typically more efficient and yields the TFPV candidates together with their slow manifolds given implicitly as the irreducible components of the corresponding affine variety $\mathcal{V}(f^{(0)}) = \{x \in \mathbb{R}^n \mid f^{(0)}(x) = 0\}$.

Computing a Gröbner basis G with an elimination ordering for the state variables of an ideal obtained from necessary conditions for the existence of a reduction yields all TFPVs implicitly as $\mathcal{V}(G)$. This may be infeasible to compute, but can reveal more complex expressions that can be considered as small parameters. However, introducing new parameters from these expressions typically allows us to use the methods for slow-fast separations of rates, particularly the computation of the reduced system.

Computing reductions

Computing a reduction for a slow-fast separation of rates as in Theorem 1 in Goeke & Walcher (2014) requires essentially two steps. Let Y be the irreducible component of $\mathcal{V}(f^{(0)})$ with dimension s corresponding to the slow manifold M_0 and $r = n - s$.

First, we need to find a parametric representation of Y as a manifold. This may be done automatically by a provided heuristic. If this fails, Y must be parameterized explicitly w.r.t. s state variables (the local coordinates) manually.

Next, we need to find a product decomposition $P\psi = f^{(0)}$, where P is a $n \times r$ matrix of rational functions and ψ is a vector of polynomials locally satisfying $\mathcal{V}(\psi) = \mathcal{V}(f^{(0)})$ and rank $P =$ rank $D\psi = r$. If Y is defined by r generators or r independent entries of $f^{(0)}$ can be found by a heuristic, P and ψ can be computed automatically.

With this, the reduced system in the sense of Tikhonov is given by

$$\dot{x} = \left[1_n - P(x)(D\psi(x)P(x))^{-1}D\psi(x)\right] f^{(1)}(x).$$

For complex systems, there may exist many TFPVs each admitting multiple reductions, which makes their analysis intractable. Thus, `TikhonovFenichelReductions.jl` contains methods for finding all unique slow manifolds (as varieties) and computing all reductions onto each of them. This relies on the heuristics for finding a parametric representations of the varieties and setting a product decomposition for $f^{(0)}$, but in most cases this means we can compute all or at least most reductions fully automatically.

Integration with the `Julia` ecosystem

The input system can be given as a reaction network defined with `Catalyst.jl` (Loman et al., 2023). Because the reduced systems are represented using types from `Oscar.jl`, the latter's functions can be used to aid the symbolic analysis. `Julia`'s support for metaprogramming allows to perform further tasks such as a numerical analysis without having to copy or parse code (see e.g. `TFRSimulations.jl`). For convenience, there are several methods for displaying the output, including printing as LaTeX source code via `Latexify.jl`.

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