$$\Phi = -\frac{GM}{\sqrt{R^2+z^2}} \Rightarrow \frac{\partial \Phi}{\partial R} = \frac{GM}{(R^2+z^2)^{3/2}} R \quad , \quad \frac{\partial \Phi}{\partial z} = \frac{GM}{(R^2+z^2)^{3/2}} z$$

Consider small displacements in $R \to R_0 + \delta R$ & $z \to 0 + \delta z$, at constant specific angular momentum h, about a circular orbit

z eqn $\frac{\partial \Phi}{\partial z} \simeq \frac{GM}{R^3}\delta z = \Omega^2 \delta z$ so we get $\underline{\ddot{\delta z} = -\Omega^2 \delta z}$

R eqn $\frac{\partial \Phi}{\partial R} \simeq \frac{GM}{\left[(R_0+\delta R)^2 + z^{\to 0^2}\right]^{3/2}}(R_0+\delta R) = \frac{GM}{(R_0+\delta R)^2} \simeq \frac{GM}{R_0^2}\left(1-2\frac{\delta R}{R_0}\right)$

$$R\dot{\phi}^2 = \frac{h^2}{R_0^3} \Rightarrow \frac{h^2}{(R_0+\delta R)^3} \simeq \frac{h^2}{R_0^3}\left(1-3\frac{\delta R}{R_0}\right) \quad , \quad \ddot{R} \to \ddot{\delta R} \text{ since } \ddot{R}_0 = 0.$$

$$\ddot{\delta R} = \underbrace{\frac{h}{R_0^3} - \frac{GM}{R_0^2}}_{\to 0} + \left[-\frac{3h^2}{R_0^4} + \frac{2GM}{R_0^3}\right]\delta R \quad , \text{ since } h = \sqrt{GMR_0} \text{ for circular orbit.}$$

$$= \left[-\frac{3GMR_0}{R_0^4} + \frac{2GM}{R_0^3}\right]\delta R = -\frac{GM}{R_0^3}\delta R = -\Omega^2 \delta R$$

$\Rightarrow \ddot{\delta R} = -\Omega^2 \delta R$, $\ddot{\delta z} = -\Omega^2 \delta z \leftarrow$ SHM motion with frequency $\Omega \to$ natural oscillation time $\to \underline{\Omega^{-1}}$