Research project on rough volatility

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Abstract

This report summarizes some results about rough volatility models, which are widely used to model financial markets. It is the result of a three-month research project by pair, during which we focused on the papers [1] and [2]. For each article, we successively mathematically analysed the model, and then implemented it in Python. All in all, this report should help anyone with basic background in stochastic calculus to understand the main ideas behind rough volatility models and verify them numerically through the provided Jupyter notebooks.

This report is organized as follows. In sections 1 and 2, we dwell on the notion of *roughness*, presenting numerous justifications for the introduction of *rough* processes in finance. In section 3, we present and thoroughly analyse the two main models of the papers [1] and [2], namely the Rough Bergomi model and the Rough Heston model. Section 4 is a transition towards section 5, which focuses on the description of the implementation of rough volatily models.

Simulations for [1]: short.binets.fr/rough_bergomi
Simulations for [2]: short.binets.fr/rough_heston

Contents

1 Introduction

1.1 Rough volatility models

Given a stochastic process S_t modeling the price of a certain asset at each time t, and denoting $X_t := \log S_t$, we usually assume that X is a semi-martingale and write the following general decomposition:

$$dX_t = \mu_t dt + \sigma_t dZ_t$$

where μ_t is the drift term, σ_t is a positive stochastic process called **volatility** and Z_t a standard borwnian motion. By denoting the **variance** $v_t := \sigma_t^2$, the last equation can be re-written as:

$$\frac{dS_t}{S_t} = \sqrt{v_t} dZ_t + (\mu_t + \frac{\sigma_t^2}{2})dt$$

As μ_t is an unknown process (typically characterizing the general behaviour of the market) we perform a Girsanov change of measure in order to obtain a martingale:

$$dZ_t^{\mathbb{Q}} := dZ_t + \frac{\mu_t + \frac{\sigma_t^2}{2}}{\sigma_t} dt$$

We obtain:

$$dS_t = S_t \sqrt{v_t} dZ_t^{\mathbb{Q}}$$

Given empirical measurements of σ_t (based on realized variance estimates from the Oxford-Man Institute of Quantitative Finance), it has been shown (cf. [5]) that $\log \sigma_t$ seems to obey the following dynamic:

$$\log \sigma_{t+\Delta t} - \log \sigma_t = \nu (W^H_{t+\Delta t} - W^H_t)$$

with ν a time-homogeneous parameter and W^H_t a fractionnal brownian motion with hurst exponent H. A fractionnal brownian motion (or fBM) is a way of generalizing the standard brownian motion. W^H_t is defined as a continuous time gaussian process satisfying:

$$\mathbb{E}[W^H_t W^H_s] = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right)$$

The Mandelbrot Van Ness representation of the fractionnal brownian motion [8] gives the following formula for a fBM:

$$W^H_t = C_H \left[\int_{-\infty}^{t} \frac{dW_s}{(t-s)^{1/2-H}} - \int_{-\infty}^{0} \frac{dW_s}{(-s)^{1/2-H}} \right]$$

with W_s is a standard brownian motion and $C_H := \sqrt{\frac{2H\Gamma(3/2-H)}{\Gamma(2-2H)\Gamma(H+1/2)}}$

The kernel $K : t \mapsto t^{H-1/2}$ is the key element for volatility models to be "rough". Indeed, it can be shown for example that the process:

$$\int_0^t K(t-s) dW_s$$

has $(H-\epsilon)$-Hölder regularity for every $\epsilon > 0$ (see 1.2). We may also use the convolution notation, which will be useful later:

$$(K * dW)_t := \int_0^t (t-s)^{H-1/2} dW_s$$

1.2 Two examples of stochastic processes with Hölder regularity properties

For a parameter $H \in (0,1)$, we are going to study two different stochastic processes sharing the same property, namely that they have continuous sample paths, that are furthermore locally γ-Hölder for all $\gamma \in (0, H)$, but not H-Hölder. That is why they are called *rough* processes.

Preliminary results

Let's start by proving that under a certain condition, a stochastic process cannot have locally H-Hölder continuous sample paths.

Proposition 1. *Let $Y = (Y_t)_{t\geqslant 0}$ be a real-valued $\mathbb{F}$-adapted stochastic process on the filtered probability space $(\Omega, \mathcal{T}, \mathbb{P}, \mathbb{F})$, such that $Y_t \sim \mathcal{N}(0, |t|^{2H})$ for all $t \geqslant 0$ for a certain parameter $H \in (0,1)$. We assume that $Y_0 = 0$ a.s., and that $\mathbb{F}$ is a continuous filtration verifying Blumenthal's zero-one law (ie if $A \in \mathcal{F}_0^+ = \mathcal{F}_0$, $\mathbb{P}(A) \in \{0,1\}$).*
Then almost surely, Y does not have locally H-Hölder continuous sample paths.

Proof. Fix $k \in \mathbb{N}^*$. For $n \in \mathbb{N}$, define $A_n = \left(\sup\limits_{0<t\leqslant 2^{-n}} \frac{Y_t}{t^H} \geqslant k\right)$. Since for $t > 0$, $\frac{Y_t}{t^H} \sim \mathcal{N}(0,1)$, if we denote by Φ the cumulative distribution function of the standard gaussian, one has $\mathbb{P}(A_n) \geqslant \mathbb{P}(\frac{Y_{2^{-n}}}{2^{-nH}} \geqslant k) = 1 - \Phi(k) > 0$. As $(A_n)_{n\in\mathbb{N}}$ is a non-increasing sequence of events, we have $\mathbb{P}(\bigcap\limits_{n\in\mathbb{N}} A_n) = \lim\limits_{n\to+\infty} \mathbb{P}(A_n) \geqslant 1 - \Phi(k)$. But $\bigcap\limits_{n\in\mathbb{N}} A_n \in \mathcal{F}_0^+ = \mathcal{F}_0$, hence $\mathbb{P}(\bigcap\limits_{n\in\mathbb{N}} A_n) = 1$ since it is positive.
In particular, $\sup\limits_{0<t\leqslant 1} \frac{Y_t}{t^H} \geqslant k$ a.s. By countable intersection of a.s. events, it follows that $\sup\limits_{0<t\leqslant 1} \frac{Y_t}{t^H} = +\infty$ a.s. As $Y_0 = 0$, for almost all $\omega \in \Omega$, $t \in \mathbb{R}_+ \mapsto Y_t(\omega)$ is not locally H-Hölder continuous. □

In order to obtain the continuity and Hölder regularity of the two aforementioned processes, the following theorem, established by Kolmogorov, will prove very useful.

Theorem 1 (Kolmogorov continuity theorem)**.** *Let $X = (X_t)_{t\in[0,1]}$ be a real-valued stochastic process on the probability space $(\Omega, \mathcal{T}, \mathbb{P})$ such that there exists $\alpha, c, \varepsilon > 0$ with*

$$\mathbb{E}[|X_s - X_t|^\alpha] \leqslant c|s-t|^{1+\varepsilon}$$

for all $s, t \in [0,1]$. Then there exists a process $\tilde{X} = (\tilde{X}_t)_{t\in[0,1]}$ such that:

(i) for all $t \in [0,1]$, $X_t = \tilde{X}_t$ a.s. In this case, $\tilde{X}$ is said to be a modification of X.

(ii) $\tilde{X}$ has continuous sample paths, that are furthermore γ-Hölder for all $\gamma \in \left(0, \frac{\varepsilon}{\alpha}\right)$.

Proof. The proof of this theorem is a combination of Markov's inequality and Borel-Cantelli lemma. Let $\mathcal{A} = \left\{\frac{k}{2^n}, n \in \mathbb{N}, 0 \leqslant k \leqslant 2^n\right\}$, and $\gamma \in \left(0, \frac{\varepsilon}{\alpha}\right)$. For $n \geqslant 1$, denote by B_n the event $\left(\max\limits_{1\leqslant k\leqslant 2^n} |X_{\frac{k}{2^n}} - X_{\frac{k-1}{2^n}}| \geqslant 2^{-\gamma n}\right)$. Then by Markov's inequality and the hypothesis, if $n \geqslant 1$

$$\mathbb{P}(B_n) \leqslant \sum_{k=1}^{2^n} \mathbb{P}(|X_{\frac{k}{2^n}} - X_{\frac{k-1}{2^n}}| \geqslant 2^{-\gamma n}) \leqslant \sum_{k=1}^{2^n} \mathbb{E}(|X_{\frac{k}{2^n}} - X_{\frac{k-1}{2^n}}|^\alpha) 2^{\gamma n\alpha} \leqslant c 2^{-n(\varepsilon-\gamma\alpha)}$$

Then by Borel-Cantelli's lemma, as $\sum_{n\geqslant 1}\mathbb{P}(B_n) < +\infty$, there exists $B \in \mathcal{T}$ with $\mathbb{P}(B) = 1$ such that all $\omega \in B$ are in finitely many B_n. To put it differently, if $\omega \in B$, there exists $N(\omega) \geqslant 1$ such that $w \notin B_n$ for $n \geqslant N(\omega)$. Consequently, defining

$$C(\omega) = \max_{\substack{0\leqslant n<N(\omega)\\ 1\leqslant k\leqslant 2^n}} 2^{n\gamma}|X_{\frac{k}{2^n}}(\omega) - X_{\frac{k-1}{2^n}}(\omega)| < +\infty$$

it follows that for all $n \geqslant 0$ and $1 \leqslant k \leqslant 2^n$, $|X_{\frac{k}{2^n}}(\omega) - X_{\frac{k-1}{2^n}}(\omega)| \leqslant C(\omega)2^{-n\gamma}$.
Let $\omega \in B$. We want to show that the process $(X_t(\omega))_{t\in\mathcal{A}}$ is γ-Hölder. Let s, t be in $\mathcal{A}$, and $n \geqslant 0$ such that $\frac{1}{2^{n+1}} \leqslant |s-t| \leqslant \frac{1}{2^n}$. One is easily convinced by a diagram that there exists two stationary sequences $(s_k)_{k\geqslant n}$ and $(t_k)_{k\geqslant n}$ converging towards s and t respectively such that $|s_n - t_n| \leqslant \frac{1}{2^n}$, $s_k, t_k \in \frac{1}{2^k}\mathbb{N}$ for $k \geqslant n$, and $|s_{k+1} - s_k|, |t_{k+1} - t_k| \in \{0, \frac{1}{2^{k+1}}\}$ for $k \geqslant n$. Subsequently (all sums being actually finite)

$$\begin{aligned}
|(X_s - X_t)(\omega)| &= |\sum_{k=n}^{+\infty}(X_{s_{k+1}} - X_{s_k})(\omega) + (X_{s_n} - X_{t_n})(\omega) - \sum_{k=n}^{+\infty}(X_{t_{k+1}} - X_{t_k})(\omega)| \\
&\leqslant C(\omega)\left(2^{-\gamma n} + 2\sum_{k=n}^{+\infty}2^{-\gamma(k+1)}\right) \\
&= 2^\gamma C(\omega)\frac{2^\gamma+1}{2^\gamma-1}2^{-\gamma(n+1)} \\
&\leqslant 2^\gamma C(\omega)\frac{2^\gamma+1}{2^\gamma-1}|s-t|^\gamma
\end{aligned}$$

If $\omega \in B$, using Cauchy sequences and the density of $\mathcal{A}$, the γ-Hölder function $t \in \mathcal{A} \mapsto X_t(\omega)$ can be uniquely extended to a γ-Hölder function $t \in [0,1] \mapsto \tilde{X}_t(\omega)$. If $\omega \notin B$, we set $\tilde{X}_t(\omega) = 0$ for all $t \in [0,1]$. Hence for all $\omega \in \Omega$, $t \in [0,1] \mapsto \tilde{X}(\omega)$ is γ-Hölder (and thus continuous).
It remains to prove *(i)*. Let $t \in [0,1]$, and $(t_n)_{n\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$ converging towards t with $|t_n - t| \leqslant \frac{1}{2^n}$ for all $n \in \mathbb{N}$. Then by Markov's inequality and the hypothesis, $\mathbb{P}(|X_t - X_{t_n}| \geqslant |t-t_n|^\gamma) \leqslant c|t-t_n|^{1+\varepsilon-\gamma\alpha} \leqslant c2^{-n(1+\eta)}$ where $\eta = \varepsilon - \gamma\alpha > 0$. As $\sum_{n\geqslant 0}2^{-n(1+\eta)} < +\infty$, Borel-Cantelli's lemma shows that for almost all $\omega \in \Omega$, $|X_t(\omega) - X_{t_n}(\omega)| < |t-t_n|^\gamma$ for sufficiently large n, hence $X_{t_n}(\omega) \underset{n\to+\infty}{\longrightarrow} X_t(\omega)$. In other words, $X_{t_n} \xrightarrow[n\to+\infty]{\text{a.s.}} X_t$. However, by construction, $X_{t_n} = \tilde{X}_{t_n}$ a.s., and $\tilde{X}_{t_n} \xrightarrow[n\to+\infty]{\text{a.s.}} \tilde{X}_t$. Therefore, $\tilde{X}_t = X_t$ a.s., and this concludes the proof. □

Remark 1. Theorem 1 can easily be extended to stochastic processes defined on $\mathbb{R}_+$, and taking values in $\mathbb{R}^d$ for $d \geqslant 1$. It will prove Hölder regularity on all compact sets $K \subset \mathbb{R}_+$, but not on $\mathbb{R}_+$ itself. This property is called *local Hölder regularity*.

Remark 2. In theorem 1, we can observe that the obtained modification $\tilde{X}$ of X does not actually depend on α, c or ε, since it is only built with the values taken by X on $\mathcal{A}$.

Application to the Riemann-Liouville process $\int\limits_0^t (t-s)^{H-\frac{1}{2}} dW_s$

Lemma 1. *Let $\delta \in (-\frac{1}{2}, \frac{1}{2})$. Then there exists some constant $K_\delta > 0$ such that for all $t, h > 0$, $I_{\delta,h,t} = \int\limits_0^t \left|(u+h)^\delta - u^\delta\right|^2 du \leqslant K_\delta h^{2\delta+1}$.*

Proof. By the change of variable $u = hv$ in $I_{\delta,h,t}$, we get

$$I_{\delta,h,t} = h^{2\delta+1} \int\limits_0^{\frac{t}{h}} \left|(v+1)^\delta - v^\delta\right|^2 du \leqslant K_\delta h^{2\delta+1}$$

with $K_\delta = \int\limits_0^{+\infty} \left|(v+1)^\delta - v^\delta\right|^2 dv$.

Besides, $K_\delta < +\infty$ since we are integrating a positive function and:

- when $v \to 0$, $|(v+1)^\delta - v^\delta|^2 \sim v^{2\delta}$ is integrable because $\delta > -\frac{1}{2}$
- when $v \to +\infty$, $|(v+1)^\delta - v^\delta|^2 \sim \delta^2 v^{2\delta-2}$ is integrable because $\delta < \frac{1}{2}$.

$\square$

Corollary 1 (Hölder regularity of the Riemann-Liouville process)**.** *Let $H \in (0,1)$. For $t \geqslant 0$, define $X_t = \int\limits_0^t (t-s)^{H-\frac{1}{2}} dW_s$, where $(W_s)_{s\geqslant 0}$ is a standard brownian motion. Then X has a modification $\tilde{X}$ with continuous sample paths, that are furthermore locally γ-Hölder for all $\gamma \in (0, H)$, but not H-Hölder.*

Proof. Note that if $s \geqslant t > 0$, $X_s - X_t = a_{s,t} + b_{s,t}$, where

$$a_{s,t} = \int\limits_0^t ((s-u)^{H-\frac{1}{2}} - (t-u)^{H-\frac{1}{2}}) dW_u \text{ and } b_{s,t} = \int\limits_t^s (s-u)^{H-\frac{1}{2}} dW_u$$

are independent centered gaussian variables.

It is easy to see that $a_{s,t} \sim \mathcal{N}(0, I_{H-\frac{1}{2},s-t,t})$ (notations of lemma 1) and $b_{s,t} \sim \mathcal{N}(0, \frac{1}{2H}(s-t)^{2H})$, so that by independence, $X_s - X_t \sim \mathcal{N}(0, I_{H-\frac{1}{2},s-t,t} + \frac{1}{2H}(s-t)^{2H})$.

Let $\gamma \in (0, H)$, and $p \in \mathbb{N}^*$ such that $\gamma < H - \frac{1}{p}$. Since $H - \frac{1}{2} \in (-\frac{1}{2}, \frac{1}{2})$, it follows that $\mathbb{E}[|X_s - X_t|^p] = (I_{H-\frac{1}{2},s-t,t} + \frac{1}{2H}(s-t)^{2H})^{\frac{p}{2}} \beta_p \leqslant (K_{H-\frac{1}{2}} + \frac{1}{2H})^{\frac{p}{2}} \beta_p |s-t|^{pH}$ where $\beta_p = \mathbb{E}(|Z|^p)$ if $Z \sim \mathcal{N}(0,1)$.

Using theorem 1 with $\alpha = p$, $c = (K_{H-\frac{1}{2}} + \frac{1}{2H})^{\frac{p}{2}}\beta_p$ and $\varepsilon = pH - 1$, we obtain the existence of a modification $\tilde{X}$ of X with continuous sample paths, that are locally γ-Hölder since $\gamma p < \varepsilon$.
Besides, remark 2 shows that the modification $\tilde{X}$ of X does not actually depend on γ.

To prove that $\tilde{X}$ does not have locally H-Hölder continuous sample paths, it suffices to see that $\tilde{X}_0 = 0$, $\tilde{X}_t \sim \mathcal{N}(0, \int_0^t (t-s)^{2H-1} ds) = \mathcal{N}(0, t^{2H})$, and apply proposition 1. $\tilde{X}$ is indeed $\mathbb{F}^W$-adapted (where $\mathbb{F}^W$ is assumed to denote the augmented canonical filtration associated to W). □

Application to fractional brownian motion with Hurst parameter H

Lemma 2. *Let $(W_t^H)_{t\geqslant 0}$ be a fractional brownian motion with Hurst exponent $H \in (0,1)$. Then the process $(W_{s+t}^H - W_s^H)_{t\geqslant 0}$ is also a fractional brownian motion with same Hurst exponent for all $s \geqslant 0$. In particular, if $s, t \geqslant 0$, $W_{s+t}^H - W_s^H$ has same distribution as W_t^H, ie $\mathcal{N}(0, |t|^{2H})$.*

Proof. $(W_{s+t}^H - W_s^H)_{t\geqslant 0}$ is still a centered gaussian process. Thus it remains to show that it has the same covariance function as a fractional brownian motion. If $t_1, t_2 \geqslant 0$,

$$\begin{aligned}\mathbb{E}((W_{s+t_1}^H - W_s^H)(W_{s+t_2}^H - W_s^H)) &= \mathbb{E}(W_{s+t_1}^H W_{s+t_2}^H) - \mathbb{E}(W_{s+t_1}^H W_s^H) - \mathbb{E}(W_{s+t_2}^H W_s^H) + \mathbb{E}((W_s^H)^2)\\ &= \frac{1}{2}(|t_1|^{2H} + |t_2|^{2H} - |t_1 - t_2|^{2H})\end{aligned}$$

after simplification, which concludes the proof. □

Corollary 2 (Hölder regularity of the fractional brownian motion)**.** *For a fractional brownian motion $(W_t^H)_{t\geqslant 0}$ with Hurst exponent $H \in (0,1)$, there exists a modification $\tilde{W}^H$ which has continuous sample paths, that are furthermore locally γ-Hölder for all $\gamma \in (0, H)$, but not H-Hölder.*

Proof. Let $\gamma \in (0, H)$, and $p \in \mathbb{N}^*$ such that $\gamma < H - \frac{1}{p}$. If $s \geqslant t \geqslant 0$, as by lemma 2, $W_s^H - W_t^H \sim \mathcal{N}(0, |s-t|^{2H})$, one easily has $\mathbb{E}[|W_s^H - W_t^H|^p] = \beta_p |s-t|^{pH}$, where $\beta_p = \mathbb{E}(|Z|^p)$ if $Z \sim \mathcal{N}(0,1)$. Hence theorem 1 with $\alpha = p$, $c = \beta_p$ and $\varepsilon = pH - 1$ shows that there is a modification $\tilde{W}^H$ of W with continuous sample paths, that are locally γ-Hölder since $\gamma p < \varepsilon$.
Besides, remark 2 proves that the modification $\tilde{W}^H$ of W does not actually depend on γ.

Likewise, to prove that $\tilde{W}^H$ does not have locally H-Hölder continuous sample paths, it suffices to see that $\tilde{W}_0^H = 0$, $\tilde{W}_t^H \sim \mathcal{N}(0, t^{2H})$, and apply proposition 1, assuming that the current filtration veritifes Blumenthal's zero-one law... In that respect, [6] proves that there exists a kernel $K_H : \mathbb{R}_+^2 \to \mathbb{R}$ such that $\left(\int\limits_0^t K_H(t,s)dW_s\right)_{t\geqslant 0}$ is a fractional brownian motion with Hurst parameter H, where $(W_t)_{t\geqslant 0}$ is a standard brownian motion. Hence $(\tilde{W}_t^H)_{t\geqslant 0}$ can be chosen to be $\mathbb{F}^W$-adapted. □

The roughness derived by this property can also be used in classical volatility models in order to "roughen" them.

This can be illustrated through the Heston model. Given an asset price S_t with variance V_t, the model proposes the following dynamic

$$dS_t = S_t\sqrt{V_t}dW_t$$
$$dV_t = \lambda(\theta - V_t)dt + \lambda\nu\sqrt{V_t}dB_t$$

Or equivalently:

$$V_t = V_0 + \int_0^t \lambda(\theta - V_u)du + \int_0^t \lambda\nu\sqrt{V_u}dB_u$$

A natural way of "roughening" the model (cf. [9]) is then given by plugging the rough kernel in the integral above to obtain:

$$V_t = V_0 + \frac{1}{\Gamma(H+1/2)}\int_0^t (t-u)^{H-1/2}\lambda(\theta - V_u)du + \frac{1}{\Gamma(H+1/2)}\int_0^t (t-u)^{H-1/2}\lambda\nu\sqrt{V_u}dB_u$$

We will discuss in section 3.2 the existence of such a process (in particular, ensuring that V_t is always non negative is not trivial).

1.3 Implied volatility

Black & Scholes formula is considered as one of the most important theoretical achievement in the task of understanding financial markets and forecasting uncertainty. However, an empirical analysis of the market's option prices reveal that the Black & Scholes prices tend to underestimate extreme cases due to the underlying gaussian model.

The Black & Scholes price, let's say for a call, is then used to measure the market prices using the notion of implied volatility. In other terms, for a fixed maturity T, a strike K and a spot S_0, we define the implied volatility $\sigma_{BS}(K,T)$ as following:

$$\sigma_{BS}(K,T) = BS^{-1}_{K,T,S_0}(\text{Market price})$$

Where the function $\sigma \mapsto BS_{K,T,S_0}(\sigma)$ is the Black-Schles call formula

$$BS_{K,T,S_0}(\sigma) = S_0\mathbf{N}(d_+) - Ke^{-rT}\mathbf{N}(d_-)$$

With:

$$d_{\pm}(s,k,v) = \frac{\ln(s/k)}{\sqrt{v}} \pm \frac{\sqrt{v}}{2}$$

And:

$$\mathbf{N}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du$$

If the Black-Scholes model was true, we should have $\sigma_{BS}(K,T)$ equals to a constant independent of K and T. We can therefore describe the market using an implied volatility surface corresponding to the map of $\sigma_{BS}(K,T)$. Generally, we plot $\sigma_{BS}(k,T)$ where $k := \log(K/S_0)$:

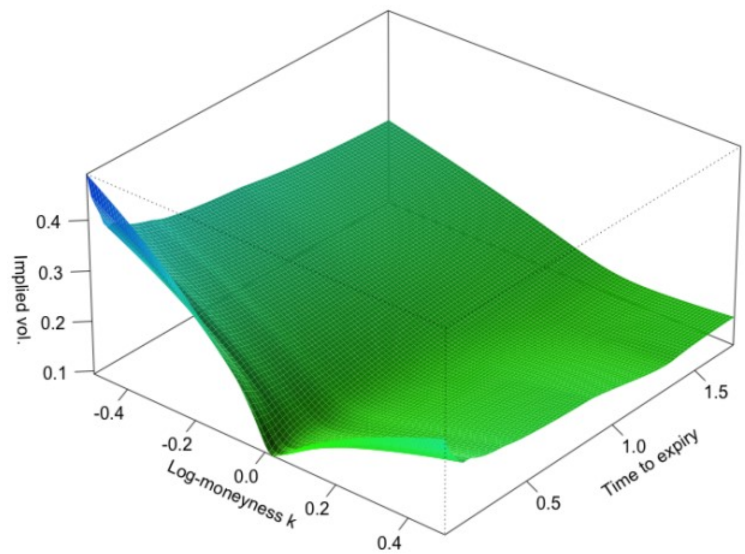

Figure 1: The SPX volatility surface as of August 14, 2013. Time measured in years (source: [1])

Something remarkable is that the following quantity:

$$\psi(\tau) = \left| \frac{\partial}{\partial k} \sigma_{BS}(k,\tau) \right|$$

Seem to have a regular behaviour

Measuring the volatility skew can be a good way to asses the quality of a model. Indeed, any given model for the dynamic of S_t is associated with an implied Black-Scholes volatility surface and one can get the volatility skew of this model by Monte-Carlo simulations or through finding an explicit formula.

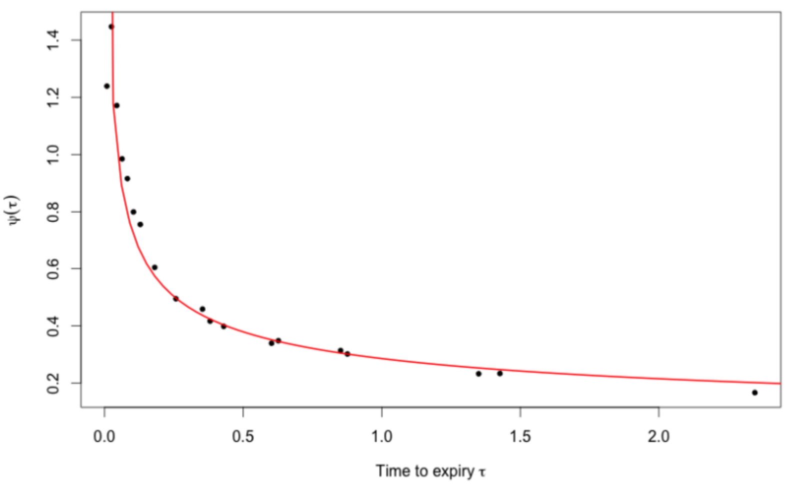

Figure 2: The black dots are non-parametric estimates of the S&P at-themoney (ATM) volatility skews as of August 14, 2013; the red curve is the power-law fit $\psi(\tau) = A\tau^{-0.407}$, τ measured in years (source: [1])

2 Foundations of rough volatiliy

2.1 Volatility skew

The first supporting argument in favor of using rough volatility models is that the ATM volatility skew generated fits well the law

$$\psi(\tau) \propto \tau^{-0.4}$$

In [4], Fukasawa also provides arguments showing that the volatility skew for rough volatility models have the form:

$$\psi(\tau) \propto \tau^{1/2-H}$$

We see here that choosing $H \equiv 0.1$ seems legitimate if we want to reproduce the empirical skewness. In contrast, classical stochastic volatility models (Hull and White, Heston, and SABR) generate a term structure of ATM skew that is constant for small τ and behaves as a sum of decaying exponentials for larger τ ([5]) .

2.2 Microstructural foundations

Another justification of the use of rough volatility models lies in a microstructural modelization of the market which can be described using Hawkes processes.

Hawkes processes are a specific type of point process. A point process is defined as a sequence of non-negative random variables $(t_i)_{i\in\mathbb{N}^*}$ such that $t_i < t_{i+1}$ (obviously, t_i will represent the time at which a transaction is made). Naming $N(t) := \sum_{i\in\mathbb{N}^*} \mathbf{1}_{t_i<t}$, we can define the intensity λ of the process N by:

$$\lambda(t) := \lim_{h\downarrow 0} \mathbb{E}\left[\frac{N(t+h)-N(t)}{h}|\mathcal{F}_t\right]$$

Where $\mathcal{F}_t$ is the filtration generated by N. Hawkes processes are such that their intensity λ verifies:

$$\lambda(t) = \lambda_0 + \sum_{t_i<t} \psi(t-t_i) = \lambda_0 + \int_0^t \psi(t-s)dN_s$$

With ψ a given Kernel (generally we take $\psi(x) \underset{+\infty}{\sim} \frac{K}{x^{1+\alpha}}$). In [11], M. Rosenbaum and T.Jaisson show that a sequence of renormalized Hawkes process converges to a differentiable process which derivatie V follows the Rough Heston dynamic.

2.3 A discussion on memory

An interesting quantity studied to have an idea of "the memory" of the fractionnal brownian motion is:

$$\sum_{k\geq 1} \mathbb{C}\mathrm{ov}(W_1^H, W_k^H - W_{k-1}^H)$$

We have:

$$\begin{aligned}\mathbb{C}\mathrm{ov}(W_1^H, W_k^H - W_{k-1}^H) &= \frac{1}{2}[k^{2H} - (k-1)^{2H}] - \frac{1}{2}[(k-1)^{2H} - (k-2)^{2H}] \\ &= \frac{k^{2H}}{2}\left(1 - \left(1-\frac{1}{k}\right)^{2H}\right) - \frac{(k-1)^{2H}}{2}\left(1-\left(1-\frac{1}{k-1}\right)^{2H}\right)\end{aligned}$$

Using this expression, we can show using a second order limited development that:

$$\mathbb{C}\mathrm{ov}(W_1^H, W_k^H - W_{k-1}^H) \sim H(2H-1)k^{2H-2}$$

Thus,

$$\sum_{k\geq 1} \mathbb{C}\mathrm{ov}(W_1^H, W_k^H - W_{k-1}^H) = +\infty \quad \text{When } H > 1/2$$

Some authors (see the discussion in section 1.2 of [5]) argue that choosing $H > 1/2$ will result in a "long-memory" process. However, using other definitions of the "memory" based on the autocorrelation function, it is shown in [5] that the long memory question can not be answered.

3 The Rough Bergomi and Rough Heston models

3.1 Rough Bergomi

This subsection provides the main ideas lying behind the introduction of the Rough Bergomi model for finance models, and draws its inspiration from [1]. In this whole part, we will assume that interest rates are zero (they could also be assumed constant).

3.1.1 Rough Bergomi model under the physical measure $\mathbb{P}$

As it happens, empirical data suggest that the time series of realized variance follow the simple pattern $\log \sigma_u - \log \sigma_t = \nu(W_u^H - W_t^H)$ if $u > t \geqslant 0$, where $(W_t^H)_{t\geqslant 0}$ is a fractional brownian motion under the physical measure $\mathbb{P}$. $H \in (0,1)$ is the Hurst exponent, and is in most applications very close to 0 (around 0.1 in general).
Denote by $(W_t^{\mathbb{P}})_{t\geqslant 0}$ a standard brownian motion under $\mathbb{P}$, with associated canonical filtration $\mathbb{F}$. Using the Mandelbrot-Van Ness representation mentioned above, and setting $v = \sigma^2$ as the instantaneous variance, it follows that if $u > t \geqslant 0$,

$$\log v_u - \log v_t = 2\nu C_H(M_t(u) + Z_t(u))$$

where $M_t(u) = \int\limits_t^u (u-s)^{H-\frac{1}{2}} dW_s^{\mathbb{P}}$ and $Z_t(u) = \int\limits_{-\infty}^t [(u-s)^{H-\frac{1}{2}} - (t-s)^{H-\frac{1}{2}}] dW_s^{\mathbb{P}}$.
As $Z_t(u)$ is $\mathcal{F}_t$-measurable, setting $\tilde{W}_t^{\mathbb{P}}(u) = \sqrt{2H} M_t(u)$ (which has distribution $\mathcal{N}(0,(u-t)^{2H})$ and is independent from $\mathcal{F}_t$) and $\eta = \frac{2\nu C_H}{\sqrt{2H}}$ gives

$$\mathbb{E}^{\mathbb{P}}[v_u|\mathcal{F}_t] = \mathbb{E}^{\mathbb{P}}\left[v_t e^{\eta \tilde{W}_t^{\mathbb{P}}(u) + 2\nu C_H Z_t(u)} |\mathcal{F}_t\right] = v_t e^{2\nu C_H Z_t(u) + \frac{1}{2}\eta^2 \mathbb{E}(|\tilde{W}_t^{\mathbb{P}}(u)|^2)}$$

Hence

$$v_u = v_t e^{\eta \tilde{W}_t^{\mathbb{P}}(u) + 2\nu C_H Z_t(u)} = \mathbb{E}^{\mathbb{P}}[v_u|\mathcal{F}_t] \mathcal{E}(\eta \tilde{W}_t^{\mathbb{P}}(u))$$

where for a random variable Y, $\mathcal{E}^{\mathbb{P}}(Y) = e^{Y - \frac{1}{2}\mathbb{E}^{\mathbb{P}}(|Y|^2)}$ is the stochastic exponential.
Hence under the physical measure $\mathbb{P}$, the following asset pricing model is natural:

$$\begin{aligned} \frac{dS_u}{S_u} &= \mu_u du + \sqrt{v_u} dB_u^{\mathbb{P}} \\ v_u &= v_t e^{\eta \tilde{W}_t^{\mathbb{P}}(u) + 2\nu C_H Z_t(u)} \end{aligned}$$

where $(W_t^{\mathbb{P}})_{t\geqslant 0}$ and $(B_t^{\mathbb{P}})_{t\geqslant 0}$ are standard brownian motions under $\mathbb{P}$ with a given correlation ρ.

3.1.2 Rough Bergomi model under the pricing measure $\mathbb{Q}$

In order to remove the drift, we would like to find a measure $\mathbb{Q} \sim \mathbb{P}$ under which $(S_u)_{u\in[t,T]}$ is a martingale. It is particularly natural to try to find $\mathbb{Q}$ such that

$$\frac{dS_u}{S_u} = \sqrt{v_u} dB_u^{\mathbb{Q}}$$

with $(B_t^{\mathbb{P}})_{t\geqslant 0}$ a standard brownian motion under $\mathbb{Q}$. This would be equivalent to having $dB_u^{\mathbb{Q}} = dB_u^{\mathbb{P}} + \frac{\mu_u}{\sqrt{v_u}} du$, which will be assumed to be a Girsanov change of measure. To that respect, using Novikov's criterion, and the fact that for $t \leqslant u \leqslant T$, $\tilde{W}_t^{\mathbb{P}}(u)$ and $Z_t(u)$ are independent, basic assumptions on μ_u such as uniform boundness would be sufficient to apply Girsanov's theorem.

Using the correlation ρ between $(W_t^{\mathbb{P}})_{t\geqslant 0}$ and $(B_t^{\mathbb{P}})_{t\geqslant 0}$, one can easily get the existence of a process $(\lambda_s)_{s\in[t,T]}$ such that $dW_s^{\mathbb{P}} = dW_s^{\mathbb{Q}} + \lambda_s ds$ with some standard brownian motion $(\tilde{W}_t^{\mathbb{Q}})_{t\geqslant 0}$ under $\mathbb{Q}$.

Setting $\tilde{W}_t^{\mathbb{Q}}(u) = \sqrt{2H} \int\limits_u^t (u-s)^{H-\frac{1}{2}} dW_s^{\mathbb{Q}}$, it follows from the previous part that

$$v_u = \mathbb{E}^{\mathbb{P}}[v_u|\mathcal{F}_t]\mathcal{E}^{\mathbb{Q}}(\eta\tilde{W}_t^{\mathbb{Q}}(u)) \exp\left[\eta\sqrt{2H}\int\limits_t^u (u-s)^{H-\frac{1}{2}}\lambda_s ds\right]$$

Hence $v_u = \xi_t(u)\mathcal{E}^{\mathbb{Q}}(\eta\tilde{W}_t^{\mathbb{Q}}(u))$ where the *forward-variance* $\xi_t(u)$ is defined by $\xi_t(u) = \mathbb{E}^{\mathbb{Q}}[v_u|\mathcal{F}_t]$ and is equal to

$$\xi_t(u) = \mathbb{E}^{\mathbb{P}}[v_u|\mathcal{F}_t] \exp\left[\eta\sqrt{2H}\int\limits_t^u (u-s)^{H-\frac{1}{2}}\lambda_s ds\right]$$

Thus, even under the risk-neutral measure $\mathbb{Q}$, $(v_u)_{u\in[t,T]}$ has some memory of the past, and is therefore not markovian. Indeed, $\xi_t(u) = \mathbb{E}^{\mathbb{Q}}[v_u|\mathcal{F}_t]$ depends on $\mathbb{E}^{\mathbb{P}}[v_u|\mathcal{F}_t]$, which itself takes into account the global previous behavior of the process $(W_u^{\mathbb{P}})$.

To sum it all up, under the pricing measure $\mathbb{Q}$, the empirical observations led us to suggest the following model with initial time t and $u \in [t,T]$:

$$\frac{dS_u}{S_u} = \sqrt{v_u} dB_u^{\mathbb{Q}}$$
$$v_u = \xi_t(u)\mathcal{E}^{\mathbb{Q}}(\eta\tilde{W}_t^{\mathbb{Q}}(u))$$

Where $(W_t^{\mathbb{Q}})_{t\geqslant 0}$ and $(B_t^{\mathbb{Q}})_{t\geqslant 0}$ are standard brownian motions under $\mathbb{Q}$ whose correlations can be explicitly computed (see [1]), $\tilde{W}_t^{\mathbb{Q}}(u)$ is defined above as a Riemann-Liouville process.

3.2 Rough Heston

Existence of the Rough Heston model

We provide in this section the main arguments supporting the proof of existence of a weak solution to the Rough Heston equation under some conditions on θ. The stochastic volatility in this model obeys to the following dynamic:

$$V = g_0 + K * (b(V)dt + \sigma(V)dW) \quad (1)$$

with:

$$g_0 : t \mapsto V_0 + \int_0^t K(t-s)\theta(s)ds$$

$$b : v \mapsto -\lambda v \text{ and } \sigma : v \mapsto \sqrt{v}$$

The major problem lies with proving that V_t will stay non-negative, and that $\sigma(V_t)$ will always be well-defined in $\mathbb{R}_+$.

The logical intution is to say that we should have $\sigma(0) = 0$ because if V_t reaches zero, there is no way that V_{t+dt} stays positive if the noise component $\sigma(V_t)dW_t$ is not equal to 0. To get a more precise demonstration, we can use the idea presented in [3] which consists in introducing:

$$b_n(v) := b((v - 1/n)^+) \underset{n\to\infty}{\to} b(v^+)$$

$$\sigma_n(v) := \sigma((v - 1/n)^+) \underset{n\to\infty}{\to} \sigma(v^+)$$

For each value of n, the Cauchy-Lipschitz theorem ensures the existence of a weak solution V^n.

Using techincal lemmas, we can prove that along a subsequence, V^n converges weakly to a solution V of:

$$V = g_0 + K * (b(V^+)dt + \sigma(V^+)dW)$$

To show that V stays non negative (therefore $V = V^+$ and it is a solution of (1)), it is enough to prove that V^n is non-negative.

Calling $dZ_t := b_n(V)dt + \sigma_n(V)dW_t$, we have for $h > 0$:

$$V^n_{t+h} = g_0(t+h) + (K * dZ)_{t+h} = g_0(t+h) + (\Delta_h K * dZ)_t + Y_t \quad (2)$$

Where $\Delta_h K : t \mapsto K(t+h)$ and $Y_t := \int_t^{t+h} K(t+h-s)dZ_s$. Another techical lemma gives us that for any stopping time τ, on $\{\tau < \infty\}$ we have the identity:

$$Y_\tau := \int_\tau^{\tau+h} K(\tau + h - s)dZ_s$$

We will take

$$\tau := \inf\{t \geq 0 : V_t < 0\}$$

We introduce here the resolvent of the first kind L of K which verifies the property:

$$L * K = K * L \equiv 1$$

In our case, $K(t) = \frac{t^{H-1/2}}{\Gamma(H+1/2)}$ and we can verifiy that $L(t) := \frac{t^{-H-1/2}}{\Gamma(1/2-H)}$ is the resolvent of K.

After verifying that $\Delta_h K * L$ is right-continuous and of locally bounded variation, we have:

$$\Delta_h K * L = d(\Delta_h K * L) * I$$

$$\Delta_h K = d(\Delta_h K * L) * K$$

Thus:

$$\begin{aligned}\Delta_h K * dZ &= d(\Delta_h K * L) * (V - g_0) \\ &= d(\Delta_h K * L)V - d(\Delta_h K * L) * g_0\end{aligned}$$

Plugging this last equation into (2) gives us:

$$V_{\tau+h} = F_{g_0}(h,\tau) + Y_\tau + \underbrace{d(\Delta_h K * L) * V_\tau}_{\mathrm{I}}$$

With:

$$F_{g_0}(h,\tau) = \big(\Delta_h g_0 - d(\Delta_h K * L) * g_0\big)(\tau)$$

We will show that I is positive. Let $t \geq s$. Calling $C := \frac{1}{\Gamma(1/2+H)\Gamma(1/2-H)}$, we have

$$\underbrace{(\Delta_h K * L)(t)}_{:=\phi(t)} = C\int_0^t (t+h-u)^{H-1/2} u^{-H-1/2} du$$

So:

$$\phi(t) - \phi(s) = C\int_0^t [\underbrace{(t+h-u)^{H-1/2} - (s+h-u)^{H-1/2}}_{\geq 0}] u^{-H-1/2} du + \int_s^t (t+h-u)^{H-1/2} u^{-H-1/2} du$$

$$\phi(t) - \phi(s) \geq 0$$

Using that and the fact that $V_t \geq 0 \ \forall t \leq \tau$, we get that I ≥ 0
If we impose the condition:

$$\boxed{F_{g_0}(h,\tau) \geq 0}$$

Then we get:

$$V_{\tau+h} \geq Y_\tau \text{ on } \{\tau < \infty\} \ (3)$$

Let us now define the event:

$$A_\epsilon := \{b(V_s) \geq 0 \text{ and } \sigma(V_s) = 0 \, \forall \, s \in [\tau, \tau + \epsilon]\}$$

We will now prove that

$$\mathbb{P}(\{\tau < \infty\} \cap A_\epsilon) = 0 \ (4)$$

By (3), we have $V_{\tau+h} \geq 0 \ \forall h \in]0, \epsilon[$. But by definition of τ, there is some $h \in]0, \epsilon[$ where $V_{\tau+h} < 0$ which proves (4).
As $V_s \leq \frac{1}{n}$ on some interval of the form $[\tau, \tau + \delta]$ (by continuity of V), we have $b_n(V_s) \geq 0$ and $\sigma_n(V_s) = 0$ for all s in this interval. In other words, we have:

$$\mathbb{P}\Bigg[\bigcup_{\epsilon \in \mathbb{Q} \cap]0,1[} A_\epsilon\Bigg] = 1$$

We have:

$$\begin{aligned}
\mathbb{P}(\{\tau < \infty\}) &= \mathbb{P}\Bigg[\{\tau < \infty\} \cap \bigcup_{\epsilon \in \mathbb{Q} \cap]0,1[} A_\epsilon\Bigg] \\
&\leq \sum_{\epsilon \in \mathbb{Q} \cap]0,1[} \mathbb{P}(\{\tau < \infty\} \cap A_\epsilon) \\
&\leq 0
\end{aligned}$$

Which means that V stays non-negative. To conclude, if we impose that θ is such that the condition $F_{g_0}(\tau, h) \geq 0$ is verified, then the Rough Heston model is well defined.

Let's check that the condition $F_{g_0}(t, h) \geq 0$ holds with our choice of θ constant. With the same notation $\phi(t) := (\Delta_h K * L)(t)$. First of all, we have:

$$g_0(t) = V_0 + \int_0^t \frac{(t-s)^{H-1/2}}{\Gamma(H+1/2)} \theta ds = V_0 + \frac{\theta t^{H+1/2}}{(H+1/2)\Gamma(H+1/2)}$$

We have:

$$\begin{aligned}
F_{g_0}(t,h) &= g_0(t+h) - \int_0^t \phi'(u)\left(V_0 + \frac{\theta(t-u)^{H+1/2}}{(H+1/2)\Gamma(H+1/2)}\right)du \\
&= g_0(t+h) - V_0\phi(t) - \frac{\theta}{(H+1/2)\Gamma(H+1/2)}\int_0^t \phi'(t)(t-u)^{H+1/2}du \\
&= V_0(1-\phi(t)) + \frac{\theta}{\Gamma(H+1/2)}\left\{\frac{(t+h)^{H+1/2}}{H+1/2} - \int_0^t \phi'(u)\frac{(t-u)^{H+1/2}}{H+1/2}du\right\} \\
&= V_0(1-\phi(t)) + \frac{\theta}{\Gamma(H+1/2)}\left\{\frac{(t+h)^{H+1/2}}{H+1/2} - \int_0^t \phi(u)(t-u)^{H-1/2}du\right\} \text{ (Int. by parts)}
\end{aligned}$$

Since K is decreasing, we have obviously:

$$0 \leq \phi(t) \leq (K * L)(t) = 1$$

So if $V_0 \geq 0$:

$$F_{g_0}(t,h) \geq V_0(1-1) + \frac{\theta}{\Gamma(H+1/2)}\left\{\frac{(t+h)^{H+1/2}}{H+1/2} - \int_0^t (t-u)^{H-1/2}du\right\}$$

$$F_{g_0}(t,h) \geq 0$$

Which is what we wanted.

3.3 Towards numerical simulations

3.3.1 Rough Bergomi model: Monte-Carlo simulations

On the one hand, in the case of the Rough Bergomi model, pricing European call options is extremely time-consuming, since in absence of an explicit formula, the only reasonable method is to simulate hundreds of thousands of sample paths $(S_t)_{t\in[0,T]}$, and use Monte-Carlo to approximate $\mathbb{E}^{\mathbb{Q}}((S_T-K)^+)$, where $\mathbb{Q}$ is the risk-neutral measure (interest rates are assumed to be zero for the sake of simplifying computations).

3.3.2 Rough Heston model: a Fourier transform formula for option pricing

On the other hand, for the Rough Heston model, the following method has been widely used to fastly price options and calibrate the model. As it happens, [9] gives an explicit formula for the *extended* characteristic function of $\log S_T$. More precisely, $\varphi_T(z) = \mathbb{E}\left(e^{iz\log S_T}\right)$ can

be obtained by solving a specific fractional Ricatti equation, for $z \in \mathbb{C}$ verifying $\mathrm{Im}(z) \in [-1, 0]$ with $\delta > 0$.
As shown in [7], in a more general framework, there is a way to price an European call option with strike K and maturity T knowing only φ_T and S_0, assuming that a certain condition on the convergence domain of φ_T is satisfied. The exact condition is that $z \mapsto \varphi_T(-z)$ has to be defined at least on a complex region of the form $\{z \in \mathbb{C}, \mathrm{Im}(z) \in (a, b)\}$ with $1 < a < b$. As the characteristic function φ_T does not apparently fulfill this condition, we are going to price another derivative, and then go back to the European call option. We cannot *a priori* use the European put option either, since $z \mapsto \varphi_T(-z)$ would have to be defined at least on a complex region of the form $\{z \in \mathbb{C}, \mathrm{Im}(z) \in (a, b)\}$ with $a < b < 0$.
However, by looking closer at the moments of S_T in the Rough Heston model, the paper [10] shows that ϕ_T can be slightly extended to $\{z \in \mathbb{C}, \mathrm{Im}(z) \in (a, b)\}$ with $a, b \in \mathbb{R}$ verifying $a < -1 < 0 < b$. This would allow a direct pricing of the European call option thanks to [7].
In this report, as aforementioned, we are going to price another derivative, defined by the payoff $\min(e^x, K)$ at maturity $T > 0$, which will not require greater regularity for ϕ_T than stated in [9].

Lemma 3 (Fourier transform of the derivative with payoff $\min(e^x, K)$ at maturity $T > 0$)**.** *Define $w : x \in \mathbb{R} \longmapsto \min(e^x, K)$ as a new derivative, specifically modified to allow Fourier transform convergence on a specific domain. Then if $z \in \mathbb{C}$ verifies $0 < Im(z) < 1$, the Fourier transform of w at point z is given by $\hat{w}(z) = \frac{K^{iz+1}}{z(z-i)}$.*

Proof. By direct calculations, if $0 < \mathrm{Im}(z) < 1$,

$$\begin{aligned}
\hat{w}(z) &= \int_{-\infty}^{+\infty} e^{izx} \min(e^x, K) dx \\
&= \int_{-\infty}^{\log K} e^{izx} e^x dx + K \int_{\log K}^{+\infty} e^{izx} dx \\
&= \left[\frac{e^{(iz+1)x}}{iz+1} \right]_{x=-\infty}^{x=\log K} + \left[K \frac{e^{izx}}{iz} \right]_{x=\log K}^{x=+\infty} \\
&= \left(\frac{K^{iz+1}}{iz+1} - K \frac{K^{iz}}{iz} \right) \text{ since } 0 < \mathrm{Im}(z) < 1 \\
&= \frac{K^{iz+1}}{z(z-i)}
\end{aligned}$$

□

We are now ready to price this derivative in the Rough Heston model. For the sake of simplicity, we are going to assume without loss of generality in the following computations that $S_0 = 1$. Denote by $P(K,T)$ (resp. $C(K,T)$) the price of a the new derivative (resp. the European call option) under the Rough Heston model, with strike $K > 0$ and maturity $T > 0$. Assuming interest rates are zero, it follows that

$$
\begin{aligned}
P(K,T) &= \mathbb{E}^{\mathbb{Q}}[w(X)] \text{ with } w \text{ is defined in lemma 3 and } X = \log S_T \\
&= \frac{1}{2\pi}\mathbb{E}^{\mathbb{Q}}\left[\int_{i\alpha-\infty}^{i\alpha+\infty} e^{-izX}\hat{w}(z)dz\right] \text{ using inverse Fourier transform with } \alpha \in \mathbb{R} \\
&= \frac{1}{2\pi}\int_{i\alpha-\infty}^{i\alpha+\infty} \mathbb{E}^{\mathbb{Q}}[e^{-izX}]\hat{w}(z)dz \text{ with Fubini theorem under the right hypotheses} \\
&= \frac{1}{2\pi}\int_{i\alpha-\infty}^{i\alpha+\infty} \varphi_T(-z)\hat{w}(z)dz
\end{aligned}
$$

Now, as long as $\alpha \in (0,1)$, $\hat{w}$ and $z \mapsto \varphi_T(-z)$ are well-defined and regular on the horizontal line $\mathbb{R} + i\alpha$. Besides, the application of Fubini theorem is justified above by the following estimation for $z \in \mathbb{C}$ with $\mathrm{Im}(z) = \alpha$

$$
\mathbb{E}^{\mathbb{Q}}[|e^{-izX}|]|\hat{w}(z)| \leqslant \mathbb{E}^{\mathbb{Q}}[e^{\alpha X}]\frac{K}{|z(z-i)|} = \mathbb{E}^{\mathbb{Q}}[S_T^{\alpha}]\frac{K}{|z(z-i)|}
$$

which is integrable when $z \longrightarrow i\alpha \pm \infty$.
Thus, for all $\alpha \in (0,1)$, if we denote by $k = \log K$ the log-strike, the following equality holds:

$$
P(K,T) = \frac{1}{2\pi}\int_{i\alpha-\infty}^{i\alpha+\infty} \varphi_T(-z)\hat{w}(z)dz = \frac{K}{2\pi}\int_{i\alpha-\infty}^{i\alpha+\infty} \varphi_T(-z)\frac{e^{izk}}{z(z-i)}dz
$$

Now, setting $\alpha = \frac{1}{2} \in (0,1)$, ie $z = u + \frac{i}{2}$ with $u \in \mathbb{R}$ gives

$$
P(K,T) = \frac{\sqrt{K}}{2\pi}\int_{-\infty}^{+\infty} \varphi_T(-u - \frac{i}{2})\frac{e^{iuk}}{u^2 + \frac{1}{4}}du
$$

Insofar as for $u \in \mathbb{R}$, $\overline{\varphi_T(-u-\frac{i}{2})e^{iuk}} = \varphi_T(u-\frac{i}{2})e^{-iuk}$, it follows that

$$P(K,T) = \frac{\sqrt{K}}{\pi} \int_0^{+\infty} \mathrm{Re}(\varphi_T(u-\frac{i}{2})e^{-iuk})\frac{du}{u^2+\frac{1}{4}}$$

Therefore,

$$C(K,T) = \mathbb{E}^{\mathbb{Q}}[(e^X-K)^+] = \mathbb{E}^{\mathbb{Q}}[e^X - \min(e^X,K)] = \mathbb{E}^{\mathbb{Q}}(S_T) - P(K,T)$$

As pointed out by [3], under $\mathbb{Q}$, $(S_t)_{t\geqslant 0}$ is not only a local martingale but it is actually a real martingale. The proof requires multiple technical intermediary results, that we will not detail here. The only thing that matters is that $\mathbb{E}^{\mathbb{Q}}(S_T) = S_0 = 1$ in our demonstration. Hence $C(K,T) = 1 - \frac{\sqrt{K}}{\pi}\int_0^{+\infty} \mathrm{Re}(\varphi_T(u-\frac{i}{2})e^{-iuk})\frac{du}{u^2+\frac{1}{4}}$.
More generally, adding $S_0 > 0$ as a parameter clearly gives

$$C(S_0,K,T) = S_0 - \frac{\sqrt{KS_0}}{\pi} \int_0^{+\infty} \mathrm{Re}(\varphi_T(u-\frac{i}{2})e^{-iuk})\frac{du}{u^2+\frac{1}{4}}$$

with $\varphi_T(z) = \mathbb{E}^{\mathbb{Q}}\left(e^{iz\log\left(\frac{S_T}{S_0}\right)}\right)$ and $k = \log\frac{K}{S_0}$.
The existence of this formula to price a European call option will eventually prove very useful when it comes to numerical simulations, and is one of the main reasons of the success of Rough Heston models.

It is also shown in [3] that the characteristic function can be expressed using the solution of a fractionnal Riccati equation. for each $z \in \mathbb{C}$ such that $\Re(z) \in [0,1]$:

$$L(t,z) := \mathbb{E}(\exp(z\log(S_t)))$$

Given by:

$$\exp\left(\int_0^t F(z,\psi(t-s,z))g(s)ds\right)$$

With ψ the unique continuous solution of the fractionnal Ricatti Volterra equation:

$$\psi(t,z) = \int_0^t K(t-s)F(z,\psi(s,z))ds$$

4 Multi-factor approximation for the Rough Heston model

One of the practical advantages of using the Rough Heston model lies in a result established in [2] which shows that it can be approximated by a multi-factor model in which the Riccati equation can be solved with a classical Euler scheme.
The idea of the approximation is to replace the kernel $K(t) = \frac{1}{\Gamma H+1/2} t^{H-1/2}$ by a sum of more regular kernels:

$$K^n(t) = \sum_{i=1}^{n} c_i^n e^{-\gamma_i^n t}$$

The idea behind that is that the original kernel can be seen as a Laplace transform of a positive measure μ:

$$K(t) = \int_0^\infty e^{-\gamma t} \mu(d\gamma)$$

With:

$$\mu(d\gamma) = \frac{\gamma^{-H-1/2}}{\Gamma(H+1/2)\Gamma(1/2-H)}$$

The approximation discussed consists in replacing μ by a sum of diracs μ_n:

$$\mu_n(d\gamma) = \sum_{i=1}^{n} c_i^n \delta_{\gamma_i^n}(d\gamma)$$

With positives weights c_i^n and γ_i^n.
To ensure that the Kernel $\int_0^T (K(s) - K^n(s))^2 ds \to 0$, it suggested in [2] to choose:

$$c_i^n = \frac{1}{\Gamma(H+\frac{1}{2})\Gamma(\frac{1}{2}-H)} \int_{(i-1)\pi_n}^{i\pi_n} \gamma^{-H-\frac{1}{2}} d\gamma$$

$$\gamma_i^n = \frac{1}{\Gamma(H+\frac{1}{2})\Gamma(\frac{1}{2}-H) c_i^n} \int_{(i-1)\pi_n}^{i\pi_n} \gamma^{\frac{1}{2}-H} d\gamma$$

Where:

$$\pi_n = \frac{n^{-\frac{1}{5}}}{T} \left(\frac{\sqrt{10}(1-2H)}{5-2H} \right)^{\frac{2}{5}}$$

Using this approximation, we obtain then the multi-factor stochastic volatility model:

$$dS_t^n = S_t^n \sqrt{V_t^n} dW_t \quad V_t^n = g^n(t) + \sum_{i=1}^{n} c_i^n V_t^{n,i}$$

Where:
$$dV_t^{n,i} = (-\gamma_i^n V_t^{n,i} - \lambda V_t^n)dt + \sigma(V_t^n)dB_t$$
And:
$$g^n(t) = V_0 + \int_0^t K^n(t-u)\theta(u)du$$

It is shown in [2] that the processes (S^n, V^n) converges to a process (S, V) which follows the Rough Heston dynamic.

Moreover, we have for each $z \in \mathbb{C}$ such that $\Re(z) \in [0,1]$:

$$L^n(t,z) := \mathbb{E}(\exp(z \log(S_t^n/S_t)))$$

Given by:
$$\exp\left(\int_0^t F(z, \psi^n(t-s,z))g^n(s)ds\right)$$

With ψ^n the unique continuoussolution of the Ricatti Volterra equation:

$$\psi^n(t,z) = \int_0^t K^n(t-s)F(z, \psi^n(s,z))ds$$

The advantage of this method is that the solution of the Riccati equation obtained can be simulated using a classical Euler scheme because it follows the dynamic:

$$\partial_t \psi^{n,i}(z,t) = -\gamma_i^n \psi^{n,i}(t,z) + F(z, \psi^n(t,z)), \;\; \psi^{n,i}(0,z) = 0$$

$$\psi^n(t,z) = \sum_{i=1}^n c_i^n \psi^{n,i}(t,z)$$

5 Numerical simulations

We show in this section the different simulations obtained for Rough Bergomi and Rough Heston models.

5.1 Rough Bergomi

We simulated the Rough Bergomi model using a discrete Euler scheme applied to the equation:
$$dS_t/S_t = \sqrt{v_t}dZ_t$$
For better numerical accuracy, we simulate $\log S_t$ using:

$$d(\log S_t) = -\frac{1}{2}v_t dt + \sqrt{v_t}dZ_t$$

v_t is Simulated using the formula:

$$v_t = \exp\left(\eta\tilde{W}_t - \frac{\eta}{2}\mathbb{E}[|\tilde{W}_t|^2]\right)$$

Where $\tilde{W}_t = 2\eta H \int_0^t (t-s)^{H-1/2} dW_s$. We used Cholesky decomposition to generate simultaneously paths of Z and $\tilde{W}$. Figure 3 shows the volatility smiles obtained for different maturities and using 50 values of k. Figure 4 Shows the volatility skew obtained. With $H = 0.07$, $\eta = 1.9$, $\rho = -0.9$ (as suggested in the article), we obtain $\psi(\tau) \propto t^{-0.383}$

5.2 Rough Heston

We used the multi-factor approximation with:

$$\pi_n = \frac{n^{-\frac{1}{5}}}{T}\left(\frac{\sqrt{10}(1-2H)}{5-2H}\right)^{\frac{2}{5}}$$

And for $0 \leqslant i \leqslant n$,

$$c_i^n = \frac{1}{\Gamma(H+\frac{1}{2})\Gamma(\frac{1}{2}-H)}\int_{(i-1)\pi_n}^{i\pi_n} \gamma^{-H-\frac{1}{2}} d\gamma = \frac{\pi_n^{\frac{1}{2}-H}}{(\frac{1}{2}-H)\Gamma(H+\frac{1}{2})\Gamma(\frac{1}{2}-H)}(i^{\frac{1}{2}-H} - (i-1)^{\frac{1}{2}-H})$$

And for $1 \leqslant i \leqslant n$,

$$\gamma_i^n = \frac{1}{\Gamma(H+\frac{1}{2})\Gamma(\frac{1}{2}-H)c_i^n}\int_{(i-1)\pi_n}^{i\pi_n} \gamma^{\frac{1}{2}-H} d\gamma = \frac{\pi_n^{\frac{3}{2}-H}}{(\frac{3}{2}-H)\Gamma(H+\frac{1}{2})\Gamma(\frac{1}{2}-H)c_i^n}(i^{\frac{3}{2}-H} - (i-1)^{\frac{3}{2}-H})$$

We used a discrete Euler scheme to simulate the factors of the Riccati equation ψ_n.

$$\partial_t \psi^{n,i}(t,z) = -\gamma_i^n \psi^{n,i}(t,z) + F\left(z, \sum_{i=1}^n c_i^n \psi^{n,i}(t,z)\right)$$

There is also a numerical "trick" which consist in taking $\gamma_i^n \psi^{n,i}(t+dt,z)$ in the right equation. The Euler scheme is therefore:

$$\psi^{n,i}(t+dt,z) = \frac{\psi^{n,i}(t,z) + F\left(z, \sum_{i=1}^n c_i^n \psi^{n,i}(t,z)\right)}{1+\gamma_i^n dt}$$

Dividing by $1+\gamma_i^n dt$ provides more numerical stability. The "true" value of ψ is computed using an Adams Scheme (detailed in [9]) to solve:

$$\psi(t,z) = \int_0^t K(t-s)F(z,\psi(s,z))ds$$

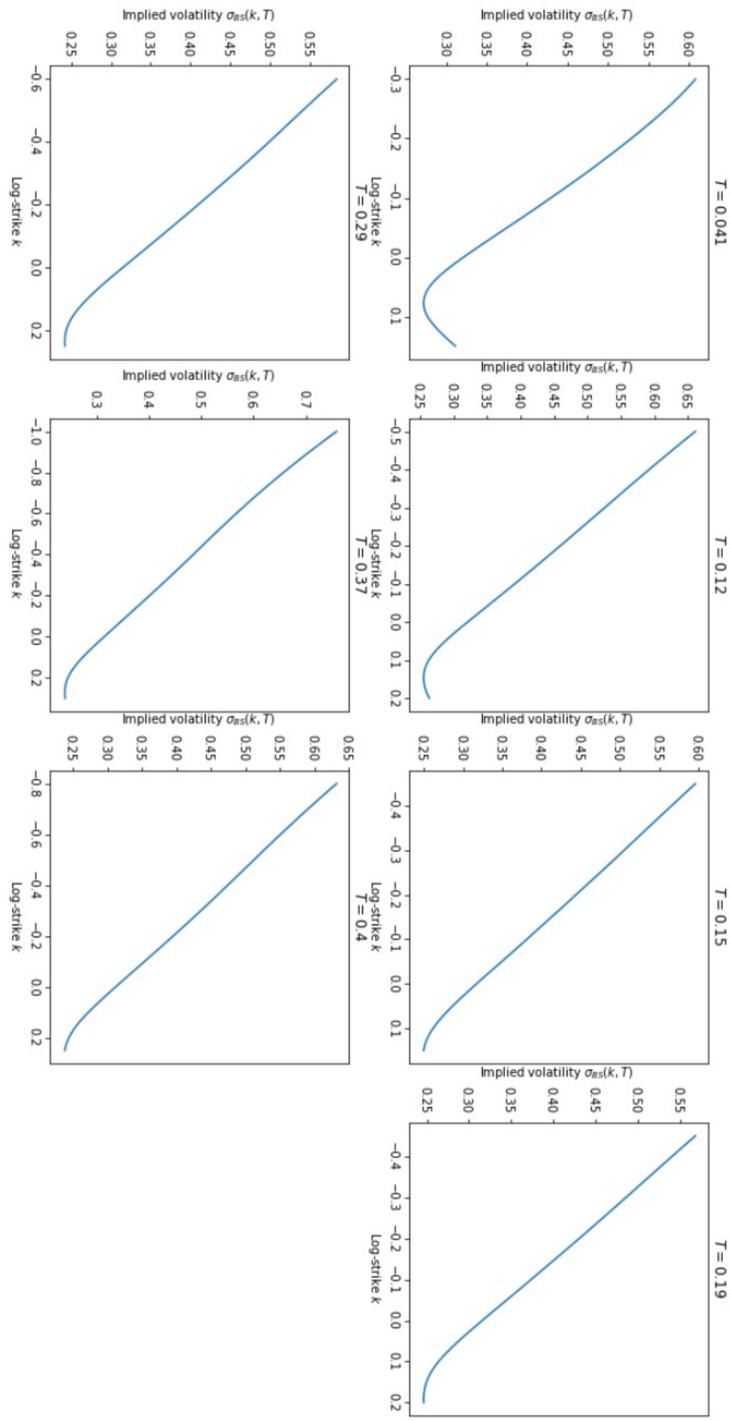

Figure 3: Volatility smiles generated from rBergomi Monte-Carlo simulations

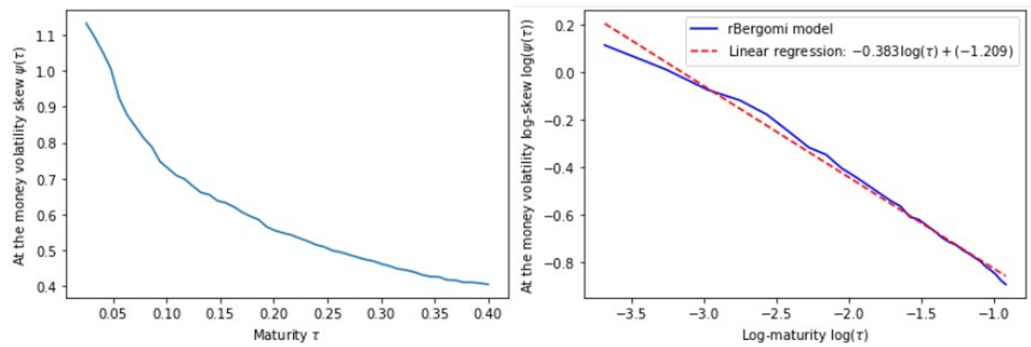

Figure 4: Left: Volatility skew. Right: linear regression on the volatiliy skew as a function of the log-maturity

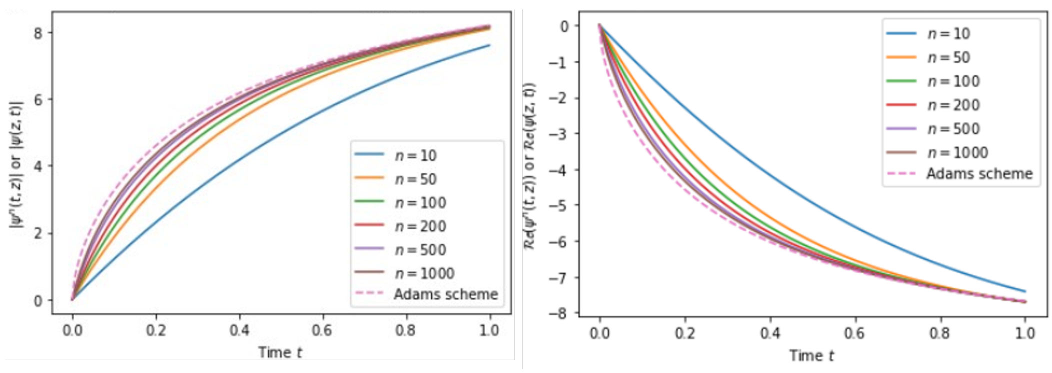

Figure 5: Left: convergence of $|\psi^n|$. Right: convergence of $\Re(\psi^n)$

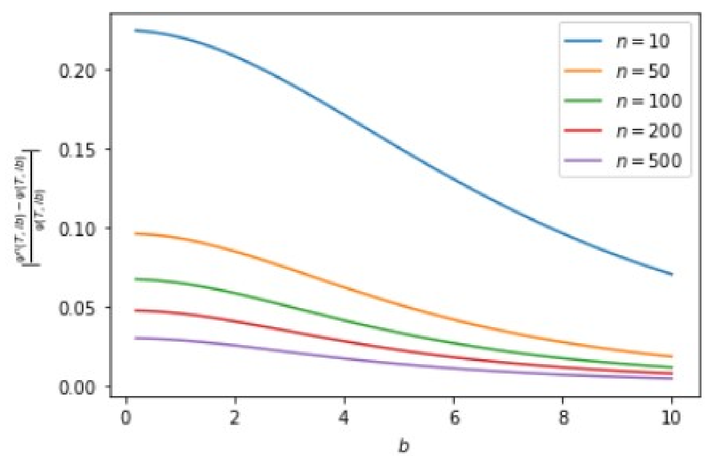

Figure 6: Relative error between $\psi(T, ib)$ and $\psi^n(T, ib)$

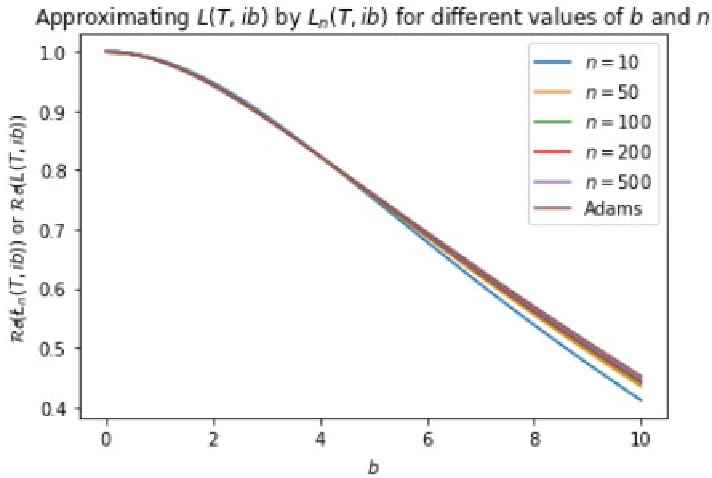

Figure 7: Approximation of the characteristic function $L(T, ib)$

Figure 6 shows the relative error between the multi-factor solution and the Adams scheme. The characteristic function can therefore be computed. Figure 7 shows the result $L^n(T, ib)$ using various values of n. The result is consistent with the Adams method.

Once we have computed L^n, we can price the call of an european option using the Lewis formula ([7]):

$$Call(S, K, T) = S_0 - \frac{\sqrt{SK}}{\pi} \int_0^\infty \Re(e^{iuk} L(i(u - i/2))) \frac{du}{u^2 + \frac{1}{4}}$$

Figure 8 shows the volatility smile obtained for different values of n.

We can then compute the volatilty ATM skew and perform a linear regression on the ATM skew as a function of the log-maturity. (Figure 9). The coefficient obtained is close to -0.400.

However, We notice that the ATM skew wierdly converges to a constant for small values of τ.

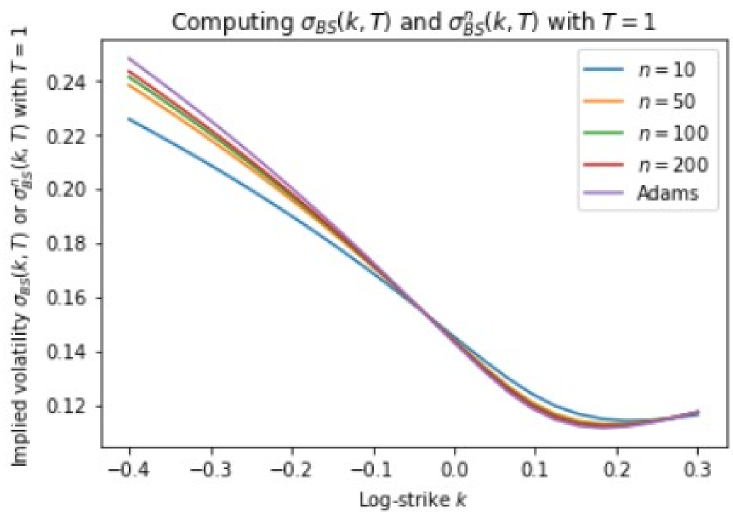

Figure 8: Implied volatility smile

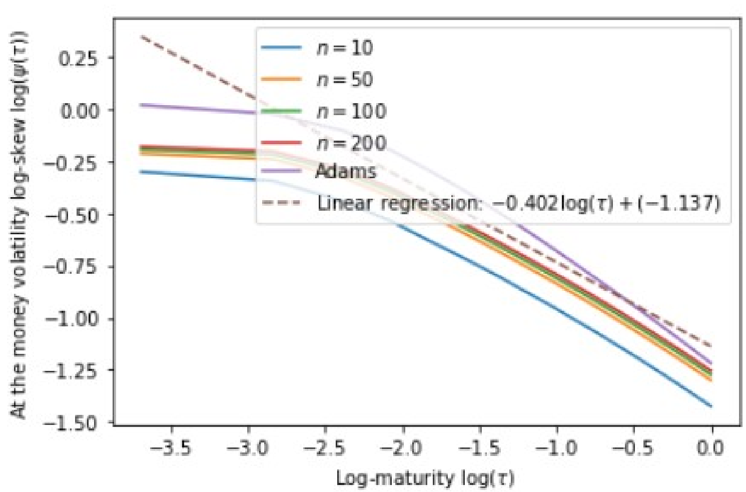

Figure 9: Volatility ATM skew as a function of the log maturity

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