Model fitting

Joachim Vandekerckhove

Winter 2025

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$$P(y|\theta, a, b_i) = \frac{1}{1 + e^{-a(\theta - b_i)}} \tag{1}$$

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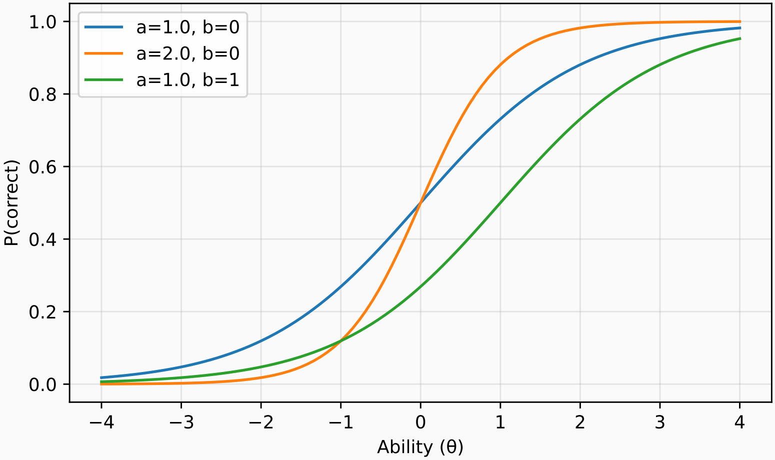
  - θ: Subject ability
- The three-parameter logistic (3PL) adds a guessing parameter c:

$$P(y|\theta, a, b_i, c) = c + (1 - c)\frac{1}{1 + e^{-a(\theta - b_i)}} \tag{2}$$

Visualizing the 2PL Model



Ordinary Least Squares

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    - Optimal when errors are normally distributed
    - Has closed-form solution for linear models
    - Simple to implement and interpret

Squared Error Estimation

- Implementing OLS for the 2PL model, parameters are estimated by minimizing squared error:

$$\text{SSE}(\theta, a, \mathbf{b}, \mathbf{y}) = \sum_{i=1}^{N} (y_i - P(y_i|\theta, a, b_i))^2 \tag{4}$$

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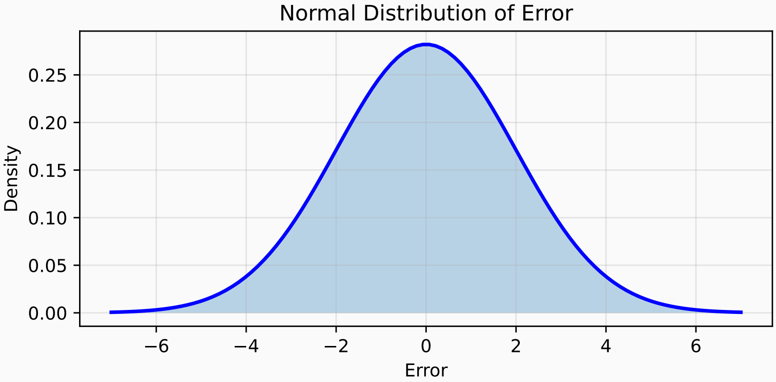
- This approach:
    - Is intuitive and widely used in simple cases
    - Assumes normally distributed errors

Normality Assumption



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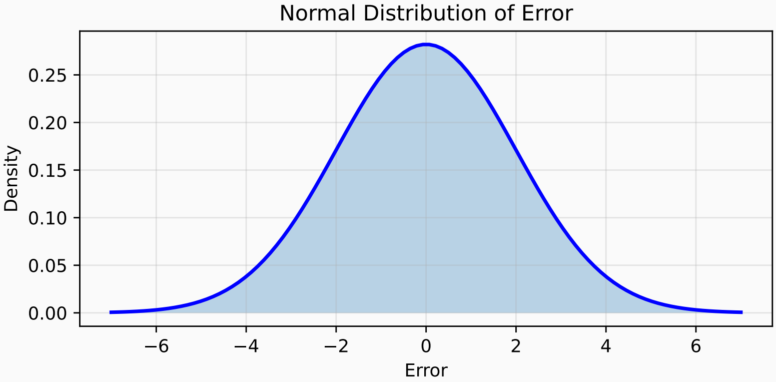

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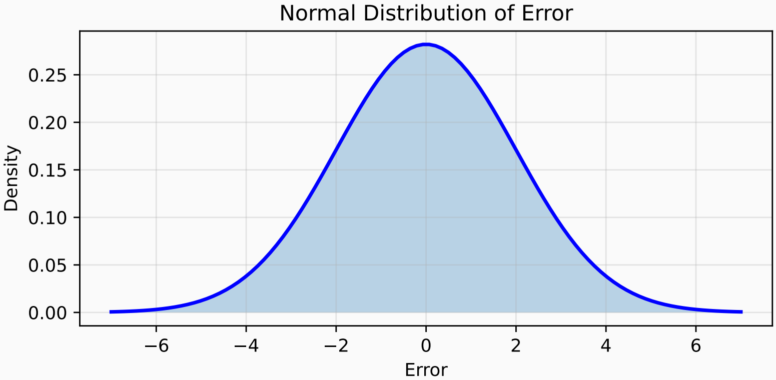

- May not be appropriate for non-normal errors
- Can give biased estimates for binary outcomes
- Fine for a first pass, but not optimal for many applications

Maximum Likelihood Estimation

- For binary outcomes, likelihood is more appropriate:

$$L(\theta, a, \mathbf{b}, \mathbf{y}) = \prod_{i=1}^{N} P(y_i|\theta, a, b_i)^{y_i}(1 - P(y_i|\theta, a, b_i))^{1-y_i} \tag{5}$$

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$$\ell(\theta, a, \mathbf{b}, \mathbf{y}) = \sum_{i=1}^{N} [y_i \log P(y_i|\theta, a, b_i) + (1 - y_i) \log(1 - P(y_i|\theta, a, b_i))] \tag{6}$$

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- Using the likelihood function instead of squared error appropriately accounts for the fact that some prediction errors are more severe than others

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$$P(y|\theta, a, x, \beta) = \frac{1}{1 + e^{-a(\theta - f(x,\beta))}} \tag{7}$$

with $f(x, \beta) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_k x_k$

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- Here, β is a vector of parameters for the predictors that replaces the item difficulty parameters **b**
- Fewer predictors is better for model parsimony and interpretability
- Certainly, we need there to be fewer predictors than conditions, so $k < N$

LASSO and Ridge Regularization

$$\ell(\theta, a, x, \beta) = \sum_{i=1}^{N} [y_i \log P(y|\theta, a, x, \beta) + (1 - y_i) \log(1 - P(y|\theta, a, x, \beta))]$$

- LASSO (Least Absolute Shrinkage and Selection Operator) equation:

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- Ridge regularization equation:

$$\ell_{\text{Ridge}}(\theta, a, x, \beta) = \ell(\theta, a, x, \beta) - \lambda \sum_{j=1}^{k} \beta_j^2 \tag{9}$$

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- Regularization is a form of Bayesian inference
- Can be combined with likelihoods or more basic loss functions (like squared error)

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- I will add a twist that is likely to confuse an AI assistant

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