Introduction to Bayesian inference

Joachim Vandekerckhove

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- ► Special issue of Psychonomic Bulletin & Review (volume 25, 2018)

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▶ Not to be confused with *"Bayes-in-the-head"*, a set of psychological theories about how lay humans perform inference in daily life

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- In another meaning—the classical, or frequentist meaning—probability is a statement of *expected frequency over many repetitions*

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- ▶ We are not usually interested in the frequency with which a well-defined process will achieve a certain outcome.

The Sum and Product Rules of probability

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These notations can be combined: $P(A, B|\neg C, \neg D)$ is the probability that A and B are both true assuming that C and D are both false.

The Product Rule of probability

With this notation in mind, we introduce the **Product Rule of probability**:

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In words: the probability that A and B are both true is equal to the probability of B multiplied by the conditional probability of A *assuming B is true*.

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  - The truth of some hypothesis H, which must be either true or false: $\{H, \neg H\}$

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▶ In words, the probability it rains today is the sum of two joint probabilities: (1) the probability it rains today and tomorrow, and (2) the probability it rains today but not tomorrow.

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That is, the probability of event A alone is the sum of all the joint probabilities between A and the elements of a disjunctive set.

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Together, these two rules allow us to calculate probabilities in an incredible variety of circumstances. One combination of the two rules in particular is useful for scientific inference is *hypothesis testing*.

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- Of interest is the probability that H is true, given the data X, or $P(H|X)$.

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This is one common formulation of **Bayes' Rule**, and analogous versions can be written for each of the other competing hypotheses; for example, Bayes' Rule for $\neg H$ is

$$P(\neg H|X) = \frac{P(\neg H)P(X|\neg H)}{P(X)}.$$

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- ▶ What remains is the denominator $P(X)$

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- The name varies by how one uses Bayes' Rule
- When one uses it to explain Bayes' Rule, *the prior predictive probability of the data P(X)* is the probability of observing a given outcome in the experiment, taking into account all the possible hypotheses we are considering

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Which gives a weighted-average probability of observing the outcome.

A more complete formulation of Bayes' Rule

$$P(H|X) = \frac{P(H)P(X|H)}{P(H)P(X|H) + P(\neg H)P(X|\neg H)}.$$

Bayes' Rule is obtained as a necessary consequence of the Product Rule and the Sum Rule of probability.

Why is Bayes better?

The Lady Tasting Tea

This story is set in the early 1900s, in a dining hall at Rothamsted Agricultural Experiment Station.

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- Now famous statistician Ronald Aylmer Fisher enjoyed tea with his colleague, Dr. Muriel Bristol
- Fisher put tea infusion into a cup and added milk
- But Dr. Muriel protested, preferring her tea prepared the other way (first milk, then infusion)

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- ▶ In the experiment, Dr. Bristol was able to make the correct discrimination five times, but erred on the sixth: RRRRRW
- ▶ So, can she truly tell the difference or not? How do we decide?

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  - ▶ Suppose the lady is completely unable to do what she claims – she is effectively guessing. Let's call this the null hypothesis, H_0
  - ▶ Then try to discredit (*nullify*) H_0 by demonstrating that it makes the data very unlikely ($p < .05$)

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- ▶ Fisher's first attempt is clearly nonsense
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  - ▶ So using this reasoning, we would reject the H_0 *no matter what the data were*
  - ▶ Fisher realized this absurdity, and made a second attempt

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  - ▶ Instead, consider the probability of observing a *single* error, but at any point in the sequence: WRRRRR, RWRRRR, RRWRRR, RRRWRR, RRRRWR, RRRRRW

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    - ▶ No longer "significant" at .05!

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  - ▶ $p = 5.7 \times 10^{75} \times 2^{-256} \approx .049$, and we again reject H_0 for *every possible outcome*

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  - If 1 error in 6 is significant, surely so is no error, or 6 R's
  - We compute p as the probability of observing data that is *at least as extreme* as the real data, assuming that H_0 is true
  - Absurdly, we now use as evidence an imaginary data pattern (RRRRRR) *that we did not observe and that no hypothesis we hold predicts* (more on this in a moment)

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- ▶ For the lady tasting tea:
  - ▶ For the outcome RRRRRW, there are 5 others as extreme and 1, with no errors, more extreme, giving 7 cases in all and a total probability of $7 \times 2^{-6} = .109$, not significant at 5%

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- The frequentist is required to consider what results are as, or more, extreme
- In this example, Fisher takes other possibilities with 6 pairs of cups
- But why fix 6? Did they decide that in advance, or did Dr. Bristol have a meeting to go to after tea? Had the cups been prepared less efficiently, might she have done fewer?

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- ▶ This is absurd! What does it matter what might have happened, but didn't?

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But $p(x|H_0)$ is not what we are after—we are interested in $p(H_0|x)$.

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- ▶ To answer this consider another lady. . .