The Lady Tasting Wine

Joachim Vandekerckhove

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- ► She is put to a similar test

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- It is possible for you to disagree and still be sensible

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- ▶ K_t and K_w are chosen such that the sum (or integral) over all possibilities is 1. This is always possible if the distribution is proper. The solution for wine here is easy enough (it is the sum of $(1 - P_R)(P_R - 0.5)$ for all values of P_R), but it isn't in general

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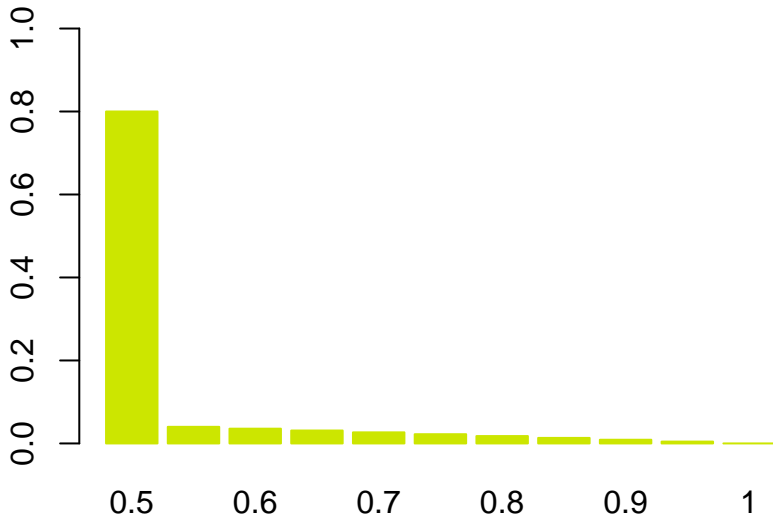
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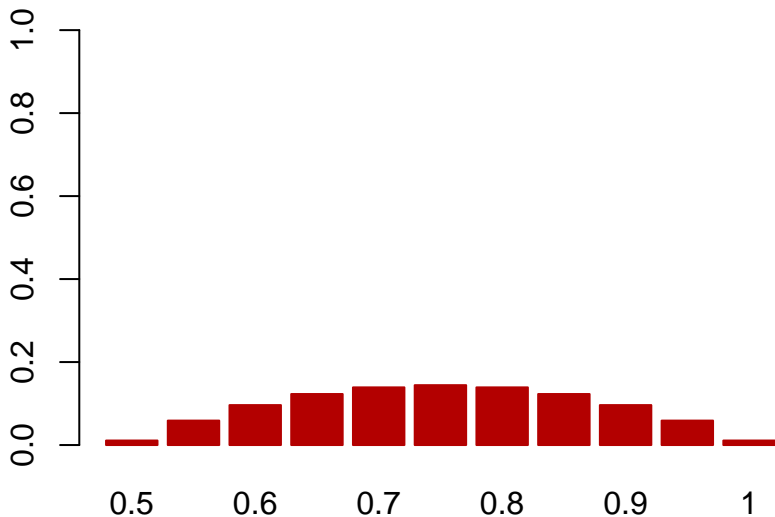- ▶ K_* is *the inverse of* the sum of everything else over values of P_R

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tea

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wine

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- I usually make the proportionality explicit to avoid confusion

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Exercise: plot these in R, using $\#R = 5$, $\#W = 1$

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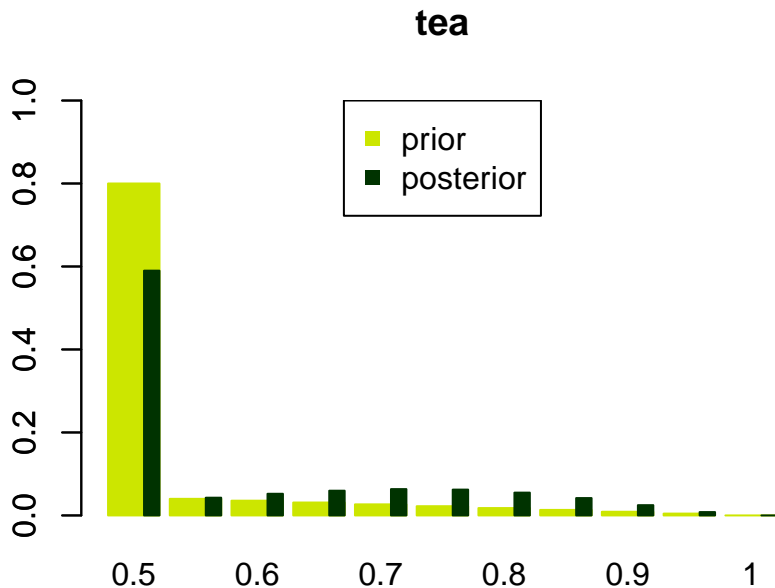
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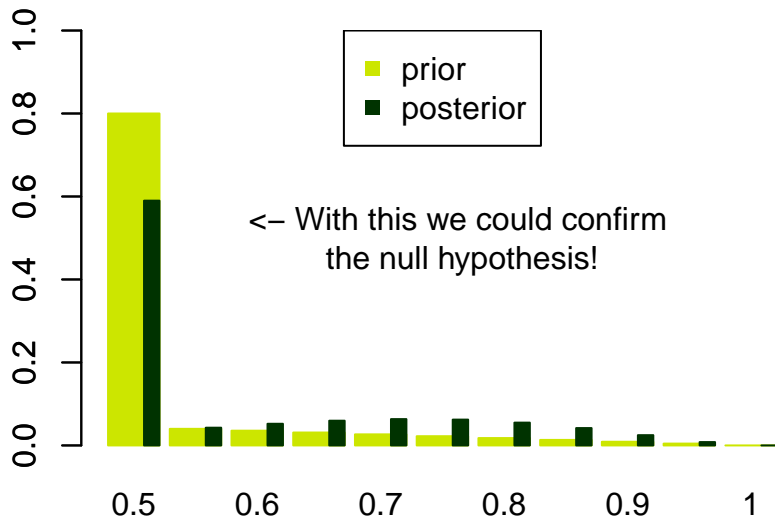- Also, make them pretty.

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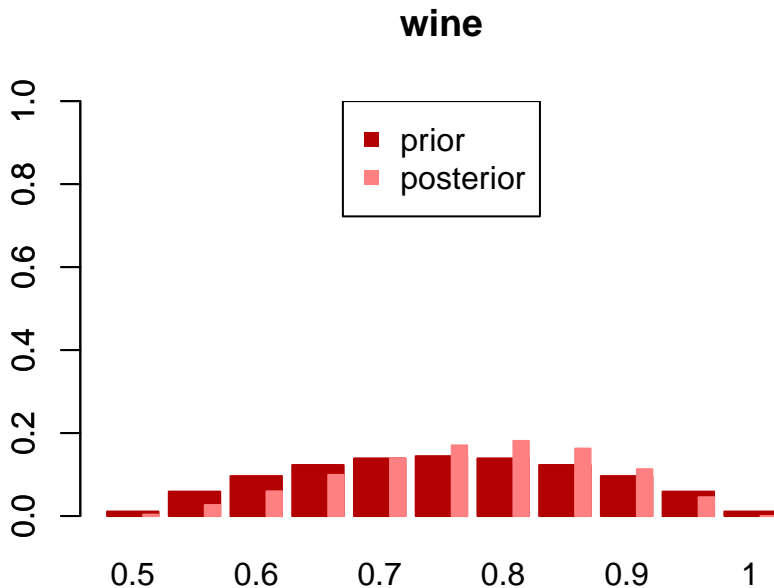
wine

The Lady Tasting Wine (RRRRRW)

tea

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tea

prior
posterior

<- With this we could confirm
the null hypothesis!

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- ▶ Update the posterior for the lady tasting wine by multiplying the old posterior with the new data:

$$p(P_R|\#R, \#W, \#R_2, \#W_2, wine) =$$
$$S_w \times K_w(1.01 - P_R)(P_R - 0.49)$$
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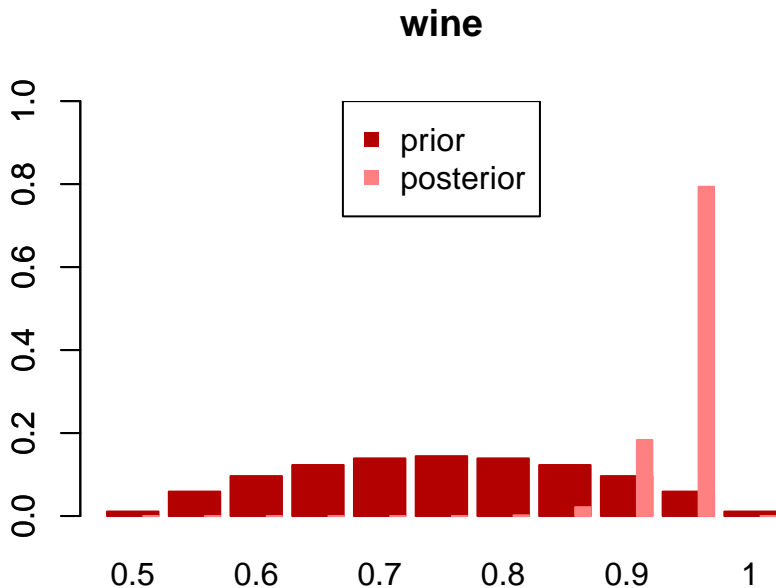
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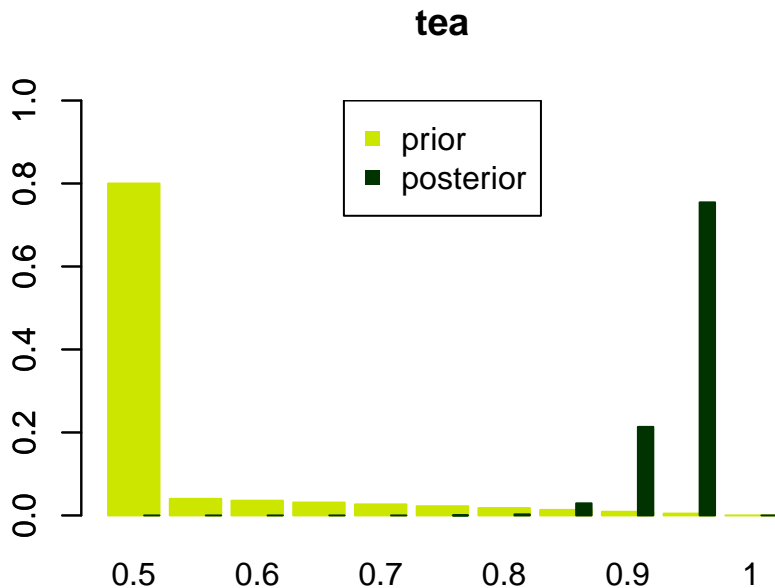
- ▶ ... which is equal to:

$$p(P_R | \#R + \#R_2, \#W + \#W_2, wine) =$$
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$$\times C'(1 - P_R)^{\#W + \#W_2} P_R^{\#R + \#R_2}$$

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  - ▶ Alright, suppose that the ladies actually made no error in the first 6 trials. How many more will it take before the posterior probability of $P_R = 1$ dominates the alternatives?
    - ▶ After 34 corrects, $p(P_R|data)$ for wine tasting accrues at $P_R = 1$
    - ▶ ... but nothing Dr. Muriel does will convince us that $P_R = 1$, because a priori, $p(P_R = 1) = 0$. Cromwell's rule is the recommendation to give a prior nonzero mass at any point that is not a logical impossibility.

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- The solution involves Bayes' theorem
  - Compare probabilities of the data under H_0 and alternatives
  - Different hypotheses weighted by prior beliefs
  - Priors are modified by the data to yield posterior beliefs
  - Then compare the various possible explanations for what has happened, and compare posterior beliefs with priors

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  - H_0 will be more easily discounted using Fisher's method than with the Bayesian approach
    - The vast number of significance tests that are used today will encourage specious beliefs in the efficacy of drugs, treatments, or experimental manipulations
    - Whenever you read some effect having been detected, remember that it probably refers to significance, which too easily suggests an effect when none exists

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    - how big an effect is (as with wine)
  - The posterior tells us exactly what we need to know
    - In contrast to the *p*-value, which is a probability for something that did not happen under the assumption of a hypothesis that may not be true

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  - The Bayesian view recognizes that ones opinion of tasting the two liquids may be different or that the ladies may have different skills

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