Random Vectors

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Random Vectors: When are they useful?

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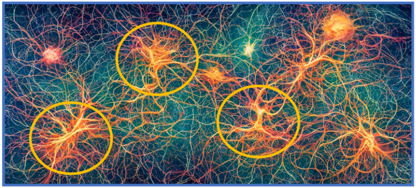

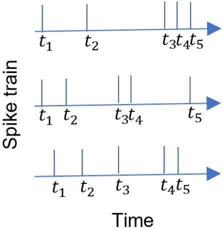

$$T_1 = \left(t_1^1, t_2^1, t_3^1, t_4^1, t_5^1\right)^T$$

$$T_2 = \left(t_1^2, t_2^2, t_3^2, t_4^2, t_5^2\right)^T$$

$$T_3 = \left(t_1^3, t_2^3, t_3^3, t_4^3, t_5^3\right)^T$$

PDF and CDF

$X = (X_1, ..., X_n)^T$: A random vector

$F_x(x)$: *Cumulative* Distribution Function(*CDF*)

$f_x(x)$: Probability *Density* function (*pdf*)

PDF and CDF

By definition, Cumulative Distribution Function(CDF) is:
$F_x(x) = P[X_1 \leq x_1, ..., X_n \leq x_n]$

$x = (x_1, ..., x_n)$ we get:

$F_x(x) = P[X \leq x]$

we associate the events:
$X \leq \infty$ with the certain event, $F_x(\infty) = 1$,and

$X \leq -\infty$ with the impossible event, $F_x(-\infty) = 0$.

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PDF and CDF

The probability *density* function (pdf) is defined as:

$f_x(x) = \frac{\partial^n F_x(x)}{\partial x_1 \ldots \partial x_n}$

Equivalently we could have defined it as:

$f_x(x) = \lim_{\Delta x_1 \to 0, \ldots, \Delta x_n \to 0} \frac{P[x_1 < X_1 \leq x_1 + \Delta x_1, \ldots, x_n < X_n \leq x_1 + \Delta x_n]}{\Delta x_1 \ldots \Delta x_n}$

Therefore,

$f_x(x) \Delta x_1 \ldots \Delta x_n \simeq P[x_1 < X_1 \leq x_1 + \Delta x_1, \ldots, x_n < X_n \leq x_1 + \Delta x_n]$

PDF and CDF

pdf is defined as:

$f_x(x) = \frac{\partial^n F_x(x)}{\partial x_1 ... \partial x_n}$

if we integrate the equation, we obtain:

$F_x(x) = \int_{-\infty}^{x_1} ... \int_{-\infty}^{x_n} f_x(x^{'}) dx_1^{'} ... dx_n^{'} = \int_{-\infty}^{x} f_x(x^{'}) dx^{'}$

more generally:

$P[B] = \int_{x \in B} f_x(x^{'}) dx^{'}$, where $B \subset R^N$

PDF and CDF

constraint: $(P[B] \neq 0)$

conditional *CDF*: $F_{x|B}(x|B) = P[X \leq x|B] = \frac{P[X \leq x, B]}{P[B]}$

mixture *CDF*: $F_x(x) = \sum_{i=1}^{n} F_{x|B_i}(x|B_i)P[B_i]$

conditional *pdf*: $f_{x|B}(x|B) = \frac{\partial^n F_{x|B}(x|B)}{\partial x_1 \ldots \partial x_n}$

mixture *pdf*:$f_x(x) = \sum_{i=1}^{n} f_{x|B_i}(x|B_i)P[B_i]$

mixture: a linear combination

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PDF and CDF

Joint distribution of *two* random vectors:

$X = (X_1, ..., X_n).T$

$Y = (Y_1, ..., Y_M).T$

$F_{XY}(x, y) = P[X \leq x, Y \leq y]$

joint density: $f_{XY}(x, y) = \frac{\partial^{n+m} F_{XY}(x,y)}{\partial x_1 ... \partial x_n \partial y_1 ... \partial y_m}$

marginal density: $f_x(x) = \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} f_{XY}(x, y) dy_1 ... dy_n$

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PDF and CDF: Here's a fun example!

pdf: $f_x(x) = \frac{exp(-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu))}{\sqrt{2\pi^k |\Sigma|}}$,

where in this example,

$X = [x, y]$, therefore,

CDF: $= F_{[x,y]}([x, y]) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_x(x^{'}, y^{'}) dx^{'} dy^{'}$

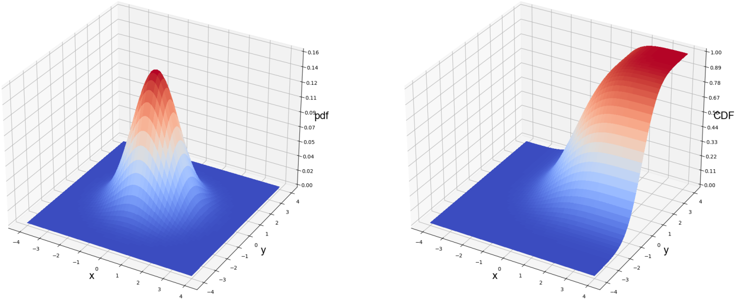
Step 1: For this interesting distribution, implement step 1 to see what pdf and CDF look like, for different mean vector and covariance matrices!

PDF and CDF: 2D

Step1 :

mean, $\mu = [0, 0]$,

covariance matrix $\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

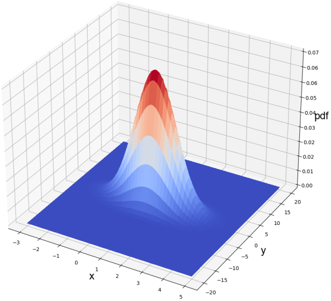


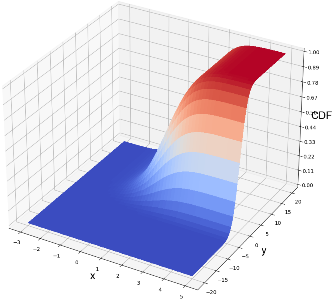
PDF and CDF: 2D

Step1 :

mean, $\mu = [1, 0]$,

covariance matrix $\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$



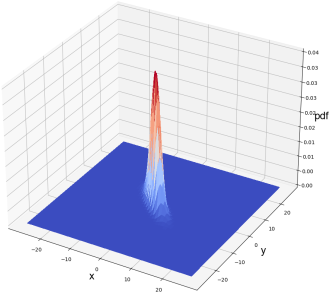


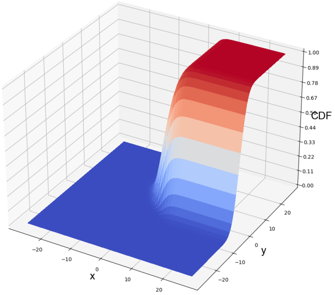
PDF and CDF: 2D

Step1 :

mean, $\mu = [0, 0]$,

covariance matrix $\Sigma = \begin{bmatrix} 9 & -8 \\ -8 & 9 \end{bmatrix}$





PDF and CDF: 3D

pdf: $f_x(x) = \frac{exp(-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu))}{\sqrt{2\pi^k |\Sigma|}}$,

where in this example,

$X = [x, y, z]$, therefore,

CDF: $= F_{[x,y,z]}([x, y, z]) = \int_{-\infty}^{x} \int_{-\infty}^{y} \int_{-\infty}^{z} f_x(x^{'}, y^{'}, z^{'}) dx^{'} dy^{'} dz^{'}$

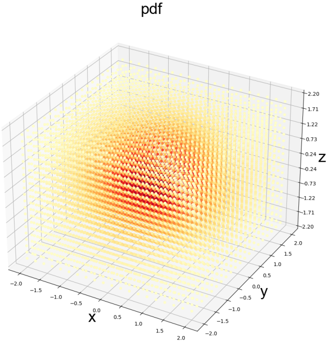
Step 2: Implement step 2 to see what pdf and CDF look like, for different mean vector and covariance matrices!

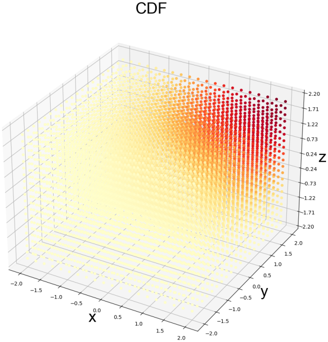
PDF and CDF: 3D

Step2 :

mean, $\mu = [0, 0, 0]$,

covariance matrix $\Sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$



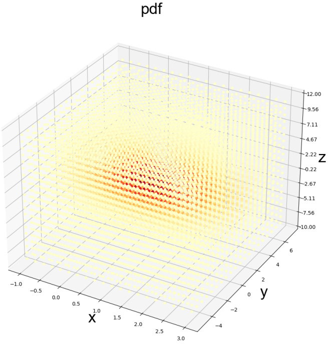


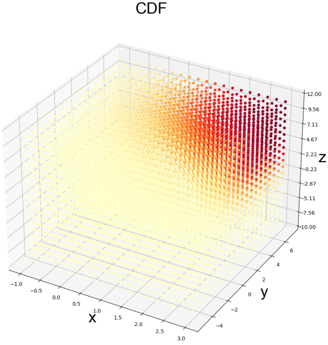
PDF and CDF: 3D

Step2 :

mean, $\mu = [1, 1, 1]$,

covariance matrix $\Sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$



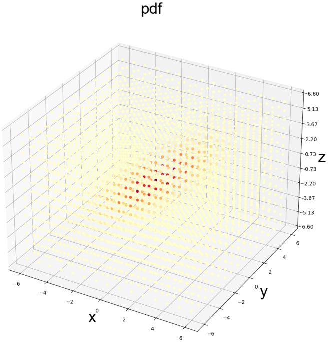


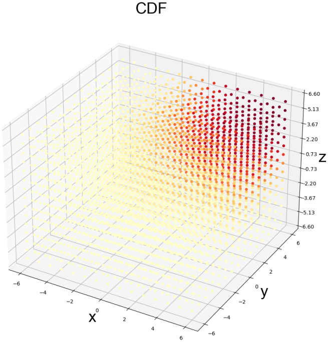
PDF and CDF: 3D

Step2 :

mean, $\mu = [0, 0, 0]$,

covariance matrix $\Sigma = \begin{bmatrix} 3 & 2 & -2 \\ 2 & 3 & 0 \\ -2 & 0 & 3 \end{bmatrix}$





Properties of Covariance Matrices

This is ...

The Multidimensional Gaussian Law

This is ...

Distribution of the Sample Mean

This is ...

Conditional Gaussian distributions

This is ...

Marginal Gaussian distributions

This is ...