Random Vectors

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Random Vectors: When are they useful?

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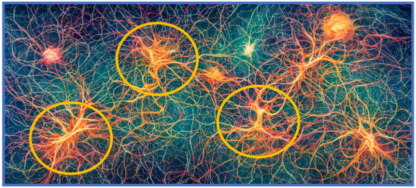

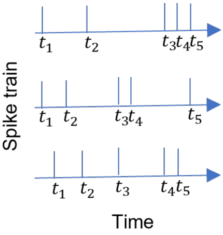

$$T_1 = \left(t_1^1, t_2^1, t_3^1, t_4^1, t_5^1\right)^T$$

$$T_2 = \left(t_1^2, t_2^2, t_3^2, t_4^2, t_5^2\right)^T$$

$$T_3 = \left(t_1^3, t_2^3, t_3^3, t_4^3, t_5^3\right)^T$$

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PDF and CDF

$X = (X_1, ..., X_n)^T$: A random vector

$F_x(x)$: *Cumulative* Distribution Function(*CDF*)

$f_x(x)$: Probability *Density* function (*pdf*)

PDF and CDF

By definition, Cumulative Distribution Function(CDF) is:
$F_x(x) = P[X_1 \leq x_1, ..., X_n \leq x_n]$

$x = (x_1, ..., x_n)$ we get:

$F_x(x) = P[X \leq x]$

we associate the events:
$X \leq \infty$ with the certain event, $F_x(\infty) = 1$,and

$X \leq -\infty$ with the impossible event, $F_x(-\infty) = 0$.

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PDF and CDF

The probability *density* function (pdf) is defined as:

$f_x(x) = \frac{\partial^n F_x(x)}{\partial x_1 ... \partial x_n}$

Equivalently we could have defined it as:

$f_x(x) = \lim_{\Delta x_1 \to 0, ..., \Delta x_n \to 0} \frac{P[x_1 < X_1 \leq x_1 + \Delta x_1, ..., x_n < X_n \leq x_1 + \Delta x_n]}{\Delta x_1 ... \Delta x_n}$

Therefore,

$f_x(x) \Delta x_1 ... \Delta x_n \simeq P[x_1 < X_1 \leq x_1 + \Delta x_1, ..., x_n < X_n \leq x_1 + \Delta x_n]$

PDF and CDF

pdf is defined as:

$$f_x(x) = \frac{\partial^n F_x(x)}{\partial x_1 ... \partial x_n}$$

if we integrate the equation, we obtain:

$$F_x(x) = \int_{-\infty}^{x_1} ... \int_{-\infty}^{x_n} f_x(x^{'})dx_1^{'}...dx_n^{'} = \int_{-\infty}^{x} f_x(x^{'})dx^{'}$$

more generally:

$$P[B] = \int_{x \in B} f_x(x^{'})dx^{'}, \text{ where } B \subset R^N$$

PDF and CDF

constraint: $(P[B] \neq 0)$

conditional *CDF*: $F_{x|B}(x|B) = P[X \leq x|B] = \frac{P[X \leq x, B]}{P[B]}$

mixture *CDF*: $F_x(x) = \sum_{i=1}^{n} F_{x|B_i}(x|B_i)P[B_i]$

conditional *pdf*: $f_{x|B}(x|B) = \frac{\partial^n F_{x|B}(x|B)}{\partial x_1 \ldots \partial x_n}$

mixture *pdf*:$f_x(x) = \sum_{i=1}^{n} f_{x|B_i}(x|B_i)P[B_i]$

mixture: a linear combination

PDF and CDF

Joint distribution of *two* random vectors:

$X = (X_1, ..., X_n).T$

$Y = (Y_1, ..., Y_M).T$

$F_{XY}(x, y) = P[X \leq x, Y \leq y]$

joint density: $f_{XY}(x, y) = \frac{\partial^{n+m} F_{XY}(x,y)}{\partial x_1 ... \partial x_n \partial y_1 ... \partial y_m}$

marginal density: $f_x(x) = \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} f_{XY}(x, y) dy_1 ... dy_n$

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PDF and CDF: Here's a fun example!

pdf: $f_x(x) = \frac{exp(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu))}{\sqrt{2\pi^k|\Sigma|}}$,

where in this example,

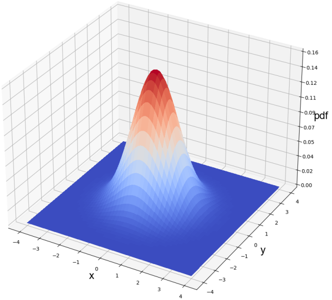
$X = [x, y]$, therefore,

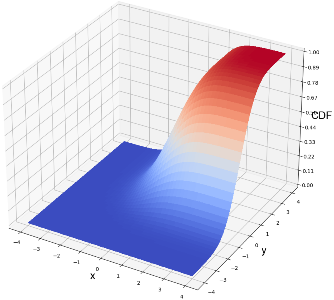
CDF: $= F_{[x,y]}([x, y]) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_x(x', y')dx'dy'$

Step 1: For this interesting distribution, implement step 1 to see what pdf and CDF look like, for different mean vector and covariance matrices!

PDF and CDF: 2D

Step1 :
mean, $\mu = [0, 0]$,
covariance matrix $\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$



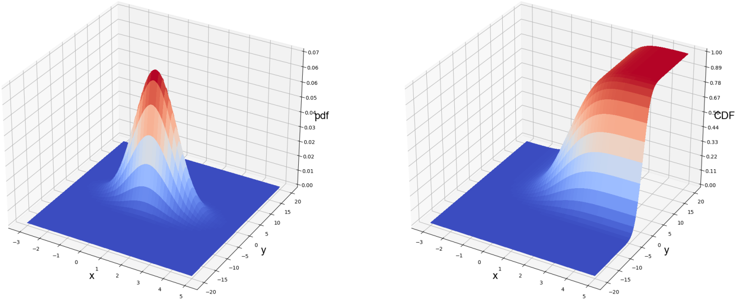


PDF and CDF: 2D

Step1 :

mean, $\mu = [1, 0]$,

covariance matrix $\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$

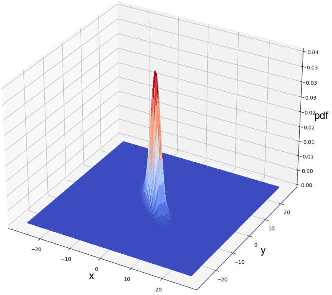


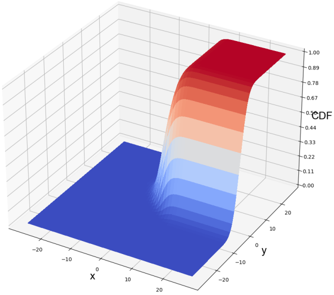
PDF and CDF: 2D

Step1 :

mean, $\mu = [0, 0]$,

covariance matrix $\Sigma = \begin{bmatrix} 9 & -8 \\ -8 & 9 \end{bmatrix}$





PDF and CDF: 3D

pdf: $f_x(x) = \frac{exp(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu))}{\sqrt{2\pi^k|\Sigma|}}$,

where in this example,

$X = [x, y, z]$, therefore,

CDF: $= F_{[x,y,z]}([x, y, z]) = \int_{-\infty}^{x}\int_{-\infty}^{y}\int_{-\infty}^{z} f_x(x', y', z')dx'dy'dz'$

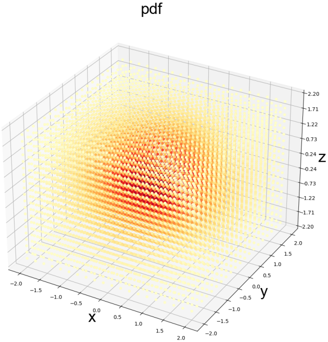
Step 2: Implement step 2 to see what pdf and CDF look like, for different mean vector and covariance matrices!

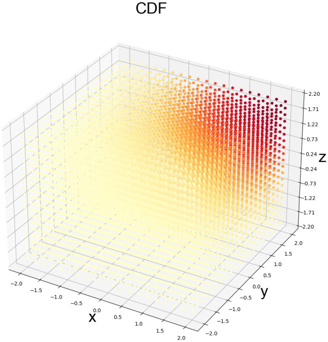
PDF and CDF: 3D

Step2 :

mean, $\mu = [0, 0, 0]$,

covariance matrix $\Sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$



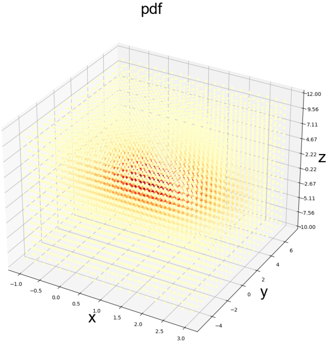


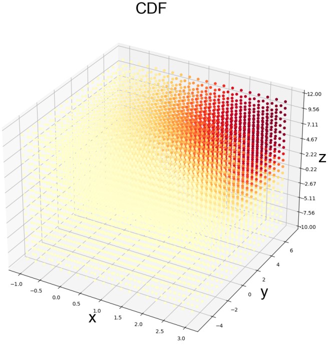
PDF and CDF: 3D

Step2 :

mean, $\mu = [1, 1, 1]$,

covariance matrix $\Sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$





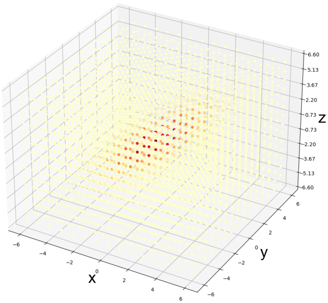
PDF and CDF: 3D

Step2 :

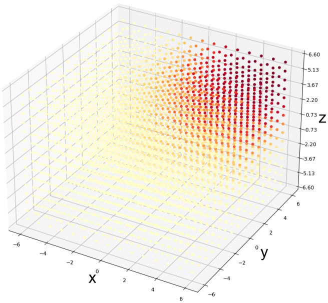
mean, $\mu = [0, 0, 0]$,

covariance matrix $\Sigma = \begin{bmatrix} 3 & 2 & -2 \\ 2 & 3 & 0 \\ -2 & 0 & 3 \end{bmatrix}$

pdf



CDF



Expectation Vector

The expectation of the vector $X = (X_1, ..., X_n)^T$ is a vector $\mu = (\mu_1, ..\mu_n)^T$ whose elements are given by:

$$\mu_i = \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} x_i f_x(x_1, ..., x_n) dx_1 ... dx_n.$$

$$f_{X_i}(x_i) = \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} f_x(x) dx_1 ... dx_{i-1} dx_{i+1} ... dx_n$$

$$\mu_i = \int_{-\infty}^{\infty} x_i f_{x_i}(x_i) dx_i$$

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Covariance Matrix

The covariance matrix associated with a real random vector$X$ is:

$$K = E[(X - \mu)(X - \mu)^T]$$

Define

$$K_{ij} = E[(X_i - \mu_i)(X_j - \mu_j)]$$

Particularly: $\sigma_i^2 = K_{ii}$, so we can write K as: $\begin{bmatrix} \sigma_1^2 & ... & K_1 n \\ ... & \sigma_i^2 & ... \\ K_{n1} & ... & \sigma_n^2 \end{bmatrix}$

1- if X is real, all the elements of K are *real*.
2- $K_{ij} = K_{ji}$, the covariance matrix is *real symmetric*!
3- Real symmetric matrices have many interesting properties! we will discuss it!

Correlation matrix

The correlation matrix R is defined by:

$R = E[XX^T]$

$R = R + \mu\mu^T$, and

$K = R - \mu\mu^T$

Definitions

Consider real n-dimensional random vectors X, Y with respective mean vectors $\mu_x, and \mu_y$:

X, and Y are *uncorrelated* if:
$E[XY^T] = \mu_x \mu_y^T$

X, and Y are *orthogonal* if:
$E[XY^T] = 0$

X, and Y are *independent* if:
$f_{XY}(x, y) = f_X(x) f_Y(y)$

*Note: Independence implies uncorrelatedness! But the converse is not *generally* true!

Expectation vector, Covariance matrix: Example!

For vectors
$Y = (X_1, X_2)^T$,
$Z = (X_3, X_4)^T$,
we write their joint vector as
$X = (X_1, X_2, X_3, X_4)^T$

and the joint PDF as:

$f_X(x) = \frac{1}{4\pi^2} exp(-\frac{1}{2} x^T x)$

Is this distribution familiar to you?

let μ_Y, and μ_Z be the mean vectors of vector Y, and Z respectively,

1- compute μ_Y, and μ_Z!, then $\mu_Y \mu_Z^T$

Expectation vector, Covariance matrix: Example!

$\mu_X = (\mu_1, \mu_2, \mu_3, \mu_4)$: the expectation of X vector, therefore

$\mu_Y = (\mu_1, \mu_2), \mu_Z = (\mu_3, \mu_4)$

$\mu_1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 f_x(x_1, x_2, x_3, x_4) dx_1 dx_2 dx_3 dx_4$

$\mu_1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 \frac{1}{4\pi^2} exp(-\frac{1}{2} x^T x) dx_1 dx_2 dx_3 dx_4$

$\mu_1 =$
$\frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 exp(-\frac{1}{2}(x_1^2 + x_2^2 + x_3^2 + x_4^2)) dx_1 dx_2 dx_3 dx_4 = 0$

Therefore, $\mu_Y = \mu_Z = (0, 0)^T$ $\mu_Y \mu_Z^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Expectation vector, Covariance matrix: Example!

$Y = (X_1, X_2)^T$
$Z = (X_3, X_4)^T$,
$X = (X_1, X_2, X_3, X_4)^T$

According to the previous part, $\mu_Y \mu_Z^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

2- Compute $E(YZ^T)$!

Are Y, and Z orthogonal? Are Y, and Z uncorrelated?

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Expectation vector, Covariance matrix: Example!

$$E(YZ^T) = \begin{bmatrix} EX_1X_3] & E[X_1X_4] \\ E[X_2X_3] & E[X_2X_4] \end{bmatrix}$$

$E[X_1X_3] =$
$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1x_3 exp(-\frac{1}{2}(x_1^2 + x_2^2 + x_3^2 + x_4^2))dx_1dx_2dx_3dx_4 =$

$\int_{-\infty}^{\infty} x_1 exp(-\frac{1}{2}x_1^2)dx_1. \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_3 exp(-\frac{1}{2}(x_2^2 + x_3^2 + x_4^2))dx_2dx_3dx_4 =$
0

therefore,

$E(YZ^T) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$: Y, and Z are orthogonal!

$E(YZ^T) = \mu_Y\mu_Z^T$: Y, and Z are uncorrelated!

Expectation vector, Covariance matrix: Example!

$Y = (X_1, X_2)^T$,
$Z = (X_3, X_4)^T$,
$X = (X_1, X_2, X_3, X_4)^T$

$f_X(x) = \frac{1}{4\pi^2} exp(-\frac{1}{2} x^T x)$

3- Are Y, and Z independent?

Expectation vector, Covariance matrix: Example!

$f_{YZ}(yz) = f_X(x) = \frac{1}{4\pi^2} exp(-\frac{1}{2}x^T x) =$

$\frac{1}{2\pi} exp(-\frac{1}{2}(x_1^2 + x_2^2)).\frac{1}{2\pi} exp(-\frac{1}{2}(x_3^2 + x_3^2)) = f_Y(y) f_Z(z)$

Therefore, Y, and Z are independent!

Expectation vector, Covariance matrix: Example!

Compute the correlation matrix, R, and covariance matrix, K for the joint vector, X!

*reminder:

$K = E[(X - \mu)(X - \mu)^T]$
Define
$K_{ij} = E[(X_i - \mu_i)(X_j - \mu_j)]$
The correlation matrix R is defined by:
$R = E[XX^T]$
$R = R + \mu\mu^T \;\; K = R - \mu\mu^T$

*hint! you will need it! $\int_{-\infty}^{\infty} x^2 exp(-ax^2) = \sqrt{\frac{\pi}{4a^3}}$

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Expectation vector, Covariance matrix: Example!

Note that we have shown that $\mu_X = (0,0,0,0,0)^T$, therefore,

$K = R - \mu\mu^T = R$

and, $K_{ij} = E[(X_i - \mu_i)(X_j - \mu_j)] = E[X_i X_j]$

and, we have shown that if $i \neq j : K_{ij} = E[X_i X_j] = 0$

So, we just need to compute $E[X_i^2]$, and since all variables are independent, and everything is symmetric,
$E[X_1^2] = E[X_2^2] = E[X_3^2] = E[X_4^2]$

Expectation vector, Covariance matrix: Example!

$E[X_1^2] = \frac{1}{\sqrt{2*\pi}} \int_{-\infty}^{\infty} x_i^2 exp(\frac{-1}{2} x_i^2) . \frac{1}{\sqrt{2*\pi^2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} exp(-\frac{1}{2}(x_2^2 + x_3^2 + x_3^2)) dx_2 dx_3 dx_4 =$

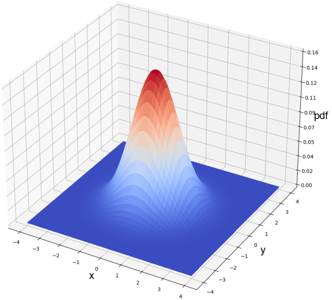
$\frac{1}{\sqrt{2*\pi}} \int_{-\infty}^{\infty} x_i^2 exp(\frac{-1}{2} x_i^2).1 = \frac{\sqrt{2\pi}}{\sqrt{2\pi}}.1 = 1$

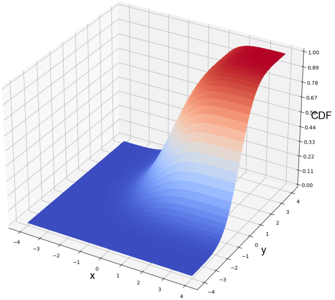
$$\text{So, } K = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

*Reminder $\sigma_i^2 = K_{ii}$, so we can write K as: $\begin{bmatrix} \sigma_1^2 & ... & K_1 n \\ ... & K_{ii}^2 & ... \\ K_{n1} & ... & \sigma_n^2 \end{bmatrix}$

Expectation vector, Covariance matrix: Example!

For both Y, and Z vectors, the PDF, is a multivariate Gaussian with:
mean, $\mu = [0, 0]$,
covariance matrix $\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$





Properties of Covariance Matrices

This is ...

The Multidimensional Gaussian Law

This is ...

Distribution of the Sample Mean

This is ...

Conditional Gaussian distributions

This is ...

Marginal Gaussian distributions

This is ...