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Foundations of probability theory

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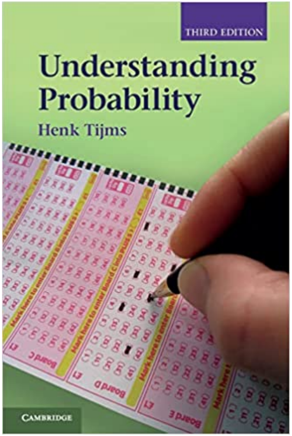
Main reference

I will mainly follow chapters seven *Foundations of probability theory* and eight *Conditional probability and Bayes* from Tijms (2012).



Main lecture goals

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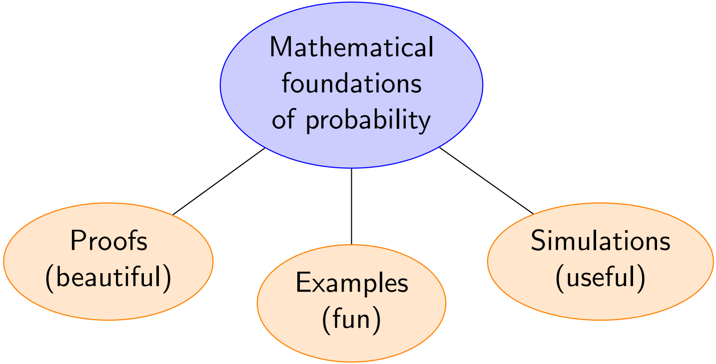

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Gerolamo Cardano (1501-1576) first mathematical analysis of games of chance, introduced probability as the ratio between favorable and the total number of possible outcomes

Pierre Fermat and Blaise Pascal (1654) correspondence about the solution of a problem posed by the Chevalier de Mere established the basic principles of probability.

Christian Huygen (1629-1695) laid the foundations of current probabity theory. Introduced the concept of expected value.

Pierre Simon Laplace (1749-1827) published *Theorie Analytique dez Probabilities*, the greatest contribution in the history of probability.

Andrej Kolmogorov (1903-1987) laid the current axiomatic foundation of probability.

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Probability model

Definition 1 (Probability model)

A **probability model** *is a mathematical representation of a random experiment.*

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Probability model

Definition 1 (Probability model)

A **probability model** *is a mathematical representation of a random experiment. It consists of a description of all possible outcomes of the experiment (i.e.,* **sample space***), a set of subsets of the sample space (i.e.,* **events***), and an assignment of probability to events (i.e.,* **probability measure***).*

Sample space

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Definition 2 (Sample space)

The set of all samples in an experiment is called the **sample space**. *It is denoted by* Ω.

Event

Definition 3 (Event)

An **event** *is a subset of the sample space. We denote the collection of all events by* $\mathcal{F}$.

Event

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Definition 3 (Event)

An **event** *is a subset of the sample space. We denote the collection of all events by $\mathcal{F}$.*

Notes:

1. We will only assign probabilities to events (i.e., to sets $A \in \mathcal{F}$).
2. For finite or countable sample spaces, we can assign probabilities to any subset of the sample space. Thus, any subset of a finite or countable sample space can be an event.
3. For uncountable sample spaces, we can only assign probabilities to well behaved subsets of the sample space (i.e., to elements in a σ algebra of subsets of the sample space). Only well-behaved subsets of an uncountable sample space can be events.

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Probability measure

Definition 4 (Probability measure)

A **probability measure** *is a function that assigns numbers between zero and one to events (i.e.,* $P : \mathcal{F} \to [0, 1]$*).*

Probability model: definitions

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Definition (Sample space)

The set of all samples in an experiment is called the **sample space**. *It is denoted by* Ω.

Definition (Event)

An **event** *is a subset of the sample space. We denote the collection of all events by* $\mathcal{F}$.

Definition (Probability measure)

A **probability measure** *is a function that assigns numbers between zero and one to events (i.e.,* $P : \mathcal{F} \to [0, 1]$*).*

Definition (Probability model)

A **probability model**, $\mathcal{M}$, *is a mathematical representation of a random experiment consisting of a sample space,* Ω*, a set of events,* $\mathcal{F}$*, and a probability measure,* P *(i.e.,* $\mathcal{M} = \{\Omega, \mathcal{F}, P\}$*).*

Examples of probability models

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For each of the following examples, let's find the sample space and propose a probability measure.

1. The experiment is to toss a fair coin once.

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For each of the following examples, let's find the sample space and propose a probability measure.

1. The experiment is to toss a fair coin once. The sample space is the set $[H, T]$.

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Examples of probability models

For each of the following examples, let's find the sample space and propose a probability measure.

1. The experiment is to toss a fair coin once. The sample space is the set $[H, T]$. We assign a probability of 0.5 to each element of the sample space.

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Examples of probability models

For each of the following examples, let's find the sample space and propose a probability measure.

1. The experiment is to toss a fair coin once. The sample space is the set $[H, T]$. We assign a probability of 0.5 to each element of the sample space.
2. The experiment is to repeatedly roll a fair die and count the number of rolls until the first six shows up.

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1. The experiment is to toss a fair coin once. The sample space is the set $[H, T]$. We assign a probability of 0.5 to each element of the sample space.
2. The experiment is to repeatedly roll a fair die and count the number of rolls until the first six shows up. The sample space is the set of positive integers.

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Examples of probability models

For each of the following examples, let's find the sample space and propose a probability measure.

1. The experiment is to toss a fair coin once. The sample space is the set $[H,\ T]$. We assign a probability of 0.5 to each element of the sample space.
2. The experiment is to repeatedly roll a fair die and count the number of rolls until the first six shows up. The sample space is the set of positive integers. The probabilities $\frac{1}{6}, \frac{5}{6} \times \frac{1}{6}, \left(\frac{5}{6}\right)^2 \times \frac{1}{6}, \ldots$ can be assigned to the outcomes $1, 2, 3, \ldots$.

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3. The experiment is to measure the time between the first and second spikes in an experimental trial.

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3. The experiment is to measure the time between the first and second spikes in an experimental trial. The sample space is the set $(0, \infty)$ of positive real numbers.

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3. The experiment is to measure the time between the first and second spikes in an experimental trial. The sample space is the set $(0, \infty)$ of positive real numbers. We can assign a probability of $1 - \exp(-\lambda t)$ to the event that the second spike is fired less than t seconds after the first spike.

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Axioms of probability theory

Axiom 1

$P(A) \geq 0, \quad \forall A \in \mathcal{F}$

Axiom 2

$P(\Omega) = 1$

Axiom 3

$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$ *for every collection of pairwise disjoint events* $A_1, A_2, \ldots$

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Some basic rules

Rule 1

For any finite number of mutually exclusive events $A_1, \ldots, A_N$,

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + \ldots + P(A_N)$$

Rule 2

For any event A,

$$P(A) = 1 - P(A^c)$$

where the event A^c consists of all outcomes that are not in A.

Rule 3

Let A, B be two events such that $A \subseteq B$. Then $P(A) \leq P(B)$.

Rule 4

For any two events A and B,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

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Experiment with equally likely outcomes

An experiment with equally likely outcomes is one with a finite number of outcomes $\omega_1, \ldots, \omega_N$, where all outcomes have the same probability (i.e., $P(\omega_i) = \frac{1}{N}$).

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An experiment with equally likely outcomes is one with a finite number of outcomes $\omega_1, \ldots, \omega_N$, where all outcomes have the same probability (i.e., $P(\omega_i) = \frac{1}{N}$).

Claim 1

For any event A, $P(A) = \frac{N(A)}{N}$, where $N(A)$ is the number of outcomes in the set A.

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Experiment with equally likely outcomes

An experiment with equally likely outcomes is one with a finite number of outcomes $\omega_1, \ldots, \omega_N$, where all outcomes have the same probability (i.e., $P(\omega_i) = \frac{1}{N}$).

Claim 1

For any event A, $P(A) = \frac{N(A)}{N}$, where $N(A)$ is the number of outcomes in the set A.

Proof.

$$A = \{a_1, \ldots, a_{N(A)}\} = \{a_1\} \cup \ldots \cup \{a_{N(A)}\}$$

$$P(A) = P(\{a_1\}) + \ldots + P(\{a_{N(A)}\}) = \sum_{i=1}^{N(A)} P(\{a_i\})$$

$$= \sum_{i=1}^{N(A)} \frac{1}{N} = \frac{N(A)}{N}$$

$\square$

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Example 7.1: John, Pedro and Rosita's dice game

Example 7.1

John, Pedro and Rosita each roll on fair die. How do we calculate the probability that the score of Rosita is equal to the sum of the scores of John and Pedro?

Analytical solution of example 7.1

$$
\begin{aligned}
\Omega &= \{(J,P,R) : 1 \le J,P,R \le 6\} \\
N &= \; 6^3 = 216 \\
A &= \{(J,P,R) \in \Omega : R = J + P)\} \\
&= \{(1,1,2),(1,2,3),(1,3,4),(1,4,5),(1,5,6), \\
&\quad (2,1,3),(2,2,4),(2,3,5),(2,4,6), \\
&\quad (3,1,4),(3,2,5),(3,3,6),(4,1,5),(4,2,6),(5,1,6)\} \\
N(A) &= \; 15 \\
P(A) &= \; \frac{N(A)}{N} = \frac{15}{216} = 0.0694
\end{aligned}
$$

Simulated solution of example 7.1

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Rule 1

Proof.

Denote by $\emptyset$ the empty set. We will first prove that $P(\emptyset) = 0$. Take $A_i = \emptyset$ for $i = 1, 2, \ldots$. Then $\emptyset = \bigcup_{i=1}^{\infty} A_i$. Next, by Axiom 3, $P(\emptyset) = \sum_{i=1}^{\infty} P(A_i) = \sum_{i=1}^{\infty} P(\emptyset)$. This implies that $P(\emptyset) = 0$.
Define $A_{N+j} = \emptyset$ for $j = 1, 2, \ldots$. Then

$$
\begin{aligned}
P(A_1 \cup A_2 \cup \ldots \cup A_N) &= P(A_1 \cup A_2 \cup \ldots \cup A_N \cup A_{N+1} \cup A_{N+2} \cup \ldots) \\
&= P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i) \\
&= \sum_{i=1}^{N} P(A_i) + \sum_{j=1}^{\infty} P(A_{N+j}) = \sum_{i=1}^{N} P(A_i)
\end{aligned}
$$

$\square$

Notes:

1. the last equality in the second line holds by Axiom 1
2. the last equality in the third line holds because $P(A_{N+j}) = P(\emptyset) = 0$.

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Proof of rule 2

Proof.

$\Omega = A \cup A^c$. A and A^c are disjoint. Then, by Rule 1, $P(\Omega) = P(A) + P(A^c)$. From Axiom 2, $P(\Omega) = 1$. Thus, $1 = P(A) + P(A^c)$.

□

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Exercise

Prove rule 3.

Proof of rule 4

Proof.

$$\begin{aligned} A \cup B &= (A \setminus B) \cup (B \setminus A) \cup (A \cap B) \\ A &= (A \setminus B) \cup (A \cap B) \\ B &= (B \setminus A) \cup (A \cap B) \end{aligned}$$

Since the sets in the right-hand-side of the above equations are pairwise disjoint, by rule 1, we obtain

$$\begin{aligned} P(A \cup B) &= P(A \setminus B) + P(B \setminus A) + P(A \cap B) \\ P(A) &= P(A \setminus B) + P(A \cap B) \to P(A \setminus B) = P(A) - P(A \cap B) \\ P(B) &= P(B \setminus A) + P(A \cap B) \to P(B \setminus A) = P(B) - P(A \cap B) \end{aligned}$$

Replacing the equations on the right of the second and third line in the equation on the first line the rule is proved.

$$\begin{aligned} P(A \cup B) &= P(A) - P(A \cap B) + P(B) - P(A \cap B) + P(A \cap B) \\ &= P(A) + P(B) - P(A \cap B) \end{aligned}$$

$\square$

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Example 7.7: Chevalier de Mere to Blaise Pascal

The gambler Chevalier de Mere posed the following problem to the famous French mathematician Blaise Pascal in 1654. This problem marks the beginning of probability theory.

Example 7.7

a. *How many rolls of a fair die are required to have at least a 50% chance of rolling at least one six?*
b. *How many rolls of two fair dice are required to have at least a 50% chance of rolling at least one double six?*

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Analytical solution of example 7.7a

(a)

- Let's fix the number of rolls r. The sample space is $\Omega = \{(i_1, \ldots, i_r) : 1 \leq i_k \leq 6\}$, where i_k is the up face of the die on the kth roll. The outcomes in Ω are equiprobable.

Analytical solution of example 7.7a

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(a)

- Let's fix the number of rolls r. The sample space is $\Omega = \{(i_1, \ldots, i_r) : 1 \leq i_k \leq 6\}$, where i_k is the up face of the die on the kth roll. The outcomes in Ω are equiprobable.
- We want to calculate the probability of the event A= "at least one six shows up in the r rolls.". When you see the keyword "at least one" in an event, it is easier to calculate the probability of the complement A^c= "no six shows up in the r rolls."

Analytical solution of example 7.7a

(a)

- Let's fix the number of rolls r. The sample space is $\Omega = \{(i_1, \ldots, i_r) : 1 \leq i_k \leq 6\}$, where i_k is the up face of the die on the kth roll. The outcomes in Ω are equiprobable.
- We want to calculate the probability of the event A="at least one six shows up in the r rolls.". When you see the keyword "at least one" in an event, it is easier to calculate the probability of the complement A^c="no six shows up in the r rolls."
- $P(A^c) = \frac{N(A^c)}{N} = \frac{5^r}{6^r} = \left(\frac{5}{6}\right)^r.$

Analytical solution of example 7.7a

(a)

- Let's fix the number of rolls r. The sample space is $\Omega = \{(i_1, \ldots, i_r) : 1 \leq i_k \leq 6\}$, where i_k is the up face of the die on the kth roll. The outcomes in Ω are equiprobable.
- We want to calculate the probability of the event A= "at least one six shows up in the r rolls.". When you see the keyword "at least one" in an event, it is easier to calculate the probability of the complement A^c= "no six shows up in the r rolls."
- $P(A^c) = \frac{N(A^c)}{N} = \frac{5^r}{6^r} = \left(\frac{5}{6}\right)^r$.
- $\frac{1}{2} < P(A) = 1 - P(A^c) = 1 - \left(\frac{5}{6}\right)^r$ iff $\left(\frac{5}{6}\right)^r < \frac{1}{2}$ iff $\log \left(\frac{5}{6}\right)^r < \log \frac{1}{2}$ iff $r \log \frac{5}{6} < \log \frac{1}{2}$ iff $r > \frac{\log \frac{1}{2}}{\log \frac{5}{6}} = 3.8$

Simulated solution of example 7.7a

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Please see code here.

Exercise

Solve example 7.7b analytically and by simulation. Answer: you need at least 25 draws.

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Example 7.9: soccer teams in quarterfinal

Example 7.9

The eight soccer teams which have reached the quarterfinals of the Championship League are formed by two teams from of the the countries England, Germany, Italy and Spain. The four matches to be played in the quarterfinal are determined by drawing lots.

a. *What is the probability that the two teams from the same country play against each other in each of the four matches?*
b. *What is the probability that there is a match between the two teams from England or between the two teams from Germany?*

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Analytical solution of example 7.9: setup

- Let T be the set of eight teams
  $T = \{E_1, E_2, G_1, G_2, I_1, I_2, S_1, S_2\}$

Analytical solution of example 7.9: setup

- Let T be the set of eight teams $T = \{E_1, E_2, G_1, G_2, I_1, I_2, S_1, S_2\}$
- A match is a set of two different elements from T (i.e., match $= \{T_i, T_j\}$, match $\subset T$, $T_i \neq Tj$).

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Analytical solution of example 7.9: setup

- Let T be the set of eight teams $T = \{E_1, E_2, G_1, G_2, I_1, I_2, S_1, S_2\}$
- A match is a set of two different elements from T (i.e., match $= \{T_i, T_j\}$, match $\subset T$, $T_i \neq Tj$).
- A quarterfinal is a set of four different matches.

Analytical solution of example 7.9: setup

- Let T be the set of eight teams $T = \{E_1, E_2, G_1, G_2, I_1, I_2, S_1, S_2\}$
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- A quarterfinal is a set of four different matches.
- Ω is the set of all possible quarterfinals (e.g., $\{\{E1, G1\}, \{I1, G2\}, \{S2, G2\}, \{I2, G1\}\} \in \Omega$).

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Analytical solution of example 7.9: setup

- Let T be the set of eight teams $T = \{E_1, E_2, G_1, G_2, I_1, I_2, S_1, S_2\}$
- A match is a set of two different elements from T (i.e., match $= \{T_i, T_j\}$, match $\subset T$, $T_i \neq Tj$).
- A quarterfinal is a set of four different matches.
- Ω is the set of all possible quarterfinals (e.g., $\{\{E1, G1\}, \{I1, G2\}, \{S2, G2\}, \{I2, G1\}\} \in \Omega$).
- Ω contains equally-likely outcomes. Thus, for any event A, $P(A) = \frac{N(A)}{N}$.

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Analytical solution of example 7.9: setup

- Let T be the set of eight teams $T = \{E_1, E_2, G_1, G_2, I_1, I_2, S_1, S_2\}$
- A match is a set of two different elements from T (i.e., $\text{match} = \{T_i, T_j\}, \text{match} \subset T, T_i \neq Tj$).
- A quarterfinal is a set of four different matches.
- Ω is the set of all possible quarterfinals (e.g., $\{\{E1, G1\}, \{I1, G2\}, \{S2, G2\}, \{I2, G1\}\} \in \Omega$).
- Ω contains equally-likely outcomes. Thus, for any event A, $P(A) = \frac{N(A)}{N}$.
- $N = \frac{\binom{8}{2}\binom{6}{2}\binom{4}{2}}{4!}$. There are $\binom{8}{2}$, $\binom{6}{2}$, $\binom{4}{2}$ and 1 ways of selecting the first, second, third and fourth matches, respectively. We divide by 4! because the order between matches does not matter.

Analytical solution of example 7.9a

- The event A=“*two teams from the same country play against each other in each of the four matches*” contains only one outcome (i.e., $A = \{\{E_1, E_2\}, \{G_1, G_2\}, \{I_1, I_2\}, \{S_1, S_2\}\}$).

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Analytical solution of example 7.9a

- The event A="*two teams from the same country play against each other in each of the four matches*" contains only one outcome (i.e., $A = \{\{E_1, E_2\}, \{G_1, G_2\}, \{I_1, I_2\}, \{S_1, S_2\}\}$).
- $P(A) = \frac{N(A)}{N} = \frac{1}{\frac{\binom{8}{2}\binom{6}{2}\binom{4}{2}}{4!}} = 0.009524$

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Analytical solution of example 7.9b

- The event B="*there is a match between the two teams from England or between the two teams from Germany*" is the union of the events B_E="*there is a match between the two teams from England*" and B_G="*there is a match between the two teams from Germany*" (i.e., $B = B_E \cup B_G$).

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Analytical solution of example 7.9b

- The event B=“*there is a match between the two teams from England or between the two teams from Germany*” is the union of the events B_E=“*there is a match between the two teams from England*” and B_G=“*there is a match between the two teams from Germany*” (i.e., $B = B_E \cup B_G$).
- Since B_E and B_G are not disjoint, we should use Rule 3 to compute $P(B_E \cup B_G)$ (i.e., $P(B_E \cup B_G) = P(B_E) + P(B_G) - P(B_E \cap B_G)$.

Analytical solution of example 7.9b

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- The event B=“*there is a match between the two teams from England or between the two teams from Germany*” is the union of the events B_E=“*there is a match between the two teams from England*” and B_G=“*there is a match between the two teams from Germany*” (i.e., $B = B_E \cup B_G$).
- Since B_E and B_G are not disjoint, we should use Rule 3 to compute $P(B_E \cup B_G)$ (i.e., $P(B_E \cup B_G) = P(B_E) + P(B_G) - P(B_E \cap B_G)$.
- $P(B_E) = P(B_G)$. Since the match $\{E_1, E_2\}$ is in all quarterfinals in B_E, to calculate $N(B_E)$ we need to computer the number of matches between six teams of three countries, as done in part (a). This gives $N(B_E) = \frac{\binom{6}{2}\binom{4}{2}}{3!}$.

  Then $P(B_E) = P(B_G) = \frac{\frac{\binom{6}{2}\binom{4}{2}}{3!}}{\frac{\binom{8}{2}\binom{6}{2}\binom{4}{2}}{4!}} = 0.142857.$

Analytical solution of example 7.9b

- The event B="*there is a match between the two teams from England or between the two teams from Germany*" is the union of the events B_E="*there is a match between the two teams from England*" and B_G="*there is a match between the two teams from Germany*" (i.e., $B = B_E \cup B_G$).
- Since B_E and B_G are not disjoint, we should use Rule 3 to compute $P(B_E \cup B_G)$ (i.e., $P(B_E \cup B_G) = P(B_E) + P(B_G) - P(B_E \cap B_G)$.
- $P(B_E) = P(B_G)$. Since the match $\{E_1, E_2\}$ is in all quarterfinals in B_E, to calculate $N(B_E)$ we need to computer the number of matches between six teams of three countries, as done in part (a). This gives $N(B_E) = \frac{\binom{6}{2}\binom{4}{2}}{3!}$.

  Then $P(B_E) = P(B_G) = \dfrac{\frac{\binom{6}{2}\binom{4}{2}}{3!}}{\frac{\binom{8}{2}\binom{6}{2}\binom{4}{2}}{4!}} = 0.142857$.
- Since the match $\{E_1, E_2\}$ and $\{G_1, G_2\}$ are in all quarterfinals in $B_E \cap B_G$, as in part (a), $N(B_E \cap B_G) = \frac{\binom{4}{2}}{2!}$. Then

  $P(B_E \cap B_G) = \dfrac{\frac{\binom{4}{2}}{2!}}{\frac{\binom{8}{2}\binom{6}{2}\binom{4}{2}}{4!}} = 0.028571$.

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Analytical solution of example 7.9b

- The event B="*there is a match between the two teams from England or between the two teams from Germany*" is the union of the events B_E="*there is a match between the two teams from England*" and B_G="*there is a match between the two teams from Germany*" (i.e., $B = B_E \cup B_G$).
- Since B_E and B_G are not disjoint, we should use Rule 3 to compute $P(B_E \cup B_G)$ (i.e., $P(B_E \cup B_G) = P(B_E) + P(B_G) - P(B_E \cap B_G)$.
- $P(B_E) = P(B_G)$. Since the match $\{E_1, E_2\}$ is in all quarterfinals in B_E, to calculate $N(B_E)$ we need to computer the number of matches between six teams of three countries, as done in part (a). This gives $N(B_E) = \frac{\binom{6}{2}\binom{4}{2}}{3!}$. Then $P(B_E) = P(B_G) = \frac{\frac{\binom{6}{2}\binom{4}{2}}{3!}}{\frac{\binom{8}{2}\binom{6}{2}\binom{4}{2}}{4!}} = 0.142857$.
- Since the match $\{E_1, E_2\}$ and $\{G_1, G_2\}$ are in all quarterfinals in $B_E \cap B_G$, as in part (a), $N(B_E \cap B_G) = \frac{\binom{4}{2}}{2!}$. Then $P(B_E \cap B_G) = \frac{\frac{\binom{4}{2}}{2!}}{\frac{\binom{8}{2}\binom{6}{2}\binom{4}{2}}{4!}} = 0.028571$.
- Thus $P(B_E \cup B_G) = P(B_E) + P(B_G) - P(B_E \cap B_G) =$ $2 \times 0.142857 - 0.028571 = 0.257143$.

Simulated solution of example 7.9a

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Exercise

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Solve example 7.7b by simulation. Answer: you should obtain a solution close to the analytical one.

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Definition of conditional probability: motivation

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- We are given a probability model $(\Omega, \mathcal{F}, P)$.

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Definition of conditional probability: motivation

- We are given a probability model $(\Omega, \mathcal{F}, P)$.
- We are interested in the probability of event $A \in \mathcal{F}$. This model provides us the unconditioned probability of A, $P(A)$.

Definition of conditional probability: motivation

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- We are given a probability model $(\Omega, \mathcal{F}, P)$.
- We are interested in the probability of event $A \in \mathcal{F}$. This model provides us the unconditioned probability of A, $P(A)$.
- Our colleague performs an experiment and tells us that event B occurred. How does the fact that B occurred changes the probability of A?

$$P(A) \xrightarrow{\text{B occurred}} P(A|B)$$

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$$P(A) \xrightarrow{\text{B occurred}} P(A|B)$$

- Definition 1: $P(\cdot|B) = P(\cdot \cap B)$, where $\cdot$ can be any event.

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- Our colleague performs an experiment and tells us that event B occurred. How does the fact that B occurred changes the probability of A?

$$P(A) \xrightarrow{\text{B occurred}} P(A|B)$$

- Definition 1: $P(\cdot|B) = P(\cdot \cap B)$, where $\cdot$ can be any event.
- Problem with Definition 1: we want $P(B|B) = 1$, but from Definition 1, $P(B|B) = P(B \cap B) = P(B) \leq 1$.

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- Problem with Definition 1: we want $P(B|B) = 1$, but from Definition 1, $P(B|B) = P(B \cap B) = P(B) \leq 1$.
- Definition 2: $P(\cdot|B) = \frac{P(\cdot \cap B)}{P(B)}$.

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- Definition 2: $P(\cdot|B) = \frac{P(\cdot \cap B)}{P(B)}$.
  - $P(B|B) = 1$ ✓

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- Problem with Definition 1: we want $P(B|B) = 1$, but from Definition 1, $P(B|B) = P(B \cap B) = P(B) \leq 1$.
- Definition 2: $P(\cdot|B) = \frac{P(\cdot \cap B)}{P(B)}$.
  - $P(B|B) = 1$ ✓
  - $P(A) = P(A|\Omega) = \frac{P(A \cap \Omega)}{P(\Omega)} = \frac{P(A)}{1} = P(A)$ ✓

Definition of conditional probability

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Definition 5 (Conditional probability)

For any to events A and B, with $P(B) > 0$, the conditional probability of A given B, $P(A|B)$, is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

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Example 8.1: conditional probability for two dice

Example 8.1

Someone has rolled two dice. You know that one of the dice turned up a face value of six. What is the probability that the other die turned up a six as well?

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Analytical solution of example 8.1

- $\Omega = \{(i_1, i_2) : 1 \leq i_1, i_2 \leq 6\}$. $N = 36$. Equally probable outcomes.

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Analytical solution of example 8.1

- $\Omega = \{(i_1, i_2) : 1 \leq i_1, i_2 \leq 6\}$. $N = 36$. Equally probable outcomes.
- B=*"one die turned up a face value of six"* $= \{(6, i), (j, 6), (6, 6) : 1 \leq i, j \leq 5\}$. $N(B) = 11$. $P(B) = \frac{N(B)}{N} = \frac{11}{36}$.

Analytical solution of example 8.1

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- B=*"one die turned up a face value of six"* $= \{(6, i), (j, 6), (6, 6) : 1 \leq i, j \leq 5\}$. $N(B) = 11$. $P(B) = \frac{N(B)}{N} = \frac{11}{36}$.
- A=*"the other die turned up a face value of six"*

Analytical solution of example 8.1

- $\Omega = \{(i_1, i_2) : 1 \leq i_1, i_2 \leq 6\}$. $N = 36$. Equally probable outcomes.
- B=*"one die turned up a face value of six"* $= \{(6, i), (j, 6), (6, 6) : 1 \leq i, j \leq 5\}$. $N(B) = 11$. $P(B) = \frac{N(B)}{N} = \frac{11}{36}$.
- A=*"the other die turned up a face value of six"*
- $A \cap B$=*"the two dice turned up a face value of six"* $= \{(6, 6)\}$. $N(A \cap B) = 1$. $P(A \cap B) = \frac{N(A \cap B)}{N} = \frac{1}{36}$.

Analytical solution of example 8.1

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- $\Omega = \{(i_1, i_2) : 1 \leq i_1, i_2 \leq 6\}$. $N = 36$. Equally probable outcomes.
- B=*"one die turned up a face value of six"*$=$ $\{(6,i),(j,6),(6,6) : 1 \leq i,j \leq 5\}$. $N(B) = 11$. $P(B) = \frac{N(B)}{N} = \frac{11}{36}$.
- A=*"the other die turned up a face value of six"*
- $A \cap B$=*"the two dice turned up a face value of six"*$=$ $\{(6,6)\}$. $N(A \cap B) = 1$. $P(A \cap B) = \frac{N(A \cap B)}{N} = \frac{1}{36}$.
- Approach 1 [definition]: $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/36}{11/36} = \frac{1}{11}$.

Analytical solution of example 8.1

- $\Omega = \{(i_1, i_2) : 1 \leq i_1, i_2 \leq 6\}$. $N = 36$. Equally probable outcomes.
- B=*"one die turned up a face value of six"*$=$ $\{(6,i),(j,6),(6,6) : 1 \leq i,j \leq 5\}$. $N(B) = 11$. $P(B) = \frac{N(B)}{N} = \frac{11}{36}$.
- A=*"the other die turned up a face value of six"*
- $A \cap B$=*"the two dice turned up a face value of six"*$=$ $\{(6,6)\}$. $N(A \cap B) = 1$. $P(A \cap B) = \frac{N(A \cap B)}{N} = \frac{1}{36}$.
- Approach 1 [definition]: $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/36}{11/36} = \frac{1}{11}$.
- Approach 2 [B as Ω]: $P(A|B) = \frac{N(A \cap B)}{N(B)} = \frac{1}{11}$.

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Exercise

Someone has rolled two dice. You know that the first die turned up a face value of six. What is the probability that the second die turned up a six as well?

Assigning probabilities by conditioning

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Rule 5

For any $n \in \mathbb{N}$, $n \geq 2$, and any sequence of events $A_1, \ldots, A_n$,

$$P(A_1 \cap \ldots \cap A_n) = P(A_n | A_{n-1} \cap \ldots \cap A_1) P(A_{n-1} | A_{n-2} \cap \ldots \cap A_1) \ldots P(A_1)$$

Proof of rule 5

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Proof.

By induction.

$$P_2: \quad P(A_1 \cap A_2) = P(A_2|A_1)P(A_1)$$

$P_n \to P_{n+1}$:

$$\begin{aligned}
P(A_1 \cap A_2 \cap \ldots \cap A_n \cap A_{n+1}) &= P(B \cap A_{n+1}) \\
&= P(A_{n+1}|B)P(B) \\
&= P(A_{n+1}|A_n \cap A_{n-1} \cap \ldots A_1) \\
&\quad P(A_n \cap A_{n-1} \cap \ldots \cap A_1) \\
&= P(A_{n+1}|A_n \cap A_{n-1} \cap \ldots A_1) \\
&\quad P(A_n|A_{n-1} \cap \ldots \cap A_1) \\
&\quad P(A_{n-1}|A_{n-2} \cap \ldots \cap A_1) \ldots P(A_1)
\end{aligned}$$

□

Notes:

1. P_2 follows from the definition of conditional probability.
2. in the first equality in $P_n \to P_{n+1}$ we defined the event $B = A_1 \cap \ldots \cap A_n$.
3. in the second equality in $P_n \to P_{n+1}$ we used the definition of conditional probability.
4. in the third equality in $P_n \to P_{n+1}$ we replaced B by its definition.
5. in the fourth equality in $P_n \to P_{n+1}$ we used the inductive hypothesis $P(A_1 \cap \ldots \cap A_n) = P(A_n|A_{n-1} \cap \ldots \cap A_1)P(A_{n-1}|A_{n-2} \cap \ldots \cap A_1) \ldots P(A_1)$.

Example 8.3: allocating tourists to hotels

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Example 8.3

A group of 15 tourists is stranded in a city with four hotels of the same class. Each of the hotels has enough room available to accommodate the 15 tourists. The group's guide, who has a good working relationship with each of the four hotels, assigns the tourists to the hotels as follows. First, he randomly determines how many tourists will go to hotel A, then how many of the remaining tourists will go to hotel B, and next how many of the still remaining tourists will go to hotel C. All remaining tourists are sent to hotel D. At each stage of the assignment the guide draws a random number between zero and the number of tourists left.

a. *Calculate the probability of any assignment of tourists to hotels.*
b. *Check that the probability of all possible assignments equals one.*
c. *What is the probability that all four hotels receive guests?*
d. *Is the selected rule fair to the four hotels?*

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Analytical solution of example 8.3

Let i, j and k be the number of tourists assigned to hotels A, B an C respectively. Then

$$\begin{aligned}\Omega = \{(i, j, k, 15-(i+j+k)) : \; & 0 \le i \le 15,\\ & 0 \le j \le 15-i;\\ & 0 \le k \le 15-(i+j)\}\end{aligned}$$

(a) Define the events $E_j(k)$=“hotel W receives k tourists,” with $W \in \{A, B, C, D\}$. The assignment $(i, j, k, 15-(i+j+k))$ is the only member of the event $E_A(i) \cap E_B(j) \cap E_C(k) \cap E_D(15-(i+j+k))$. Then

$$\begin{aligned}P(\{(i, j, k, 15-(i+j+k))\}) &= P(E_A(i) \cap E_B(j) \cap E_C(k))\\ &= P(E_C(k)|E_A(i) \cap E_B(j))\; P(E_B(j)|E_A(i))\; P(E_A(i))\\ P(E_A(i)) &= \frac{1}{16}\\ P(E_B(j)|E_A(i)) &= \frac{1}{16-i}\\ P(E_C(k)|E_B(j) \cap E_A(i)) &= \frac{1}{16-(i+j)}\end{aligned}$$

Therefore

$$P(\{(i, j, k, 15-(i+j+k))\}) = \frac{1}{16-(i+j)}\,\frac{1}{16-i}\,\frac{1}{16}$$

Analytical solution of example 8.3

Let i, j and k be the number of tourists assigned to hotels A, B an C respectively. Then

$$\begin{aligned}\Omega = \{(i,j,k,15-(i+j+k)) : \; & 0 \le i \le 15,\\ & 0 \le j \le 15-i;\\ & 0 \le k \le 15-(i+j)\}\end{aligned}$$

b

$$\begin{aligned}
P(\Omega) &= \sum_{\omega\in\Omega} P(\omega) = \sum_{i=0}^{15}\sum_{j=0}^{15-i}\sum_{k=0}^{15-(i+j)} P(\{(i,j,k,15-(i+j+k))\})\\
&= \sum_{i=0}^{15}\sum_{j=0}^{15-i}\sum_{k=0}^{15-(i+j)} \frac{1}{16-(i+j)}\,\frac{1}{16-i}\,\frac{1}{16}\\
&= \frac{1}{16}\sum_{i=0}^{15}\frac{1}{16-i}\sum_{j=0}^{15-i}\frac{1}{16-(i+j)}\sum_{k=0}^{15-(i+j)} 1\\
&= \frac{1}{16}\sum_{i=0}^{15}\frac{1}{16-i}\sum_{j=0}^{15-i}\frac{1}{16-(i+j)}(16-(i+j))\\
&= \frac{1}{16}\sum_{i=0}^{15}\frac{1}{16-i}\sum_{j=0}^{15-i} 1 = \frac{1}{16}\sum_{i=0}^{15}\frac{1}{16-i}(16-i) = \frac{1}{16}\sum_{i=0}^{15} 1 = \frac{1}{16}16 = 1
\end{aligned}$$

Analytical solution of example 8.3

(c) Define the event E="all hotels receive guests." Then

$$E = \{(i, j, k, 15-(i+j+k)) : 1 \le i \le 12, \; 1 \le j \le 13-i; \; 1 \le k \le 14-(i+j)\}$$

$$\begin{aligned} P(E) &= \sum_{i=1}^{12} \sum_{j=1}^{13-i} \sum_{k=1}^{14-(i+j)} P(\{(i, j, k, 15-(i+j+k))\}) \\ &= \sum_{i=1}^{12} \sum_{j=1}^{13-i} \sum_{k=1}^{14-(i+j)} \frac{1}{16-(i+j)} \frac{1}{16-i} \frac{1}{16} \\ &= 0.2856 \end{aligned}$$

(d)

$$P(\{(15, 0, 0, 0)\}) = \frac{1}{16}$$

$$P(\{(0, 15, 0, 0)\}) = \frac{1}{16^2}$$

Thus, the selected rule is unfair to the four hotels.

Simulated solution of example 8.3c

Please see code here.

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Independent events

If event A is independent from event B, then knowing that B has occurred does not change the probability that event A occurs.

Definition 6

Two events A and B are independent if

$$P(A|B) = P(A)$$

Independent events

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If event A is independent from event B, then knowing that B has occurred does not change the probability that event A occurs.

Definition 6

Two events A and B are independent if

$$P(A|B) = P(A)$$

Claim

If events A and B are independent if and only if

$$P(A \cap B) = P(A)P(B)$$

Independent events

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If event A is independent from event B, then knowing that B has occurred does not change the probability that event A occurs.

Definition 6

Two events A and B are independent if

$$P(A|B) = P(A)$$

Claim

If events A and B are independent if and only if

$$P(A \cap B) = P(A)P(B)$$

Proof.

$\rightarrow$: $P(A \cap B) = P(A|B)P(B) = P(A)P(B)$

$\leftarrow$: $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$ $\square$

Example 8.5: independent dice events?

Example 8.5

Two fair dice are thrown. Let A be the event that the number shown in die one is even, and B the event that the sum of the two dice is odd. Are A and B independent?

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Analytical solution of example 8.5

The sample space is

$$
\begin{aligned}
\Omega &= \{(d_1, s) : d_1 \text{ number shown in die one, } s \text{ sum of the two dice}\} \\
&= \{(1,2),(1,3),\ldots,(1,7), \\
&\quad (2,3),(2,4),\ldots,(2,8), \\
&\quad \ldots, \\
&\quad (6,7),(6,8),\ldots,(6,12)\} \\
N(\Omega) &= 36, N(A) = N(B) = 18, N(A \cap B) = 9 \\
P(B) &= \frac{N(B)}{N(\Omega)} = \frac{18}{36} = \frac{1}{2} \\
P(B|A) &= \frac{N(A \cap B)}{N(A)} = \frac{9}{18} = \frac{1}{2}
\end{aligned}
$$

Since $P(B) = P(B|A)$, events A and B are independent.

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Exercise: independence-related proofs

Prove the following claims:

Claim (disjoint events are generally not independent)

Let A and B be two events and P a probability measure such that $P(A) \neq 0$ and $P(B) \neq 0$. Then, it is always true that the fact that A and B are disjoint implies that A and B are not independent.

Claim (symmetry of independence)

$P(A|B) = P(A)$ *if and only if* $P(B|A) = P(B)$.

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Conditionally independent events

Definition 7

Events A and B are conditionally independent given event C if and only if $P(A|B \cap C) = P(A|C)$.

Example on conditional independence

Example tossing a possibly unfair coin

Suppose both Martin and Norman toss the same coin. Let A be the event "Norman's toss resulted in heads", and B the event "Martin's toss resulted in heads." Assume also that there is a possibility that the coin in biased towards heads but we do not know this for certain. In this case events A and B are not independent. For example, observing that B occurred causes us to increase our belief that A occurred (i.e., $P(A|B) > P(A)$).

In the above, events A and B are dependent on event C, "the coin is biased towards heads." Although events A and B are not independent, it turns out that they are conditionally independent given event C, i.e., once we know for certain that event C occurred then any evidence about B cannot change our belief about A (i.e., $P(A|C) = P(A|B \cap C)$).

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Example tossing a possibly unfair coin

Let's assign probabilities to the events above and verify the claims that (a) events A and B are dependent (i.e., $P(A|B) > P(A)$) and that (b) events A and B are conditionally independent given event C (i.e., $P(A|B \cap C) = P(A|B)$).

a. *list the elements of the sample space Ω, and of events A, B and C.*
b. *assign probabilities to events containing only one element of Ω, so that*
    - *events A and B and events A and B^c are conditionally independent given that the coin is biased or given that the coin is unbiased,*
    - *the probability of heads given that the coin is biased is 0.7*
    - *the coin is equally likely biased or unbiaed.*
c. *with the previous assigned probabilities, check that events A and B are not independent but that they are conditionally independent given C.*

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Analytical solution of example on conditional independence

(a)

$$\begin{aligned}\Omega &= \{\text{HHU,HTU,THU,TTU,HHB,HTB,THB,TTB}\}\\ A &= \{\text{HHU,HTU,HHB,HHB}\}\\ B &= \{\text{HHU,THU,HHB,HHB}\}\\ C &= \{\text{HHB,HTB,THB,TTB}\}\end{aligned}$$

In the above, for example, HTB means that Norman tossed a heads, Martin tossed a tail and the coin was biased towards heads.

Analytical solution of example on conditional independence

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(b) Given the example setup we know that

$$
\begin{aligned}
P(A|C^c) = P(B|C^c) &= \frac{1}{2} \\
P(A|C) = P(B|C) &= \frac{1}{7} \\
P(C) = P(C^c) &= \frac{1}{2} \\
P(A|B, C^c) = P(A|B^c, C^c) &= P(A|C^c) \\
P(A|B, C) = P(A|B^c, C) &= P(A|C)
\end{aligned}
$$

Analytical solution of example on conditional independence

b. Next we will assign probabilities uing Rule 5,

$$P(\{HHU\}) = P(A \cap B \cap C^c) = P(A|B \cap C^c)P(B|C^c)P(C^c)$$
$$= P(A|C^c)P(B|C^c)P(C^c) = \frac{1}{2}\frac{1}{2}\frac{1}{2} = \frac{1}{8}$$

The second equality above uses Rule 5, and the third equality uses the fact that A is conditionally independent from B given C^c.
In the same way we can see that
$P(\{HTU\}) = P(\{THU\}) = P(\{TTU\}) = \frac{1}{8}$.

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Analytical solution of example on conditional independence

(b) For a biased coin we have

$$
\begin{aligned}
P(\{HHB\}) &= P(A \cap B \cap C) = P(A|B \cap C)P(B|C)P(C) \\
&= P(A|C)P(B|C)P(C) = \frac{7}{10}\frac{7}{10}\frac{1}{2} = \frac{49}{200} \\
P(\{HTB\}) &= P(A \cap B^c \cap C) = P(A|B^c \cap C)P(B^c|C)P(C) \\
&= P(A|C)P(B^c|C)P(C) = P(A|C)(1 - P(B|C))P(C) \\
&= \frac{7}{10}\frac{3}{10}\frac{1}{2} = \frac{21}{200} \\
P(\{THB\}) &= P(A^c \cap B \cap C) = P(A^c|B \cap C)P(B|C)P(C) \\
&= P(A^c|C)P(B|C)P(C) = (1 - P(A|C))P(B|C)P(C) \\
&= \frac{3}{10}\frac{7}{10}\frac{1}{2} = \frac{21}{200} \\
P(\{TTB\}) &= P(A^c \cap B^c \cap C) = P(A^c|B^c \cap C)P(B^c|C)P(C) \\
&= P(A^c|C)P(B^c|C)P(C) = (1 - P(A|C))(1 - P(B|C))P(C) \\
&= \frac{3}{10}\frac{3}{10}\frac{1}{2} = \frac{9}{200}
\end{aligned}
$$

Analytical solution of example on conditional independence

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b) Thus, the probabilities of single-outcome events are:

$\{HHU\}$	$\{HTU\}$	$\{THU\}$	$\{TTU\}$	$\{HHB\}$	$\{HTB\}$	$\{THB\}$	$\{TTB\}$
$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{49}{200}$	$\frac{21}{200}$	$\frac{21}{200}$	$\frac{9}{200}$

c)

$$
\begin{aligned}
P(A|B) = \frac{P(A \cap B)}{P(B)} &= \frac{P((A \cap B) \cap (C \cup C^c))}{P(B \cap ((A \cap C) \cup (A \cap C^c) \cup (A^c \cap C) \cup (A^c \cap C^c))} \\
&= \frac{P((A \cap B \cap C) \cup (A \cap B \cap C^c))}{P((B \cap A \cap C) \cup (B \cap A \cap C^c) \cup (B \cap A^c \cap C) \cup (B \cap A^c \cap C^c))} \\
&= \frac{P(A \cap B \cap C) + P(A \cap B \cap C^c)}{P(B \cap A \cap C) + P(B \cap A \cap C^c) + P(B \cap A^c \cap C) + P(B \cap A^c \cap C^c))} \\
&= \frac{P(\{HHB\}) + P(\{HHU\})}{P(\{HHB\}) + P(\{HHU\}) + P(\{THB\}) + P(\{THU\})} \\
&= \frac{\frac{49}{200} + \frac{1}{8}}{\frac{49}{200} + \frac{1}{8} + \frac{21}{200} + \frac{1}{8}} = \frac{\frac{74}{200}}{\frac{120}{200}} = \frac{74}{120} = 0.62
\end{aligned}
$$

$$P(A) = P(B) = \frac{120}{200} = 0.6 < 0.62 = P(A|B)$$ Therefore, A and B are not independent.

Analytical solution of example on conditional independence

- c

$$
\begin{aligned}
P(A|B \cap C) &= \frac{P(A \cap B \cap C)}{P(B \cap C)} = \frac{P(A \cap B \cap C)}{P((B \cap C) \cap (A \cup A^c))} = \frac{P(A \cap B \cap C)}{P((A \cap B \cap C) \cup (A^c \cap B \cap C))} \\
&= \frac{P(A \cap B \cap C)}{P(A \cap B \cap C) + P(A^c \cap B \cap C)} = \frac{P(\{HHB\})}{P(\{HHB\} + P(\{THB\})} = \frac{\frac{49}{200}}{\frac{49}{200} + \frac{21}{200}} = \frac{7}{10} \\
P(A|C) &= \frac{P(A \cap C)}{P(C)} = \frac{P((A \cap C) \cap (B \cup B^c))}{P(C \cap ((A \cap B) \cup (A \cap B^c) \cup (A^c \cap B) \cup (A^c \cap B^c))} \\
&= \frac{P((A \cap B \cap C) \cup (A \cup B^c \cap C))}{P((A \cap B \cap C) \cup (A \cap B^c \cap C) \cup (A^c \cap B \cap C) \cup (A^c \cap B^c \cap C))} \\
&= \frac{P(A \cap B \cap C) + P(A \cup B^c \cap C)}{P(A \cap B \cap C) + P(A \cap B^c \cap C) + P(A^c \cap B \cap C) + P(A^c \cap B^c \cap C)} \\
&= \frac{P(\{HHB\}) + P(\{HTB\})}{P(\{HHB\}) + P(\{HTB\}) + P(\{THB\}) + P(\{TTB\})} \\
&= \frac{\frac{49}{200} + \frac{21}{200}}{\frac{49}{200} + \frac{21}{200} + \frac{21}{200} + \frac{9}{200}} \\
&= \frac{\frac{70}{200}}{\frac{100}{200}} = \frac{7}{10} = P(A|B \cap C) \quad \text{Therefore, A and B are conditionally independent given C.}
\end{aligned}
$$

Example p86: dice followed by coin tosses

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Example page 266

A fair die is rolled to yield a number between one and six, and then a coin is tossed that many times. What is the probability that heads will not appear?

Analytical solution of example p86

$$\begin{aligned}
P(\text{no_heads}) &= P(\text{no_heads} \cap \Omega) \\
&= P(\text{no_heads} \cap (\text{die_equals_1} \cup \text{die_equals_2} \cup \cdots \cup \text{die_equals_6})) \\
&= P((\text{no_heads} \cap \text{die_equals_1}) \cup (\text{no_heads} \cap \text{die_equals_2}) \cup \ldots \cup (\text{no_heads} \cap \text{die_equals_6}))) \\
&= \sum_{i=1}^{6} P(\text{no_heads} \cap \text{die_equals_i}) = \sum_{i=1}^{6} P(\text{no_heads}|\text{die_equals_i})P(\text{die_equals_i}) \\
&= P(T)P(\text{die_equals_1}) + P(TT)P(\text{die_equals_2}) + \ldots + P(TTTTTT)P(\text{die_equals_6}) \\
&= \frac{1}{2}\frac{1}{6} + \left(\frac{1}{2}\right)^2 \frac{1}{6} + \ldots + \left(\frac{1}{2}\right)^6 \frac{1}{6} = 0.1640625
\end{aligned}$$

Simulated solution of example p86

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Law of conditional probability

Rule 6

Law of conditional probability *Let A be an event that can only occur if one of the mutually exclusive events $B_1, \ldots, B_n$ occurs. Then*

$$P(A) = P(A|B_1)P(B_1) + \ldots + P(A|B_n)P(B_n)$$

Exercise: proof of law of conditional probability

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Prove the law of conditional probability. Hint: use induction as in the proof of rule 5.

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Exercise: tour de France

Example 8.6

The upcoming Tour the France bicycle tournament will take place from July 1 through July 23. One hundred and eighty cyclists will participate in event. What is the probability that two or more participating cyclists will have birthdays on the same day during the tournament?

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Bayesian inference

- In Bayesian statistics parameters of a model, θ are considered random quantities. They are assigned a (subjective) probability $P(\theta)$.
- The probability of parameters is subject to change as new data arrives, $P(\theta|D)$.

$$P(\theta) \xrightarrow{\text{data observed}} P(\theta|D)$$

- How to calculate the posterior probability $P(\theta|D)$?

$$P(\theta|D) = \frac{P(\theta \cap D)}{P(D)}$$

$$P(\theta \cap D) = P(D|\theta)P(\theta)$$

$$P(\theta|D) = \frac{P(D|\theta)P(\theta)}{P(D)} \propto P(D|\theta)P(\theta)$$

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Bayesian inference

- Sequential update

$$\begin{aligned}
P(\theta|D_1 \cap D_2) &= \frac{P(\theta \cap D_1 \cap D_2)}{P(D_1 \cap D_2)} \\
&= \frac{P(D_2|D_1 \cap \theta)P(D_1 \cap \theta)}{P(D_1 \cap D_2)} \\
&= \frac{P(D_2|\theta)P(D_1|\theta)P(\theta)}{P(D_1 \cap D_2)} \\
&= K\ P(D_2|\theta)P(\theta|D_1) \propto P(D_2|\theta)P(\theta|D_1)
\end{aligned}$$

Notes:

1. the first and second equalities use the definition of conditional probability.
2. the third equality assumes that the datasets D_1 and D_2 are conditionally independent given θ.
3. the last equality groups all factors that don't depend on θ in the proportionality constant K.

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Exercise: sequential posterior probability for arbitrary number of observations

For any $n \in \mathbb{N}$, prove that

$$P(\theta|D_1 \cap \ldots \cap D_{n+1}) = P(D_{n+1}|\theta)P(\theta|D_1 \cap D_2 \cap \ldots \cap D_n)$$

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Example 8.13: disease treatment effect

Example 8.13

A new treatment is tried out for a disease. A standard treatment has a success probability of 35%. The discrete uniform distribution on 0.00, 0.01, . . . , 0.99, 1.00 is taken as prior for the success probability of the new treatment. The experiment design is to make exactly ten observations by treating ten patients. The experimental study yield seven successes and three failures (Berry, 2006).

a. *Plot the prior of the success probability of the new treatment.*
b. *Plot the likelihood of the success probability of the new treatment.*
c. *Plot the posterior of the success probability of the new treatment.*
d. *What is the posterior probability that the new treatment is more effective than the standard one?*

Analytical solution of example 8.13

a. prior:

$$p(\theta) = \frac{1}{101}, \quad \text{for } \theta = 0.00, 0.01, \ldots, 0.99, 1.00$$

Analytical solution of example 8.13

b) likelihood:

$$\Omega = \{(r_1, \ldots, r_{10}) : r_i \in \{S, F\}\}$$

Define the events $k\_successes$="experiments where k out of the 10 observations were successes", with $k \in \{0, 1, \ldots, 10\}$.

$$\begin{aligned}
0_\text{successes} &= \{(F, F, F, F, F, F, F, F, F, F)\} \\
1_\text{successes} &= \{(S, F, F, F, F, F, F, F, F, F), (F, S, F, F, F, F, F, F, F, F), \ldots, \\
&\quad (F, F, F, F, F, F, F, F, F, S)\} \\
2_\text{successes} &= \{(S, S, F, F, F, F, F, F, F, F), (S, F, S, F, F, F, F, F, F, F), \ldots, \\
&\quad (F, F, F, F, F, F, F, F, S, S)\}
\end{aligned}$$

Calculate the events probabilities

$$\begin{aligned}
P(0_successes|\theta) =& P(\{(F, F, F, F, F, F, F, F, F, F)\}) = (1-\theta)^{10} \\
P(1_successes|\theta) =& P(\{(S, F, F, F, F, F, F, F, F, F), (F, S, F, F, F, F, F, F, F, F), \ldots, \\
& (F, F, F, F, F, F, F, F, F, S)\}\}) \\
=& P(\{(S, F, F, F, F, F, F, F, F, F)\}) + P(\{(F, S, F, F, F, F, F, F, F, F)\}) + \ldots + \\
& P(\{(F, F, F, F, F, F, F, F, F, S)\}) = 10P(\{(S, F, F, F, F, F, F, F, F, F)\}) \\
=& 10 \;\; \theta(1-\theta)^9
\end{aligned}$$

Analytical solution of example 8.13

- b likelihood:

$$
\begin{aligned}
P(2\_successes|\theta) = & P(\{ \ (S,S,F,F,F,F,F,F,F,F),(S,F,S,F,F,F,F,F,F,F),\ldots, \\
& \quad (F,F,F,F,F,F,F,F,S,S)\}\}) \\
= & P(\{ \ (S,S,F,F,F,F,F,F,F,F)\})+ \\
& P(\{ \ (S,F,S,F,F,F,F,F,F,F)\})+\ldots+ \\
& P(\{ \ (F,F,F,F,F,F,F,F,S,S)\}) \\
= & \binom{10}{2} P(\{(S,F,F,F,F,F,F,F,F,F)\}) \\
= & \binom{10}{2} \theta^2(1-\theta)^8 \\
P(k\_successes|\theta) = & \binom{10}{k} \theta^k(1-\theta)^{10-k}, \quad k \in \{0,1,\ldots,10\}
\end{aligned}
$$

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c posterior:

$$P(\theta|7\_\text{successes}) = K_1 P(7\_\text{successes}|\theta)P(\theta) = K_1 \binom{10}{7} \theta^7 (1-\theta)^3 \frac{1}{101} = K_2 \theta^7 (1-\theta)^3$$

$$1 = \sum_{k=0}^{100} P(\theta_k|7\_\text{successes}) = K_2 \sum_{k=0}^{100} \theta_k^7 (1-\theta_k)^3; \quad \theta_k = \frac{k}{100}$$

$$K_2 = \frac{1}{\sum_{k=0}^{100} \theta_k^7 (1-\theta_k)^3}$$

$$P(\theta|7\_\text{successes}) = \frac{\theta^7 (1-\theta)^3}{\sum_{k=0}^{100} \theta_k^7 (1-\theta_k)^3}$$

d

$$P(\text{new treatment more effective than standard one}) = \sum_{k=36}^{100} P(\theta_k|7\_\text{successes})$$

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Exercise: sequential posterior probability

Continuing example 8.13, calculate the posterior probability of the new treatment after a second observation with six successes and a first observation with seven successes.

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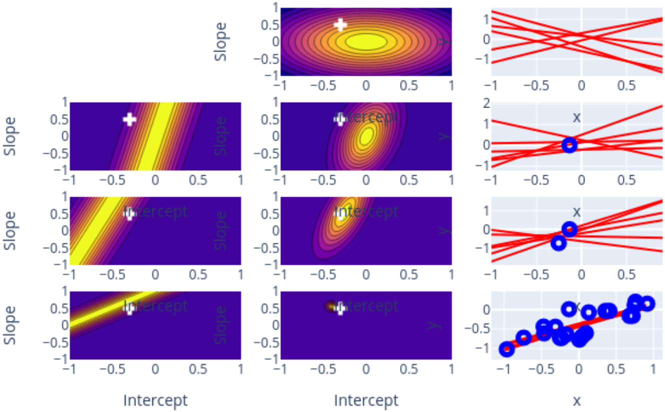

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