GATSBY

Gatsby Bridging Program

Probability: Discrete Distributions

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Gatsby Computational Neuroscience Unit

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Random Variables, Probability Mass Functions, and Cumulative Distribution Functions

What is a Random Variable?

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Example 1

Consider a bag containing 3 red balls (R) and 5 green balls (G). In our experiment, we are going to draw 2 balls from the bag, with replacement. The sample space is $\Omega = \{\mathrm{RR}, \mathrm{RG}, \mathrm{GR}, \mathrm{GG}\}$.

We are interested in determining the probabilities of drawing various numbers of red balls. To do this, we could start by defining a random variable $X : \Omega \to \{0, 1, 2\}$ such that

$$X(\omega) = \begin{cases} 0 & \text{if } \omega = \mathrm{GG} \\ 1 & \text{if } \omega \in \{\mathrm{RG}, \mathrm{GR}\} \\ 2 & \text{if } \omega = \mathrm{RR} \end{cases} .$$

What is a Random Variable?

Random variables are often utilised without any explicit reference to the sample space. Instead of writing $X(\omega)$, we will typically just write X. Instead of writing $\mathbb{P}(E)$ for some event E, we typically write $\mathbb{P}(X \in A)$ for some set of numerical values A.

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Example 1 Cont.

What is the probability of drawing one red ball? We could express this as $\mathbb{P}(\{\mathrm{RG}, \mathrm{GR}\})$, but it is common and accepted notation to instead write $\mathbb{P}(X \in \{1\})$, or $\mathbb{P}(0 < X < 2)$, or most preferably in this case $\mathbb{P}(X = 1)$.

$$\begin{aligned}\mathbb{P}(X = 1) &= \frac{3}{8}\frac{5}{8} + \frac{5}{8}\frac{3}{8} \\ &= \frac{15}{32}.\end{aligned}$$

The Chain Rule and Independence

In a previous lecture, we covered the following relationship for events E and F:

$$\begin{aligned}\mathbb{P}(E \cap F) &= \mathbb{P}(E \mid F)\,\mathbb{P}(F) \\ &= \mathbb{P}(F \mid E)\,\mathbb{P}(E)\ .\end{aligned}$$

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This also applies to random variables. Here we are looking at the probability of random variables X and Y taking values in sets A and B respectively:

Chain Rule for Two Random Variables

$$\begin{aligned}\overbrace{\mathbb{P}(X \in A, Y \in B)}^{\substack{\text{Joint}\\ \text{Probability}}} &= \overbrace{\mathbb{P}(X \in A \mid Y \in B)}^{\substack{\text{Conditional}\\ \text{Probability}}}\overbrace{\mathbb{P}(Y \in B)}^{\substack{\text{Marginal}\\ \text{Probability}}} \\ &= \mathbb{P}(Y \in B \mid X \in A)\,\mathbb{P}(X \in A)\end{aligned}$$

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A future lecture will cover joint, conditional, and marginal distributions and the chain rule, independence, and marginalisation in more detail.

The Chain Rule and Independence

In this lecture, we will only consider independent random variables. For two random variables to be independent, the realisation of one must have no effect on the distribution of the other. E.g. a coin flip is independent of the outcome of the previous coin flip.

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For N independent random variables, this generalises to:

Independence of N Random Variables

$$\mathbb{P}(X_1 \in A_1, \ldots, X_N \in A_N) = \prod_{n=1}^{N} \mathbb{P}(X_n \in A_n)$$

The Chain Rule and Independence

Example 1 Cont.

In our red ball example, we could also use separate random variables for the outcomes of each draw from the bag. Let $X_1 \in \{0, 1\}$ be the random variable representing the first draw and $X_2 \in \{0, 1\}$ be the random variable representing the second draw (0 for a green ball and 1 for red ball). Then it is straightforward to see that $X = X_1 + X_2$.

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Are X_1 and X_2 independent? Yes, they are independent random variables, as the first draw has no effect on the second draw.

What if we had the same experimental setup, but without replacing the ball after the first draw? In this case they are not independent, as the colour of the first drawn ball will dictate the probabilities of the second drawn ball.

Discrete and Continuous Random Variables

Discrete random variables can take a countable number of values. For example:

- $X \in \{0, 1\}$
- $X \in \{0, 1, 2, \dots\}$
- X representing the number of buses arriving within an hour.

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Continuous random variables can take values in continuous ranges. For example:

- $X \in [0, 1]$
- $X \in \mathbb{R}$
- X representing the waiting time until the next bus.

Probability Mass Functions

Discrete distributions can be defined by their probability mass function (PMF). The PMF of random variable X is often denoted by f_X or p_X, and is defined as:

Probability Mass Functions

$$f_X(x) = \mathbb{P}(X = x)$$

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f_X is simply a function name. It is fine to use a different name, as long as it is clear how the function is defined. Occasionally, the same name is used for PMFs if it is clear from context how these are defined, but I'd advise against this practice for the sake of clarity. E.g. it is clearer to write $p_X(x)$ and $p_Y(y)$ than $p(x)$ and $p(y)$ if these correspond to 2 different PMFs.

Probability Mass Functions

Probabilities can't be negative and must sum to 1 over the set of all possible values $\mathcal{X}$, so we have the following constraints:

$$f_X(x) \geq 0 \text{ for all } x$$

and

$$\sum_{x \in \mathcal{X}} f_X(x) = 1 .$$

$\mathcal{X}$ is referred to as the support of X. $f_X(x) = 0$ for $x \notin \mathcal{X}$.

Probability Mass Functions

Example 1 Cont.

Let's continue with our red ball example. What is the PMF of X? First, we recognise that $\mathcal{X} = \{0, 1, 2\}$. You can verify for yourself that

$$f_X(x) = \begin{cases} \frac{25}{64} & \text{if } x = 0 \\ \frac{15}{32} & \text{if } x = 1 \\ \frac{9}{64} & \text{if } x = 2 \\ 0 & \text{otherwise} \end{cases},$$

and that this function satisfies the conditions for a PMF on the previous slide.

Probability Mass Functions

Often, we use a $\sim$ to mean "is distributed as". It is very common to see the notation

$$X \sim f_x \, ,$$

which, in the discrete case, means that X has the PMF f_x. Variations of this notation exist, but there should never be any ambiguity about the distribution of a random variable when using a $\sim$.

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Sometimes you will see

$$X_1, \ldots, X_N \overset{\text{i.i.d.}}{\sim} f_x \, ,$$

which implies that all N random variables are independent and identically distributed (i.i.d.) with PMF f_x.

Cumulative Distribution Functions

A probability distribution can also be defined in terms of its cumulative distribution function (CDF). The CDF of a random variable X is a monotonically increasing function defined as:

Cumulative Distribution Functions

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Cumulative Distribution Functions

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For discrete random variables, we can relate this definition to the PMF as follows:

$$F_X(x) = \sum_{y \leq x} f_X(y) \ .$$

Cumulative Distribution Functions

As probabilities must sum to 1 over the support, we have the following emergent properties for CDFs:

$$\lim_{x \to -\infty} F_X(x) = 0$$

$$\lim_{x \to \infty} F_X(x) = 1$$

Cumulative Distribution Functions

As probabilities must sum to 1 over the support, we have the following emergent properties for CDFs:

$$\lim_{x \to -\infty} F_X(x) = 0$$

$$\lim_{x \to \infty} F_X(x) = 1$$

We can also see that the following is true:

$$\mathbb{P}(a < X \leq b) = F_X(b) - F_X(a) \ .$$

Cumulative Distribution Functions

Example 1 Cont.

Back to the red ball example. What is the CDF of X? Previously, we found that

$$f_X(x) = \begin{cases} \frac{25}{64} & \text{if } x = 0 \\ \frac{15}{32} & \text{if } x = 1 \\ \frac{9}{64} & \text{if } x = 2 \\ 0 & \text{otherwise} \end{cases}.$$

Hence, the CDF is

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{25}{64} & \text{if } 0 \leq x < 1 \\ \frac{55}{64} & \text{if } 1 \leq x < 2 \\ 1 & \text{if } x \geq 2 \end{cases}.$$

PDF and CDF Plots

Example 1 Cont.

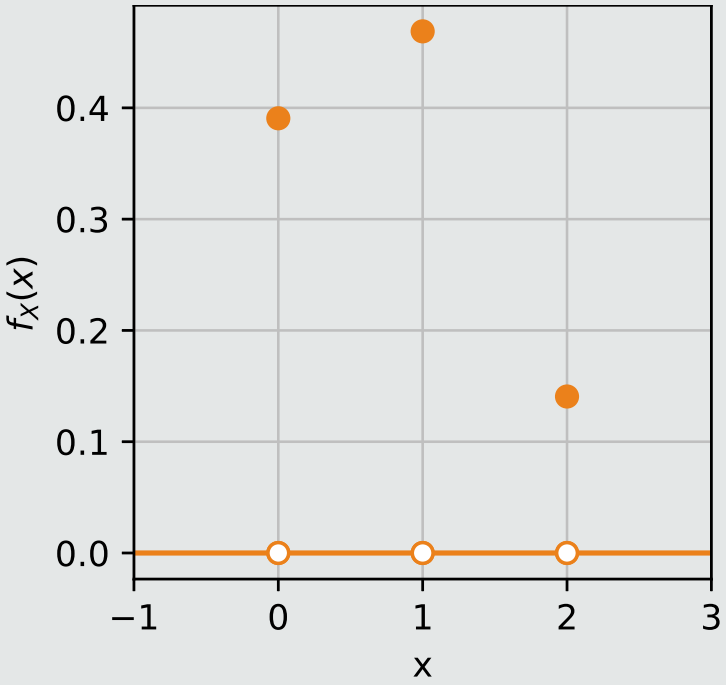

Figure 1: Probability Mass Function

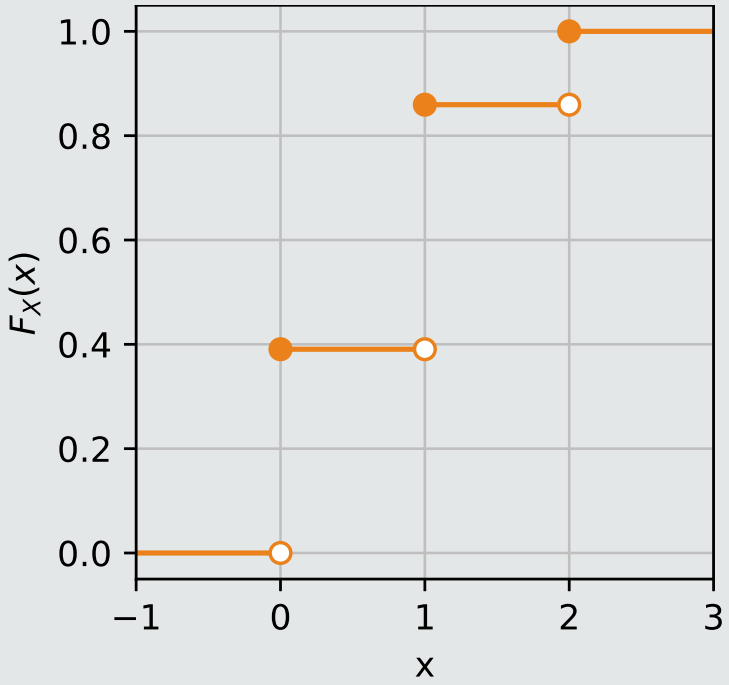

Figure 2: Cumulative Distribution Function

Sampling

Similarly to how we refer to realisations of random variables, we can also talk about samples of distributions. Sampling from a discrete distribution with PMF f_x simply implies that we randomly generate a number x with probability $f_x\left(x\right)$.

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To be entirely correct, drawing N samples from f_x involves running N random experiments with N associated random variables $X_1, \ldots, X_N \overset{\text{i.i.d.}}{\sim} f_x$ and generating N realisations $x_1, \ldots, x_N$.

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As $N \to \infty$, we will observe that

$$\frac{\sum_{n=1}^{N} [x_n = x]}{N} \to f_x(x)$$

Intermission

Expectation and Variance

Expectations

Suppose we want to know the average value of a function $g(x)$ evaluated at samples from a distribution. More precisely, we wish to draw realisations of $X_1, \ldots, X_N \overset{\text{i.i.d.}}{\sim} f_x$ and then compute the mean of $\{g(x_1), \ldots, g(x_N)\}$. As N increases, this will converge to a value which we call the expectation (or expected value). This theorem is referred to as the law of large numbers.

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We can estimate the expectation for finite N as follows:

Empirical Estimates of Expectations

$$\mathop{\mathbb{E}}_{X \sim f_X}[g(X)] \approx \frac{1}{N} \sum_{n=1}^{N} g(x_n) \qquad \text{for large } N$$

Expectations

If X is discrete with support $\mathcal{X}$ and PMF f_X, we can also define this expectation as a weighted average:

Expectations on Discrete Distributions

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When talking about the mean of a distribution defined by f_X, this specifically refers to the value given by

$$\mathop{\mathbb{E}}_{X \sim f_X} [X] \ .$$

Properties of Expectations

- Linearity:

$$\mathbb{E}\left[ag\left(X\right)+bh\left(X\right)\right]=a\mathbb{E}\left[g\left(X\right)\right]+b\mathbb{E}\left[h\left(X\right)\right]\ ,$$

  for constants a, b and functions g, h.

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Additional properties exist for multivariate distributions.

Expectations

Example 2

Suppose we roll a six-sided die. Let $X \in \{1, 2, 3, 4, 5, 6\}$ represent this roll. What is the expected value of X?

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$$f_X(x) = \begin{cases} \frac{1}{6} & \text{if } x \in \mathcal{X} \\ 0 & \text{otherwise} \end{cases}.$$

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$$f_X(x) = \begin{cases} \frac{1}{6} & \text{if } x \in \mathcal{X} \\ 0 & \text{otherwise} \end{cases}.$$

Then the expected value of X is given by

$$\begin{aligned} \mathbb{E}[X] &= \sum_{x \in \mathcal{X}} f_X(x)\, x \\ &= \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \frac{1}{6} \cdot 3 + \frac{1}{6} \cdot 4 + \frac{1}{6} \cdot 5 + \frac{1}{6} \cdot 6 \\ &= 3.5 \end{aligned}$$

Expectations

Why care about expectations?

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- A huge number of statistical estimation problems, where we try to estimate model parameters given some data, can be written in terms of expectations.
- This includes machine learning problems. Training a machine learning model involves minimising a loss function. The loss function and its gradient is very often written in terms of expectations.
- We can estimate these expectations simply by drawing samples!

Variance

How about if we want to know how ”spread out” a distribution is? A way of measuring this is with the variance. It measures the expected deviation from the mean.

Variance of a Random Variable

$$\begin{aligned}\operatorname{Var}(X) &= \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] \\ &= \mathbb{E}\left[X^2 - 2X\mathbb{E}[X] + \mathbb{E}[X]^2\right] \\ &= \mathbb{E}\left[X^2\right] - 2\mathbb{E}[X]\,\mathbb{E}[X] + \mathbb{E}[X]^2 \\ &= \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2\end{aligned}$$

Properties of Variance

- $\mathrm{Var}\left(aX + b\right) = a^2\mathrm{Var}\left(X\right)$ for constants a, b.

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Variance

Example 2 Cont.

Continuing with the die roll example. What is the variance of X?

Variance

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We will compute the variance of X in two separate ways. Method 1:

$$\begin{aligned}
\operatorname{Var}(X) &= \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] \\
&= \mathbb{E}\left[(X - 3.5)^2\right] \\
&= \sum_{x \in \mathcal{X}} f_X(x)(x - 3.5)^2 \\
&= \frac{1}{6}(1 - 3.5)^2 + \frac{1}{6}(2 - 3.5)^2 + \frac{1}{6}(3 - 3.5)^2 + \frac{1}{6}(4 - 3.5)^2 \\
&\quad + \frac{1}{6}(5 - 3.5)^2 + \frac{1}{6}(6 - 3.5)^2 \\
&= 2.91\dot{6}
\end{aligned}$$

Variance

Example 2 Cont.

Method 2:

$$\begin{aligned}
\operatorname{Var}(X) &= \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2 \\
&= \sum_{x \in \mathcal{X}} f_X(x)\, x^2 - 3.5^2 \\
&= \frac{1}{6} \cdot 1^2 + \frac{1}{6} \cdot 2^2 + \frac{1}{6} \cdot 3^2 + \frac{1}{6} \cdot 4^2 + \frac{1}{6} \cdot 5^2 + \frac{1}{6} \cdot 6^2 - 3.5^2 \\
&= 2.91\dot{6}
\end{aligned}$$

Notation

It is typical to see the notations μ, σ^2, and σ used for the mean, variance, and standard deviation of a distribution respectively. Clearly, the standard deviation is just the square root of the variance. In other words, we have:

$$\mu = \mathbb{E}[X]$$
$$\sigma^2 = \mathrm{Var}(X) \ .$$

Intermission

Important Discrete Distributions

Why Study Specific Distributions?

- Many real-world scenarios can be modelled with a few simple distributions. E.g. modelling the chance of success, the number of successes in a finite number of trials, the number of failures until a success, the waiting time until an event happens, and the uncertainty in an estimate after multiple measurements.

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- A large number of these simple distributions have very nice mathematical properties, including important relationships between them.

After the lecture on continuous distributions, I'd recommend doing some exploring of these distributions and their relationships.

Discrete Uniform Distribution

If a random variable takes any integer between a and b inclusive with equal probability, then it follows a discrete uniform distribution.

Discrete Uniform Distribution

Discrete uniform random variable $X \sim \mathcal{U}\{a, b\}$ has the following properties:

Parameters	$a, b \in \mathbb{Z} \mid b \geq a$	
Support	$\mathcal{X} = \{a, \ldots, b\}$	
PMF	$f_X(x) = \frac{1}{b-a+1}$	for $x \in \mathcal{X}$
Mean	$\mathbb{E}[X] = \frac{a+b}{2}$	
Variance	$\mathrm{Var}(X) = \frac{(b-a+1)^2-1}{12}$	

Discrete Uniform Distribution

Example 2 Cont.

Back to the die roll example. Is X uniformly distributed?

Discrete Uniform Distribution

Example 2 Cont.

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Yes, it is uniformly distributed with $a = 1$ and $b = 6$.

Discrete Uniform Distribution

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Back to the die roll example. Is X uniformly distributed?

Yes, it is uniformly distributed with $a = 1$ and $b = 6$. We can easily confirm our previous calculations with the formulas on the last slide:

$$\begin{aligned}\mathbb{E}[X] &= \frac{1+6}{2} \\ &= 3.5\end{aligned}$$

and

$$\begin{aligned}\operatorname{Var}(X) &= \frac{(6-1+1)^2-1}{12} \\ &= 2.91\dot{6}\,.\end{aligned}$$

Bernoulli Distribution

If a random variable takes the value 1 with probability p, and 0 otherwise, then it follows a Bernoulli distribution. It models the probability of "success".

Bernoulli Distribution

Bernoulli random variable $X \sim \text{Bern}(p)$ has the following properties:

Parameters	$p \in [0, 1]$	
Support	$\mathcal{X} = \{0, 1\}$	
PMF	$f_X(x) = p^x (1-p)^{1-x}$	for $x \in \mathcal{X}$
Mean	$\mathbb{E}[X] = p$	
Variance	$\text{Var}(X) = p(1-p)$	

Binomial Distribution

If $X_1, \ldots, X_N \overset{\text{i.i.d.}}{\sim} \text{Bern}(p)$ and $X = X_1 + \cdots + X_N$, then X is a binomial random variable. It models the number of successes in N Bernoulli trials.

Binomial Distribution

Binomial random variable $X \sim \text{Bin}(N, p)$ has the following properties:

Parameters	$n \in \{0, 1, 2 \ldots\}$ and $p \in [0, 1]$	
Support	$\mathcal{X} = \{0, \ldots, N\}$	
PMF	$f_X(x) = \binom{N}{x} p^x (1-p)^{N-x}$	for $x \in \mathcal{X}$
Mean	$\mathbb{E}[X] = Np$	
Variance	$\text{Var}(X) = Np(1-p)$	

Clearly, if $X \sim \text{Bin}(1, p)$ then X is also Bernoulli distributed.

Binomial Distribution

Why is the binomial PMF given by $\binom{N}{x} p^x \, (1-p)^{N-x}$?

Binomial Distribution

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In N trials we have x successes and $N-x$ failures. The $p^x \left(1-p\right)^{N-x}$ term gives the probability of observing the successes at specific times. E.g. the probability of running 3 trials and having only trials 1 and 3 produce successes.

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More precisely,

$$\begin{aligned}
\mathbb{P}\left(X_1 = x_1, \ldots, X_N = x_N\right) &= \prod_{n=1}^{N} \mathbb{P}\left(X_n = x_n\right) \\
&= \prod_{n=1}^{N} p^{x_n} \left(1-p\right)^{1-x_n} \\
&= p^{x_1 + \cdots + x_N} \left(1-p\right)^{N-(x_1+\cdots+x_N)} .
\end{aligned}$$

Binomial Distribution

However, we have to consider all possible ways in which x successes can occur. If we run 3 trials and get 2 successes, we have three different possibilities:

- Trials 1 and 2 were successes.

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- Trials 1 and 2 were successes.
- Trials 1 and 3 were successes.
- Trials 2 and 3 were successes.

In general, we have $\binom{N}{x}$ ways in which N trials can produce x successes. Adding the probabilities from these cases together gives the binomial PMF.

Binomial Distribution

Example 3

An avid card collector wishes to purchase 10 packs of trading cards in hopes of finding a rare card. A rare card is known to exist in 1% of card packs. No more than 1 rare card is ever in a pack. What is the probability the collector receives at least 1 rare card?

Binomial Distribution

Example 3

An avid card collector wishes to purchase 10 packs of trading cards in hopes of finding a rare card. A rare card is known to exist in 1% of card packs. No more than 1 rare card is ever in a pack. What is the probability the collector receives at least 1 rare card?

We can model the distribution of received rare cards with a binomial random variable. Let $X \sim \text{Bin}(10, 0.01)$ with PMF f_X. Then the probability of receiving at least 1 rare card is given by:

$$\begin{aligned}
\mathbb{P}(X \geq 1) &= 1 - \mathbb{P}(X = 0) \\
&= 1 - f_X(0) \\
&= 1 - \binom{10}{0} 0.01^0 (1 - 0.01)^{10-0} \\
&\approx 0.0956 .
\end{aligned}$$

Geometric Distribution

If $X_1, X_2, \ldots \overset{\text{i.i.d.}}{\sim} \text{Bern}(p)$ and $X = \min\{n \in \mathbb{N} \mid X_n = 1\}$, then X is a geometric random variable. It models the number of sequential Bernoulli trials performed until getting a success.

Geometric Distribution

Geometric random variable $X \sim \text{Geo}(p)$ has the following properties:

Parameters	$p \in [0, 1]$	
Support	$\mathcal{X} = \mathbb{N}$	
PMF	$f_X(x) = (1-p)^{x-1} p$	for $x \in \mathcal{X}$
Mean	$\mathbb{E}[X] = \frac{1}{p}$	
Variance	$\text{Var}(X) = \frac{1-p}{p^2}$	

Geometric Distribution

Example 3 Cont.

The card collector was disappointed to find no rare cards in the 10 opened packs. They decide to continue purchasing packs until they find a rare card. What is the expected number of packs they will open from this point onward?

Geometric Distribution

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The geometric distribution is perfect for modelling this problem. Let $Y \sim \mathrm{Geo}\,(0.01)$. Then the expected number of packs opened is simply

$$\begin{aligned}\mathbb{E}\,[Y] &= \frac{1}{0.01} \\ &= 100\,,\end{aligned}$$

which makes sense intuitively.

Geometric Distribution

Example 3 Cont.

The card collector was disappointed to find no rare cards in the 10 opened packs. They decide to continue purchasing packs until they find a rare card. What is the expected number of packs they will open from this point onward?

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which makes sense intuitively.

The collector opens 100 packs, but still doesn't find a rare card. What is the expected number of packs they will open from this point onward? Still 100; the geometric distribution is memoryless.

Poisson Distribution

If $X_1, \ldots, X_N \overset{\text{i.i.d.}}{\sim} \text{Bern}\left(\frac{\lambda}{N}\right)$, $X = X_1 + \cdots + X_N$, and $N \to \infty$, then X is a Poisson random variable. It models the number of independently occurring events in a fixed interval of time or space, where λ is the mean rate of occurrence.

Poisson Distribution

Poisson random variable $X \sim \text{Pois}(\lambda)$ has the following properties:

Parameters	$\lambda \in (0, \infty)$	
Support	$\mathcal{X} = \mathbb{N}$	
PMF	$f_X(x) = \frac{\lambda^x e^{-\lambda}}{x!}$	for $x \in \mathcal{X}$
Mean	$\mathbb{E}[X] = \lambda$	
Variance	$\text{Var}(X) = \lambda$	

Clearly, we can also say if $X \sim \text{Bin}\left(N, \frac{\lambda}{N}\right)$ and $N \to \infty$ then X is Poisson distributed. The proof of this is a problem set question.

Poisson Distribution

Example 4

Consider a single neuron in isolation, which we inject with some current. We find the timings of the spikes to be highly random and almost independent of one another. The timings aren't entirely independent (partly due to a refractory period after a spike), but we consider this a negligible factor. The mean firing rate is 2 Hz. What is the probability of observing at least 1 spike in a 3 second window?

Poisson Distribution

Example 4

Given our assumptions, we can model the number of spikes in this window with a Poisson distribution. Let $X \sim \text{Pois}(6)$ with PMF f_X, as the average number of spikes occurring in this window is 3×2. Then the probability of observing at least 1 spike is:

$$\begin{aligned}
\mathbb{P}(X \geq 1) &= 1 - \mathbb{P}(X = 0) \\
&= 1 - f_X(0) \\
&= 1 - \frac{6^0 e^{-6}}{0!} \\
&\approx 0.998\,.
\end{aligned}$$

Intermission

Introduction to Stochastic Processes