Probability: Functions of Random Variables Problem Set

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1 The CDF Method for One-Dimensional Codomains

1. Given $X \sim \text{Exp}(\lambda)$, what is the support and PDF of $Y = X^2$?
2. The log-normal distribution has support $[0, \infty)$, parameters $\mu \in \mathbb{R}, \sigma > 0$, PDF
$$f(x) = \frac{1}{x\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(\log(x) - \mu)^2}{2\sigma^2}\right),$$
and CDF
$$F(x) = \frac{1}{2}\left(1 + \text{erf}\left(\frac{\log(x) - \mu}{\sqrt{2}\sigma}\right)\right).$$
Here we denote it by $\mathcal{LN}(\mu, \sigma^2)$. Find the support and PDF of $Y = X^a$, where $X \sim \mathcal{LN}(\mu, \sigma^2)$ and $a > 0$. What can we say about the distribution of Y?
3. Given $X \sim \mathcal{U}[0, 2]$, what is the support and PDF of $Y = -(X - 1)^2$?
4. Use a convolution to find the PMF of $X + Y$ given independent $X \sim \text{Bin}(n, p)$ and $Y \sim \text{Bin}(m, p)$. What is this distribution? You may use Vandermonde's identity:
$$\sum_{x=0}^{y} \binom{n}{x}\binom{m}{y-x} = \binom{m+n}{y}.$$
5. What is the PDF for the sum of two i.i.d. exponential random variables?
6. Prove that the minimum of a set of independent exponential random variables is also exponential.

2 The Method of Transformations for Bijective Functions

1. $$f(x) = \frac{\lambda e^{-x}}{(1 + e^{-x})^{\lambda+1}}$$
is the PDF for the skew-logistic distribution. It has support $\mathbb{R}$ and a single parameter $\lambda > 0$. Find the PDF for $Y = \log\left(1 + e^{-X}\right)$, where X is skew-logistic. What is this distribution?
2. Use the method of transformations to solve question (1.2).
3. If $X \sim \text{Exp}(\lambda)$, $Y \sim \text{Exp}(\mu)$, and $Y = g(X)$, what is g? *Hint*: g takes the form $g(x) = ax$ for some constant a. What must a be?
4. Given $X \sim \mathcal{N}(0, 1)$ and $Y = X^3$, what is the PDF of Y? Seriously consider how the domain and codomain of this function must be restricted for the PDF of Y to make sense.
5. Given a 2D standard Gaussian with PDF $f_{\boldsymbol{X}}(\boldsymbol{x}) = \frac{1}{2\pi}\exp\left(-\frac{1}{2}\boldsymbol{x}^T\boldsymbol{x}\right)$, what is the PDF of
$$\boldsymbol{Y} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \boldsymbol{X}\,?$$
What does this imply about rotating a 2D standard Gaussian about the origin?

6. Consider some arbitrary multivariate Gaussian random variable $\boldsymbol{X} \in \mathbb{R}^n$ with PDF

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = (2\pi)^{-n/2} \det(\Sigma)^{-1/2} \exp\left(-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T \Sigma^{-1} (\boldsymbol{x}-\boldsymbol{\mu})\right).$$

What is the PDF of $\boldsymbol{Y} = A\boldsymbol{X}$, where A is an orthogonal matrix? What are the mean and covariance for $\boldsymbol{Y}$?