Foundations of probability theory

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Main reference

I will mainly follow chapters seven *Foundations of probability theory* and eight *Conditional probability and Bayes* from Tijms (2012).

Main lecture goals

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Foundations
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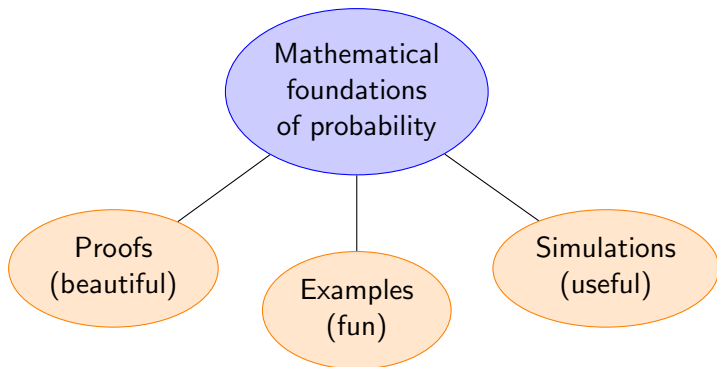Historical notes

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References

1. Foundations of probability theory

2. Conditional probability

3. Statistical inference

Historical notes

- explain the frequency-based interpretation of probability.
- constructing the mathematical foundations of probability
theory has proven to be a long-lasting process of trial an error.
- the approach of defining probability as relative frequencies of
repeatable experiments lead to unsatisfactory theory (why?)
https://www.jstor.org/stable/pdf/20115155.pdf
- the frequency view of probability has a long history that goes
back to Aristotle.
- in 1933 the Russian mathematician Andrej Kolmogrov
(1903-1987) laid a satisfactory mathematical foundation of
probability theory.
He created a set of axioms. Axioms state a number of minimal
requirements that the probability objects should satisfy. From
these few algorithms all claims of probability can be derived, as
we will see.

Probability model

Definition 1 (Probability model)

A **probability model** *is a mathematical representation of a random experiment.*

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*A **probability model** is a mathematical representation of a random experiment. It consists of a description of all possible outcomes of the experiment (i.e., **sample space**), a set of subsets of the sample space (i.e., **events**), and an assignment of probability to events (i.e., **probability measure**).*

Definition 2 (Sample space)

*The set of all samples in an experiment is called the **sample space**. It is denoted by Ω.*

Event

Definition 3 (Event)

*An **event** is a subset of the sample space. We denote the collection of all events by $\mathcal{F}$.*

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Notes:

1. We will only assign probabilities to events (i.e., to sets $A \in \mathcal{F}$).

2. For finite or countable sample spaces, we can assign probabilities to any subset of the sample space. Thus, any subset of a finite or countable sample space can be an event.

3. For uncountable sample spaces, we can only assign probabilities to well behaved subsets of the sample space (i.e., to elements in a σ algebra of subsets of the sample space). Only well-behaved subsets of an uncountable sample space can be events.

Probability measure

Definition 4 (Probability measure)

A **probability measure** *is a function that assigns numbers between zero and one to events (i.e., $P : \mathcal{F} \rightarrow [0, 1]$).*

Probability model: definitions

Definition (Sample space)

The set of all samples in an experiment is called the **sample space***. It is denoted by* Ω.

Definition (Event)

An **event** *is a subset of the sample space. We denote the collection of all events by* $\mathcal{F}$.

Definition (Probability measure)

A **probability measure** *is a function that assigns numbers between zero and one to events (i.e.,* $P : \mathcal{F} \rightarrow [0, 1]$*).*

Definition (Probability model)

A **probability model**, $\mathcal{M}$, *is a mathematical representation of a random experiment consisting of a sample space,* Ω, *a set of events,* $\mathcal{F}$, *and a probability measure,* P *(i.e.,* $\mathcal{M} = \{\Omega, \mathcal{F}, P\}$*).*

Examples of probability models

For each of the following examples, let's find the sample space and propose a probability measure.

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2. The experiment is to repeatedly roll a fair die and count the number of rolls until the first six shows up.

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Examples of probability models

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2. The experiment is to repeatedly roll a fair die and count the number of rolls until the first six shows up. The sample space is the set of positive integers. The probabilities $\frac{1}{6}, \frac{5}{6} \times \frac{1}{6}, \left(\frac{5}{6}\right)^2 \times \frac{1}{6}, \ldots$ can be assigned to the outcomes $1, 2, 3, \ldots$.

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3. The experiment is to measure the time between the first and second spikes in an experimental trial.

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For each of the following examples, let's find the sample space and propose a probability measure.

1. The experiment is to toss a fair coin once. The sample space is the set $[H, T]$. We assign a probability of 0.5 to each element of the sample space.

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3. The experiment is to measure the time between the first and second spikes in an experimental trial. The sample space is the set $(0, \infty)$ of positive real numbers. We can assign a probability of $1 - \exp(-\lambda t)$ to the event that the second spike is fired less than t seconds after the first spike.

Axioms of probability theory

Axiom 1

$P(A) \geq 0, \quad \forall A \in \mathcal{F}$

Axiom 2

$P(\Omega) = 1$

Axiom 3

$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$ for every collection of pairwise disjoint events $A_1, A_2, \ldots$

Some basic rules

Rule 1

For any finite number of mutually exclusive events $A_1, \ldots, A_N$,

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + \ldots + P(A_N)$$

Rule 2

For any event A,

$$P(A) = 1 - P(A^c)$$

where the event A^c consists of all outcomes that are not in A.

Rule 3

Let A, B be two events such that $A \subseteq B$. Then $P(A) \leq P(B)$.

Rule 4

For any two events A and B,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

An experiment with equally likely outcomes is one with a finite number of outcomes $\omega_1, \ldots, \omega_N$, where all outcomes have the same probability (i.e., $P(\omega_i) = \frac{1}{N}$).

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Claim 1

For any event A, $P(A) = \frac{N(A)}{N}$, where $N(A)$ is the number of outcomes in the set A.

Experiment with equally likely outcomes

An experiment with equally likely outcomes is one with a finite number of outcomes $\omega_1, \ldots, \omega_N$, where all outcomes have the same probability (i.e., $P(\omega_i) = \frac{1}{N}$).

Claim 1

For any event A, $P(A) = \frac{N(A)}{N}$, where $N(A)$ is the number of outcomes in the set A.

Proof.

$$A = \{a_1, \ldots, a_{N(A)}\} = \{a_1\} \cup \ldots \cup \{a_{N(A)}\}$$

$$P(A) = P(\{a_1\}) + \ldots + P(\{a_{N(A)}\}) = \sum_{i=1}^{N(A)} P(\{a_i\})$$

$$= \sum_{i=1}^{N(A)} \frac{1}{N} = \frac{N(A)}{N}$$

$\square$

Example 7.1

John, Pedro and Rosita each roll on fair die. How do we calculate the probability that the score of Rosita is equal to the sum of the scores of John and Pedro?

$$\Omega = \{(J, P, R) : 1 \le J, P, R \le 6\}$$
$$N = 6^3 = 216$$
$$A = \{(J, P, R) \in \Omega : R = J + P\}$$
$$= \{(1, 1, 2), (1, 2, 3), (1, 3, 4), (1, 4, 5), (1, 5, 6),$$
$$(2, 1, 3), (2, 2, 4), (2, 3, 5), (2, 4, 6),$$
$$(3, 1, 4), (3, 2, 5), (3, 3, 6), (4, 1, 5), (4, 2, 6), (5, 1, 6)\}$$
$$N(A) = 15$$
$$P(A) = \frac{N(A)}{N} = \frac{15}{216}$$

Rule 1

Proof.

Denote by $\emptyset$ the empty set. We will first prove that $P(\emptyset) = 0$. Take $A_i = \emptyset$ for $i = 1, 2, \ldots$. Then $\emptyset = \bigcup_{i=1}^{\infty} A_i$. Next, by Axiom 3, $P(\emptyset) = \sum_{i=1}^{\infty} P(A_i) = \sum_{i=1}^{\infty} P(\emptyset)$. This implies that $P(\emptyset) = 0$.

Define $A_{N+j} = \emptyset$ for $j = 1, 2, \ldots$. Then

$$P(A_1 \cup A_2 \cup \ldots \cup A_N) = P(A_1 \cup A_2 \cup \ldots \cup A_N \cup A_{N+1} \cup A_{N+2} \cup \ldots)$$

$$= P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$$

$$= \sum_{i=1}^{N} P(A_i) + \sum_{j=1}^{\infty} P(A_{N+j}) = \sum_{i=1}^{N} P(A_i)$$

□

Notes:

1. the last equality in the second line holds by Axiom 1

2. the last equality in the third line holds because $P(A_{N+j}) = P(\emptyset) = 0$.

Proof of rule 2

Proof.

$\Omega = A \cup A^c$. A and A^c are disjoint. Then, by Rule 1,
$P(\Omega) = P(A) + P(A^c)$. From Axiom 2, $P(\Omega) = 1$. Thus,
$1 = P(A) + P(A^c)$.

$\square$

Exercise

Prove rule 3.

Proof of rule 4

Proof.

$$A \cup B = (A \setminus B) \cup (B \setminus A) \cup (A \cap B)$$
$$A = (A \setminus B) \cup (A \cap B)$$
$$B = (B \setminus A) \cup (A \cap B)$$

Since the sets in the right-hand-side of the above equations are pairwise disjoint, by rule 1, we obtain

$$P(A \cup B) = P(A \setminus B) + P(B \setminus A) + P(A \cap B)$$
$$P(A) = P(A \setminus B) + P(A \cap B) \rightarrow P(A \setminus B) = P(A) - P(A \cap B)$$
$$P(B) = P(B \setminus A) + P(A \cap B) \rightarrow P(B \setminus A) = P(B) - P(A \cap B)$$

Replacing the equations on the right of the second and third line in the equation on the first line the rule is proved.

$$P(A \cup B) = P(A) - P(A \cap B) + P(B) - P(A \cap B) + P(A \cap B)$$
$$= P(A) + P(B) - P(A \cap B)$$

□

Example 7.7: Chevalier de Mere to Blaise Pascal

The gambler Chevalier de Mere posed the following problem to the famous French mathematician Blaise Pascal in 1654. This problem marks the beginning of probability theory.

Example 7.7

- *How many rolls of a fair die are required to have at least a 50% chance of rolling at least one six?*
- *How many rolls of two fair dice are required to have at least a 50% chance of rolling at least one double six?*

(a)

- Let's fix the number of rolls r. The sample space is
  $\Omega = \{(i_1, \ldots, i_r) : 1 \leq i_k \leq 6\}$, where i_k is the up face of
  the die on the kth roll. The outcomes in Ω are
  equiprobable.

(a)

- Let's fix the number of rolls r. The sample space is $\Omega = \{(i_1, \ldots, i_r) : 1 \leq i_k \leq 6\}$, where i_k is the up face of the die on the kth roll. The outcomes in Ω are equiprobable.

- We want to calculate the probability of the event $A=$"at least one six shows up in the r rolls.". When you see the keyword "at least one" in an event, it is easier to calculate the probability of the complement $A^c=$"no six shows up in the r rolls."

Analytical solution of example 7.7a

(a)

- Let's fix the number of rolls r. The sample space is $\Omega = \{(i_1, \ldots, i_r) : 1 \leq i_k \leq 6\}$, where i_k is the up face of the die on the kth roll. The outcomes in Ω are equiprobable.

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- $P(A^c) = \frac{N(A^c)}{N} = \frac{5^r}{6^r} = \left(\frac{5}{6}\right)^r$.

(a)

- Let's fix the number of rolls r. The sample space is $\Omega = \{(i_1, \ldots, i_r) : 1 \leq i_k \leq 6\}$, where i_k is the up face of the die on the kth roll. The outcomes in Ω are equiprobable.

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- $P(A^c) = \frac{N(A^c)}{N} = \frac{5^r}{6^r} = \left(\frac{5}{6}\right)^r$.

- $\frac{1}{2} < P(A) = 1 - P(A^c) = 1 - \left(\frac{5}{6}\right)^r$ iff $\left(\frac{5}{6}\right)^r < \frac{1}{2}$ iff $\log\left(\frac{5}{6}\right)^r < \log\frac{1}{2}$ iff $r\log\frac{5}{6} < \log\frac{1}{2}$ iff $r > \frac{\log\frac{1}{2}}{\log\frac{5}{6}} = 3.8$

Please see code here.

Solve example 7.7b analytically and by simulation. Answer: you
need at least 25 draws.

Example 7.9

The eight soccer teams which have reached the quarterfinals of the Championship League are formed by two teams from of the the countries England, Germany, Italy and Spain. The four matches to be played in the quarterfinal are determined by drawing lots.

a *What is the probability that the two teams from the same country play against each other in each of the four matches?*

b *What is the probability that there is a match between the two teams from England or between the two teams from Germany?*

Analytical solution of example 7.9: setup

- Let T be the set of eight teams
  $T = \{E_1, E_2, G_1, G_2, I_1, I_2, S_1, S_2\}$

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  match $= \{T_i, T_j\}$, match $\subset T$, $T_i \neq Tj$).

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  $T = \{E_1, E_2, G_1, G_2, I_1, I_2, S_1, S_2\}$
- A match is a set of two different elements from T (i.e.,
  match $= \{T_i, T_j\}$, match $\subset T$, $T_i \neq Tj$).
- A quarterfinal is a set of four different matches.
- Ω is the set of all possible quarterfinals (e.g.,
  $\{\{E1, G1\}, \{I1, G2\}, \{S2, G2\}, \{I2, G1\}\} \in \Omega$).

- Let T be the set of eight teams
  $T = \{E_1, E_2, G_1, G_2, I_1, I_2, S_1, S_2\}$
- A match is a set of two different elements from T (i.e., match $= \{T_i, T_j\}$, match $\subset T$, $T_i \neq Tj$).
- A quarterfinal is a set of four different matches.
- Ω is the set of all possible quarterfinals (e.g., $\{\{E1, G1\}, \{I1, G2\}, \{S2, G2\}, \{I2, G1\}\} \in \Omega$).
- Ω contains equally-likely outcomes. Thus, for any event A, $P(A) = \frac{N(A)}{N}$.

- Let T be the set of eight teams
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- Ω is the set of all possible quarterfinals (e.g.,
  $\{\{E1, G1\}, \{I1, G2\}, \{S2, G2\}, \{I2, G1\}\} \in \Omega$).
- Ω contains equally-likely outcomes. Thus, for any event A,
  $P(A) = \frac{N(A)}{N}$.
- $N = \frac{\binom{8}{2}\binom{6}{2}\binom{4}{2}}{4!}$. There are $\binom{8}{2}$, $\binom{6}{2}$, $\binom{4}{2}$ and 1 ways of
  selecting the first, second, third and fourth matches,
  respectively. We divide by 4! because the order between
  matches does not matter.

- The event A=*"two teams from the same country play against each other in each of the four matches"* contains only one outcome (i.e.,
  $A = \{\{E_1, E_2\}, \{G_1, G_2\}, \{I_1, I_2\}, \{S_1, S_2\}\}$).

Analytical solution of example 7.9a

- The event A="*two teams from the same country play against each other in each of the four matches*" contains only one outcome (i.e., $A = \{\{E_1, E_2\}, \{G_1, G_2\}, \{I_1, I_2\}, \{S_1, S_2\}\}$).

- $P(A) = \frac{N(A)}{N} = \frac{1}{\frac{\binom{8}{2}\binom{6}{2}\binom{4}{2}}{4!}} = 0.009524$

Analytical solution of example 7.9b

- The event $B=$"*there is a match between the two teams from England or between the two teams from Germany*" is the union of the events $B_E=$"*there is a match between the two teams from England*" and $B_G=$"*there is a match between the two teams from Germany*" (i.e., $B = B_E \cup B_G$).

Analytical solution of example 7.9b

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- Since B_E and B_G are not disjoint, we should use Rule 3 to compute $P(B_E \cup B_G)$ (i.e., $P(B_E \cup B_G) = P(B_E) + P(B_G) - P(B_E \cap B_G)$).

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- $P(B_E) = P(B_G)$. Since the match $\{E_1, E_2\}$ is in all quarterfinals in B_E, to calculate $N(B_E)$ we need to computer the number of matches between six teams of three countries, as done in part (a). This gives $N(B_E) = \frac{\binom{6}{2}\binom{4}{2}}{3!}$.

  Then $P(B_E) = P(B_G) = \frac{\frac{\binom{6}{2}\binom{4}{2}}{3!}}{\frac{\binom{8}{2}\binom{6}{2}\binom{4}{2}}{4!}} = 0.142857$.

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  Then $P(B_E) = P(B_G) = \frac{\frac{\binom{6}{2}\binom{4}{2}}{3!}}{\frac{\binom{8}{2}\binom{6}{2}\binom{4}{2}}{4!}} = 0.142857$.

- Since the match $\{E_1, E_2\}$ and $\{G_1, G_2\}$ are in all quarterfinals in $B_E \cap B_G$, as in part (a), $N(B_E \cap B_G) = \frac{\binom{4}{2}}{2!}$. Then

  $P(B_E \cap B_G) = \frac{\frac{\binom{4}{2}}{2!}}{\frac{\binom{8}{2}\binom{6}{2}\binom{4}{2}}{4!}} = 0.028571$.

Analytical solution of example 7.9b

- The event $B=$"*there is a match between the two teams from England or between the two teams from Germany*" is the union of the events $B_E=$"*there is a match between the two teams from England*" and $B_G=$"*there is a match between the two teams from Germany*" (i.e., $B = B_E \cup B_G$).

- Since B_E and B_G are not disjoint, we should use Rule 3 to compute $P(B_E \cup B_G)$ (i.e., $P(B_E \cup B_G) = P(B_E) + P(B_G) - P(B_E \cap B_G)$).

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  Then $P(B_E) = P(B_G) = \frac{\frac{\binom{6}{2}\binom{4}{2}}{3!}}{\frac{\binom{8}{2}\binom{6}{2}\binom{4}{2}}{4!}} = 0.142857$.

- Since the match $\{E_1, E_2\}$ and $\{G_1, G_2\}$ are in all quarterfinals in $B_E \cap B_G$, as in part (a), $N(B_E \cap B_G) = \frac{\binom{4}{2}}{2!}$. Then

  $P(B_E \cap B_G) = \frac{\frac{\binom{4}{2}}{2!}}{\frac{\binom{8}{2}\binom{6}{2}\binom{4}{2}}{4!}} = 0.028571$.

- Thus $P(B_E \cup B_G) = P(B_E) + P(B_G) - P(B_E \cap B_G) = 2 \times 0.142857 - 0.028571 = 0.257143$.

Please see code here.

Solve example 7.7b by simulation. Answer: you should obtain a solution close to the analytical one.

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- We are interested in the probability of event $A \in \mathcal{F}$. This model provides us the unconditioned probability of A, $P(A)$.
- Our colleague performs an experiment and tells us that event B occurred. How does the fact that B occurred changes the probability of A?

$$P(A) \xrightarrow{\text{B occurred}} P(A|B)$$

- We are given a probability model $(\Omega, \mathcal{F}, P)$.
- We are interested in the probability of event $A \in \mathcal{F}$. This model provides us the unconditioned probability of A, $P(A)$.
- Our colleague performs an experiment and tells us that event B occurred. How does the fact that B occurred changes the probability of A?

$$P(A) \xrightarrow{\quad B \text{ occurred} \quad} P(A|B)$$

- Definition 1: $P(\cdot|B) = P(\cdot \cap B)$, where $\cdot$ can be any event.

- We are given a probability model $(\Omega, \mathcal{F}, P)$.
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- Definition 2: $P(\cdot|B) = \frac{P(\cdot \cap B)}{P(B)}$.

Definition of conditional probability: motivation

- We are given a probability model $(\Omega, \mathcal{F}, P)$.
- We are interested in the probability of event $A \in \mathcal{F}$. This model provides us the unconditioned probability of A, $P(A)$.
- Our colleague performs an experiment and tells us that event B occurred. How does the fact that B occurred changes the probability of A?

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  - $P(B|B) = 1$ ✓

- We are given a probability model $(\Omega, \mathcal{F}, P)$.
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- Definition 1: $P(\cdot|B) = P(\cdot \cap B)$, where $\cdot$ can be any event.
- Problem with Definition 1: we want $P(B|B) = 1$, but from Definition 1, $P(B|B) = P(B \cap B) = P(B) \leq 1$.
- Definition 2: $P(\cdot|B) = \frac{P(\cdot \cap B)}{P(B)}$.
  - $P(B|B) = 1$ ✓
  - $P(A) = P(A|\Omega) = \frac{P(A \cap \Omega)}{P(\Omega)} = \frac{P(A)}{1} = P(A)$ ✓

Definition 5 (Conditional probability)

For any to events A and B, with $P(B) > 0$, the conditional probability of A given B, $P(A|B)$, is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Example 8.1

Someone has rolled two dice. You know that one of the dice turned up a face value of six. What is the probability that the other die turned up a six as well?

- $\Omega = \{(i_1, i_2) : 1 \le i_1, i_2 \le 6\}$. $N = 36$. Equally probable outcomes.

Analytical solution of example 8.1

- $\Omega = \{(i_1, i_2) : 1 \leq i_1, i_2 \leq 6\}$. $N = 36$. Equally probable outcomes.
- $B =$ "*one die turned up a face value of six*" $=$ $\{(6, i), (j, 6), (6, 6) : 1 \leq i, j \leq 5\}$. $N(B) = 11$. $P(B) = \frac{N(B)}{N} = \frac{11}{36}$.

Analytical solution of example 8.1

- $\Omega = \{(i_1, i_2) : 1 \leq i_1, i_2 \leq 6\}$. $N = 36$. Equally probable outcomes.
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- $A =$ "*the other die turned up a face value of six*"

- $A \cap B =$ "*the two dice turned up a face value of six*" $=$ $\{(6, 6)\}$. $N(A \cap B) = 1$. $P(A \cap B) = \frac{N(A \cap B)}{N} = \frac{1}{36}$.

- $\Omega = \{(i_1, i_2) : 1 \leq i_1, i_2 \leq 6\}$. $N = 36$. Equally probable outcomes.
- $B =$ "*one die turned up a face value of six*" $=$ $\{(6, i), (j, 6), (6, 6) : 1 \leq i, j \leq 5\}$. $N(B) = 11$. $P(B) = \frac{N(B)}{N} = \frac{11}{36}$.
- $A =$ "*the other die turned up a face value of six*"
- $A \cap B =$ "*the two dice turned up a face value of six*" $=$ $\{(6, 6)\}$. $N(A \cap B) = 1$. $P(A \cap B) = \frac{N(A \cap B)}{N} = \frac{1}{36}$.
- Approach 1 [definition]: $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/36}{11/36} = \frac{1}{11}$.

- $\Omega = \{(i_1, i_2) : 1 \le i_1, i_2 \le 6\}$. $N = 36$. Equally probable outcomes.
- $B =$ "one die turned up a face value of six" $=$ $\{(6, i), (j, 6), (6, 6) : 1 \le i, j \le 5\}$. $N(B) = 11$. $P(B) = \frac{N(B)}{N} = \frac{11}{36}$.
- $A =$ "the other die turned up a face value of six"
- $A \cap B =$ "the two dice turned up a face value of six" $=$ $\{(6, 6)\}$. $N(A \cap B) = 1$. $P(A \cap B) = \frac{N(A \cap B)}{N} = \frac{1}{36}$.
- Approach 1 [definition]: $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/36}{11/36} = \frac{1}{11}$.
- Approach 2 [B as Ω]: $P(A|B) = \frac{N(A \cap B)}{N(B)} = \frac{1}{11}$.

Simulated solution of example 8.1

Please see code here.

Someone has rolled two dice. You know that the first die
turned up a face value of six. What is the probability that the
second die turned up a six as well?

Assigning probabilities by conditioning

Rule 5

For any $n \in \mathbb{N}$, $n \geq 2$, and any sequence of events $A_1, \ldots, A_n$,

$$P(A_1 \cap \ldots \cap A_n) = P(A_n|A_{n-1} \cap \ldots \cap A_1)P(A_{n-1}|A_{n-2} \cap \ldots \cap A_1) \ldots P(A_1)$$

Proof of rule 5

Proof.

By induction.

$$P_2: \quad P(A_1 \cap A_2) = P(A_2|A_1)P(A_1)$$

$P_n \rightarrow P_{n+1}:$

$$
\begin{aligned}
P(A_1 \cap A_2 \cap \ldots \cap A_n \cap A_{n+1}) &= P(B \cap A_{n+1}) \\
&= P(A_{n+1}|B)P(B) \\
&= P(A_{n+1}|A_n \cap A_{n-1} \cap \ldots A_1) \\
&\quad P(A_n \cap A_{n-1} \cap \ldots \cap A_1) \\
&= P(A_{n+1}|A_n \cap A_{n-1} \cap \ldots A_1) \\
&\quad P(A_n|A_{n-1} \cap \ldots \cap A_1) \\
&\quad P(A_{n-1}|A_{n-2} \cap \ldots \cap A_1) \ldots P(A_1)
\end{aligned}
$$

□

Notes:

1. P_2 follows from the definition of conditional probability.
2. in the first equality in $P_n \rightarrow P_{n+1}$ we defined the event $B = A_1 \cap \ldots \cap A_n$.
3. in the second equality in $P_n \rightarrow P_{n+1}$ we used the definition of conditional probability.
4. in the third equality in $P_n \rightarrow P_{n+1}$ we replaced B by its definition.
5. in the fourth equality in $P_n \rightarrow P_{n+1}$ we used the inductive hypothesis
   $P(A_1 \cap \ldots \cap A_n) = P(A_n|A_{n-1} \cap \ldots \cap A_1)P(A_{n-1}|A_{n-2} \cap \ldots \cap A_1) \ldots P(A_1)$.

Example 8.3: allocating tourists to hotels

Example 8.3

A group of 15 tourists is stranded in a city with four hotels of the same class. Each of the hotels has enough room available to accommodate the 15 tourists. The group's guide, who has a good working relationship with each of the four hotels, assigns the tourists to the hotels as follows. First, he randomly determines how many tourists will go to hotel A, then how many of the remaining tourists will go to hotel B, and next how many of the still remaining tourists will go to hotel C. All remaining tourists are sent to hotel D. At each stage of the assignment the guide draws a random number between zero and the number of tourists left.

a *Calculate the probability of any assignment of tourists to hotels.*

b *Check that the probability of all possible assignments equals one.*

c *What is the probability that all four hotels receive guests?*

d *Is the selected rule fair to the four hotels?*

Let i, j and k be the number of tourists assigned to hotels A, B an C respectively. Then

$$\Omega = \{(i, j, k, 15 - (i + j + k)) : 0 \leq i \leq 15,$$
$$0 \leq j \leq 15 - i;$$
$$0 \leq k \leq 15 - (i + j)\}$$

ⓐ Define the events $E_j(k)=$"hotel W receives k tourists," with $W \in \{A, B, C, D\}$.
The assignment $(i, j, k, 15 - (i + j + k))$ is the only member of the event
$E_A(i) \cap E_B(j) \cap E_C(k) \cap E_D(15 - (i + j + k))$. Then

$$P(\{(i, j, k, 15 - (i + j + k))\}) = P(E_A(i) \cap E_B(j) \cap E_C(k))$$
$$= P(E_C(k)|E_A(i) \cap E_B(j)) \, P(E_B(j)|E_A(i)) \, P(E_A(i))$$

$$P(E_A(i)) = \frac{1}{16}$$

$$P(E_B(j)|E_A(i)) = \frac{1}{16 - i}$$

$$P(E_C(k)|E_B(j) \cap E_A(i)) = \frac{1}{16 - (i + j)}$$

Therefore

$$P(\{(i, j, k, 15 - (i + j + k))\}) = \frac{1}{16 - (i + j)} \frac{1}{16 - i} \frac{1}{16}$$

Analytical solution of example 8.3

Let i, j and k be the number of tourists assigned to hotels A, B an C respectively. Then

$$\Omega = \{(i, j, k, 15 - (i + j + k)) : 0 \leq i \leq 15,$$
$$0 \leq j \leq 15 - i;$$
$$0 \leq k \leq 15 - (i + j)\}$$

b

$$P(\Omega) = \sum_{\omega \in \Omega} P(\omega) = \sum_{i=0}^{15} \sum_{j=0}^{15-i} \sum_{k=0}^{15-(i+j)} P(\{(i, j, k, 15 - (i + j + k))\})$$

$$= \sum_{i=0}^{15} \sum_{j=0}^{15-i} \sum_{k=0}^{15-(i+j)} \frac{1}{16 - (i + j)} \frac{1}{16 - i} \frac{1}{16}$$

$$= \frac{1}{16} \sum_{i=0}^{15} \frac{1}{16 - i} \sum_{j=0}^{15-i} \frac{1}{16 - (i + j)} \sum_{k=0}^{15-(i+j)} 1$$

$$= \frac{1}{16} \sum_{i=0}^{15} \frac{1}{16 - i} \sum_{j=0}^{15-i} \frac{1}{16 - (i + j)} (16 - (i + j))$$

$$= \frac{1}{16} \sum_{i=0}^{15} \frac{1}{16 - i} \sum_{j=0}^{15-i} 1 = \frac{1}{16} \sum_{i=0}^{15} \frac{1}{16 - i} (16 - i) = \frac{1}{16} \sum_{i=0}^{15} 1 = \frac{1}{16} 16 = 1$$

c Define the event $E=$"all hotels receive guests." Then

$$E = \{(i, j, k, 15 - (i + j + k)) : 1 \le i \le 12,$$
$$1 \le j \le 13 - i;$$
$$1 \le k \le 14 - (i + j)\}$$

$$P(E) = \sum_{i=1}^{12} \sum_{j=1}^{13-i} \sum_{k=1}^{14-(i+j)} P(\{(i, j, k, 15 - (i + j + k))\})$$

$$= \sum_{i=1}^{12} \sum_{j=1}^{13-i} \sum_{k=1}^{14-(i+j)} \frac{1}{16 - (i + j)} \frac{1}{16 - i} \frac{1}{16}$$

$$= 0.2856$$

d

$$P(\{(15, 0, 0, 0)\}) = \frac{1}{16}$$

$$P(\{(0, 15, 0, 0)\}) = \frac{1}{16^2}$$

Thus, the selected rule is unfair to the four hotels.

Please see code here.

Independent events

If event A is independent from event B, then knowing that B has occurred does not change the probability that event A occurs.

Definition 6

Two events A and B are independent if

$$P(A|B) = P(A)$$

Independent events

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Two events A and B are independent if

$$P(A|B) = P(A)$$

Claim

If events A and B are independent if and only if

$$P(A \cap B) = P(A)P(B)$$

Independent events

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Definition 6

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Claim

If events A and B are independent if and only if

$$P(A \cap B) = P(A)P(B)$$

Proof.

$\rightarrow: P(A \cap B) = P(A|B)P(B) = P(A)P(B)$

$\leftarrow: P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$ □

Example 8.5: independent dice events?

Example 8.5

Two fair dice are thrown. Let A be the event that the number shown in die one is even, and B the event that the sum of the two dice is odd. Are A and B independent?

Analytical solution of example 8.5

The sample space is

$$\Omega = \{(d_1, s) : d_1 \text{ number shown in die one, } s \text{ sum of the two dice}\}$$
$$= \{(1, 2), (1, 3), \dots, (1, 7),$$
$$(2, 3), (2, 4), \dots, (2, 8),$$
$$\dots,$$
$$(6, 7), (6, 8), \dots, (6, 12)\}$$

$$N(\Omega) = 36, N(A) = N(B) = 18, N(A \cap B) = 9$$

$$P(B) = \frac{N(B)}{N(\Omega)} = \frac{18}{36} = \frac{1}{2}$$

$$P(B|A) = \frac{N(A \cap B)}{N(A)} = \frac{9}{18} = \frac{1}{2}$$

Since $P(B) = P(B|A)$, events A and B are independent.

Exercise: independence-related proofs

Prove the following claims:

Claim (disjoint events are generally not independent)

Let A and B be two events and P a probability measure such that $P(A) \neq 0$ and $P(B) \neq 0$. Then, it is always true that the fact that A and B are disjoint implies that A and B are not independent.

Claim (symmetry of independence)

$P(A|B) = P(A)$ if and only if $P(B|A) = P(B)$.

Example page 266

A fair die is rolled to yield a number between one and six, and then a coin is tossed that many times. What is the probability that heads will not appear?

Analytical solution of example p86

$$P(\text{no\_heads}) = P(\text{no\_heads} \cap \Omega)$$

$$= P(\text{no\_heads} \cap (\text{die\_equals\_1} \cup \text{die\_equals\_2} \cup \cdots \cup \text{die\_equals\_6}))$$

$$= P((\text{no\_heads} \cap \text{die\_equals\_1}) \cup (\text{no\_heads} \cap \text{die\_equals\_2}) \cup \ldots \cup (\text{no\_heads} \cap \text{die\_equals\_6})))$$

$$= \sum_{i=1}^{6} P(\text{no\_heads} \cap \text{die\_equals\_i}) = \sum_{i=1}^{6} P(\text{no\_heads}|\text{die\_equals\_i})P(\text{die\_equals\_i})$$

$$= P(T)P(\text{die\_equals\_1}) + P(TT)P(\text{die\_equals\_2}) + \ldots + P(TTTTTT)P(\text{die\_equals\_6})$$

$$= \frac{1}{2}\frac{1}{6} + \left(\frac{1}{2}\right)^2 \frac{1}{6} + \ldots + \left(\frac{1}{2}\right)^6 \frac{1}{6} = 0.1640625$$

Please see code here.

Rule 6

Law of conditional probability *Let A be an event that can only occur if one of the mutually exclusive events $B_1, \ldots, B_n$ occurs. Then*

$$P(A) = P(A|B_1)P(B_1) + \ldots + P(A|B_n)P(B_n)$$

Prove the law of conditional probability. Hint: use induction as
in the proof of rule 5.

Exercise: tour de France

Example 8.6

The upcoming Tour the France bicycle tournament will take place from July 1 through July 23. One hundred and eighty cyclists will participate in event. What is the probability that two or more participating cyclists will have birthdays on the same day during the tournament?

Contents

1 Foundations of probability theory

2 Conditional probability

3 Statistical inference

Bayesian inference

- In Bayesian statistics parameters of a model, θ are considered random quantities. They are assigned a (subjective) probability $P(\theta)$.
- The probability of parameters is subject to change as new data arrives, $P(\theta|D)$.

$$P(\theta) \xrightarrow{\text{data observed}} P(\theta|D)$$

- How to calculate the posterior probability $P(\theta|D)$?

$$P(\theta|D) = \frac{P(\theta \cap D)}{P(D)}$$

$$P(\theta \cap D) = P(D|\theta)P(\theta)$$

$$P(\theta|D) = \frac{P(D|\theta)P(\theta)}{P(D)} \propto P(D|\theta)P(\theta)$$

Bayesian inference

- Sequential update

$$
\begin{aligned}
P(\theta | D_1 \cap D_2) &= \frac{P(\theta \cap D_1 \cap D_2)}{P(D_1 \cap D_2)} \\
&= \frac{P(D_2 | D_1 \cap \theta) P(D_1 \cap \theta)}{P(D_1 \cap D_2)} \\
&= \frac{P(D_2 | \theta) P(D_1 | \theta) P(\theta)}{P(D_1 \cap D_2)} \\
&\propto P(D_2 | \theta) P(\theta | D_1)
\end{aligned}
$$

Exercise: sequential posterior probability for arbitrary number of observations

For any $n \in \mathbb{N}$, prove that

$$P(\theta|D_1 \cap \ldots \cap D_{n+1}) = P(D_{n+1}|\theta)P(\theta|D_1 \cap D_2 \cap \ldots \cap D_n)$$

Example 8.13: disease treatment effect

Example 8.13

A new treatment is tried out for a disease. A standard treatment has a success probability of 35%. The discrete uniform distribution on 0.00, 0.01, . . . , 0.99, 1.00 is taken as prior for the success probability of the new treatment. The experiment design is to make exactly ten observations by treating ten patients. The experimental study yield seven successes and three failures (Berry, 2006).

a *Plot the prior of the success probability of the new treatment.*

b *Plot the likelihood of the success probability of the new treatment.*

c *Plot the posterior of the success probability of the new treatment.*

d *What is the posterior probability that the new treatment is more effective than the standard one?*

Analytical solution of example 8.13

a prior:

$$p(\theta) = \frac{1}{101}, \quad \text{for } \theta = 0.00, 0.01, \ldots, 0.99, 1.00$$

b likelihood:

$$\Omega = \{(r_1, \ldots, r_{10}) : r_i \in \{S, F\}\}$$

Define the events $k\_successes=$"experiments where k out of the 10 observations were successes", with $k \in \{0, 1, \ldots, 10\}$.

$$0\_successes = \{(F, F, F, F, F, F, F, F, F, F)\}$$
$$1\_successes = \{(S, F, F, F, F, F, F, F, F, F), (F, S, F, F, F, F, F, F, F, F), \ldots,$$
$$(F, F, F, F, F, F, F, F, F, S)\}$$
$$2\_successes = \{(S, S, F, F, F, F, F, F, F, F), (S, F, S, F, F, F, F, F, F, F), \ldots,$$
$$(F, F, F, F, F, F, F, F, S, S)\}$$

Calculate the events probabilities

$$P(0\_successes|\theta) = P(\{(F, F, F, F, F, F, F, F, F, F)\}) = (1 - \theta)^{10}$$
$$P(1\_successes|\theta) = P(\{(S, F, F, F, F, F, F, F, F, F), (F, S, F, F, F, F, F, F, F, F), \ldots,$$
$$(F, F, F, F, F, F, F, F, F, S)\}\})$$
$$= P(\{(S, F, F, F, F, F, F, F, F, F)\}) + P(\{(F, S, F, F, F, F, F, F, F, F)\}) + \ldots +$$
$$P(\{(F, F, F, F, F, F, F, F, F, S)\}) = 10 P(\{(S, F, F, F, F, F, F, F, F, F)\})$$
$$= 10 \ \theta(1 - \theta)^9$$

b likelihood:

$$
\begin{aligned}
P(2\_successes|\theta) =& P(\{\ \ (S,S,F,F,F,F,F,F,F,F),(S,F,S,F,F,F,F,F,F,F),\ldots, \\
& (F,F,F,F,F,F,F,F,S,S)\}\}) \\
=& P(\{\ \ (S,S,F,F,F,F,F,F,F,F)\})+ \\
& P(\{\ \ (S,F,S,F,F,F,F,F,F,F)\})+\ldots+ \\
& P(\{\ \ (F,F,F,F,F,F,F,F,S,S)\}) \\
=& \binom{10}{2} P(\{(S,F,F,F,F,F,F,F,F,F)\}) \\
=& \binom{10}{2} \theta^2 (1-\theta)^8 \\
P(k\_successes|\theta) =& \binom{10}{k} \theta^k (1-\theta)^{10-k}, \quad k \in \{0,1,\ldots,10\}
\end{aligned}
$$

c posterior:

$$P(\theta|7\_\text{successes}) = K_1 P(7\_\text{successes}|\theta)P(\theta) = K_1 \binom{10}{7} \theta^7 (1-\theta)^3 \frac{1}{101} = K_2 \theta^7 (1-\theta)^3$$

$$1 = \sum_{k=0}^{100} P(\theta_k|7\_\text{successes}) = K_2 \sum_{k=0}^{100} \theta_k^7 (1-\theta_k)^3; \quad \theta_k = \frac{k}{100}$$

$$K_2 = \frac{1}{\sum_{k=0}^{100} \theta_k^7 (1-\theta_k)^3}$$

$$P(\theta|7\_\text{successes}) = \frac{\theta^7 (1-\theta)^3}{\sum_{k=0}^{100} \theta_k^7 (1-\theta_k)^3}$$

d

$$P(\text{new treatment more effective than standard one}) = \sum_{k=36}^{100} P(\theta_k|7\_\text{successes})$$

Continuing example 8.13, calculate the posterior probability of
the new treatment after a second observation with six
successes and a first observation with seven successes.

References

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Tijms, H. (2012). *Understanding probability*. Cambridge University Press.