Continuous Random Variables

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Gatsby Bridging Programme 2023

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Objectives

- Introduce the concept and formal definition of a continuous random variable X and a probability density function.
- Learn how to find the probability that a continuous random variable falls in some interval [a, b].
- Learn that if X is continuous, the probability that X takes on any specific value is 0.
- Introduce the concept and formal definition of a cumulative distribution function of a continuous random variable.
- Learn how to find the cumulative distribution function of a continuous random variable X from the probability density function of X.

Discrete vs. continuous random variables

Unlike discrete random variables, which can take on a countable number of possible values (e.g. faces of a die or cards of a deck), continuous random variables can take on an uncountable number of possible values (e.g. all the real numbers in an interval).

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Examples

- the voltage membrane potential of a cell
- the interspike interval of a neuron
- the force generated by a muscle
- the velocity of an eye movement

Continuous random variables

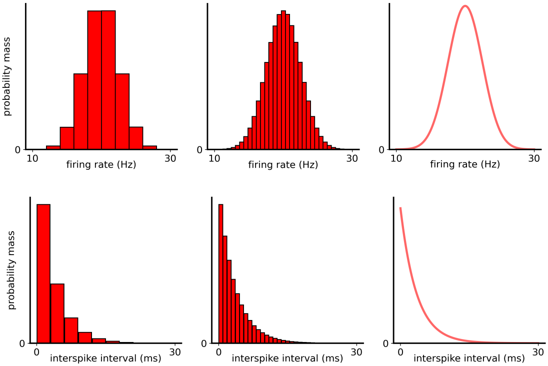
Definition

A random variable X is continuous if:

1. possible values comprise either a single interval on the number line (i.e. for some $a < b$, any number x between a and b is a possible value) or a union of disjoint intervals, and
2. $P(X = c) = 0$ for any number c that is a possible value of X.

Discrete probability distributions in the limit

Continuous random variables can be discretised into bins to form a discrete distribution that can be viewed as a probability histogram. As the bins become narrower, the histogram approaches a smooth curve.



The probability density function

Definition

The **probability density function** (PDF) of a continuous random variable X is a function $f(x)$ defined on the interval $(-\infty, \infty)$ such that for any two numbers a and b with $a \leq b$,

$$P(a \leq X \leq b) = \int_a^b f(x)dx.$$

That is, the probability that X takes on a value in the interval [a, b] is the area under the graph of the density function above this interval.

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A valid probability density function $f(x)$ has the following properties:

$$f(x) \geq 0 \text{ for all } x \tag{1}$$

$$\int_{-\infty}^{\infty} f(x)dx = 1. \tag{2}$$

Probabilities as integrals

The probability that a continuous random variable X takes on a value in the interval [a, b] is given by the area under the probability density function $f(x)$.

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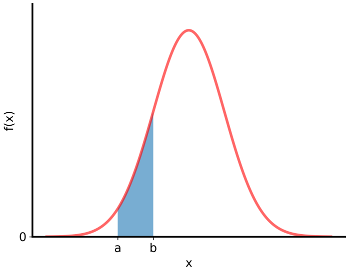

Density as probability per unit length

The probability of a small interval δ is approximately the density $\times$ δ:

$$\begin{aligned} P(x \leq X \leq x + \delta) &= \int_x^{x+\delta} f(t)dt \\ &\approx f(x) \times \delta \end{aligned}$$

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Thus density is probability per unit length (rate of probability accumulation):

$$\frac{P(x \leq X \leq x+\delta)}{\delta} \approx f(x)$$

Each possible value has zero probability

The probability that X takes on a particular value a is 0, as

$$\begin{aligned} P(X = a) &= \int_a^a f(x)dx \\ &= \lim_{\epsilon \to 0} \int_{a-\epsilon}^{a+\epsilon} f(x)dx \\ &= 0. \end{aligned}$$

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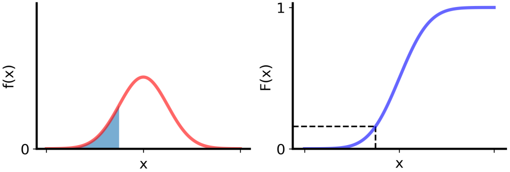
This implies that probabilities don't depend on interval end points:

$$P(a \leq X \leq b) = P(a < X < b) = P(a < X \leq b) = P(a \leq X < b),$$

as $P(X = a) = P(X = b) = 0$.

The cumulative distribution function

The cumulative distribution function (CDF) $F(x)$ is the area under the probability density function $f(x)$ to the left of x.



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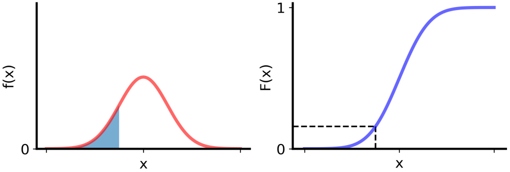


The CDF is a monotonically-increasing continuous function $F : \mathbb{R} \mapsto [0, 1]$ satisfying $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$.

The cumulative distribution function

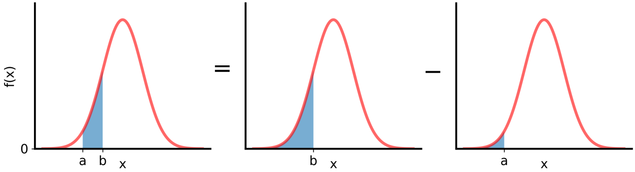
Definition

Let X be a continuous random variable with probability density function $f(x)$, then the **cumulative distribution function** $F(x)$ is defined as

$$\begin{aligned} F(x) &= P(X \leq x) \\ &= \int_{-\infty}^{x} f(t)dt. \end{aligned}$$

Computing probabilities using the CDF

$$P(a \leq X \leq b) = F(b) - F(a).$$



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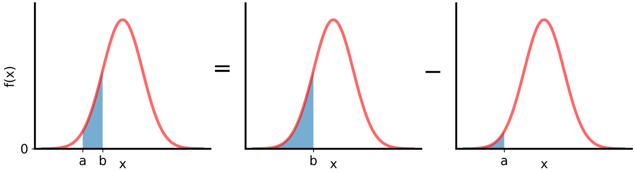
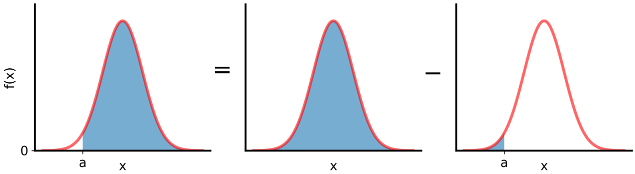


$$\begin{aligned} P(X > a) &= F(\infty) - F(a) \\ &= 1 - F(a). \end{aligned}$$



Obtaining the PDF from the CDF

At every x at which the derivative $F'(x)$ exists, $F'(x) = f(x)$.

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Examples

When X has a uniform distribution, for $a < x < b$:

$$F'(x) = \frac{d}{dx}\left(\frac{x-a}{b-a}\right) = \frac{1}{b-a} = f(x)$$

Percentiles of a continuous distribution

Definition

Let p be a number between 0 and 1. The **(100p)th percentile** of the distribution of a continuous random variable X, denoted by $\eta(p)$, is defined by

$$p = F(\eta(p)) = \int_{-\infty}^{\eta(p)} f(x)dx$$

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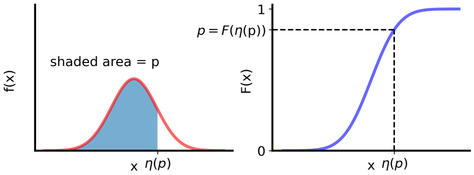
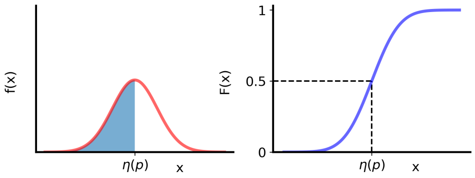


Median

Definition

The **median** of a continuous distribution, denoted by $\tilde{\mu}$, is the 50th percentile, so $\tilde{\mu}$ satisfies $F(\tilde{\mu}) = 0.5$. That is, half the are area under the probability density function is to the left of $\tilde{\mu}$ and half is to the right of $\tilde{\mu}$.



Mean and variance

The expected value (mean) of a continuous random variable X is:

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The variance of X is:

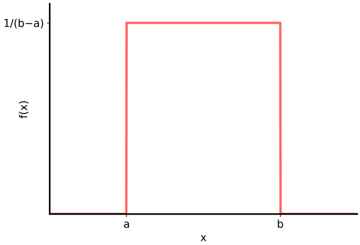
$$\begin{aligned} \mathrm{Var}[X] &= \mathbb{E}[(x - \mathbb{E}[X])^2] \\ &= \int_{-\infty}^{\infty} (x - \mathbb{E}[X])^2 f(x) \mathrm{d}x. \end{aligned}$$

Example: the uniform distribution

Definition

X is said to have a **uniform distribution** on the interval [a, b] if the PDF of X is:

$$f(x; a, b) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise.} \end{cases}$$



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When X has a uniform distribution, its expected value is:

$$\begin{aligned}\mathbb{E}[X] &= \int_{-\infty}^{\infty} x f(x) \mathrm{d}x \\ &= \int_a^b x \frac{1}{b-a} \mathrm{d}x \\ &= \frac{a+b}{2}.\end{aligned}$$

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Its variance is:

$$\begin{aligned}\mathrm{Var}[X] &= \int_{-\infty}^{\infty} (x - \mathbb{E}[X])^2 f(x) \mathrm{d}x \\ &= \int_a^b \left(x - \frac{a+b}{2}\right)^2 \frac{1}{b-a} \mathrm{d}x \\ &= \frac{(b-a)^2}{12}.\end{aligned}$$

The inverse transform method

Theorem

Let $U \sim \mathrm{U}(0,1)$ *be a continuous random variable having a standard uniform distribution on the interval [0,1]. Then, the random variable*

$$X = F^{-1}(U)$$

is distributed as the cumulative distribution function F*, that is* $P(X \leq x) = F(x)$*.*

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Proof.

$$\begin{aligned} P(X \leq x) &= P(F^{-1}(U) \leq x) \\ &= P(U \leq F(x)) \\ &= F(x). \end{aligned}$$

because $\{F^{-1}(U) \leq x\} = \{U \leq F(x)\}$ (equality of events) and $P(U \leq F(x)) = F(x)$ when $U \sim \mathrm{U}(0,1)$. □

Sampling using the CDF

The inverse transform method can be used to sample a continuous random variable given the inverse of its cumulative distribution function.

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The inverse transform method can be used to sample a continuous random variable given the inverse of its cumulative distribution function.

To draw a sample $x \sim f(x)$:

1. Sample $u \sim \mathrm{U}(0, 1)$ (recall that $F : \mathbb{R} \mapsto [0, 1]$)
2. Let $x = F^{-1}(u)$

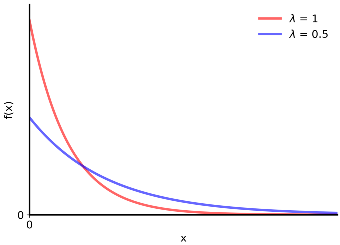
Example: the exponential distribution

Definition

X is said to have an **exponential distribution** on the interval $[0, \infty)$ if the PDF of X is:

$$f(x; \lambda) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$

where λ is a rate parameter that governs the rate of decay of $f(x)$.



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and the inverse of the CDF is:

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Hence, to sample $x \sim f(x)$:

1. Sample $u \sim \mathrm{U}(0, 1)$ (using a pseudo-random number generator)
2. Let $x = -\frac{\log(1-u)}{\lambda}$

The normal (Gaussian) distribution