Continuous Random Variables

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Objectives

- Introduce the concept and formal definition of a continuous random variable X and a probability density function.
- Learn how to find the probability that a continuous random variable falls in some interval [a, b].
- Learn that if X is continuous, the probability that X takes on any specific value is 0.
- Introduce the concept and formal definition of a cumulative distribution function of a continuous random variable.
- Learn how to find the cumulative distribution function of a continuous random variable X from the probability density function of X.

Continuous random variables

Definition

A random variable (RV) X is continuous if:

1. possible values comprise either a single interval on the number line (i.e. for some $a < b$, any number x between a and b is a possible value) or a union of disjoint intervals, and
2. $P(X = c) = 0$ for any number c that is a possible value of X.

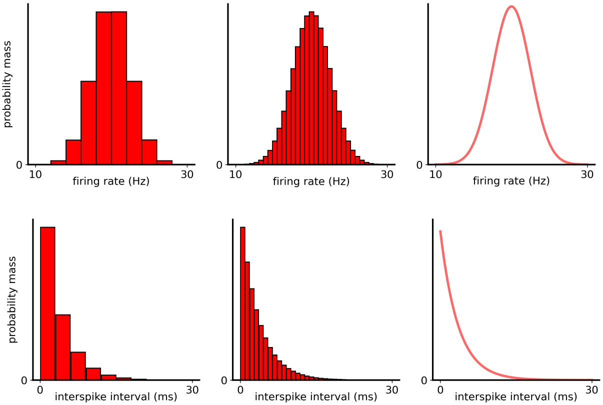
Unlike discrete RVs, which have a finite number of outcomes, continuous RVs can take on an infinite continuous range of possible values.

Examples

- the voltage membrane potential of a cell
- the interspike interval of a neuron
- the force generated by a muscle
- the velocity of an eye movement

Discrete probability distributions in the limit

Continuous random variables can be discretised into bins to form a probability mass function. As the bins become narrower, the probability mass function approaches a smooth curve.



The probability density function (PDF)

Definition

A random variable X is continuous if there exists a nonnegative function $f(x)$ defined on the interval $(-\infty, \infty)$, such that for any interval $[a, b]$ we have

$$P(a \leq X \leq b) = \int_a^b f(x)dx.$$

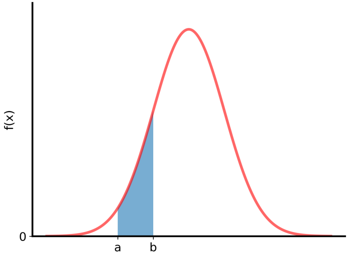
A valid probability density function $f(x)$ has the following properties:

$$f(x) \geq 0 \text{ for all } x \tag{1}$$

$$\int_{-\infty}^{\infty} f(x)dx = 1. \tag{2}$$

Probabilities as integrals

The probability that X takes on a value in the interval [a, b] is given by the area under the probability density function $f(x)$.



$$P(a \leq X \leq b) = \int_a^b f(x)dx$$

Density as probability per unit length

The probability of a small interval δ is approximately the density $\times$ δ:

$$P(x \leq X \leq x + \delta) = \int_{x}^{x+\delta} f(t)dt$$
$$\approx f(x) \times \delta$$

Thus density is probability per unit length (probability accumulation rate):

$$\frac{P(x \leq X \leq x + \delta)}{\delta} \approx f(x)$$

Each possible value has zero probability

The probability that X takes on a particular value a is 0, as

$$\begin{aligned} P(X = a) &= \int_a^a f(x)dx \\ &= \lim_{\epsilon \to 0} \int_{a-\epsilon}^{a+\epsilon} f(x)dx \\ &= 0. \end{aligned}$$

This implies that probabilities don't depend on interval end points:

$$P(a \leq X \leq b) = P(a < X < b) = P(a < X \leq b) = P(a \leq X < b),$$

as $P(X = a) = P(X = b) = 0$.

Mean and variance

The expected value (mean) of a continuous random variable X is:

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x) \mathrm{d}x,$$

the expected value of a function $g(x)$ of X is:

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f(x) \mathrm{d}x,$$

the variance of X is:

$$\begin{aligned}
\mathrm{Var}[X] &= \mathbb{E}[(x - \mathbb{E}[X])^2] \\
&= \int_{-\infty}^{\infty} (x - \mathbb{E}[X])^2 f(x) \mathrm{d}x.
\end{aligned}$$

Example: the uniform distribution

When X has a uniform distribution, the PDF is:

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise,} \end{cases}$$

the expected value of X is:

$$\mathbb{E}[X] = \int_a^b x \frac{1}{b-a} \mathrm{d}x = \frac{a+b}{2},$$

the variance of X is:

$$\mathrm{Var}[X] = \int_a^b \left(x - \frac{a+b}{2} \right)^2 \frac{1}{b-a} \mathrm{d}x = \frac{(b-a)^2}{12}.$$

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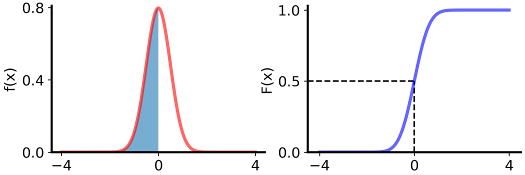
The cumulative distribution function (CDF)

The cumulative distribution function $F(x)$ is the area under the probability density function $f(x)$ to the left of x.



The cumulative distribution function (CDF)

Definition

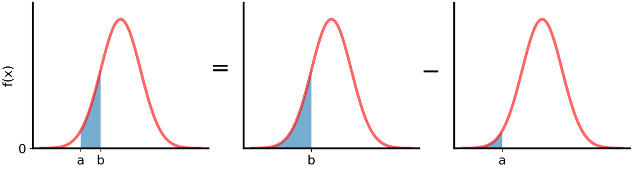
Let X be a continuous random variable with probability density function $f(x)$, then the cumulative distribution function is defined as

$$\begin{aligned} F(x) &= P(X \leq x) \\ &= \int_{-\infty}^{x} f(t)dt. \end{aligned}$$

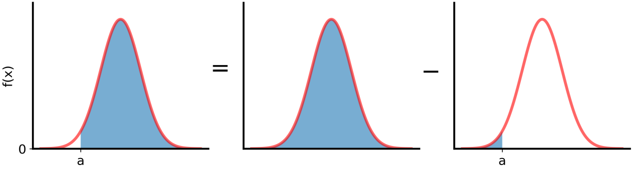
The CDF is a monotonically-increasing continuous function $F : \mathbb{R} \mapsto [0, 1]$ satisfying $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$.

Computing probabilities using the CDF

$$P(a \leq X \leq b) = F(b) - F(a).$$



$$\begin{aligned} P(X > a) &= F(\infty) - F(a) \\ &= 1 - F(a). \end{aligned}$$



Obtaining the PDF from the CDF

At every x at which the derivative $F'(x)$ exists, $F'(x) = f(x)$.

Examples

When X has a uniform distribution, for $a < x < b$:

$$F'(x) = \frac{d}{dx}\left(\frac{x-a}{b-a}\right) = \frac{1}{b-a} = f(x)$$

Sampling using the CDF

The inverse transform sampling algorithm can be used to sample a continuous random variable using the inverse of its cumulative distribution function.

Recall that $F : \mathbb{R} \mapsto [0, 1]$.

To draw a sample $x \sim f(x)$:

1. Sample $u \sim \mathrm{U}(0, 1)$ (using a pseudo-random number generator)
2. Let $x = F^{-1}(u)$

Example: the exponential distribution

The PDF of the exponential distribution is:

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$

which implies that the CDF is:

$$F(x) = 1 - e^{-\lambda x} = u,$$

and the inverse of the CDF is:

$$F^{-1}(u) = -\frac{\log(1-u)}{\lambda} = x.$$

Hence, to sample $x \sim f(x)$:

1. Sample $u \sim \mathrm{U}(0, 1)$
2. Let $x = -\frac{\log(1-u)}{\lambda}$

The normal (Gaussian) distribution