Continuous Random Variables

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Objectives

Introduce the concept and formal definition of a continuous random variable X and a probability density function.

Learn how to find the probability that a continuous random variable falls in some interval [a, b].

Learn that if X is continuous, the probability that X takes on any specific value is 0.

Introduce the concept and formal definition of a cumulative distribution function of a continuous random variable.

Learn how to find the cumulative distribution function of a continuous random variable X from the probability density function of X.

Discrete vs. continuous random variables

Unlike discrete random variables, which can take on a countable number of possible values (e.g. faces of a die or cards of a deck), continuous random variables can take on an uncountable number of possible values (e.g. all the real numbers in an interval).

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Examples

- the voltage membrane potential of a cell
- the interspike interval of a neuron
- the force generated by a muscle
- the velocity of an eye movement

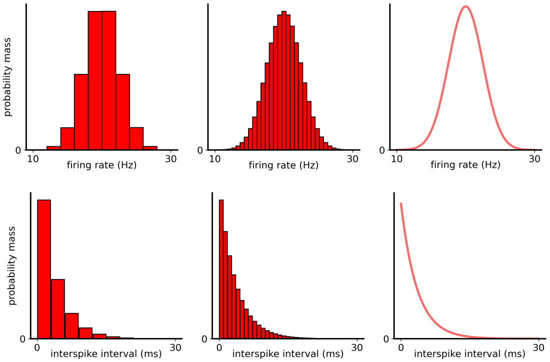
Definition

A random variable X is continuous if:

1. possible values comprise either a single interval on the number line (i.e. for some $a < b$, any number x between a and b is a possible value) or a union of disjoint intervals, and
2. $P(X = c) = 0$ for any number c that is a possible value of X.

Discrete probability distributions in the limit

Continuous random variables can be discretised into bins to form a discrete distribution that can be viewed as a probability histogram. As the bins become narrower, the histogram approaches a smooth curve.



The probability density function

Definition

The **probability density function** (PDF) of a continuous random variable X is a function $f(x)$ defined on the interval $(-\infty, \infty)$ such that for any two numbers a and b with $a \leq b$,

$$P(a \leq X \leq b) = \int_a^b f(x)dx.$$

That is, the probability that X takes on a value in the interval [a, b] is the area under the graph of the density function above this interval.

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A valid probability density function $f(x)$ must have the following properties to respect the axioms of probability:

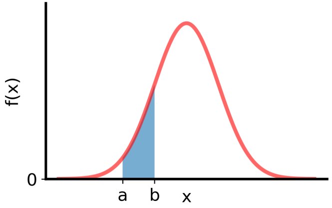
$$f(x) \geq 0 \text{ for all } x \tag{1}$$

$$\int_{-\infty}^{\infty} f(x)dx = 1. \tag{2}$$

Probabilities as integrals

The probability that a continuous random variable X takes on a value in the interval [a, b] is given by the area under the probability density function $f(x)$:

$$P(a \leq X \leq b) = \int_a^b f(x)dx$$



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Thus, density is probability per unit of x (rate of probability accumulation):

$$\frac{P(x \leq X \leq x + dx)}{dx} = f(x)$$

Each possible value has zero probability

The probability that X takes on a particular value a is 0, as

$$\begin{aligned} P(X = a) &= \int_a^a f(x)dx \\ &= \lim_{\epsilon \to 0} \int_{a-\epsilon}^{a+\epsilon} f(x)dx \\ &= 0. \end{aligned}$$

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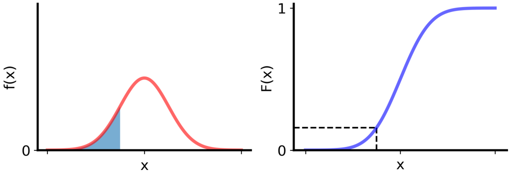
This implies that probabilities don't depend on interval end points:

$$P(a \leq X \leq b) = P(a < X < b) = P(a < X \leq b) = P(a \leq X < b),$$

as $P(X = a) = P(X = b) = 0$.

The cumulative distribution function

The cumulative distribution function (CDF) $F(x)$ is the area under the probability density function $f(x)$ to the left of x.



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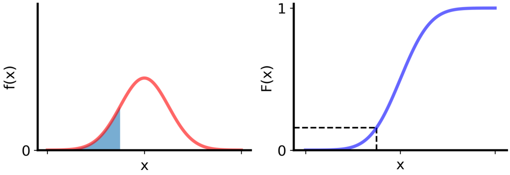


The CDF $F(x)$ has the following properties:

a. $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$

b. $F(x)$ is a non-decreasing (monotonic) function of x.

Collectively, a. and b. imply that $F : \mathbb{R} \mapsto [0, 1]$.

The cumulative distribution function

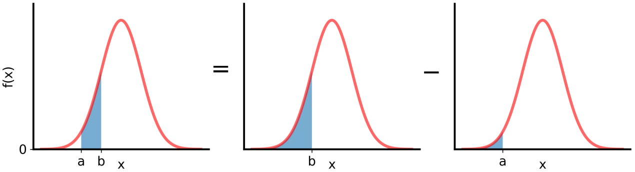
Definition

Let X be a continuous random variable with probability density function $f(x)$, then the **cumulative distribution function** $F(x)$ is defined as

$$\begin{aligned} F(x) &= P(X \leq x) \\ &= \int_{-\infty}^{x} f(t)dt. \end{aligned}$$

Computing probabilities using the CDF

$$P(a \leq X \leq b) = F(b) - F(a).$$



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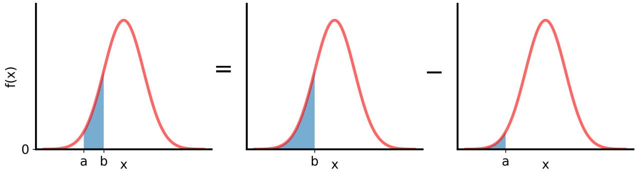
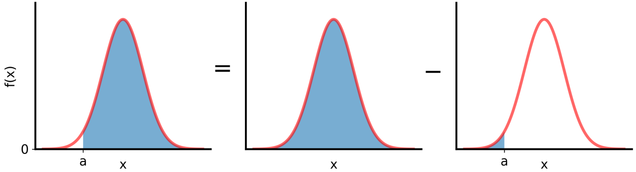


$$\begin{aligned} P(X > a) &= F(\infty) - F(a) \\ &= 1 - F(a). \end{aligned}$$



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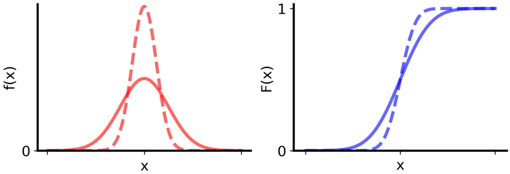

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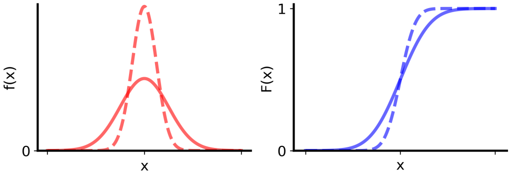

At every x at which the derivative $\frac{\delta F(x)}{\delta x}$ exists, $\frac{\delta F(x)}{\delta x} = f(x)$.

Example: the uniform distribution

Definition

A random variable X is said to have a **uniform distribution** on the interval $[a, b]$ if the PDF of X is

$$f(x; a, b) = \begin{cases} \frac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise.} \end{cases}$$

When X has a uniform distribution on the interval $[a, b]$, we write this as $X \sim \mathrm{U}(a, b)$. The CDF of X is given by

$$F(x; a, b) = \begin{cases} 0 & \text{if } x < a \\ \frac{x-a}{b-a} & \text{if } a \le x \le b \\ 1 & \text{if } x > b \end{cases}$$

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When X has a uniform distribution on the interval [a, b], for $a \leq x \leq b$:

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Percentiles of a continuous distribution

Definition

Let p be a number between 0 and 1. The **(100p)th percentile** of the distribution of a continuous random variable X, denoted by $\eta(p)$, is defined by

$$p = F(\eta(p)) = \int_{-\infty}^{\eta(p)} f(x)dx$$

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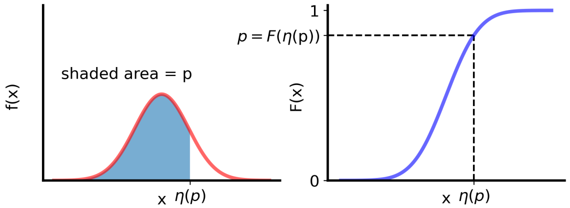
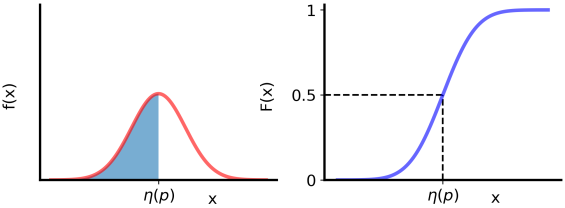


Median

Definition

The **median** of a continuous distribution, denoted by $\tilde{\mu}$, is the 50th percentile, so $\tilde{\mu}$ satisfies $F(\tilde{\mu}) = 0.5$. That is, half the are area under the probability density function is to the left of $\tilde{\mu}$ and half is to the right of $\tilde{\mu}$.



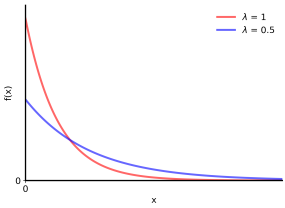
Example: the exponential distribution

Definition

A random variable X is said to have an **exponential distribution** on the interval $[0, \infty)$ if the PDF of X is:

$$f(x; \lambda) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$

where $\lambda > 0$ is a rate parameter that governs the rate of decay of $f(x)$. When X has an exponential distribution with parameter λ, we write $X \sim \mathrm{Exp}(\lambda)$.



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$$\eta(p) = -\frac{\log(1-p)}{\lambda}$$

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and

$$\eta(0.5) = -\frac{\log(0.5)}{\lambda}.$$

Mean and variance

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The variance of X is:

$$\begin{aligned} \mathrm{Var}[X] &= \int_{-\infty}^{\infty} (x - \mathbb{E}[X])^2 f(x) \mathrm{d}x \\ &= \mathbb{E}[(x - \mathbb{E}[X])^2] \\ &= \mathbb{E}[X^2] - \mathbb{E}[X]^2. \end{aligned}$$

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The inverse transform method

Theorem

Let $U \sim \mathrm{U}(0,1)$ be a continuous random variable having a standard uniform distribution on the interval [0,1]. Then, the random variable

$$X = F^{-1}(U)$$

is distributed as the cumulative distribution function F, that is $P(X \leq x) = F(x)$.

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Proof.

$$P(X \leq x) = P(F^{-1}(U) \leq x)$$

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Proof.

$$\begin{aligned} P(X \leq x) &= P(F^{-1}(U) \leq x) \\ &= P(F(F^{-1}(U)) \leq F(x)) \\ &= P(U \leq F(x)) \\ &= F(x). \quad \square \end{aligned}$$

because 1) F is non-decreasing (monotonic) and 2) $P(U \leq F(x)) = F(x)$ when $U \sim \mathrm{U}(0,1)$.

Sampling using the CDF

The inverse transform method can be used to sample a continuous random variable given the inverse of its cumulative distribution function.

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To draw a sample $x \sim f(x)$:

1. Sample $u \sim \mathrm{U}(0, 1)$ (recall that $F : \mathbb{R} \mapsto [0, 1]$)
2. Let $x = F^{-1}(u)$

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Hence, to sample $x \sim f(x)$:

1. Sample $u \sim \mathrm{U}(0, 1)$ (using a pseudo-random number generator)
2. Let $x = -\frac{\log(1-u)}{\lambda}$

The normal distribution

Definition

A continuous random variable X has a **normal distribution** with parameters μ and σ (or σ^2), where $-\infty < \mu < \infty$ and $0 < \sigma$, if the probability density function of X is:

$$f(x; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad -\infty < x < \infty$$

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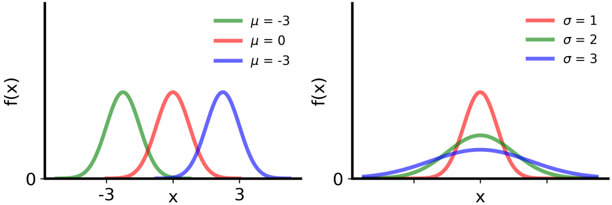

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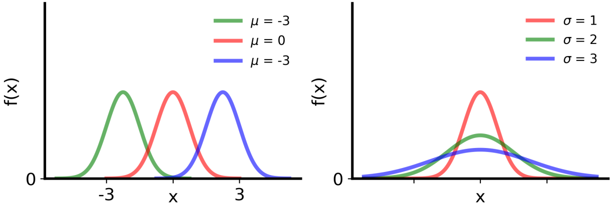

The normal distribution is the most important distribution in all of probability theory. It is ubiquitous in statistical analysis (central limit theorem).

Linear transformation of a normal random variable

Claim

If X is normally distributed with parameters μ and σ, then $Y = aX + b$ is normally distributed with parameters $a\mu + b$ and $a\sigma$.

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where F_X is the CDF of X.

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where F_X is the CDF of X. By differentiation, the PDF of Y is

$$f_Y(x) = \frac{1}{a} f_X((x - b)/a)$$

$\square$

Linear transformation of a normal random variable

Claim

If X is normally distributed with parameters μ and σ, then $Y = aX + b$ is normally distributed with parameters $a\mu + b$ and $a\sigma$.

Proof.

Let F_Y denote the CDF of $Y = aX + b$, then

$$\begin{aligned} F_Y(x) &= P(aX + b \leq x) \\ &= P(X \leq (x - b)/a) \\ &= F_X((x - b)/a), \end{aligned}$$

where F_X is the CDF of X. By differentiation, the PDF of Y is

$$\begin{aligned} f_Y(x) &= \frac{1}{a} f_X((x - b)/a) \\ &= \frac{1}{a\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\frac{x-b}{a} - \mu}{\sigma}\right)^2} \end{aligned}$$

$\square$

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The standard normal random variable

When X is a normal random variable with parameters μ and σ, the computation of $P(a \leq X \leq b)$ requires evaluating

$$\int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx.$$

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Definition

The normal distribution with parameter values $\mu = 0$ and $\sigma = 1$ is called the **standard normal distribution**. A random variable Z having a standard normal distribution is called a **standard normal random variable** and has probability density function given by

$$f(z; 0, 1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}.$$

Standardising a normal random variable

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Proposition

If X has a normal distribution with mean μ and standard deviation σ, then the **standardised variable** $Z = (X - \mu)/\sigma$ *has a standard normal distribution. Thus*

$$\begin{aligned} P(a \leq X \leq b) &= P\left(\frac{a-\mu}{\sigma} \leq Z \leq \frac{b-\mu}{\sigma}\right) \\ &= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right), \end{aligned}$$

where Φ is the cumulative distribution function of a standard normal random variable.