

# Markov and Chebyshev, Convergence in Probability, Weak Law of Large Numbers

**Q1.** You place an electrode on your head and you measure the voltage.

Will you get the exact same number any time you measure?

Can we think of the Voltage measured as a Random Variable?

How can we figure out (estimate) the expected value of the RV Voltage?

**A1.**

No.

Yes.

Average many measurements -> this works because of the Weak Law of Large Numbers.

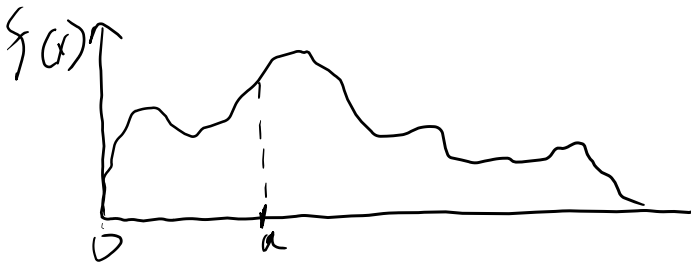
## Markov Inequality

**Q2.** Let's say I tell you that the expected value of the Voltage RV is  $10\mu\text{V}$  (small, i.e. close to zero) and that due to the way you've placed the electrode you will only read positive values. If I ask you whether you will ever measure a value of  $10\text{mV}$ , what can you tell me? How about  $1\text{V}$ ?

**A2.**

$E[X]$  is small and  $X \geq 0$

$$E[X] = \int_{-\infty}^{\infty} xf(x)dx \geq \int_a^{\infty} xf(x)dx \geq \int_a^{\infty} af(x)dx = a \int_a^{\infty} f(x)dx = aP(X \geq a)$$



$$E[X] = aP(X \geq a) \Rightarrow P(X \geq a) = \frac{E[X]}{a}$$

OR

$Y = 0$  if  $X < a$

$Y = a$  if  $X \geq a$

$$E[Y] \geq E[X] \Rightarrow aP(X \geq a) \geq E[X]$$

**Q3.** Can we use the Markov inequality to estimate the probability that a RV with an exponential distribution with  $\lambda=1$  will be bigger or equal than a number  $a$ ?

**A3.**

$$P(x \geq a) = 1/a$$

but we know that  $P(x \geq a) = e^{-a}$

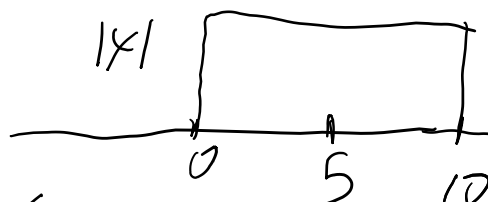
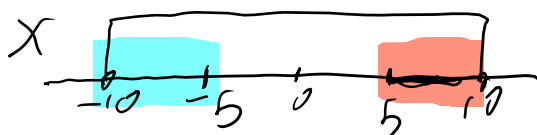
So the M.I. gives an upper bound (not a very good one). It is though a first rough estimation.



**Q4.** What is the M.I. of a uniform RV between -10 and 10 for  $a=5$ ? What is the  $P(X \geq 5)$

**A4.**

$$P(X \geq 5) = 5/20 = 1/4$$



$$P(|X| \geq 5) = \frac{E[|X|]}{5} = \frac{5}{5} = 1$$

But  $P(|X| \geq 5) = 2 * P(X \geq 5)$



Talk about the probability of the square of a variable being larger than the square of a number  
 $(P(X \geq c) = P(X^2 \geq c^2))$

### Chebyshev Inequality

**Q5.** Let's think about the variance of a probability distribution. For small variance what are the chances of a number being far away from the mean?

Show:

$$P(|X - \mu| \geq c) = \frac{\sigma^2}{c^2}$$

**A5.**

$$P(|X - \mu| \geq c) = P((X - \mu)^2 \geq c^2) \geq \frac{E[(X - \mu)^2]}{c^2} = \frac{\sigma^2}{c^2}$$

**Q6.** Calculate the C.I. for  $c = k \sigma$

**A6.**

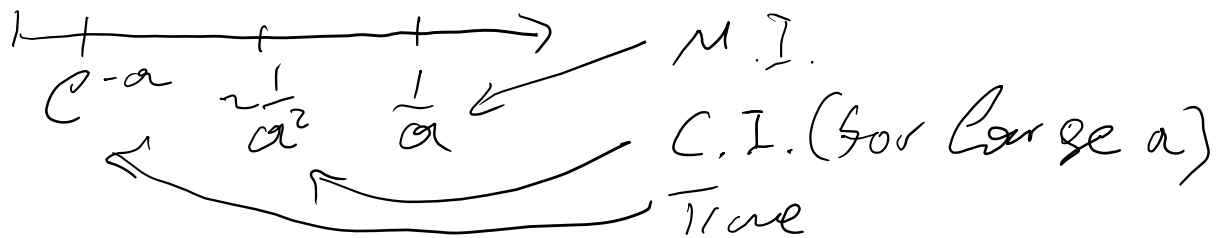
$$P(|X - \mu| \geq k\sigma) = \frac{1}{k^2}$$

**Q7.** Calculate the C.I. for the exponential distribution with  $\lambda=1$  for  $a > 1$ .

**A7.**

$$P(X \geq a) = P((X - 1) \geq a - 1) \leq P(|X - 1| \geq a - 1) \leq \frac{1}{(a - 1)^2}$$





So, for large  $a$  the C.I. is a stronger bound. Is this true for smaller values of  $a$  (how about  $a=1.2$  in the above example)?

## Weak Law of Large Numbers

Back to **Q1**. Why does averaging many samples of the same RV result in an estimate of the RV's expectation value with an ever increasing accuracy?

### A1 extended.

Assume you have measured the voltage from the electrode  $n$  times. Each measurement can be considered a RV. They all have the same mean and the same Variance (why?) and let's assume they are independent (is that true in this case?)

Define the RVs as  $X_1, X_2, X_3, \dots, X_n$

(You can also think of each RV as  $X_i = \mu + W_i$  with  $E[W_i] = 0$  and independent  $W_i$ )

Define Sample mean  $M_n = \frac{X_1 + X_2 + X_3 + \dots + X_n}{n}$  and call  $\mu = E[X_i]$

What is  $M_n$  (another RV)? What does  $E[M_n]$  mean (the theoretical expected value of the sample mean if we were to do multiple experiments).

I would like to show that as  $n$  becomes really big,  $M_n$  becomes closer and closer to  $\mu$ !

**Q8.** Show that  $E[M_n] = \mu$

**A8.**

$$E[M_n] = \frac{E[X_1 + X_2 + X_3 + \dots + X_n]}{n} = \frac{n\mu}{n} = \mu$$

**Q9.** Calculate the  $Var(M_n)$

**A9.**

$$Var(M_n) = \frac{Var(X_1 + X_2 + X_3 + \dots + X_n)}{n^2} = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

**Q10.** Use C.I. to calculate a bound of the probability that the  $M_n$  is not  $\mu$  by  $\epsilon$ . What happens if  $n$  goes to infinity?

**A10.**

$$P(|M_n - \mu| \geq \epsilon) \leq \frac{Var(M_n)}{\epsilon^2} = \frac{\sigma^2}{n\epsilon^2} \xrightarrow{\text{for } n \rightarrow \infty} 0$$

(Maybe talk here about the frequentist view of probability defining event A,  $p = P(A)$  and  $X_i$  indicator of A so the  $M_n$  becomes the empirical frequency of event A)

## Accuracy of estimations using the C.I.

**Q11.** You use your voltage measuring technology to build a brain computer interface. But you realise that your algorithm works only for some people. Before you market the product you need to know the percentage of people ( $p$ ) the BCI will work on. You try your tech on  $n$  people to find out. Can you say with absolute confidence what that  $p$  is? Also what do we need to do to keep the iid assumption alive here while we are trying this on some people?

**A11.**

No. One gets unlucky (or lucky some times) since the number of people you can sample is limited.

Do not the same person more than once.

**Q12.** Can you guarantee a very small error in the  $p$  (say 1%).

**A12.**

No. See A11.

**Q13.** Given an  $n$  (let's say 5000) what is probability that the error of  $p$  will be 5%?

**A13.**

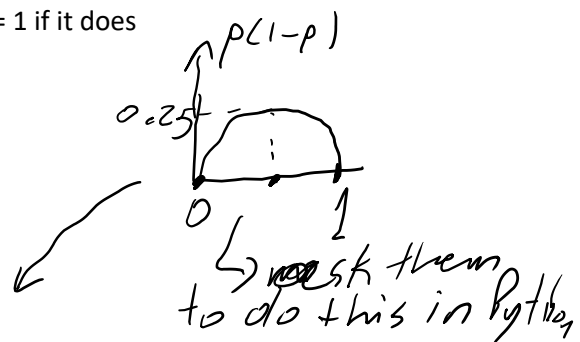
$X_i = 0$  if the algo doesn't work on the  $i$ th person and  $X_i = 1$  if it does

$$M_n = \frac{X_1 + X_2 + X_3 + \dots + X_n}{n}$$

We want  $|M_n - p| < 0.05$

$X_i$  is Bernoulli so the variance is  $p(1-p)$

$$\text{C.I. : } P(|M_n - p| < 0.01) \leq \frac{\sigma^2}{n\varepsilon^2} = \frac{p(1-p)}{5 \cdot 10^3 \cdot 5 \cdot 10^{-4}} \leq 0.1$$



**Q14.** How many people should you try the algo on to have a probability of being wrong by 5% equal to 1%?

**A14.**

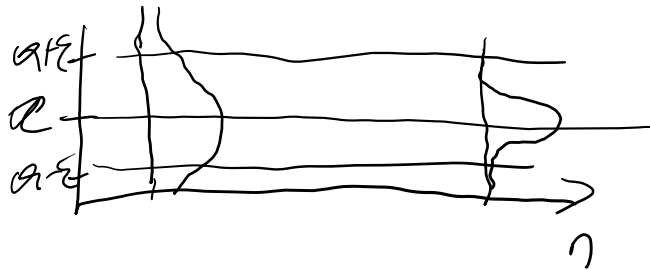
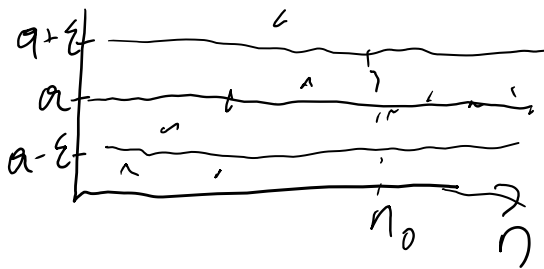
$$\frac{0.25}{n \cdot 5 \cdot 10^{-4}} \leq 0.01 \Rightarrow n \geq 50000$$

## Convergence in probability

A sequence  $Y_n$  converges in probability to a number  $a$  if:

$$\text{For any } \varepsilon > 0, \lim_{n \rightarrow \infty} P(|Y_n - a| \geq \varepsilon) = 0$$

So the WLLN says that  $M_n \xrightarrow[n \rightarrow \infty]{} \mu$  in probability



Here the limit is not a number but a distribution with an ever decreasing variance!

**Q15.** Find an example where a sequence  $Y_n$  converges in probability to a number but the expectation value of any  $Y_n$  ( $E[Y_n]$ ) converges to a different number

**A15.**

$Y_n = 0$  with  $p = 1 - (1/n)$  and  $n^2$  with  $p = 1/n$

$$P(|Y_n - 0| \geq \epsilon) = \frac{1}{n} \rightarrow 0 \text{ in prob.}$$

$$\text{But } E[Y_n] = n^2 \frac{1}{n} = n \xrightarrow[n \rightarrow \infty]{} \infty$$

