

Continuous
random variables

Probability
density functions

Cumulative
distribution
functions

Common
distributions

Continuous Random Variables

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Gatsby Bridging Programme 2023



Objectives

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- Introduce the concept and formal definition of a continuous random variable X and a probability density function.
- Learn how to find the probability that a continuous random variable falls in some interval $[a, b]$.
- Learn that if X is continuous, the probability that X takes on any specific value is 0.
- Introduce the concept and formal definition of a cumulative distribution function of a continuous random variable.
- Learn how to find the cumulative distribution function of a continuous random variable X from the probability density function of X .

Discrete vs. continuous random variables

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Unlike discrete random variables, which can take on a countable number of possible values (e.g. faces of a die or cards in a deck), continuous random variables can take on an uncountable number of possible values (e.g. all the real numbers in an interval).

Discrete vs. continuous random variables

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Unlike discrete random variables, which can take on a countable number of possible values (e.g. faces of a die or cards in a deck), continuous random variables can take on an uncountable number of possible values (e.g. all the real numbers in an interval).

Examples

- the voltage membrane potential of a cell
- the interspike interval of a neuron
- the force generated by a muscle
- the velocity of an eye movement

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Definition

A random variable X is continuous if:

- ① possible values comprise either a single interval on the number line (i.e. for some $a < b$, any number x between a and b is a possible value) or a union of disjoint intervals, and
- ② $P(X = c) = 0$ for any number c that is a possible value of X .

Discrete probability distributions in the limit

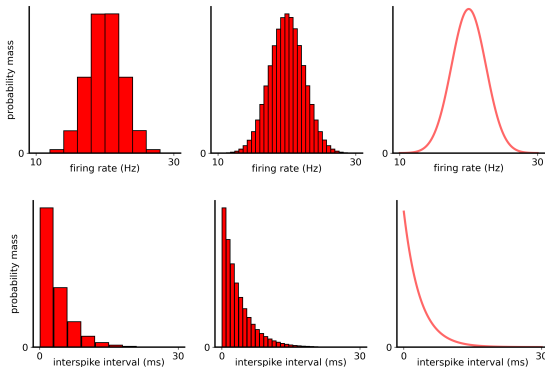
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Continuous random variables can be discretised into bins to form a discrete distribution that can be viewed as a probability histogram. As the bins become narrower, the histogram approaches a smooth curve.



The probability density function

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Definition

The **probability density function** (PDF) of a continuous random variable X is a function $f(x)$ defined on the interval $(-\infty, \infty)$ such that for any two numbers a and b with $a \leq b$,

$$P(a \leq X \leq b) = \int_a^b f(x) dx.$$

That is, the probability that X takes on a value in the interval $[a, b]$ is the area under the graph of the density function above this interval.

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That is, the probability that X takes on a value in the interval $[a, b]$ is the area under the graph of the density function above this interval.

A valid probability density function $f(x)$ has the following properties:

$$f(x) \geq 0 \text{ for all } x \quad (1)$$

$$\int_{-\infty}^{\infty} f(x) dx = 1. \quad (2)$$

Probabilities as integrals

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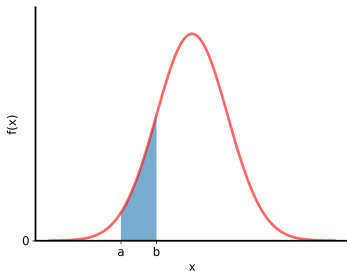
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The probability that a continuous random variable X takes on a value in the interval $[a, b]$ is given by the area under the probability density function $f(x)$.

$$P(a \leq X \leq b) = \int_a^b f(x) dx$$



Density as probability per unit length

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The probability of a small interval δ is approximately the density $\times \delta$:

$$\begin{aligned}P(x \leq X \leq x + \delta) &= \int_x^{x+\delta} f(t) dt \\ &\approx f(x) \times \delta\end{aligned}$$

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The probability of a small interval δ is approximately the density $\times \delta$:

$$\begin{aligned}P(x \leq X \leq x + \delta) &= \int_x^{x+\delta} f(t) dt \\ &\approx f(x) \times \delta\end{aligned}$$

Thus density is probability per unit length (probability accumulation rate):

$$\frac{P(x \leq X \leq x + \delta)}{\delta} \approx f(x)$$

Each possible value has zero probability

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The probability that X takes on a particular value a is 0, as

$$\begin{aligned}P(X = a) &= \int_a^a f(x)dx \\&= \lim_{\epsilon \rightarrow 0} \int_{a-\epsilon}^{a+\epsilon} f(x)dx \\&= 0.\end{aligned}$$

Each possible value has zero probability

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The probability that X takes on a particular value a is 0, as

$$\begin{aligned}P(X = a) &= \int_a^a f(x)dx \\&= \lim_{\epsilon \rightarrow 0} \int_{a-\epsilon}^{a+\epsilon} f(x)dx \\&= 0.\end{aligned}$$

This implies that probabilities don't depend on interval end points:

$$P(a \leq X \leq b) = P(a < X < b) = P(a < X \leq b) = P(a \leq X < b),$$

as $P(X = a) = P(X = b) = 0$.

Mean and variance

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The expected value (mean) of a continuous random variable X is:

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} xf(x)dx.$$

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The expected value (mean) of a continuous random variable X is:

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} xf(x)dx.$$

The expected value of a function $g(x)$ of X is:

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx.$$

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$$\mathbb{E}[X] = \int_{-\infty}^{\infty} xf(x)dx.$$

The expected value of a function $g(x)$ of X is:

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx.$$

The variance of X is:

$$\begin{aligned}\text{Var}[X] &= \mathbb{E}[(x - \mathbb{E}[X])^2] \\ &= \int_{-\infty}^{\infty} (x - \mathbb{E}[X])^2 f(x)dx.\end{aligned}$$

Example: the uniform distribution

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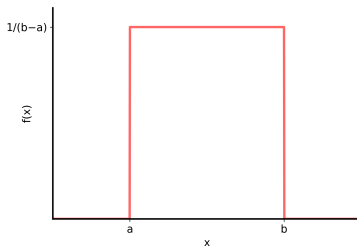
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Definition

X is said to have a **uniform distribution** on the interval $[a, b]$ if the PDF of X is:

$$f(x; a, b) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise.} \end{cases}$$



Example: the uniform distribution

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When X has a uniform distribution, the expected value of X is:

$$\begin{aligned}\mathbb{E}[X] &= \int_{-\infty}^{\infty} xf(x)dx \\ &= \int_a^b x \frac{1}{b-a} dx \\ &= \frac{a+b}{2}.\end{aligned}$$

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When X has a uniform distribution, the expected value of X is:

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The variance of X is:

$$\begin{aligned}\text{Var}[X] &= \int_{-\infty}^{\infty} (x - \mathbb{E}[X])^2 f(x) dx \\ &= \int_a^b \left(x - \frac{a+b}{2}\right)^2 \frac{1}{b-a} dx \\ &= \frac{(b-a)^2}{12}.\end{aligned}$$

The cumulative distribution function

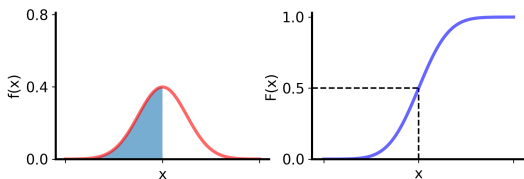
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The cumulative distribution function $F(x)$ is the area under the probability density function $f(x)$ to the left of x .



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Definition

Let X be a continuous random variable with probability density function $f(x)$, then the **cumulative distribution function** (CDF) $F(x)$ is defined as

$$\begin{aligned} F(x) &= P(X \leq x) \\ &= \int_{-\infty}^x f(t) dt. \end{aligned}$$

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Definition

Let X be a continuous random variable with probability density function $f(x)$, then the **cumulative distribution function** (CDF) $F(x)$ is defined as

$$\begin{aligned} F(x) &= P(X \leq x) \\ &= \int_{-\infty}^x f(t) dt. \end{aligned}$$

The CDF is a monotonically-increasing continuous function $F : \mathbb{R} \mapsto [0, 1]$ satisfying $\lim_{x \rightarrow -\infty} F(x) = 0$ and $\lim_{x \rightarrow \infty} F(x) = 1$.

Computing probabilities using the CDF

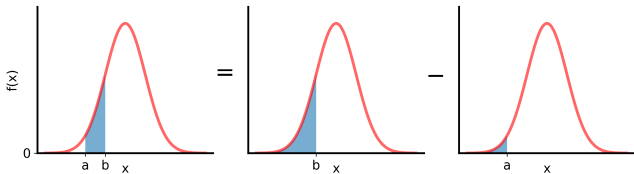
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$$P(a \leq X \leq b) = F(b) - F(a).$$



Computing probabilities using the CDF

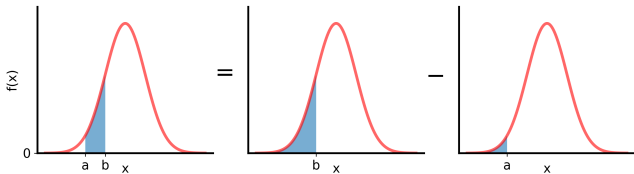
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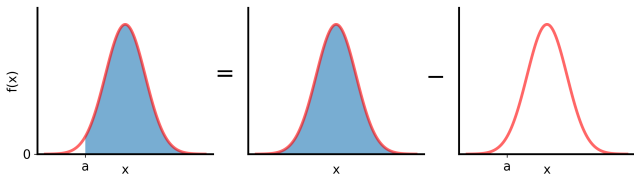
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$$P(a \leq X \leq b) = F(b) - F(a).$$



$$\begin{aligned} P(X > a) &= F(\infty) - F(a) \\ &= 1 - F(a). \end{aligned}$$



Obtaining the PDF from the CDF

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At every x at which the derivative $F'(x)$ exists, $F'(x) = f(x)$.

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At every x at which the derivative $F'(x)$ exists, $F'(x) = f(x)$.

Examples

When X has a uniform distribution, for $a < x < b$:

$$F'(x) = \frac{d}{dx} \left(\frac{x - a}{b - a} \right) = \frac{1}{b - a} = f(x)$$

Sampling using the CDF

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The inverse transform sampling algorithm can be used to sample a continuous random variable using the inverse of its cumulative distribution function.

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The inverse transform sampling algorithm can be used to sample a continuous random variable using the inverse of its cumulative distribution function.

Recall that $F : \mathbb{R} \mapsto [0, 1]$.

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The inverse transform sampling algorithm can be used to sample a continuous random variable using the inverse of its cumulative distribution function.

Recall that $F : \mathbb{R} \mapsto [0, 1]$.

To draw a sample $x \sim f(x)$:

- 1 Sample $u \sim U(0, 1)$ (using a pseudo-random number generator)
- 2 Let $x = F^{-1}(u)$

Example: the exponential distribution

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The PDF of the exponential distribution is:

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$

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The PDF of the exponential distribution is:

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$

which implies that the CDF is:

$$F(x) = 1 - e^{-\lambda x} = u,$$

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The PDF of the exponential distribution is:

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$

which implies that the CDF is:

$$F(x) = 1 - e^{-\lambda x} = u,$$

and the inverse of the CDF is:

$$F^{-1}(u) = -\frac{\log(1-u)}{\lambda} = x.$$

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The PDF of the exponential distribution is:

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$

which implies that the CDF is:

$$F(x) = 1 - e^{-\lambda x} = u,$$

and the inverse of the CDF is:

$$F^{-1}(u) = -\frac{\log(1-u)}{\lambda} = x.$$

Hence, to sample $x \sim f(x)$:

- 1 Sample $u \sim U(0, 1)$
- 2 Let $x = -\frac{\log(1-u)}{\lambda}$

The normal (Gaussian) distribution

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