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Continuous Random Variables

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Gatsby Bridging Programme 2023



Objectives

- Introduce the concept and formal definition of a continuous random variable and a probability density function.

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- Introduce the concept and formal definition of a continuous random variable and a probability density function.
- Learn how to find the probability that a continuous random variable falls in some interval $[a, b]$.

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- Introduce the concept and formal definition of a continuous random variable and a probability density function.
- Learn how to find the probability that a continuous random variable falls in some interval $[a, b]$.
- Introduce the concept and formal definition of a cumulative distribution function of a continuous random variable.

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- Learn how to find the probability that a continuous random variable falls in some interval $[a, b]$.
- Introduce the concept and formal definition of a cumulative distribution function of a continuous random variable.
- Learn how to find the cumulative distribution function of a continuous random variable from the probability density function (and vice versa).

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- Introduce the concept and formal definition of the expected value of a function of a continuous random variable.

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- Introduce the concept and formal definition of the expected value of a function of a continuous random variable.
- Learn how to find the mean and variance of a continuous random variable.

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- Introduce the concept and formal definition of the expected value of a function of a continuous random variable.
- Learn how to find the mean and variance of a continuous random variable.
- Learn how to sample a continuous random variable using the inverse transform method.

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- Introduce the concept and formal definition of a cumulative distribution function of a continuous random variable.
- Learn how to find the cumulative distribution function of a continuous random variable from the probability density function (and vice versa).
- Introduce the concept and formal definition of the expected value of a function of a continuous random variable.
- Learn how to find the mean and variance of a continuous random variable.
- Learn how to sample a continuous random variable using the inverse transform method.
- Introduce some important probability distributions for a continuous random variable (the uniform, normal and exponential distributions).

Discrete vs. continuous random variables

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Unlike discrete random variables, which can take on a finite or countable number of possible values (e.g. faces of a die or cards of a deck), continuous random variables can take on an uncountable number of possible values (e.g. all the real numbers in an interval).

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Unlike discrete random variables, which can take on a finite or countable number of possible values (e.g. faces of a die or cards of a deck), continuous random variables can take on an uncountable number of possible values (e.g. all the real numbers in an interval).

Examples

- the voltage membrane potential of a cell
- the interspike interval of a neuron
- the force generated by a muscle
- the velocity of an eye movement

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Examples

- the voltage membrane potential of a cell
- the interspike interval of a neuron
- the force generated by a muscle
- the velocity of an eye movement

Many concepts introduced for discrete random variables (e.g. probability mass functions, cumulative distributions functions) have analogs in the continuous domain (finite sums replaced by integrals).

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A random variable X is continuous if:

1. possible values comprise either a single interval on the number line (i.e. for some $a < b$, any number x between a and b is a possible value) or a union of disjoint intervals, and
2. $P(X = c) = 0$ for any number c that is a possible value of X .

Discrete probability distributions in the limit

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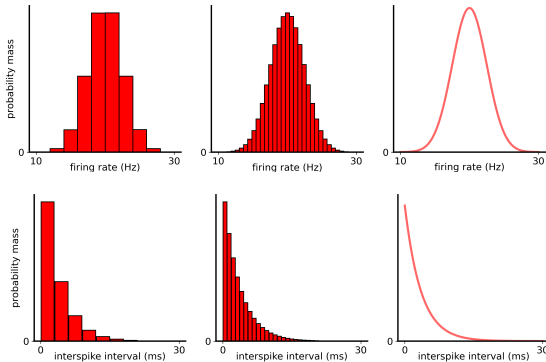
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Continuous random variables can be discretised into bins to form a discrete distribution that can be viewed as a probability histogram. As the bins become narrower, the histogram approaches a smooth curve.



The probability density function

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The **probability density function** (PDF) of a continuous random variable X is a function $f_X(x)$ defined on the interval $(-\infty, \infty)$ such that for any two numbers a and b with $a \leq b$,

$$P(a \leq X \leq b) = \int_a^b f_X(x) dx.$$

That is, the probability that X takes on a value in the interval $[a, b]$ is the area under the graph of the density function above this interval.

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$$P(a \leq X \leq b) = \int_a^b f_X(x) dx.$$

That is, the probability that X takes on a value in the interval $[a, b]$ is the area under the graph of the density function above this interval.

A valid probability density function $f_X(x)$ must have the following properties:

$$f_X(x) \geq 0 \text{ for all } x \quad (1)$$

$$\int_{-\infty}^{\infty} f_X(x) dx = 1. \quad (2)$$

Probabilities as integrals

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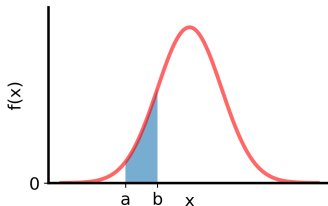
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The probability that a continuous random variable X takes on a value in the interval $[a, b]$ is given by the area under the probability density function $f_X(x)$:

$$P(a \leq X \leq b) = \int_a^b f_X(x) dx$$



Density as probability per unit of x

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If $f_X(x)$ is not the probability of x , what is it?

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If $f_X(x)$ is not the probability of x , what is it?

The probability that X will lie in an infinitesimal interval dx about x is $f_X(x)dx$:

$$\begin{aligned} P(x \leq X \leq x + dx) &= \int_x^{x+dx} f_X(t) dt \\ &= f_X(x) dx \end{aligned}$$

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If $f_X(x)$ is not the probability of x , what is it?

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$$\begin{aligned}P(x \leq X \leq x + dx) &= \int_x^{x+dx} f_X(t)dt \\ &= f_X(x)dx\end{aligned}$$

Thus, density is probability per unit of x (rate of probability accumulation):

$$\frac{P(x \leq X \leq x + dx)}{dx} = f_X(x)$$

Each possible value has zero probability

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The probability that X takes on a particular value a is 0, as

$$\begin{aligned}P(X = a) &= \int_a^a f_X(x) dx \\&= \lim_{\epsilon \rightarrow 0} \int_{a-\epsilon}^{a+\epsilon} f_X(x) dx \\&= 0.\end{aligned}$$

Each possible value has zero probability

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This implies that probabilities don't depend on interval end points:

$$P(a \leq X \leq b) = P(a < X < b) = P(a < X \leq b) = P(a \leq X < b),$$

as $P(X = a) = P(X = b) = 0$.

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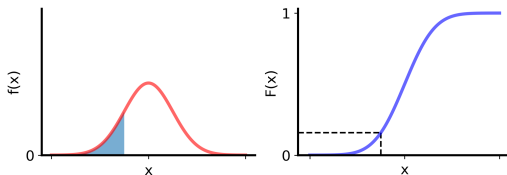
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The cumulative distribution function (CDF) $F_X(x)$ is the area under the probability density function $f_X(x)$ to the left of x .



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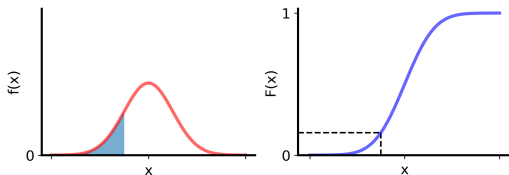
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The cumulative distribution function (CDF) $F_X(x)$ is the area under the probability density function $f_X(x)$ to the left of x .



The CDF $F_X(x)$ has the following properties:

- $F_X(x)$ is a non-decreasing (monotonic) function of x
- $\lim_{x \rightarrow -\infty} F_X(x) = 0$ and $\lim_{x \rightarrow \infty} F_X(x) = 1$.

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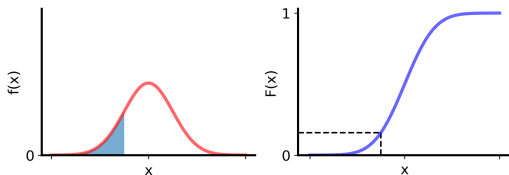
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- $F_X(x)$ is a non-decreasing (monotonic) function of x
- $\lim_{x \rightarrow -\infty} F_X(x) = 0$ and $\lim_{x \rightarrow \infty} F_X(x) = 1$.

Collectively, a. and b. imply that $F_X : \mathbb{R} \mapsto [0, 1]$.

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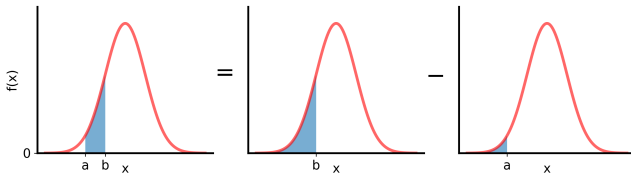
Definition

Let X be a continuous random variable with probability density function $f_X(x)$, then the **cumulative distribution function** $F_X(x)$ is defined as

$$\begin{aligned} F_X(x) &= P(X \leq x) \\ &= \int_{-\infty}^x f_X(t) dt. \end{aligned}$$

Computing probabilities using the CDF

$$P(a \leq X \leq b) = F_X(b) - F_X(a).$$



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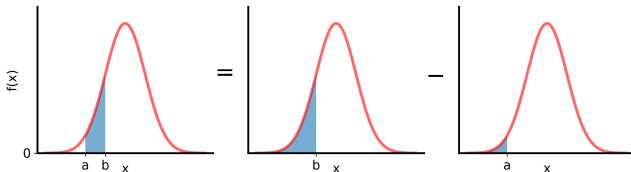
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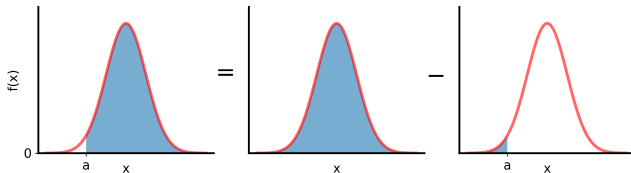
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$$P(a \leq X \leq b) = F_X(b) - F_X(a).$$



$$\begin{aligned} P(X > a) &= F_X(\infty) - F_X(a) \\ &= 1 - F_X(a). \end{aligned}$$



Obtaining the PDF from the CDF

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How would you obtain the PDF from the CDF?

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How would you obtain the PDF from the CDF?

The CDF is the integral of the PDF, and so the PDF is the derivative of the CDF.

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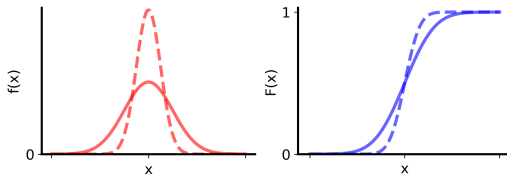
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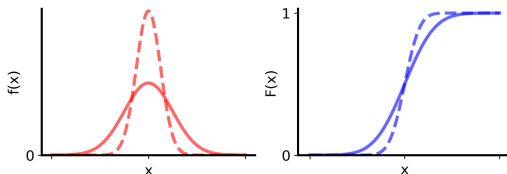
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How would you obtain the PDF from the CDF?

The CDF is the integral of the PDF, and so the PDF is the derivative of the CDF.



At every x at which the derivative $\frac{\delta F_X(x)}{\delta x}$ exists, $\frac{\delta F_X(x)}{\delta x} = f_X(x)$.

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A continuous random variable X is said to have a **uniform distribution** on the interval $[a, b]$ if the PDF of X is

$$f_X(x; a, b) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise,} \end{cases}$$

and the CDF of X is

$$F_X(x; a, b) = \begin{cases} 0 & \text{if } x < a \\ \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ 1 & \text{if } x > b. \end{cases}$$

When X has a uniform distribution on $[a, b]$, we write this as $X \sim U(a, b)$.

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When X has a uniform distribution on $[a, b]$, we write this as $X \sim U(a, b)$.

Example

When $X \sim U(a, b)$, for $a \leq x \leq b$:

$$\frac{\delta F_X(x)}{\delta x} = \frac{\delta}{\delta x} \left(\frac{x-a}{b-a} \right) = \frac{1}{b-a} = f_X(x)$$

Percentiles of a continuous distribution

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Definition

Let p be a number between 0 and 1. The **(100 p)th percentile** of the distribution of a continuous random variable X , denoted by $\eta(p)$, is defined by

$$p = F_X(\eta(p)) = \int_{-\infty}^{\eta(p)} f_X(x) dx$$

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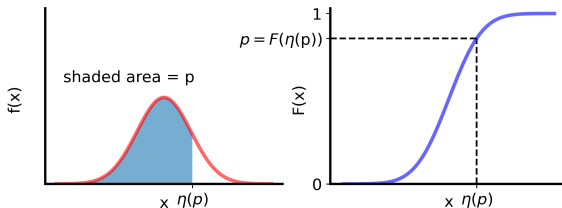
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$$p = F_X(\eta(p)) = \int_{-\infty}^{\eta(p)} f_X(x) dx$$



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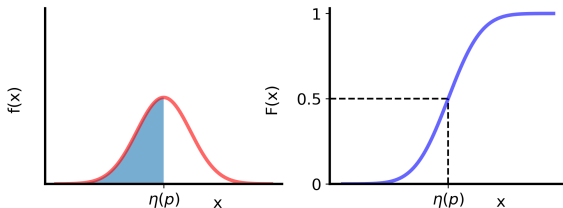
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Definition

The **median** of a continuous distribution, denoted by $\tilde{\mu}$, is the 50th percentile, so $\tilde{\mu}$ satisfies $F_X(\tilde{\mu}) = 0.5$. That is, half the area under the probability density function is to the left of $\tilde{\mu}$ and half is to the right of $\tilde{\mu}$.



Example: the exponential distribution

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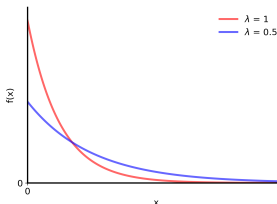
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Definition

A random variable X is said to have an **exponential distribution** on the interval $[0, \infty)$ if the PDF of X is:

$$f_X(x; \lambda) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$

where $\lambda > 0$ is a rate parameter that governs the rate of decay of $f_X(x)$. When X has an exponential distribution with parameter λ , we write $X \sim \text{Exp}(\lambda)$.



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Example

Write down the formula for the $(100p)$ th percentile of the distribution of $X \sim \text{Exp}(\lambda)$, and use it to find the median of X .

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Example

Write down the formula for the $(100p)$ th percentile of the distribution of $X \sim \text{Exp}(\lambda)$, and use it to find the median of X .

$$p = \int_0^{\eta(p)} \lambda e^{-\lambda x} dx$$

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Write down the formula for the $(100p)$ th percentile of the distribution of $X \sim \text{Exp}(\lambda)$, and use it to find the median of X .

$$\begin{aligned} p &= \int_0^{\eta(p)} \lambda e^{-\lambda x} dx \\ &= -e^{-\lambda x} \Big|_0^{\eta(p)} \end{aligned}$$

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$$\begin{aligned} p &= \int_0^{\eta(p)} \lambda e^{-\lambda x} dx \\ &= -e^{-\lambda x} \Big|_0^{\eta(p)} \\ &= 1 - e^{-\lambda \eta(p)}. \end{aligned}$$

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$$\begin{aligned} p &= \int_0^{\eta(p)} \lambda e^{-\lambda x} dx \\ &= -e^{-\lambda x} \Big|_0^{\eta(p)} \\ &= 1 - e^{-\lambda \eta(p)}. \end{aligned}$$

Therefore,

$$\eta(p) = -\frac{\log(1-p)}{\lambda}$$

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$$\begin{aligned} p &= \int_0^{\eta(p)} \lambda e^{-\lambda x} dx \\ &= -e^{-\lambda x} \Big|_0^{\eta(p)} \\ &= 1 - e^{-\lambda \eta(p)}. \end{aligned}$$

Therefore,

$$\eta(p) = -\frac{\log(1-p)}{\lambda}$$

and

$$\eta(0.5) = -\frac{\log(0.5)}{\lambda}.$$

Expected value of a function of a continuous random variable

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The expected value (mean) of a continuous random variable X is:

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} xf_X(x)dx.$$

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The expected value (mean) of a continuous random variable X is:

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx.$$

The expected value of a function $g(x)$ of X is:

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx.$$

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The expected value (mean) of a continuous random variable X is:

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx.$$

The expected value of a function $g(x)$ of X is:

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx.$$

The variance of X is:

$$\begin{aligned}\text{Var}[X] &= \int_{-\infty}^{\infty} (x - \mathbb{E}[X])^2 f_X(x) dx \\ &= \mathbb{E}[(x - \mathbb{E}[X])^2] \\ &= \mathbb{E}[X^2] - \mathbb{E}[X]^2.\end{aligned}$$

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Example

When $X \sim U(a, b)$, the expected value of X is:

$$\mathbb{E}[X] = \int_a^b x \frac{1}{b-a} dx$$

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Example

When $X \sim U(a, b)$, the expected value of X is:

$$\begin{aligned}\mathbb{E}[X] &= \int_a^b x \frac{1}{b-a} dx \\ &= \frac{b^2 - a^2}{2(b-a)}\end{aligned}$$

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Example

When $X \sim U(a, b)$, the expected value of X is:

$$\begin{aligned}\mathbb{E}[X] &= \int_a^b x \frac{1}{b-a} dx \\ &= \frac{b^2 - a^2}{2(b-a)} \\ &= \frac{a+b}{2},\end{aligned}$$

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When $X \sim U(a, b)$, the expected value of X is:

$$\begin{aligned}\mathbb{E}[X] &= \int_a^b x \frac{1}{b-a} dx \\ &= \frac{b^2 - a^2}{2(b-a)} \\ &= \frac{a+b}{2},\end{aligned}$$

and the variance of X is:

$$\text{Var}[X] = \int_a^b x^2 \frac{1}{b-a} dx - \left(\frac{a+b}{2} \right)^2$$

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When $X \sim U(a, b)$, the expected value of X is:

$$\begin{aligned}\mathbb{E}[X] &= \int_a^b x \frac{1}{b-a} dx \\ &= \frac{b^2 - a^2}{2(b-a)} \\ &= \frac{a+b}{2},\end{aligned}$$

and the variance of X is:

$$\begin{aligned}\text{Var}[X] &= \int_a^b x^2 \frac{1}{b-a} dx - \left(\frac{a+b}{2}\right)^2 \\ &= \frac{b^3 - a^3}{3(b-a)} - \left(\frac{a+b}{2}\right)^2\end{aligned}$$

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When $X \sim U(a, b)$, the expected value of X is:

$$\begin{aligned}\mathbb{E}[X] &= \int_a^b x \frac{1}{b-a} dx \\ &= \frac{b^2 - a^2}{2(b-a)} \\ &= \frac{a+b}{2},\end{aligned}$$

and the variance of X is:

$$\begin{aligned}\text{Var}[X] &= \int_a^b x^2 \frac{1}{b-a} dx - \left(\frac{a+b}{2}\right)^2 \\ &= \frac{b^3 - a^3}{3(b-a)} - \left(\frac{a+b}{2}\right)^2 \\ &= \frac{(b-a)^2}{12}.\end{aligned}$$

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Theorem

Suppose X is a continuous random variable having PDF $f_X(x)$ and CDF $F_X(x)$, and suppose further that $f_X(x) > 0$ on an interval (a, b) and $f_X(x) = 0$ otherwise. Then, if $U \sim U(0, 1)$, the random variable $Y = F_X^{-1}(U)$ has the same distribution as X , that is its CDF F_Y satisfies $F_Y(y) = F_X(y)$ for all y .

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Proof.

$$F_Y(y) = P(Y \leq y)$$

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Proof.

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(F_X^{-1}(U) \leq y) \end{aligned}$$

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Proof.

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(F_X^{-1}(U) \leq y) \\ &= P(F_X(F_X^{-1}(U)) \leq F_X(y)) \end{aligned}$$

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Proof.

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(F_X^{-1}(U) \leq y) \\ &= P(F_X(F_X^{-1}(U)) \leq F_X(y)) \\ &= P(U \leq F_X(y)) \end{aligned}$$

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Proof.

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(F_X^{-1}(U) \leq y) \\ &= P(F_X(F_X^{-1}(U)) \leq F_X(y)) \\ &= P(U \leq F_X(y)) \\ &= F_X(y). \quad \square \end{aligned}$$

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Suppose X is a continuous random variable having PDF $f_X(x)$ and CDF $F_X(x)$, and suppose further that $f_X(x) > 0$ on an interval (a, b) and $f_X(x) = 0$ otherwise. Then, if $U \sim U(0, 1)$, the random variable $Y = F_X^{-1}(U)$ has the same distribution as X , that is its CDF F_Y satisfies $F_Y(y) = F_X(y)$ for all y .

Proof.

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(F_X^{-1}(U) \leq y) \\ &= P(F_X(F_X^{-1}(U)) \leq F_X(y)) \\ &= P(U \leq F_X(y)) \\ &= F_X(y). \quad \square \end{aligned}$$

This proof makes use of the fact that

- F_X is strictly increasing on (a, b) , and
- $P(U \leq F_X(y)) = F_X(y)$ when $U \sim U(0, 1)$.

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The probability integral transform forms the basis of the inverse transform method, a procedure for sampling a continuous random variable given the inverse of its cumulative distribution function.

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The probability integral transform forms the basis of the inverse transform method, a procedure for sampling a continuous random variable given the inverse of its cumulative distribution function.

Algorithm The inverse transform method

Sample $u \sim U(0, 1)$

Let $x = F_X^{-1}(u)$

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The probability integral transform forms the basis of the inverse transform method, a procedure for sampling a continuous random variable given the inverse of its cumulative distribution function.

Algorithm The inverse transform method

Sample $u \sim U(0, 1)$

Let $x = F_X^{-1}(u)$

The inverse transform method draws samples that are distributed as the cumulative distribution function F_X .

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When $X \sim \text{Exp}(\lambda)$, for $x \geq 0$, the PDF of X is

$$f_X(x) = \lambda e^{-\lambda x},$$

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Example

When $X \sim \text{Exp}(\lambda)$, for $x \geq 0$, the PDF of X is

$$f_X(x) = \lambda e^{-\lambda x},$$

the CDF of X is:

$$F_X(x) = 1 - e^{-\lambda x} = u,$$

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Example

When $X \sim \text{Exp}(\lambda)$, for $x \geq 0$, the PDF of X is

$$f_X(x) = \lambda e^{-\lambda x},$$

the CDF of X is:

$$F_X(x) = 1 - e^{-\lambda x} = u,$$

and the inverse of the CDF of X is:

$$F_X^{-1}(u) = -\frac{\log(1-u)}{\lambda} = x.$$

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Example

When $X \sim \text{Exp}(\lambda)$, for $x \geq 0$, the PDF of X is

$$f_X(x) = \lambda e^{-\lambda x},$$

the CDF of X is:

$$F_X(x) = 1 - e^{-\lambda x} = u,$$

and the inverse of the CDF of X is:

$$F_X^{-1}(u) = -\frac{\log(1-u)}{\lambda} = x.$$

Algorithm The inverse transform method for the exponential distribution

Sample $u \sim U(0, 1)$

Let $x = -\frac{\log(1-u)}{\lambda}$

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Definition

A continuous random variable X has a **normal distribution** with parameters μ and σ (or σ^2), where $-\infty < \mu < \infty$ and $0 < \sigma$, if the probability density function of X is:

$$f_X(x; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad -\infty < x < \infty$$

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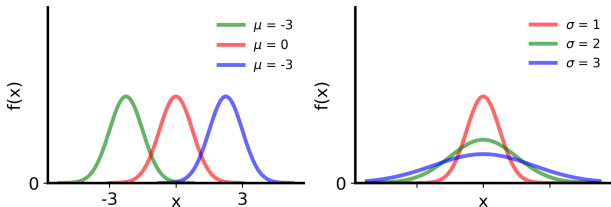
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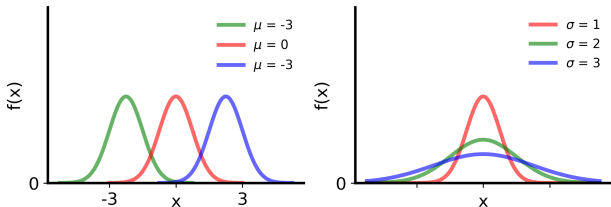
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A continuous random variable X has a **normal distribution** with parameters μ and σ (or σ^2), where $-\infty < \mu < \infty$ and $0 < \sigma$, if the probability density function of X is:

$$f_X(x; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad -\infty < x < \infty$$



The normal distribution is the most important distribution in all of probability theory. It is ubiquitous in statistical analysis (see central limit theorem).

Linear transformation of a normal random variable

Claim

If X is normally distributed with parameters μ and σ , then $Y = aX + b$ is normally distributed with parameters $a\mu + b$ and $a\sigma$.

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Claim

If X is normally distributed with parameters μ and σ , then $Y = aX + b$ is normally distributed with parameters $a\mu + b$ and $a\sigma$.

Proof.

Let F_Y denote the CDF of $Y = aX + b$, then

$$F_Y(x) = P(aX + b \leq x)$$

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Claim

If X is normally distributed with parameters μ and σ , then $Y = aX + b$ is normally distributed with parameters $a\mu + b$ and $a\sigma$.

Proof.

Let F_Y denote the CDF of $Y = aX + b$, then

$$\begin{aligned} F_Y(x) &= P(aX + b \leq x) \\ &= P(X \leq (x - b)/a) \end{aligned}$$

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If X is normally distributed with parameters μ and σ , then $Y = aX + b$ is normally distributed with parameters $a\mu + b$ and $a\sigma$.

Proof.

Let F_Y denote the CDF of $Y = aX + b$, then

$$\begin{aligned} F_Y(x) &= P(aX + b \leq x) \\ &= P(X \leq (x - b)/a) \\ &= F_X((x - b)/a), \end{aligned}$$

where F_X is the CDF of X .

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If X is normally distributed with parameters μ and σ , then $Y = aX + b$ is normally distributed with parameters $a\mu + b$ and $a\sigma$.

Proof.

Let F_Y denote the CDF of $Y = aX + b$, then

$$\begin{aligned} F_Y(x) &= P(aX + b \leq x) \\ &= P(X \leq (x - b)/a) \\ &= F_X((x - b)/a), \end{aligned}$$

where F_X is the CDF of X . By differentiation, the PDF of Y is

$$f_Y(x) = \frac{1}{a} f_X((x - b)/a)$$



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If X is normally distributed with parameters μ and σ , then $Y = aX + b$ is normally distributed with parameters $a\mu + b$ and $a\sigma$.

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Let F_Y denote the CDF of $Y = aX + b$, then

$$\begin{aligned}F_Y(x) &= P(aX + b \leq x) \\&= P(X \leq (x - b)/a) \\&= F_X((x - b)/a),\end{aligned}$$

where F_X is the CDF of X . By differentiation, the PDF of Y is

$$\begin{aligned}f_Y(x) &= \frac{1}{a} f_X((x - b)/a) \\&= \frac{1}{a\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-b}{a\sigma}\right)^2}\end{aligned}$$



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If X is normally distributed with parameters μ and σ , then $Y = aX + b$ is normally distributed with parameters $a\mu + b$ and $a\sigma$.

Proof.

Let F_Y denote the CDF of $Y = aX + b$, then

$$\begin{aligned}F_Y(x) &= P(aX + b \leq x) \\&= P(X \leq (x - b)/a) \\&= F_X((x - b)/a),\end{aligned}$$

where F_X is the CDF of X . By differentiation, the PDF of Y is

$$\begin{aligned}f_Y(x) &= \frac{1}{a} f_X((x - b)/a) \\&= \frac{1}{a\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-b-\mu}{\sigma}\right)^2} \\&= \frac{1}{a\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-(a\mu+b)}{a\sigma}\right)^2}.\end{aligned}$$



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When X is a normal random variable with parameters μ and σ , the computation of $P(a \leq X \leq b)$ requires evaluating

$$\int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx.$$

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When X is a normal random variable with parameters μ and σ , the computation of $P(a \leq X \leq b)$ requires evaluating

$$\int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx.$$

This cannot be calculated in closed form.

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When X is a normal random variable with parameters μ and σ , the computation of $P(a \leq X \leq b)$ requires evaluating

$$\int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx.$$

This cannot be calculated in closed form.

However, for $\mu = 0$ and $\sigma = 1$, this integral has been approximated and tabulated for certain values of a and b . This table can also be used to compute probabilities for any other values of μ and σ .

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When X is a normal random variable with parameters μ and σ , the computation of $P(a \leq X \leq b)$ requires evaluating

$$\int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx.$$

This cannot be calculated in closed form.

However, for $\mu = 0$ and $\sigma = 1$, this integral has been approximated and tabulated for certain values of a and b . This table can also be used to compute probabilities for any other values of μ and σ .

Definition

The normal distribution with parameter values $\mu = 0$ and $\sigma = 1$ is called the **standard normal distribution**. A random variable Z having a standard normal distribution is called a **standard normal random variable** and has probability density function given by

$$f_Z(z; 0, 1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}.$$

Tabulated CDF of the standard normal distribution

TABLE 5.1: AREA $\Phi(x)$ UNDER THE STANDARD NORMAL CURVE TO THE LEFT OF x

[illegible]

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Probabilities involving a nonstandard normal random variable are computed by standardising.

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Probabilities involving a nonstandard normal random variable are computed by standardising.

Proposition

*If X has a normal distribution with mean μ and standard deviation σ , then the **standardised variable** $Z = (X - \mu)/\sigma$ has a standard normal distribution.*

Thus

$$\begin{aligned}P(a \leq X \leq b) &= P\left(\frac{a - \mu}{\sigma} \leq Z \leq \frac{b - \mu}{\sigma}\right) \\&= \Phi\left(\frac{b - \mu}{\sigma}\right) - \Phi\left(\frac{a - \mu}{\sigma}\right),\end{aligned}$$

where Φ is the cumulative distribution function of a standard normal random variable.

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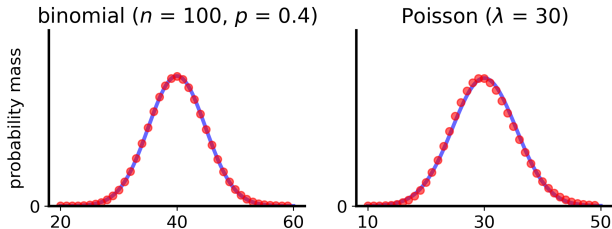
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The binomial and Poisson distributions are approximately normal for large n or large λ .



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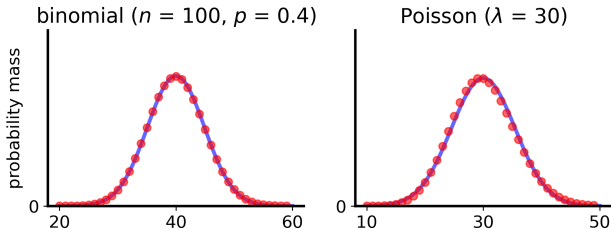
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References

The binomial and Poisson distributions are approximately normal for large n or large λ .



For the binomial with parameters n and p , $\mu = np$ and $\sigma = \sqrt{np(1-p)}$, and for the Poisson with parameter λ , $\mu = \lambda$ and $\sigma = \sqrt{\lambda}$.

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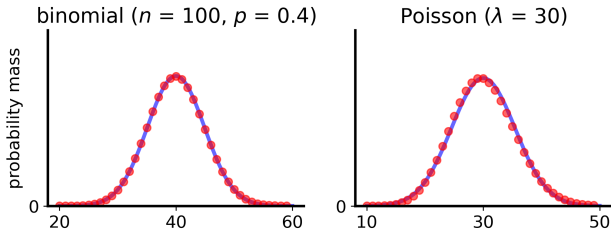
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The binomial and Poisson distributions are approximately normal for large n or large λ .



For the binomial with parameters n and p , $\mu = np$ and $\sigma = \sqrt{np(1-p)}$, and for the Poisson with parameter λ , $\mu = \lambda$ and $\sigma = \sqrt{\lambda}$.

These approximations are a great convenience, especially in conjunction with the '68% – 95% rule'.

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A random variable X satisfies $X \sim \text{Exp}(\lambda)$ for some $\lambda > 0$ if for all positive t and h the following equation is satisfied:

$$P(X > t + h | X > t) = P(X > h),$$

that is, if X is memoryless.

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$$P(X > t + h | X > t) = P(X > h),$$

that is, if X is memoryless.

Proof.

$$P(X > t + h | X > t) = \frac{P(X > t + h, X > t)}{P(X > t)}$$



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Proof.

$$\begin{aligned} P(X > t + h | X > t) &= \frac{P(X > t + h, X > t)}{P(X > t)} \\ &= \frac{P(X > t + h)}{P(X > t)} \end{aligned}$$



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that is, if X is memoryless.

Proof.

$$\begin{aligned} P(X > t + h | X > t) &= \frac{P(X > t + h, X > t)}{P(X > t)} \\ &= \frac{P(X > t + h)}{P(X > t)} \\ &= \frac{e^{-\lambda(t+h)}}{e^{-\lambda t}} \end{aligned}$$



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- Introduced the concept and formal definition of a continuous random variable and a probability density function.

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- Introduced the concept and formal definition of a continuous random variable and a probability density function.
- Learnt how to find the probability that a continuous random variable falls in some interval $[a, b]$.

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- Introduced the concept and formal definition of a continuous random variable and a probability density function.
- Learnt how to find the probability that a continuous random variable falls in some interval $[a, b]$.
- Introduced the concept and formal definition of a cumulative distribution function of a continuous random variable.

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- Introduced the concept and formal definition of a cumulative distribution function of a continuous random variable.
- Learnt how to find the cumulative distribution function of a continuous random variable from the probability density function (and vice versa).

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- Introduced the concept and formal definition of the expected value of a function of a continuous random variable.

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- Learnt how to find the cumulative distribution function of a continuous random variable from the probability density function (and vice versa).
- Introduced the concept and formal definition of the expected value of a function of a continuous random variable.
- Learnt how to find the mean and variance of a continuous random variable.

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- Introduced the concept and formal definition of the expected value of a function of a continuous random variable.
- Learnt how to find the mean and variance of a continuous random variable.
- Learnt how to sample a continuous random variable using the inverse transform method.

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- Learnt how to find the cumulative distribution function of a continuous random variable from the probability density function (and vice versa).
- Introduced the concept and formal definition of the expected value of a function of a continuous random variable.
- Learnt how to find the mean and variance of a continuous random variable.
- Learnt how to sample a continuous random variable using the inverse transform method.
- Introduced some important probability distributions for a continuous random variable (the uniform, normal and exponential distributions).

This lecture was based on the following textbooks:

1. Ross, S. (2010). A first course in probability. Pearson.
2. Kass, R. E., Eden, U. T., & Brown, E. N. (2014). Analysis of neural data (Vol. 491). New York: Springer.
3. Devore, J. L. (2011). Probability and Statistics for Engineering and the Sciences. Cengage learning.

Problems (with solutions) can be found in Chapter 5 of [A first course In probability](#).