# Problem Set: Continuous Random Variables

James Heald
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**Problem 1.** Suppose $X$ is a continuous random variable whose probability density function is given by

$$f_X(x) = \begin{cases} C(4x - 2x^2) & \text{if } 0 < x < 2 \\ 0 & \text{otherwise.} \end{cases}$$

(a) What is the value of $C$?

(b) Find $P(X > 1)$.

**Problem 2.** The amount of time in hours that a computer functions before breaking down is a continuous random variable with probability density function given by

$$f_X(x) = \begin{cases} \lambda e^{-x/100} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

What is the probability that

(a) a computer will function between 50 and 150 hours before breaking down?

(b) it will function for fewer than 100 hours?

**Problem 3.** Buses arrive at a specified stop at 15-minute intervals starting at 7 A.M. That is, they arrive at 7, 7:15, 7:30, 7:45, and so on. If a passenger arrives at the stop at a time that is uniformly distributed between 7 and 7:30, find the probability that they wait

(a) less than 5 minutes for a bus;

(b) more than 10 minutes for a bus.

**Problem 4.** In commuting to work, a professor must first get on a bus near her house and then transfer to a second bus. If the waiting time (in minutes) at each stop has a uniform distribution on the interval $[a, b]$, then it can be shown that the total waiting time X has the probability density function

$$f_X(x) = \begin{cases} \frac{x}{25} & \text{if } 0 \leq x < 5 \\ \frac{2}{5} - \frac{x}{25} & \text{if } 5 \leq x \leq 10 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Obtain an expression for the $(100p)$th percentile $\eta(p)$ [Hint: consider $0 \leq \eta(p) < 5$ and $5 \leq \eta(p) \leq 10$ separately];

(b) Find the median.

**Problem 5.** The probability density function of a continuous random variable $X$ is given by

$$f_X(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find $\mathbb{E}[e^X]$.

**Problem 6.** Suppose the probability density function of the magnitude $X$ of a dynamic load on a bridge (in Newtons) is

$$f_X(x) = \begin{cases} \frac{3x}{8} + \frac{1}{8} & \text{if } 0 \le x \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

Find

(a) $\mathbb{E}[X]$;

(b) $\mathrm{Var}[X]$.

**Problem 7.** The breakdown voltage of a randomly chosen diode is a random variable $X$ that is known to be normally distributed. Using Table 1, find the probability that a diode's breakdown voltage is within 1.5 standard deviations of its mean value.

AREA $\Phi(x)$ UNDER THE STANDARD NORMAL CURVE TO THE LEFT OF $X$

| $X$ | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
|---|---|---|---|---|---|---|---|---|---|---|
| .0 | .5000 | .5040 | .5080 | .5120 | .5160 | .5199 | .5239 | .5279 | .5319 | .5359 |
| .1 | .5398 | .5438 | .5478 | .5517 | .5557 | .5596 | .5636 | .5675 | .5714 | .5753 |
| .2 | .5793 | .5832 | .5871 | .5910 | .5948 | .5987 | .6026 | .6064 | .6103 | .6141 |
| .3 | .6179 | .6217 | .6255 | .6293 | .6331 | .6368 | .6406 | .6443 | .6480 | .6517 |
| .4 | .6554 | .6591 | .6628 | .6664 | .6700 | .6736 | .6772 | .6808 | .6844 | .6879 |
| .5 | .6915 | .6950 | .6985 | .7019 | .7054 | .7088 | .7123 | .7157 | .7190 | .7224 |
| .6 | .7257 | .7291 | .7324 | .7357 | .7389 | .7422 | .7454 | .7486 | .7517 | .7549 |
| .7 | .7580 | .7611 | .7642 | .7673 | .7704 | .7734 | .7764 | .7794 | .7823 | .7852 |
| .8 | .7881 | .7910 | .7939 | .7967 | .7995 | .8023 | .8051 | .8078 | .8106 | .8133 |
| .9 | .8159 | .8186 | .8212 | .8238 | .8264 | .8289 | .8315 | .8340 | .8365 | .8389 |
| 1.0 | .8413 | .8438 | .8461 | .8485 | .8508 | .8531 | .8554 | .8577 | .8599 | .8621 |
| 1.1 | .8643 | .8665 | .8686 | .8708 | .8729 | .8749 | .8770 | .8790 | .8810 | .8830 |
| 1.2 | .8849 | .8869 | .8888 | .8907 | .8925 | .8944 | .8962 | .8980 | .8997 | .9015 |
| 1.3 | .9032 | .9049 | .9066 | .9082 | .9099 | .9115 | .9131 | .9147 | .9162 | .9177 |
| 1.4 | .9192 | .9207 | .9222 | .9236 | .9251 | .9265 | .9279 | .9292 | .9306 | .9319 |
| 1.5 | .9332 | .9345 | .9357 | .9370 | .9382 | .9394 | .9406 | .9418 | .9429 | .9441 |
| 1.6 | .9452 | .9463 | .9474 | .9484 | .9495 | .9505 | .9515 | .9525 | .9535 | .9545 |
| 1.7 | .9554 | .9564 | .9573 | .9582 | .9591 | .9599 | .9608 | .9616 | .9625 | .9633 |
| 1.8 | .9641 | .9649 | .9656 | .9664 | .9671 | .9678 | .9686 | .9693 | .9699 | .9706 |
| 1.9 | .9713 | .9719 | .9726 | .9732 | .9738 | .9744 | .9750 | .9756 | .9761 | .9767 |
| 2.0 | .9772 | .9778 | .9783 | .9788 | .9793 | .9798 | .9803 | .9808 | .9812 | .9817 |
| 2.1 | .9821 | .9826 | .9830 | .9834 | .9838 | .9842 | .9846 | .9850 | .9854 | .9857 |
| 2.2 | .9861 | .9864 | .9868 | .9871 | .9875 | .9878 | .9881 | .9884 | .9887 | .9890 |
| 2.3 | .9893 | .9896 | .9898 | .9901 | .9904 | .9906 | .9909 | .9911 | .9913 | .9916 |
| 2.4 | .9918 | .9920 | .9922 | .9925 | .9927 | .9929 | .9931 | .9932 | .9934 | .9936 |
| 2.5 | .9938 | .9940 | .9941 | .9943 | .9945 | .9946 | .9948 | .9949 | .9951 | .9952 |
| 2.6 | .9953 | .9955 | .9956 | .9957 | .9959 | .9960 | .9961 | .9962 | .9963 | .9964 |
| 2.7 | .9965 | .9966 | .9967 | .9968 | .9969 | .9970 | .9971 | .9972 | .9973 | .9974 |
| 2.8 | .9974 | .9975 | .9976 | .9977 | .9977 | .9978 | .9979 | .9979 | .9980 | .9981 |
| 2.9 | .9981 | .9982 | .9982 | .9983 | .9984 | .9984 | .9985 | .9985 | .9986 | .9986 |
| 3.0 | .9987 | .9987 | .9987 | .9988 | .9988 | .9989 | .9989 | .9989 | .9990 | .9990 |
| 3.1 | .9990 | .9991 | .9991 | .9991 | .9992 | .9992 | .9992 | .9992 | .9993 | .9993 |
| 3.2 | .9993 | .9993 | .9994 | .9994 | .9994 | .9994 | .9994 | .9995 | .9995 | .9995 |
| 3.3 | .9995 | .9995 | .9995 | .9996 | .9996 | .9996 | .9996 | .9996 | .9996 | .9997 |
| 3.4 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9998 |

Table 1

**Problem 8.** $X$ is a normal random variable with parameters $\mu = 3$ and $\sigma^2 = 9$. Using Table 1, find

(a) $P(2 < X < 5)$;

(b) $P(X > 0)$;

(c) $P(|X - 3| > 6)$.

**Problem 9.** The continuous random variable $X$ is said to have a Weibull distribution with scale parameter $k > 0$ and scale parameter $\sigma > 0$ if the cumulative distribution function of $X$ is

$$F_X(x) = \begin{cases} 1 - e^{-(x/\sigma)^k} & \text{if } x \ge 0 \\ 0 & \text{otherwise.} \end{cases}$$

Find a function $\Phi$ such that the random variable $Y = \Phi(U)$ has the same distribution as $X$, where $U$ is a continuous random variable having a uniform distribution on the interval $[0, 1]$.

**Problem 10.** The continuous random variable $X$ is said to have a pareto distribution with scale parameter $\beta > 0$ and shape parameter $\alpha > 0$ if the probability density function of $X$ is

$$f_X(x) = \begin{cases} \frac{\alpha\beta^\alpha}{x^{\alpha+1}} & \text{if } \beta \le x < \infty \\ 0 & \text{otherwise.} \end{cases}$$

Find a function $\Phi$ such that the random variable $Y = \Phi(U)$ has the same distribution as $X$, where $U$ is a continuous random variable having a uniform distribution on the interval $[0, 1]$.

*Solution: Problem 1.*

(a) Since $f_X$ is a probability density function, we must have $\int_{-\infty}^{\infty} f_X(x)dx = 1$, implying that

$$C\int_0^2 (4x - 2x^2)dx = 1$$
$$C\left[2x^2 - \frac{2x^3}{3}\right]\Bigg|_{x=0}^{x=2} = 1,$$

and hence $C = \frac{3}{8}$.

(b) $P(X > 1) = \int_1^{\infty} f_X(x)dx = \frac{3}{8}\int_1^2 (4x - 2x^2)dx = \frac{1}{2}$.

*Solution: Problem 2.*

(a) First calculate the value of $\lambda$ that ensures that $\int_{-\infty}^{\infty} f_X(x)dx = 1$:

$$\lambda \int_0^{\infty} e^{-x/100}dx = 1$$
$$-\lambda(100)e^{-x/100}\Bigg|_{x=0}^{x=\infty} = 1$$
$$\lambda(100) = 1,$$

which implies that $\lambda = 1/100$. Hence,

$$\begin{aligned} P(50 < X < 150) &= \int_{50}^{150} \frac{1}{100}e^{-x/100}dx \\ &= -e^{-x/100}\Bigg|_{50}^{150} \\ &= e^{-1/2} - e^{-3/2} \\ &\approx 0.384 \end{aligned}$$

(b) Similarly, $P(X < 100) = \int_0^{100} \frac{1}{100}e^{-x/100}dx = -e^{-x/100}\Big|_0^{100} = 1 - e^{-1} \approx 0.633$. In other words, approximately 63.3 percent of the time, a computer will fail before registering 100 hours of use.

*Solution: Problem 3.*

Let $X$ denote the number of minutes past 7 that the passenger arrives at the stop. Since $X$ is a uniform random variable over the interval $(0, 30)$, it follows that the passenger will have to wait less than 5 minutes if (and only if) they arrive between 7:10 and 7:15 or between 7:25 and 7:30. Hence, the desired probability for part (a) is

$$P(10 < X < 15) + P(25 < X < 30) = \int_{10}^{15} \frac{1}{30}dx + \int_{25}^{30} \frac{1}{30}dx = 1/3.$$

Similarly, they would have to wait more than 10 minutes if they arrive between 7 and 7:05 or between 7:15 and 7:20, so the probability for part (b) is

$$P(0 < X < 5) + P(15 < X < 20) = \int_0^5 \frac{1}{30}dx + \int_{15}^{20} \frac{1}{30}dx = 1/3.$$

Why are the answers to (a) and (b) the same?

*Solution: Problem 4.*

(a) The $(100p)$th percentile is the value of $\eta(p)$ that satisfies $p = \int_{-\infty}^{\eta(p)} f_X(x)dx$. It is helpful to first calculate the probability that the total waiting time is less than 5 minutes:

$$\begin{aligned}
P(0 \leq X < 5) &= \int_0^5 \frac{x}{25} dx \\
&= \frac{1}{50} x^2 \bigg|_0^5 \\
&= 0.5.
\end{aligned}$$

Therefore, when $0 \leq p \leq 0.5$, we find the value of $\eta(p)$ that satisfies

$$\begin{aligned}
p &= \int_0^{\eta(p)} \frac{x}{25} dx \\
&= \frac{1}{50} x^2 \bigg|_0^{\eta(p)},
\end{aligned}$$

which is $\eta(p) = \sqrt{50p}$. When $0.5 < p \leq 1$, we find the value of $\eta(p)$ that satisfies

$$\begin{aligned}
p &= 0.5 + \int_5^{\eta(p)} \left(\frac{2}{5} - \frac{x}{25}\right) dx \\
&= 0.5 + \left[\frac{2x}{5} - \frac{x^2}{50}\right]\bigg|_5^{\eta(p)} \\
&= 0.5 + \left(\frac{2\eta(p)}{5} - \frac{\eta(p)^2}{50}\right) - \left(2 - \frac{1}{2}\right).
\end{aligned}$$

The term 0.5 is added to the right-hand side as this is the probability that the total waiting time is less than 5 minutes (calculated above). After some algebraic rearranegment, the quadratic formula can be used to solve the equation $\eta(p)^2 - 20\eta(p) + 50p + 50 = 0$ to give $\eta(p) = 10 - \sqrt{50(1-p)}$. Therefore, the expression for the $(100p)$th percentile is

$$\eta(p) = \begin{cases} \sqrt{50p} & \text{if } 0 \leq p \leq 0.5 \\ 10 - \sqrt{50(1-p)} & \text{if } 0.5 < p \leq 1. \end{cases}$$

(b) The median is $\eta(0.5) = 5$.

*Solution: Problem 5.*

The expected value of $e^X$ is

$$\begin{aligned}
\mathbb{E}[e^X] &= \int_{-\infty}^{\infty} e^X f_X(x) dx \\
&= \int_0^1 e^X dx \\
&= e^X \big|_0^1 \\
&= e - 1.
\end{aligned}$$

*Solution: Problem 6.*

(a) The expected value of $X$ is

$$\begin{aligned}\mathbb{E}[X] &= \int_0^2 x\left(\frac{3x}{8}+\frac{1}{8}\right)dx \\ &= \left[\frac{x^3}{8}+\frac{x^2}{16}\right]\Bigg|_0^2 \\ &= \frac{5}{4}.\end{aligned}$$

(b) The variance of $X$ is

$$\begin{aligned}\mathrm{Var}[X] &= \mathbb{E}[X^2]-\mathbb{E}[X]^2 \\ &= \int_0^2 x^2\left(\frac{3x}{8}+\frac{1}{8}\right)dx-\left(\frac{5}{4}\right)^2 \\ &= \left[\frac{3x^4}{32}+\frac{x^3}{24}\right]\Bigg|_0^2-\left(\frac{5}{4}\right)^2 \\ &= \frac{13}{48}.\end{aligned}$$

*Solution: Problem 7.*

This question can be answered without knowing either $\mu$ or $\sigma$, as long as the distribution is known to be normal; the answer is the same for any normal distribution:

$$\begin{aligned}P(X \text{ is within 1.5 standard deviations from the mean}) &= P(\mu-1.5\sigma \le X \le \mu+1.5\sigma) \\ &= P\left(\frac{\mu-1.5\sigma-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{\mu+1.5\sigma-\mu}{\sigma}\right) \\ &= P(-1.5 \le Z \le 1.5) \\ &= \Phi(1.5)-\Phi(-1.5) \\ &= \Phi(1.5)-[1-\Phi(1.5)] \\ &\approx 0.8664,\end{aligned}$$

where $Z=(X-\mu)/\sigma$ is a standard normal random variable with cumulative distribution function $\Phi$.

*Solution: Problem 8.*

(a)

$$\begin{aligned}P(2<X<5) &= P\left(\frac{2-\mu}{\sigma}<\frac{X-\mu}{\sigma}<\frac{5-\mu}{\sigma}\right) \\ &= P\left(\frac{2-3}{3}<\frac{X-3}{3}<\frac{5-3}{3}\right) \\ &= P\bigg(-0.33<Z<0.66\bigg) \\ &= \Phi(0.66)-\Phi(-0.33) \\ &= \Phi(0.66)-[1-\Phi(0.33)] \\ &\approx 0.3747.\end{aligned}$$

(b)

$$\begin{aligned}
P(X > 0) &= P\left(\frac{X-3}{3} > \frac{0-3}{3}\right) \\
&= P\left(Z > -1\right) \\
&= 1 - P\left(Z \leq -1\right) \\
&= 1 - \Phi(-1) \\
&= 1 - [1 - \Phi(1)] \\
&\approx 0.8413.
\end{aligned}$$

(c)

$$\begin{aligned}
P(|X-3| > 6) &= P(X - 3 > 6) + P(X - 3 < -6) \\
&= P(X > 9) + P(X < -3) \\
&= P\left(\frac{X-3}{3} > \frac{9-3}{3}\right) + P(\frac{X-3}{3} < \frac{-3-3}{3}\Big) \\
&= P(Z > 2) + P(Z < -2) \\
&= 1 - P(Z \leq 2) + P(Z < -2) \\
&= 1 - \Phi(2) + [1 - \Phi(2)] \\
&\approx 0.0456.
\end{aligned}$$

*Solution: Problem 9.*

According to the probability integral transform, $\Phi = F_X^{-1}$, the inverse of the cumulative distribution function. Therefore, to find $\Phi = F_X^{-1}$, set $U = F_X(x)$ and solve for $x$ as a function of $U$. This gives $\Phi = \sigma(-\log(1 - U))^{1/k}$.

*Solution: Problem 10.*

First find the cumulative distribution function $F_X$ and then find its inverse $F_X^{-1}$.

$$\begin{aligned}
F_X(x) &= \int_{-\infty}^{x} f_X(t)dt \\
&= \int_{-\infty}^{x} \frac{\alpha\beta^\alpha}{t^{\alpha+1}} dt \\
&= \frac{-\beta^\alpha}{t^\alpha}\bigg|_{\beta}^{x} \\
&= 1 - \left(\frac{\beta}{x}\right)^\alpha.
\end{aligned}$$

To find $\Phi = F_X^{-1}$, set $U = F_X(x)$ and solve for $x$ as a function of $U$. This gives $\Phi = \beta(1-U)^{-1/\alpha}$.