Discrete Wiener process acceleration model for tacking

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Abstract

Starting with the equations of motion of a single particle, here I derive the discrete Wiener process acceleration (DWPA) model (Bar-Shalom et al., 2004, Section 6.3.3; a linear dynamical system for tracking an object moving in one dimension). Then I extend this model for tracking an object moving in two dimensions (Section 2).

1 DWPA model in one dimension

The DWPA model for tracking the motion of an object in one dimension is a linear dynamical system

$$
\begin{aligned}
\mathbf{x}_n &= A\mathbf{x}_{n-1} + \mathbf{w}_n \quad &\text{with} \quad &\mathbf{w}_n \sim N(\mathbf{0}, Q)\\
y_n &= C\mathbf{x}_n + v_n \quad &\text{with} \quad &v_n \sim N(0, \sigma^2)
\end{aligned}
$$

where

- $\mathbf{x}_n = \left[\xi[n], \dot{\xi}[n], \ddot{\xi}[n]\right]^\intercal$ and $\xi[n], \dot{\xi}[n], \ddot{\xi}[n]$ are random variables representing the position, velocity and acceleration of the object at sample time n,
- $$A = \begin{bmatrix} 1 & T & T^2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix}$$

  and T is the sample period,
- $$Q = \gamma^2 \begin{bmatrix} \frac{1}{4}T^4 & \frac{1}{2}T^3 & \frac{1}{2}T^2 \\ \frac{1}{2}T^3 & T^2 & T \\ \frac{1}{2}T^2 & T & 1 \end{bmatrix}$$

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$$C = [1, 0, 0]$$

We next derive the forms of A and Q from the equations of motion of a particle. Matrix C simply extracts the position component from the state vector x_n.

Derivation of the form of matrices A and Q

Consider the Taylor series expansion of the position as a function of time, $\xi(t)$, up to second order (Eq. 1). Approximations of the velocity and acceleration are derived from Eq. 1 by succesive differentation (with respect to T) in Eqs. 2 and 3.

$$\xi(t+T) = \xi(t) + \dot{\xi}(t)T + \frac{\ddot{\xi}(t)}{2}T^2 \tag{1}$$

$$\dot{\xi}(t+T) = \dot{\xi}(t) + \ddot{\xi}(t)T \tag{2}$$

$$\ddot{\xi}(t+T) = \ddot{\xi}(t) \tag{3}$$

According to Eq. 3 the approximation of the acceleration, $\ddot{\xi}(t)$ is constant across all times. The DWPA model generalizes this by assumming that acceletations are constant only during each sampling period of length T, with value equal to the second derivative of the position at the start of the sampling period (i.e., $\ddot{\xi}(kT)$) plus a random value $v(k) \sim \mathcal{N}(0, \gamma^2)$ (Eq. 4).

$$\ddot{\xi}_a(t) = \ddot{\xi}(kT) + v(k) \quad t \in [kT, (k+1)T) \tag{4}$$

Replacing $\ddot{\xi}_a(t)$ by $\ddot{\xi}(t)$ in Eqs. 1, 2 and 3 and discretizingng we obtain in Eqs. 5, 6 and 7 the motion equations for the DWPA model.

$$\xi(k+1) = \xi(k) + \dot{\xi}(k)T + \frac{\ddot{\xi}(k)}{2}T^2 + \frac{v(k)}{2}T^2 \tag{5}$$

$$\dot{\xi}(k+1) = \dot{\xi}(k) + \ddot{\xi}(k)T + v(k)T \tag{6}$$

$$\ddot{\xi}(k+1) = \ddot{\xi}(k) + v(k) \tag{7}$$

Calling $x(k) = [\xi(k), \dot{\xi}(k), \ddot{\xi}(k)]^\intercal$, Eq. 8 rewrites the previous equations in matrix form.

$$\begin{aligned} x(k) &= \begin{bmatrix} 1 & T & T^2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix} x(k-1) + \begin{bmatrix} \frac{1}{2}T^2 \\ T \\ 1 \end{bmatrix} v(k) \\ &= Ax(k-1) + \Gamma v(k) \\ &= Ax(k-1) + w(k) \end{aligned} \tag{8}$$

with

$$A = \begin{bmatrix} 1 & T & T^2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Gamma = \begin{bmatrix} \frac{1}{2}T^2 \\ T \\ 1 \end{bmatrix}$$

$$w(k) = \Gamma v(k)$$

Because $v(k) \sim \mathcal{N}(0, \gamma^2)$ then $w(k)$ is also Gaussian with mean m (Eq. 9) and covariance Q (Eq. 10).

$$m = E\{w(k)\} = \Gamma E\{v(k)\} = 0 \tag{9}$$

$$\begin{aligned} Q &= E\{w(k)w(k)^\intercal\} = \Gamma E\{v(k)^2\}\Gamma^\intercal = \Gamma\gamma^2\Gamma^\intercal = \gamma^2\Gamma\Gamma^\intercal \\ &= \gamma^2 \begin{bmatrix} \frac{1}{2}T^2 \\ T \\ 1 \end{bmatrix} [\tfrac{1}{2}T^2, T, 1] = \gamma^2 \begin{bmatrix} \frac{1}{4}T^4 & \frac{1}{2}T^3 & \frac{1}{2}T^2 \\ \frac{1}{2}T^3 & T^2 & T \\ \frac{1}{2}T^2 & T & 1 \end{bmatrix} \end{aligned} \tag{10}$$

2 DWPA model in two dimensions

We extend the DWPA model from one to two dimensions considering each of the two dimensions independent of each other.

$$\tilde{\mathbf{x}}_n = \tilde{A}\mathbf{x}_{n-1} + \tilde{\mathbf{w}}_n \quad \text{with} \quad \tilde{\mathbf{w}}_n \sim N(\mathbf{0}, \tilde{Q})$$
$$\tilde{\mathbf{y}}_n = \tilde{C}\tilde{\mathbf{x}}_n + \tilde{\mathbf{v}}_n \quad \text{with} \quad \tilde{\mathbf{v}}_n \sim N(0, R)$$

where

- $\mathbf{x}_n = \left[\xi[n], \dot{\xi}[n], \ddot{\xi}[n]\right]^\intercal$ and $\xi[n], \dot{\xi}[n], \ddot{\xi}[n]$ are random variables representing the position, velocity and acceleration of the object at sample time n,
- $$A = \begin{bmatrix} 1 & T & T^2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix}$$

  and T is the sample period,
- $$Q = \gamma^2 \begin{bmatrix} \frac{1}{4}T^4 & \frac{1}{2}T^3 & \frac{1}{2}T^2 \\ \frac{1}{2}T^3 & T^2 & T \\ \frac{1}{2}T^2 & T & 1 \end{bmatrix}$$
- $$C = [1, 0, 0]$$

References

Bar-Shalom, Y., Li, X. R., and Kirubarajan, T. (2004). *Estimation with applications to tracking and navigation: theory algorithms and software*. John Wiley & Sons.