Discrete Wiener process acceleration model for tracking

Joaquin Rapela*

March 2, 2025

Abstract

Starting with the equations of motion of a single particle, here I derive the discrete Wiener process acceleration (DWPA) model (Bar-Shalom et al., 2004, Section 6.3.3), a linear dynamical system for tracking an object moving in one dimension. Then I extend this model for tracking an object moving in two dimensions (Section 2).

1 DWPA model in one dimension

The DWPA model for tracking the motion of an object in one dimension is a linear dynamical system

$$
\begin{aligned}
\mathbf{x}_n &= A\mathbf{x}_{n-1} + \mathbf{w}_n \quad \text{with} \quad \mathbf{w}_n \sim N(\mathbf{0}, Q) \quad \text{and} \quad \mathbf{x}_0 \sim N(\mathbf{m}_0, V_0) \\
y_n &= C\mathbf{x}_n + v_n \quad \text{with} \quad v_n \sim N(0, \sigma^2)
\end{aligned}
$$

where

- y_n is the position of the object at sample time n,
- $\mathbf{x}_n = \left[\xi[n], \dot{\xi}[n], \ddot{\xi}[n]\right]^\intercal$ and $\xi[n], \dot{\xi}[n], \ddot{\xi}[n]$ are random variables representing the position, velocity and acceleration of the object at sample time n,
-
$$
A = \begin{bmatrix} 1 & T & \frac{1}{2}T^2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix} \tag{1}
$$
and T is the sample period,
-
$$
Q = \gamma^2 \begin{bmatrix} \frac{1}{4}T^4 & \frac{1}{2}T^3 & \frac{1}{2}T^2 \\ \frac{1}{2}T^3 & T^2 & T \\ \frac{1}{2}T^2 & T & 1 \end{bmatrix} \tag{2}
$$

*j.rapela@ucl.ac.uk

–

$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

We next derive the forms of A and Q from the equations of motion of a particle. Matrix C simply extracts the position component from the state vector x_n.

Derivation of the form of matrices A and Q

Consider the Taylor series expansion of the position as a function of time, $\xi(t)$, up to second order (Eq. 3). Approximations of the velocity and acceleration are derived from Eq. 3 by successive differentiation (with respect to T) in Eqs. 4 and 5.

$$\xi(t+T) = \xi(t) + \dot{\xi}(t)T + \frac{\ddot{\xi}(t)}{2}T^2 \tag{3}$$
$$\dot{\xi}(t+T) = \dot{\xi}(t) + \ddot{\xi}(t)T \tag{4}$$
$$\ddot{\xi}(t+T) = \ddot{\xi}(t) \tag{5}$$

According to Eq. 5 the approximation of the acceleration, $\ddot{\xi}(t)$ is constant across all times. The DWPA model generalises this by assuming that accelerations are constant only during each sampling period of length T, with value equal to the second derivative of the position at the start of the sampling period (i.e., $\ddot{\xi}(kT)$) plus a random value $v(k) \sim \mathcal{N}(0, \gamma^2)$ (Eq. 6).

$$\ddot{\xi}_a(t) = \ddot{\xi}(kT) + v(k) \quad t \in [kT, (k+1)T) \tag{6}$$

Replacing $\ddot{\xi}_a(t)$ by $\ddot{\xi}(t)$ in Eqs. 3, 4 and 5 and discretising we obtain in Eqs. 7, 8 and 9 the motion equations for the DWPA model.

$$\xi(k+1) = \xi(k) + \dot{\xi}(k)T + \frac{\ddot{\xi}(k)}{2}T^2 + \frac{v(k)}{2}T^2 \tag{7}$$
$$\dot{\xi}(k+1) = \dot{\xi}(k) + \ddot{\xi}(k)T + v(k)T \tag{8}$$
$$\ddot{\xi}(k+1) = \ddot{\xi}(k) + v(k) \tag{9}$$

Calling $x(k) = [\xi(k), \dot{\xi}(k), \ddot{\xi}(k)]^\intercal$, Eq. 10 rewrites the previous equations in matrix form.

$$\begin{aligned} x(k) &= \begin{bmatrix} 1 & T & \frac{1}{2}T^2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix} x(k-1) + \begin{bmatrix} \frac{1}{2}T^2 \\ T \\ 1 \end{bmatrix} v(k) \\ &= Ax(k-1) + \Gamma v(k) \\ &= Ax(k-1) + w(k) \end{aligned} \tag{10}$$

with

$$A = \begin{bmatrix} 1 & T & \frac{1}{2}T^2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix} \tag{11}$$

$$\Gamma = \begin{bmatrix} \frac{1}{2}T^2 \\ T \\ 1 \end{bmatrix}$$

$$w(k) = \Gamma v(k)$$

Because $v(k) \sim \mathcal{N}(0, \gamma^2)$ then $w(k)$ is also Gaussian with mean zero and covariance Q (Eq. 2).

$$\begin{aligned} E\{w(k)\} &= \Gamma E\{v(k)\} = 0 \\ E\{w(k)w(k)^\intercal\} &= \Gamma E\{v(k)^2\}\Gamma^\intercal = \Gamma\gamma^2\Gamma^\intercal = \gamma^2\Gamma\Gamma^\intercal \\ &= \gamma^2 \begin{bmatrix} \frac{1}{2}T^2 \\ T \\ 1 \end{bmatrix} [\tfrac{1}{2}T^2, T, 1] \\ &= \gamma^2 \begin{bmatrix} \frac{1}{4}T^4 & \frac{1}{2}T^3 & \frac{1}{2}T^2 \\ \frac{1}{2}T^3 & T^2 & T \\ \frac{1}{2}T^2 & T & 1 \end{bmatrix} = Q \end{aligned} \tag{12}$$

with $Q = \gamma^2 Q_e$ and

$$Q_e = \begin{bmatrix} \frac{1}{4}T^4 & \frac{1}{2}T^3 & \frac{1}{2}T^2 \\ \frac{1}{2}T^3 & T^2 & T \\ \frac{1}{2}T^2 & T & 1 \end{bmatrix} \tag{13}$$

2 DWPA model in two dimensions

We extend the DWPA model from one to two dimensions considering each of the two dimensions independent of each other. This independence yields the following linear dynamical system.

$$\tilde{\mathbf{x}}_n = \tilde{A}\tilde{\mathbf{x}}_{n-1} + \tilde{\mathbf{w}}_n \quad \text{with} \quad \tilde{\mathbf{w}}_n \sim N(\mathbf{0}, \tilde{Q}) \quad \text{and} \quad \tilde{\mathbf{x}}_0 \sim N(\tilde{\mathbf{m}}_0, \tilde{V}_0) \tag{14}$$

$$\tilde{\mathbf{y}}_n = \tilde{C}\tilde{\mathbf{x}}_n + \tilde{\mathbf{v}}_n \quad \text{with} \quad \tilde{\mathbf{v}}_n \sim N(0, \tilde{R}) \tag{15}$$

where

- $\tilde{\mathbf{x}}_n = \left[\xi_1[n], \dot{\xi}_1[n], \ddot{\xi}_1[n], \xi_2[n], \dot{\xi}_2[n], \ddot{\xi}_2[n]\right]^\intercal$ and $\xi_i[n], \dot{\xi}_i[n], \ddot{\xi}_i[n]$ are random variables representing the position, velocity and acceleration of the object along dimension i at sample time n,

– $\tilde{\mathbf{y}}_n = [y_{n1}, y_{n2}]^\intercal$ is the measured position of the object at sample time n,

–

$$\tilde{A} = \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & A \end{bmatrix}$$

and A is given in Eq. 1

–

$$\tilde{Q} = \begin{bmatrix} \gamma_1^2 Q_e & \mathbf{0} \\ \mathbf{0} & \gamma_2^2 Q_e \end{bmatrix} \tag{16}$$

and Q_e is given in Eq. 13

–

$$\tilde{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

–

$$\tilde{R} = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} \tag{17}$$

3 DWPA model in two dimensions and a circular variable

We add a circular variable θ to the DWPA model in two dimensions. We assume that θ evolves with a constant angular velocity ω:

$$\theta(t) = \omega t$$

Due to their singularity, circular variables are difficult to model with linear dynamical systems. To overcome this problem we will model $\cos\theta$ and $\sin\theta$. Then

$$\frac{\partial}{\partial t}\cos\theta(t) = -\omega\sin\theta(t) \tag{18}$$

$$\frac{\partial}{\partial t}\sin\theta(t) = \omega\cos\theta(t) \tag{19}$$

When mice move at high velocity, their head orientation tends to align with the mouse velocity, or

$$\cos\theta(t) = \frac{v_x(t)}{v(t)}$$

$$\sin\theta(t) = \frac{v_y(t)}{v(t)}$$

with $v_x(t), v_y(t)$ the x- and y-components of $\mathbf{v}(t)$, and $v(t)$ the magnitude of the velocity vector; i.e., $v(t) = |\mathbf{v}(t)| = \sqrt{v_x^2(t) + v_y^2(t)}$.

Next we incorporate these constraints into Eqs. 18 and 19

$$\frac{\partial}{\partial t}\cos\theta(t) = -\omega \sin\theta(t) + \alpha\left(v_x(t) - v(t)\cos\theta(t)\right) \tag{20}$$
$$\frac{\partial}{\partial t}\sin\theta(t) = \omega \cos\theta(t) + \alpha\left(v_y(t) - v(t)\sin\theta(t)\right) \tag{21}$$

We now discretise Eqs. 20 and 21, with the Euler method, and model the angular velocity, ω, as a random walk with drift

$$\cos\theta[n+1] = (1 - \alpha v[n]T)\cos\theta[n] - w[n]T\sin\theta[n] + \alpha T\ v_x[n] \tag{22}$$
$$\sin\theta[n+1] = (1 - \alpha v[n]T)\sin\theta[n] + w[n]T\cos\theta[n] + \alpha T\ v_y[n] \tag{23}$$
$$\omega[n+1] = \omega[k] \tag{24}$$

With the addition of the circular variable, the components of Eqs. 14 and 15 become:

- $\tilde{\mathbf{x}}_n = \left[\xi_1[n], \dot{\xi}_1[n], \ddot{\xi}_1[n], \xi_2[n], \dot{\xi}_2[n], \ddot{\xi}_2[n], \cos\theta[n], \sin\theta[n], \omega[n]\right]^\intercal$,
- $\tilde{\mathbf{y}}_n = [y_{n1}, y_{n2}, \cos_m[n], \sin_m[n]]^\intercal$, with $[y_{n1}, y_{n2}]$ the measured position of the object at time n and $[\cos_m[n], \sin_m[n]]$ the sine and cosine of the measured orientation at time n.
-
$$\tilde{A} = \begin{bmatrix} 1 & T & \frac{1}{2}T^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & T & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & T & \frac{1}{2}T^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \alpha T & 0 & 0 & 0 & 0 & 1-\alpha v[n]T & -\omega[n]T & 0 \\ 0 & 0 & 0 & 0 & \alpha T & 0 & \omega[n]T & 1-\alpha v[n]T & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$
-
$$\tilde{Q} = \begin{bmatrix} \gamma_1^2 Q_e & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \gamma_2^2 Q_e & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \gamma_{cos}^2 & 0 & 0 \\ \mathbf{0} & \mathbf{0} & 0 & \gamma_{sin}^2 & 0 \\ \mathbf{0} & \mathbf{0} & 0 & 0 & \gamma_\omega^2 \end{bmatrix}$$

  and Q_e is given in Eq. 13

–

$$\tilde{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

–

$$\tilde{R} = \begin{bmatrix} \sigma_1^2 & 0 & 0 & 0 \\ 0 & \sigma_2^2 & 0 & 0 \\ 0 & 0 & \sigma_{cos}^2 & 0 \\ 0 & 0 & 0 & \sigma_{sin}^2 \end{bmatrix} \tag{25}$$

References

Bar-Shalom, Y., Li, X. R., and Kirubarajan, T. (2004). *Estimation with applications to tracking and navigation: theory algorithms and software*. John Wiley & Sons.