Discrete Wiener process acceleration model for tacking

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To motivate the discrete Wiener process acceleration (DWPA) model consider the Taylor series expansion of the position as a function of time, $\xi(t)$, up to second order (Eq. 1). Approximations of the derivative and accelerations are derived from Eq. 1 by succesive differentation (with respect to T) in Eqs. 2 and 3.

$$\xi(t+T) = \xi(t) + \dot{\xi}(t)T + \frac{\ddot{\xi}(t)}{2}T^2 \tag{1}$$
$$\dot{\xi}(t+T) = \dot{\xi}(t) + \ddot{\xi}(t)T \tag{2}$$
$$\ddot{\xi}(t+T) = \ddot{\xi}(t) \tag{3}$$

According to Eq. 3 the approximation of the acceleration, $\ddot{\xi}(t)$ is constant across all times. The DWPA model generalizes this by assumming that acceletations are constant only during each sampling period of length T, with value equal to the second derivative of the position at the start of the sampling period (i.e., $\ddot{\xi}(kT)$) plus a random value $v(k) \sim \mathcal{N}(0, \sigma^2)$ (Eq. 4).

$$\ddot{\xi}_a(t) = \ddot{\xi}(kT) + v(k) \quad t \in [kT, (k+1)T) \tag{4}$$

Replacing $\ddot{\xi}_a(t)$ by $\ddot{\xi}(t)$ in Eqs. 1, 2 and 3 and discretizingng we obtain in Eqs. 5, 6 and 7 the motion equations for the DWPA model.

$$\xi(k+1) = \xi(k) + \dot{\xi}(k)T + \frac{\ddot{\xi}(k)}{2}T^2 + \frac{v(k)}{2}T^2 \tag{5}$$
$$\dot{\xi}(k+1) = \dot{\xi}(k) + \ddot{\xi}(k)T + v(t)T \tag{6}$$
$$\ddot{\xi}(k+1) = \ddot{\xi}(k) + v(t) \tag{7}$$

Calling $x(k) = [\xi(k), \dot{\xi}(k), \ddot{\xi}(k)]^\intercal$, Eq. 8 rewrites the previous equations in matrix form.

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$$
\begin{aligned}
x(k) &= \begin{bmatrix} 1 & T & T^2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix} x(k-1) + \begin{bmatrix} \frac{1}{2}T^2 \\ T \\ 1 \end{bmatrix} v(k) \\
&= Fx(k-1) + \Gamma v(k) \\
&= Fx(k-1) + w(k) \qquad (8)
\end{aligned}
$$

with

$$
\begin{aligned}
F &= \begin{bmatrix} 1 & T & T^2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix} \\
\Gamma &= \begin{bmatrix} \frac{1}{2}T^2 \\ T \\ 1 \end{bmatrix} \\
w(k) &= \Gamma v(k)
\end{aligned}
$$

Because $v(k) \sim \mathcal{N}(0, \sigma^2)$ then $w(k)$ is also Gaussian with mean m (Eq. 9) and covariance Q (Eq. 10).

$$
\begin{aligned}
m &= E\{w(k)\} = \Gamma E\{v(k)\} = 0 \qquad (9) \\
Q &= E\{w(k)w(k)^\intercal\} = \Gamma E\{v(k)^2\}\Gamma^\intercal = \Gamma\sigma^2\Gamma^\intercal = \sigma^2\Gamma\Gamma^\intercal \\
&= \sigma^2 \begin{bmatrix} \frac{1}{2}T^2 \\ T \\ 1 \end{bmatrix} [\tfrac{1}{2}T^2, T, 1] = \sigma^2 \begin{bmatrix} \frac{1}{4}T^4 & \frac{1}{2}T^3 & \frac{1}{2}T^2 \\ \frac{1}{2}T^3 & T^2 & T \\ \frac{1}{2}T^2 & T & 1 \end{bmatrix} \qquad (10)
\end{aligned}
$$