

Discrete Wiener process acceleration model for tacking

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To motivate the discrete Wiener process acceleration (DWPA) model consider the Taylor series expansion of the position as a function of time, $\xi(t)$, up to second order (Eq. 1). Approximations of the derivative and accelerations are derived from Eq. 1 by successive differentiation (with respect to T) in Eqs. 2 and 3.

$$\xi(t+T) = \xi(t) + \dot{\xi}(t)T + \frac{\ddot{\xi}(t)}{2}T^2 \quad (1)$$

$$\dot{\xi}(t+T) = \dot{\xi}(t) + \ddot{\xi}(t)T \quad (2)$$

$$\ddot{\xi}(t+T) = \ddot{\xi}(t) \quad (3)$$

According to Eq. 3 the approximation of the acceleration, $\ddot{\xi}(t)$ is constant across all times. The DWPA model generalizes this by assuming that accelerations are constant only during each sampling period of length T , with value equal to the second derivative of the position at the start of the sampling period (i.e., $\ddot{\xi}(kT)$) plus a random value $v(k) \sim \mathcal{N}(0, \sigma^2)$ (Eq. 4).

$$\ddot{\xi}_a(t) = \ddot{\xi}(kT) + v(k) \quad t \in [kT, (k+1)T) \quad (4)$$

Replacing $\ddot{\xi}_a(t)$ by $\ddot{\xi}(t)$ in Eqs. 1, 2 and 3 and discretizingng we obtain in Eqs. 5, 6 and 7 the motion equations for the DWPA model.

$$\xi(k+1) = \xi(k) + \dot{\xi}(k)T + \frac{\ddot{\xi}(k)}{2}T^2 + \frac{v(k)}{2}T^2 \quad (5)$$

$$\dot{\xi}(k+1) = \dot{\xi}(k) + \ddot{\xi}(k)T + v(k)T \quad (6)$$

$$\ddot{\xi}(k+1) = \ddot{\xi}(k) + v(k) \quad (7)$$

Calling $x(k) = [\xi(k), \dot{\xi}(k), \ddot{\xi}(k)]^\top$, Eq. 8 rewrites the previous equations in matrix form.

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$$\begin{aligned}
x(k) &= \begin{bmatrix} 1 & T & T^2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix} x(k-1) + \begin{bmatrix} \frac{1}{2}T^2 \\ T \\ 1 \end{bmatrix} v(k) \\
&= Fx(k-1) + \Gamma v(k) \\
&= Fx(k-1) + w(k)
\end{aligned} \tag{8}$$

with

$$\begin{aligned}
F &= \begin{bmatrix} 1 & T & T^2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix} \\
\Gamma &= \begin{bmatrix} \frac{1}{2}T^2 \\ T \\ 1 \end{bmatrix} \\
w(k) &= \Gamma v(k)
\end{aligned}$$

Because $v(k) \sim \mathcal{N}(0, \sigma^2)$ then $w(k)$ is also Gaussian with mean m (Eq. 9) and covariance Q (Eq. 10).

$$m = E\{w(k)\} = \Gamma E\{v(k)\} = 0 \tag{9}$$

$$\begin{aligned}
Q &= E\{w(k)w(k)^\top\} = \Gamma E\{v(k)^2\} \Gamma^\top = \Gamma \sigma^2 \Gamma^\top = \sigma^2 \Gamma \Gamma^\top \\
&= \sigma^2 \begin{bmatrix} \frac{1}{2}T^2 \\ T \\ 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2}T^2, T, 1 \end{bmatrix} = \sigma^2 \begin{bmatrix} \frac{1}{4}T^4 & \frac{1}{2}T^3 & \frac{1}{2}T^2 \\ \frac{1}{2}T^3 & T^2 & T \\ \frac{1}{2}T^2 & T & 1 \end{bmatrix}
\end{aligned} \tag{10}$$