Report worksheet 2

Joaquín Rapela

February 12, 2023

Appendix A Fourier transform of a continuous periodic signal

We prove in Lemma 1 that the Fourier transform of a continuous periodic signal is a sum of scaled delta functions at multiples of the frequency of this signal.

Definition 1 (continuous signal). *A continuous signal $x(t)$ is periodic if and only if there exists a period $T > 0$ such that*

$$x(t) = x(t+T), \forall t \in \Re \tag{1}$$

Lemma 1 (Fourier transform of a periodic signal). *If $x(t)$ is periodic, with period T, then*

$$\mathcal{FT}\{x(t)\}(j\Omega) = 2\pi \sum_{k=-\infty}^{\infty} X^S[k]\, \delta\left(\Omega - \frac{2\pi k}{T}\right) \tag{2}$$

with $X^S[k]$ the Fourier series coefficient at frequency k (Eq. 4).

Proof. Because $x(t)$ is a periodic signal, it admits a Fourier series representation (Porat, 1997, Section 2.3)

$$x(t) = \sum_{k=-\infty}^{\infty} X^S[k] \exp\left(\frac{j2\pi kt}{T}\right) \tag{3}$$

with

$$X^S[k] = \frac{1}{T}\int_{-T/2}^{T/2} x(t) \exp\left(-\frac{j2\pi kt}{T}\right) \tag{4}$$

By the linearity of the Fourier transform (Porat, 1997, Eq. 2.4), from Eq. 3, we have

$$\mathcal{FT}\{x(t)\}(j\Omega) = \sum_{k=-\infty}^{\infty} X^S[k]\, \mathcal{FT}\left\{\exp\left(\frac{j2\pi kt}{T}\right)\right\}(j\Omega) \tag{5}$$

We next compute the Fourier transform of the exponential in the right hand side of Eq. 5

$$\mathcal{FT}\left\{\exp\left(\frac{j2\pi kt}{T}\right)\right\}(j\Omega) = \mathcal{FT}\left\{1\ \exp\left(\frac{j2\pi kt}{T}\right)\right\}(j\Omega) \tag{6}$$

$$= \mathcal{FT}\left\{1\right\}\left(j\left(\Omega - \frac{2\pi kt}{T}\right)\right) \tag{7}$$

$$= 2\pi\ \delta\left(\Omega - \frac{2\pi k}{T}\right) \tag{8}$$

Notes:

1. Eq. 7 follows from Eq. 6 by the the frequency shift property of the Fourier transform[1].
2. Eq. 8 follows from Eq. 7 by the Fourier transform of the DC function (Lemma 2).

Replacing Eq. 8 into Eq. 5 yields Eq. 2.

□

Lemma 2 (Fourier transform of the DC function)**.**

$$\mathcal{FT}\{1\}(j\Omega) = 2\pi\ \delta(\Omega) \tag{9}$$

Proof. We start by computing the Fourier transform of the delta function.

$$\mathcal{FT}\{\delta(t)\}(j\Omega) = \int_{-\infty}^{\infty} \delta(t)\exp\left(-j\Omega t\right)\ dt = \exp\left(-j\Omega t\right)\big|_{t=0} = 1 \tag{10}$$

Then by the duality property of the Fourier transform (Lemma 3) we have

$$\mathcal{FT}\{1\}(j\Omega) = 2\pi\ \delta(-\Omega) = 2\pi\ \delta(\Omega) \tag{11}$$

Notes:

1. The last equality in Eq. 11 holds because the delta function is even.

□

Lemma 3 (Duality of the Fourier transform)**.** *Let $x(t)$ be a signal and $X(j\Omega)$ be its Fourier transform, then*

$$\mathcal{FT}\{X(jt)\}(j\Omega) = 2\pi\ x(-\Omega) \tag{12}$$

[1] $y(t) = e^{j\Omega_0 t}x(t) \leftrightarrow Y(j\Omega) = X\left(j(\Omega - \Omega_0)\right)$, Porat (1997, Section 2.1)

Proof. If $x(t)$ is a signal, with real or complex values, and $X(j\Omega)$ is its Fourier transform, then they are related by the following equations (Porat, 1997, Section 2.1)

$$X(j\Omega) = \mathcal{FT}\{x(t)\}(j\Omega) = \int_{-\infty}^{\infty} x(t) \exp\left(-j\Omega t\right) dt \tag{13}$$

$$x(t) = \mathcal{IFT}\{X(j\Omega)\}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) \exp\left(j\Omega t\right) d\Omega \tag{14}$$

Then

$$\begin{aligned} \mathcal{FT}\left\{X(jt)\right\}(j\Omega) &= \int_{-\infty}^{\infty} X(jt) \exp\left(-j\Omega t\right) dt & (15)\\ &= 2\pi \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} X(jt) \exp\left(jt(-\Omega)\right) dt \right) & (16)\\ &= 2\pi \; x(-\Omega) & (17) \end{aligned}$$

Notes:

1. in Eq. 15 we applied the Fourier transform (Eq. 13) to the complex signal $X(jt)$
2. in Eq. 17 we used the inverse Fourier transform (Eq. 14) with the change of variables Ω in Eq. 14 to t in Eq. 16 and t in Eq. 14 to $-\Omega$ in Eq. 16.

□

References

Porat, B. (1997). *A course in digital signal processing.* John Wiley & Sons, Inc.